COMPUTER-ASSISTED AND COMPUTER-GENERATED RESEARCH IN COMBINATORIAL GAMES AND PATTERN-AVOIDANCE

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ABSTRACT

COMPUTER-ASSISTED AND COMPUTER-GENERATED RESEARCH IN COMBINATORIAL GAMES AND PATTERN-AVOIDANCE

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The combinatorial game Chomp was studied and some computer-proved results on the P-positions and the values of the Sprague-Grundy function are listed. In particular, a formula was proved for a specify type of P-positions, which extended the formula given in the modern classic *Winning Ways for Your Mathematical Plays* by Berlekemp et al. For Chomp positions with k rows, the periodic properties of the P-positions with the top $k-2$ rows fixed, and those of the values of the Sprague-Grundy function with the top $k-1$ rows fixed were also studied. It was further conjectured that those properties are true for any positions with the corresponding bottom rows fixed, which was later proved by Steve Byrnes.

Fraenkel proposed two conjectures on the multi-heap Wythoff's game. A partial proof to the conjectures was given when the smallest heap has up to 10 pieces. Furthermore, by studying the properties of (special) Wythoff's sequence, the two conjectures are proved to be equivalent, and an easy way to predict the behavior of the sequence was also given.

The "connective constant" for ternary square-free words was proved to be at least $110^{1/42} \approx 1.118419\ldots$, the best known yet.

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I want to thank my advisor Prof. Doron Zeilberger whose patience and encouragement made this possible.

To my family,

whose support

helped me through thick and thin.

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CHAPTER 1

INTRODUCTION

1.1 Combinatorial Games

A combinatorial game is a game, usually but possibly with some exceptions, that satisfies the following conditions:

- The game is played by *two players*.
- There are usually finite many *positions*, and one of them is the *starting position*, i.e., where the game starts.
- The *rules* clearly define the *moves* that each player can make from any given position. The result of each move is one of the positions we have just defined.
- The players move alternately.
- A player wins a game by making the last move in *normal play*, and loses by making such a move in *misère play*.
- In general, the game always ends because some player cannot make a move, and it cannot be drawn by endless moves.

- Both players have *complete* information about the game, i.e., with infinite intelligence, either player should be able to see all the ramifications from a given position, and therefore there is no bluffing.

- There are no chance moves, e.g., no dealing of cards, no rolling of dice.

Here are some of the games that we are familiar with:

Snakes-and-Ladders has complete information, but uses dice.

Scissors-Paper-Stone has no chance move, but the players do not have complete information about the game, namely, the disposition of their opponent's fingers.

Tic-Tac-Toe is not a combinatorial game because the last player who cannot make a move is not necessarily the loser.

Chess also fails the same rule because the game can be tied by stalemate (in which the last player does not lose), or drawn by infinite play (in which the game does not end).

Monopoly fails to be a combinatorial game because of several reasons. It uses dice; Players do not know the arrangement of the cards; and the game can theoretically go on forever.

Poker and *Bridge* are games in which the players do not have complete information of the cards, and so bluffing is a big part of the games.

Tennis is not a combinatorial game because it is hard to define positions and moves.

Solitaire is not either because it is a one-player game and the arrangement of the cards is random, which is sometimes unknown to the player.

Nim is a game played with piles of beans where players take turns removing any positive number of beans from any single pile. Nim satisfies all the conditions listed above, and so it is a combinatorial game.

Another combinatorial game is *Nimble*, which is played with coins on a strip of squares. Players take turns sliding just one coin to the left. A coin can jump onto or over other coins, even clear off the strip. Any number of coins can be stacked on a square. The last player wins.

N. G. de Bruijn's *Silver Dollar Game* is played liked the Nimble except the following rules: players cannot slide coins onto or over other coins; there is a coin worth much more than the sum of all the others, and whoever slides the coin off the strip loses because he must hand the coin to his opponent.

Domineering, also called Crosscram and Dominoes, was first considered by Göran Andersson. Left and Right take turns in placing dominoes on a checker-board. Left orients his dominoes vertically and Right horizontally. Each domino covers exactly two squares of the board and no overlap is permitted. The first player who cannot place his next domino loses.

1.2 Impartial Games and the Sprague-Grundy Function

An *impartial* game is a game whose positions are available to both players, regardless of whose turn it is to move. Otherwise, it is called *partizan*. Nim, Nimble and Silver Dollar Game are all impartial games, while Domineering is a partizan game.

Here we only consider impartial games with finitely many positions (so the game will eventually end) in normal play convention, and we call the two players *Left* and *Right*.

We call an *option* of a position to be a possible move from the given position, and without confusion, we also identify a move with its resulting position.

Each game G can be inductively defined as

$$G = \{G^L \mid G^R\},$$

where G^L and G^R are the options for Left and Right from the starting position, respectively. For impartial games, since all the options are available to either player, G^L is always the same as G^R. Thus we can also write

$$G = \{G^L\} = \{G^R\}.$$

For the game of Nim, if we write $*n$ for the game of a pile of n beans, we have

$$*0 = \{\},$$
$$*1 = \{*0\},$$
$$*2 = \{*0, *1\}, \ldots,$$
$$*n = \{*0, *1, \ldots, *(n-1)\}.$$

We define 0 as the set of all games where the first player always loses regardless of which move he makes. For such a game G, we write, by convention, $G = 0$. So $*0 = 0$. We define the sum, or addition, of games as follows. To play the sum of arbitrary games $G_1, \ldots, G_k$ is to play a game G of which the games $\{G_i\}_{1 \leq i \leq k}$ are independent components. Once it is his turn, a player picks one the components and makes a legal move for him in that game. He loses by not able to make a move in any of the components. We denote this game G by $G_1 + \cdots + G_k$. For example, $*n_1 + \cdots + *n_k$ is the game of k piles of beans with $n_1, \ldots, n_k$ beans in the piles, where players can remove any number of beans from a single pile. This game is also called a k-heap Nim game. Since the order of the games $G_1, \ldots, G_k$ is irrelevant to the final outcome of G, the addition is clearly commutative and associative. Two impartial games G_1 and G_2 are said to be equal if $G_1 + G_2 = 0$, i.e., the first player will lose the sum of the two games. In such a case, we will use the notation $G_1 = G_2$. In this sense, 0 can also be seen as *any* game in which the first player loses, or more precisely as *the* game in which neither player can make a move. Without confusion, we will refer any of such games as a "0 game" or simply 0.

It is easy to see that a player is about to lose if he is to play from a position of two identical piles of beans: no matter what move he makes, his opponent can make the same move on the other pile, therefore he will be the first one *not* to be able to make a move. This is the same as saying $*n + *n = 0$, i.e., $*n$ is its own negative. We have also proved $*m + *n \neq 0$ if $m \neq n$, since the first player can win by removing $|m - n|$ beans from the larger pile to create two equal piles. However the winning strategies for the two separate games $*m$ and $*n$ are the same: taking everything away. So having the same winning strategies does not always imply the two games are equal.

We now claim that $*1 + *2 = *3$. The game $*1 + *2$ is the game having two piles of beans with 1 and 2 beans in the piles. Both players have the same options from the game: either to remove 1 bean from the first pile, or to remove 1 or 2 beans from the second. The moves leave us with the positions $*2$, $*1 + *1 = 0$ and $*1$, respectively. So by the inductive definition $*1 + *2 = \{*0, *1, *2\} = *3$.

A more intuitive way of proving this is that we can first prove that the first player loses when playing the game with three piles of beans with 1, 2, and 3 beans respectively, or equivalently, $*1 + *2 + *3 = 0$. Since $*3$ is its own negative, $*1 + *2 = *3$. Similarly, we have $*1 + *3 = *2$ and $*2 + *3 = *1$.

With enough time and patience, we can prove that $*1 + *4 + *5 = 0$ and $*2 + *4 + *6 = 0$ by playing the corresponding three-pile games. Therefore we can deduce another set of equations

$$*3 + *5 = *2 + *1 + *5 = *2 + *4 = *6,$$

without playing the tedious, or at least it will become so after several experiments, games.

In fact, people have realized that the sum of any two of such nim heaps is just another nim heap, and the result is the same as the XOR binary operation, namely, binary addition without carry. For example,

$$*5 + *7 = 101 + 111 = 010 = *2.$$

Table 1.1 lists some values of the nim addition.

On the other hand, if we study $*1 + *5$ by listing all of its options, we have

$$*1 + *5 = \{*5, *1, *1 + *1, *1 + *2, *1 + *3, *1 + *4\} = \{0, *1, *2, *3, *5\},$$

which, as we have seen, should be $*4$. Observe that 4 is the least non-negative number *not* in the last set. This leads us to the definition of *mex*, which stands

	0	1	2	3	4	5	6	7	8
0	0	1	2	3	4	5	6	7	8
1	1	0	3	2	5	4	7	6	9
2	2	3	0	1	6	7	4	5	10
3	3	2	1	0	7	6	5	4	11
4	4	5	6	7	0	1	2	3	12
5	5	4	7	6	1	0	3	2	13
6	6	7	4	5	2	3	0	1	14
7	7	6	5	4	3	2	1	0	15
8	8	9	10	11	12	13	14	15	0

Table 1.1: A Table of Nim-Addition

for **M**inimal **Ex**clusive. For a set of non-negative integers S, $\text{mex}(S)$ equals to the least non-negative number *not* in S, i.e., $\min(\mathbb{Z}_{\geq 0} - S)$.

R. P. Sprague in 1936 and P. M. Grundy in 1939 independently discovered the following theorem, as quoted from WW:

> Every impartial game is just a bogus Nim-heap. The Mex Rule gives the size of the heap for G as the least possible number that is not the size of any of the heaps corresponding to the options of G. [3]

The theorem states two important things: it assures the existence and uniqueness of the equivalent Nim-heap; and it gives an intuitive way of calculating the size of the Nim-heap. However the calculation is inductive, or recursive, which insists that you have the previous knowledge of the corresponding values of all possible positions reachable from the current position. Such a task usually takes exponential time with respect to the size of the input, and is not always useful in practice.

For each impartial game, people define the value of its *Sprague-Grundy function*, or simply the *Grundy function*, as the size of the corresponding Nim-heap mentioned in the theory above. Sometimes people also call it the *nim value* of the game, and call the addition of the nim values the nim sum or nim addition.

We say a position is a *P-position* if the player who makes the *previous* move is going to win, otherwise it is an *N-position*. A position with nim value 0 is a P-position, while all the others are N-positions, and no other outcome is possible for impartial games. The reason is that for a position with a non-zero nim value, a player can always make a move to the option whose nim value is 0, while from nim value 0, all the options have non-zero nim value, because of the Mex Rule. And the final position, i.e., the end of the game, always has the nim value 0, since $\text{mex}(\emptyset) = 0$. There is another interpretation of the relationship between these two kinds of positions: A P-positionis a position whose options are all N-positions, while an N-positionis a position that has a P-positionas an option. So studying the nim values of all the positions of a game will automatically produce the winning strategy, and will also allow us to determine that of the sum of several different games.

1.3 Why Games?

Combinatorial games are interesting to us because with a slight change of the rules, a game will require totally different winning strategy. Such strategies can vary from trivial to very hard, from tractable to intractable, and sometimes even unsolvable.

Secondly, they are quite unlike the traditional existential decision and optimization problems. Whereas in the existential decision problems area, there are only a few problems whose complexity has not yet been determined, the complexity of the majority of combinatorial game is still unknown. A study of the precise nature of the complexity of those games enables us to attain a deeper understanding of the difficulties involved in certain new and old open game problems, which is a key to their solution. Research of these games can also lead us to new and interesting algorithmic challenges, in addition to the fun of playing games.

Thirdly, while there are only a limited number of games known to be in the set of NP-problems, there are a large set of those that are Pspace-complete and Exptime-complete. Therefore the study of the combinatorial games gave us an opportunity to investigate higher complexity classes.

CHAPTER 2

THE GAME OF CHOMP

2.1 The Game of Chomp

Chomp [20] is a two-player game that starts out with an M by N chocolate bar, in which the square on the top-left corner is poisonous. A player must name a remaining square, and eat it together with all the squares below and to the right of it. Whoever eats the poisonous one (top-left) loses. The game can also be interpreted as two players alternately naming a divisor of a given number N, which may not be multiples of previously named numbers. Whoever names 1 loses.

An example, given in [3], shows if $N = 16 \times 27$, we have a chocolate bar like the following:

1	2	4	8	16
3	6	12	24	48
9	18	36	72	144
27	54	108	216	432

Thus, naming number 24 is to eat all the squares that are multiples of 24, i.e., the squares below and to the right of it.

With infinite two-dimensional Chomp, the result is obvious, the first player eats the square at 2×2, then mimics his opponent's move by eating the same number of squares on the X-axis as his opponent eats on the Y-axis, and vice versa. David Gale offers a prize of \$100.00 for the first complete analysis of 3D-Chomp, i.e., where N has three distinct prime divisors, raised to arbitrary high powers [31]. As easy and lucrative as it seems, nobody has a complete analysis of FINITE 2D-Chomp yet!

2.2 What Do We Already Know

Some trivial results can be easily seen. For example, any rectangular position is an N-position (See in [3]); any P-position with two rows must look like $(\alpha, \alpha - 1)$ $(\alpha > 1)$, i.e., positions with α squares in the first row, and $\alpha - 1$ squares the second. An N-position does not necessarily have a unique winning move either. For example, (3, 2, 1) has three winning moves, (3, 1, 1), (2, 2, 1), and (3, 2). Ken Thompson found that 4×5 and 5×2 both are the winning moves for 8×10 [3].

Some of the P-positions are given in [3]. I calculated some three-rowed and four-rowed P-positions with the help of computers. Doron Zeilberger gives a computer program that finds patterns for three-rowed Chomp with the third row fixed in [36]. The most complete result for special cases of Chomp with more than three rows was given in [3]: for a Chomp position with x rows, and a squares in the first row, b squares in the second, and one square for the rest of the rows, to be a P-position it must have the following form:

$$x = \begin{cases} \lfloor \frac{2a+b}{2} \rfloor & if\ a+b\ even; \\ min\{\lceil \frac{2a-b}{2} \rceil, \lceil \frac{3(a-b)}{2} \rceil\} & if\ a+b\ odd. \end{cases}$$

2.3 Formulae for P-positions

In this section we adopt the notation used in both [3] and [36], namely we use $[a, b, c]$ to represent a three-rowed Chomp position with $a + b + c$ squares in the first row, $a + b$ in the second, a in the third. So we are writing Chomp positions "upside down".

As we know, the formula for two-rowed P-positions are trivial, i.e., $[a, 1]$ ($a > 0$), by induction. And the formulae look simple for three-rowed, when the third row is fixed, since it always appears to be $[a, b + x, c]$, where a, b, c are fixed integers and x is a symbolic variable for all non-negative integers. The simplest example might be [2, x, 2], which characterizes [2, 0, 2], [2, 1, 2], [2, 2, 2], [2, 3, 2], ... as P-positions, as they can be verified by brute force and induction.

I created a Maple program to calculate Chomp P-positions of arbitrary number of rows and columns (to the limit of computers, of course). To our pleasure, things do get more complicated with more rows. Instead of having symbolic variables whose coefficients are 1 as in three-rowed Chomp, we can have coefficients of x larger than 1. For example, if the value of the bottom two rows of a four-rowed Chomp position is [2, 2], it is a P-position only when it is one of the following: [2, 2, 1, 3], [2, 2, 2, 3], [2, 2, $3 + 2x$, 4], [2, 2, $4 + 2x$, 2]. So in this case, instead of having one pattern for positions with a large number of squares, we have patterns that compliment each other, and yield the final formula. A brief explanation of the algorithm follows.

To minimize the computational complexity, we fix the bottom $k - 2$ rows of positions with k rows, and try to find the appropriate formula. As shown in [36], we use formal power series to calculate P-positions. Every Chomp position can be represented as a monomial in the formal power series $1/(1 - x_1)...(1 - x_k)$. For example, [2, 0, 2] will be denoted by $x_1^2 x_2^2 x_3^4$ (remember we are reading the Chomp positions upside down). For a position $[x_1, ..., x_k]$, we can define its weight w to be

$$w([x_1, ..., x_k]) = \sum_{i=1}^{k} (k - i + 1)x_i = kx_1 + (k - 1)x_2 + \cdots + x_k.$$

It is easy to see that the total weight is exactly the number of squares the position has. Since we are fixing all the rows except two, we are creating a generating function of the form of a formal power series that requires only two variables, and we can use the generating function to calculate the P-positions recursively. The generating function is the following:

$$\frac{1}{(1-x_1)(1-x_2)} - 1 - \sum_{\substack{[y_1,y_2,\ldots,y_{k-2},w_1,w_2]\in\mathcal{P}\\ [y_1,y_2,\ldots,y_{k-2},w_1,w_2]\in\mathcal{N}}} x_1^{w_1}x_2^{w_2},$$

where $[y_1, y_2, \ldots, y_{k-2}]$ are the fixed bottom rows, $\mathcal{P}$ is the set of all currently known P-positions, $\mathcal{N}$ is the set of all N-positions that can reach one of those P-positions with one legal move. While there are finitely many positions in $\mathcal{P}$, there are infinitely many positions in $\mathcal{N}$. However we can manage to represent those infinitely many positions using finitely many *rational functions*, and thus make the generating function above a finite sum. With a given generating function, we look for the next P-position by searching for the monomial with a positive coefficient and the least weight in the Taylor expansion of the generating function. If there are more than one such monomials, we can choose any one of them. Once we find a new P-position, we update the generating function by inserting the new position into $\mathcal{P}$ and updating $\mathcal{N}$ correspondingly, and repeat the process again. Of course, during the calculation, positions might be subtracted from the formal series multiple times, e.g., [1, 1, 1] ((3, 2, 1) if we use the original notation), but all this does is to change the coefficient from 1 to a negative number instead of 0, which does not have any effect on our result, since we are only looking for monomials with *positive* coefficients.

Since we do not know the coefficient β of the patterns $\alpha + \beta x$ with x as the symbolic variable for all the non-negative integers, we have to make educated guesses for the values in the Maple program. Whenever the program finds a pre-defined number of positions with the same number of rows, such that the values of the second rows form an arithmetic series whilst the other rows are identical, it tries to validate the formula. For example, positions like [2, 2, 3, 4], [2, 2, 5, 4], [2, 2, 7, 4] and [2, 2, 9, 4] suggest the formula [2, 2, $3 + 2x$, 4]. The newly generated formula will be checked against the existing P-positions by verifying that the formula will not create any known N-positions. Of course the formulae can still be erroneous and the program is capable of correcting itself by eliminating the formula when that happens.

The formulae are validated when no more P-positions can be generated. The life cycle of a conjecture is completed once a formula is validated or corrected.

The computer program was written in Maple, and can be downloaded at *http://www.math.temple.edu/~xysun* along with some pre-calculated results. Type in *Help()* for help. People can also play Chomp against a computer. Type in *PlayChomp(POS)* where *POS* is the initial position. The user and computer will take turns to name the piece they want to erase from the position. The program uses pre-calculated results. If the computer cannot find a winning move, or if the position is beyond the range of the results, the program will randomly eliminate a piece from the position so the game will go on.

With the help of the program, we have the following theorems.

Theorem 2.1 *For a k-rowed Chomp P-position $[x_1, \ldots, x_k]$ with $x_1, \ldots, x_{k-2}$ fixed, if*

$$\begin{cases} k = 3 \quad \text{and } x_1 \leq 100, \text{ or} \\ k = 4 \quad \text{and } x_1 + x_2 \leq 9, \text{except when } [x_1, x_2] = [6, 3], \end{cases}$$

then $x_k - x_{k-1}$ will eventually be periodic.

Proof: The theorem was completely proved by computer using the method explained above. Details of the proof are in the Maple package. Some of the periodic properties for four-rowed P-positions are listed in Table 2.1, where T is the value of the bottom two rows, p is the length of the period, and S is the value of x_3 when the periodic property starts for the given top rows. The exception is due to the fact that the calculation exceeded a predefined limit and thus could not finish the proof. However there is still strong evidence to support the theorem. The complete list of all the values can be found at the website specified above too.

Theorem 2.2 *For a Chomp position with x rows, a squares in the first row, b squares in the second, two squares in the third, and one square for the rest of the rows, to be a P-position, it must satisfy the following formula:*

T	p	S	T	p	S
[1, 0]			[3, 4]	8	13
[2, 0]			[3, 5]	16	12
[2, 1]	1	4	[3, 6]		
[2, 2]	2	3	[4, 0]	2	4
[2, 3]			[4, 1]		
[2, 4]			[4, 2]	8	8
[2, 5]			[4, 3]		
[2, 6]	1	4	[4, 4]		
[2, 7]			[4, 5]		
[3, 0]			[5, 0]		
[3, 1]	2	4	[5, 1]	8	14
[3, 2]	4	4	[5, 2]		
[3, 3]	8	11			

Table 2.1: Periodic Properties of Some Four-rowed P-positions

$$x = \begin{cases} 1 & if\ a = 1; \\ 2 & if\ a = b+1; \\ \lfloor \frac{2a+b}{2} \rfloor & if\ a+b\ odd\ and\ a \neq b+1; \\ \lfloor \frac{3a}{2} \rfloor + 1 & if\ a = b; \\ min\{\lceil \frac{2a-b}{2} \rceil, \frac{3(a-b)}{2}\} & if\ a+b\ even\ and\ a \neq b. \end{cases}$$

The theorem extends the formula given in [3] as listed in Section 2.2. Before we prove the theorem, let us first prove that such an x as specified in the theorem exists and is unique.

Definition 2.1 *We call the* height h *of a Chomp position* $[a_1, \ldots, a_n]$ *to be the number* x *such that either* $\underbrace{[1, 0, \ldots, 0, a_1 - 1, a_2, \ldots, a_n]}_{x}$ $(x > n)$*, or* $[\sum_{k=1}^{n-x+1} a_k, a_{n-x+2}, \ldots, a_n]$ $(x \leq n)$ *is a* P*-position, and we write* $h([a_1, \ldots, a_n]) = x$*.*

We are trying to either chomp the position to have only x rows left, or add $x-n$ one-squared rows to the position so that the result is a P-position, e.g., $h([1,0,\ldots,0]) = 1$, $h([\underbrace{2,0,\ldots,0}_{n}]) = n+1$, and $h([a_1,\ldots,a_{n-1},a_{n-1}+1]) = 2$. The following lemma assures us the existence and the uniqueness of the height for any Chomp position. Let us first denote $F([a_1,\ldots,a_n])$ to be the set of followers of $[a_1,\ldots,a_n]$, i.e., all the positions that can be derived from the position by one chomp, and $\mathrm{mex}(\{b_1,\ldots,b_m\})$, the *M*inimal *EX*clusion, the least nonnegative integer that is *not* in the set $\{b_1,\ldots,b_m\}$.

Lemma 2.1 *For any Chomp position* $[a_1,\ldots,a_n]$, $h([a_1,\ldots,a_n])$ *uniquely exists, and* $h([a_1,\ldots,a_n]) = x$, *if there exists an* $x \le n$ *such that* $[\sum_{k=1}^{n-x+1} a_k, a_{n-x+2},\ldots,a_n]$ *is a P-position; otherwise* $h([a_1,\ldots,a_n]) = \mathrm{mex}\{h([a'_1,\ldots,a'_n]) \mid [a'_1,\ldots,a'_n] \in F([a_1,\ldots,a_n])\}$.

Proof: The case for $x \le n$ is trivial by the definition of h. If such an x does not exist, and $y = \mathrm{mex}\{h([a'_1,\ldots,a'_n]) \mid [a'_1,\ldots,a'_n] \in F([a_1,\ldots,a_n])\}$, then $y > n$ and the followers of $\underbrace{[1,0,\ldots,0,a_1-1,\ldots,a_n]}_{y})$ are all N-positions by the definition of mex, therefore the position itself is a P-position. For the uniqueness part, we only have to notice that for any two positive integers $x_1 < x_2$, the position generated by the methods above using x_1 is the follower of the one generated using x_2, thus at most one of them is a P-position.

Now we can rewrite the theorem as

Theorem 2.2′:

$$h([2,b-2,a-b]) = \begin{cases} 1 & if\ a=1; \\ 2 & if\ a=b+1; \\ \lfloor \frac{2a+b}{2} \rfloor & if\ a+b\ odd\ and\ a \neq b+1; \\ \lfloor \frac{3a}{2} \rfloor + 1 & if\ a=b; \\ min\{\lceil \frac{2a-b}{2} \rceil, \frac{3(a-b)}{2}\}; & if\ a+b\ even\ and\ a \neq b. \end{cases}$$

Proof of Theorem 2.2: Notice that the result is strikingly similar to the one given in [3] except when $a = b$ and $a = b + 1$, and the two results compliment each other perfectly.

We denote the number of squares of the first, second, and third rows as α, β and γ respectively, and those of the first and second columns x and y. In our case, we only consider the positions with $y = 3$ and $\gamma = 2$:

X	X	X	X	X	X	X	X	X	X	X	X	X	α
X	X	X	X	X									β
X	X												γ
X													
X													
X													
x	y												

The first two scenarios for $x = 1, 2$ are trivial and we will avoid those cases in the discussion below. By Lemma 1, we have to consider the height of all the followers of $[2, b-2, a-b]$.

From the position, we can chomp to

$$\textit{the first row up to } b+1 \qquad (b < \alpha < a,\ \beta = b,\ \textit{and } \gamma = 2); \tag{2.1}$$

$$\textit{the first and the second row} \qquad (\alpha = \beta,\ 2 \leq \beta \leq b,\ \textit{and } \gamma = 2); \tag{2.2}$$

$$\textit{the second row only} \qquad (\alpha = a,\ 2 \leq \beta < b,\ \textit{and } \gamma = 2); \tag{2.3}$$

$$\textit{the second and third row} \qquad (\alpha = a,\ \beta = 1 \textit{ and } \gamma = 1); \tag{2.4}$$

$$\textit{the third row only} \qquad (\alpha = a,\ \beta = b \textit{ and } \gamma = 1). \tag{2.5}$$

By proper induction on a and b, we can deduce the following equations:

$$x = \min\left\{\tfrac{3(\alpha-b)}{2}, \left\lceil\tfrac{2\alpha-b}{2}\right\rceil\right\} \quad if\ b<\alpha<a,\ \alpha+b\ even; \tag{2.6}$$
$$x = \left\lfloor\tfrac{2\alpha+b}{2}\right\rfloor \quad if\ b<\alpha<a,\ \alpha+b\ odd; \tag{2.7}$$
$$x = \left\lfloor\tfrac{3\beta}{2}\right\rfloor+1 \quad if\ 1<\beta\leq b; \tag{2.8}$$
$$x = \min\left\{\tfrac{3(a-\beta)}{2}, \left\lceil\tfrac{2a-\beta}{2}\right\rceil\right\} \quad if\ 2\leq\beta<b,\ a+\beta\ even; \tag{2.9}$$
$$x = \left\lfloor\tfrac{2a+\beta}{2}\right\rfloor \quad if\ 2\leq\beta<b,\ a+\beta\ odd; \tag{2.10}$$
$$x = a; \tag{2.11}$$
$$x = \left\lfloor\tfrac{2a+b}{2}\right\rfloor \quad if\ a+b\ even; \tag{2.12}$$
$$x = \min\left\{\left\lceil\tfrac{2a-b}{2}\right\rceil, \left\lceil\tfrac{3(a-b)}{2}\right\rceil\right\} \quad if\ a+b\ odd, \tag{2.13}$$

where equation 2.1 yields equations 2.6 and 2.7, 2.2 yields 2.8, 2.3 yields 2.9 and 2.10, 2.4 yields 2.11, 2.5 yields 2.12 and 2.13

It is easy to see that $3(\alpha-\beta)\leq(2\alpha-\beta)$ iff $\alpha\leq 2\beta$

So from equation 2.6 we can have:

$$\textbf{either}\ x = \frac{3(\alpha-b)}{2}\ \ when\ \alpha\leq 2b\ and\ \alpha+b\ even,$$

which yields: (α is always incremented by 2 in the following arguments)

$$x = 3,\ldots,\tfrac{3b}{2}\ incremented\ by\ 3 \quad if\ \ b\ even\ and\ a\geq 2b\ and\ \alpha=b+2,\ldots,2b; \tag{2.14}$$
$$x = 3,\ldots,\tfrac{3(b-1)}{2}\ incremented\ by\ 3 \quad if\ \ b\ odd\ and\ a\geq 2b\ and\ \alpha=b+2,\ldots,2b-1; \tag{2.15}$$
$$x = 3,\ldots,\tfrac{3(a-b-2)}{2}\ incremented\ by\ 3 \quad if\ a+b\ even\ and\ a\leq 2b\ and\ \alpha=b+2,\ldots,a-2; \tag{2.16}$$
$$x = 3,\ldots,\tfrac{3(a-b-1)}{2}\ incremented\ by\ 3 \quad if\ a+b\ odd\ and\ a\leq 2b\ and\ \alpha=b+3,\ldots,a-1. \tag{2.17}$$

$$\textbf{or}\ x = \left\lceil\frac{2\alpha-b}{2}\right\rceil\ \ when\ \alpha\geq 2b\ and\ \alpha+b\ even,$$

which yields:

$$x = \tfrac{3b}{2}, \ldots, \tfrac{2a-b-4}{2} \textit{ incremented by } 2 \quad \textit{if } a \textit{ even}, \ b \textit{ even and } \alpha = 2b, \ldots, a-2; \tag{2.18}$$

$$x = \tfrac{3b+3}{2}, \ldots, \tfrac{2a-b-3}{2} \textit{ incremented by } 2 \quad \textit{if } a \textit{ odd}, \ b \textit{ odd and } \alpha = 2b+1, \ldots, a-2; \tag{2.19}$$

$$x = \tfrac{3b+3}{2}, \ldots, \tfrac{2a-b-1}{2} \textit{ incremented by } 2 \quad \textit{if } a \textit{ even}, \ b \textit{ odd and } \alpha = 2b+1, \ldots, a-1; \tag{2.20}$$

$$x = \tfrac{3b}{2}, \ldots, \tfrac{2a-b-2}{2} \textit{ incremented by } 2 \quad \textit{if } a \textit{ odd}, \ b \textit{ even and } \alpha = 2b, \ldots, a-1. \tag{2.21}$$

From equation 2.7:

$$x = \tfrac{3b+6}{2}, \ldots, \tfrac{2a+b-2}{2} \textit{ incremented by } 2 \quad \textit{if } a \textit{ even and } b \textit{ even and } \alpha = b+3, \ldots, a-1; \tag{2.22}$$

$$x = \tfrac{3b+5}{2}, \ldots, \tfrac{2a+b-3}{2} \textit{ incremented by } 2 \quad \textit{if } a \textit{ odd and } b \textit{ odd and } \alpha = b+3, \ldots, a-1; \tag{2.23}$$

$$x = \tfrac{3b+5}{2}, \ldots, \tfrac{2a+b-5}{2} \textit{ incremented by } 2 \quad \textit{if } a \textit{ even and } b \textit{ odd and } \alpha = b+3, \ldots, a-2; \tag{2.24}$$

$$x = \tfrac{3b+6}{2}, \ldots, \tfrac{2a+b-4}{2} \textit{ incremented by } 2 \quad \textit{if } a \textit{ odd and } b \textit{ even and } \alpha = b+3, \ldots, a-2. \tag{2.25}$$

Note that we are avoiding the case $\alpha = \beta + 1$.

From equation 2.8 we have

$$x = 4, \ 5, \ 7, \ 8, \ \ldots \ , \ \left\lfloor \frac{3b}{2} \right\rfloor + 1, \tag{2.26}$$

which are all the numbers from 4 to $\left\lfloor \frac{3b}{2} \right\rfloor + 1$ that are *NOT* divisible by 3.

From equation 2.9 the result is similar to that from equation 2.6.

From equation 2.10

$$x = \quad a+1,\ldots,a+\tfrac{b-2}{2} \quad if\ a\ even\ and\ b\ even; \tag{2.27}$$
$$x = \quad a+1,\ldots,a+\tfrac{b-1}{2} \quad if\ a\ odd\ and\ b\ odd; \tag{2.28}$$
$$x = \quad a+1,\ldots,a+\tfrac{b-2}{2} \quad if\ a\ odd\ and\ b\ even; \tag{2.29}$$
$$x = \quad a+1,\ldots,a+\tfrac{b-3}{2} \quad if\ a\ even\ and\ b\ odd. \tag{2.30}$$

Equations 2.11, 2.12 and 2.13 result in constant numbers that require no further discussion.

Now we can deduce our result from the equations.

Assuming $a \geq 2b$, we have:

x always covers $3,\ldots,\lfloor\frac{3b+2}{2}\rfloor$ from equations 2.14, 2.15 and 2.26.

x covers $\frac{3b+4}{2},\ldots,\frac{2a-b-2}{2}$ but not $\frac{2a-b}{2} = \lceil\frac{2a-b}{2}\rceil$ if a even and b even from equations 2.18 and 2.22. Note that the values from the two equations are mutually exclusive, and had x covered $\frac{2a-b}{2}$, it would have appeared in equation 2.18, which does not.

x covers $\frac{3b+3}{2},\ldots,\frac{2a-b-1}{2}$ but not $\frac{2a-b+1}{2} = \lceil\frac{2a-b}{2}\rceil$ if a odd and b odd from equations 2.19 and 2.23, by similar reasoning as shown above.

All the other equations are either dealing with $a+b$ odd, or have result bigger than $\lceil\frac{2a-b}{2}\rceil$. So the height of the position, which is the least number NOT in the above numbers, is $\lceil\frac{2a-b}{2}\rceil$ when $a+b$ even.

If a even and b odd, x covers $\frac{3b+3}{2},\ldots,\frac{2a-b+1}{2}$ from equations 2.20 and 2.24; $\frac{2a-b+1}{2},\ldots,a-1$ from 2.9 since $a \geq 2b$; a from 2.11; $a+1,\ldots,a+\frac{b-3}{2}$ from 2.30. And the other equations are insignificant as discussed above. So the height of the position is $a+\frac{b-1}{2} = \lfloor\frac{2a+b}{2}\rfloor$.

If a odd and b even, x covers $\frac{3b+4}{2},\ldots,\frac{2a-b}{2}$ from equations 2.21 and 2.25; $\frac{2a-b}{2},\ldots,a-1$ from 2.9 since $a \geq 2b$; a from 2.11; $a+1,\ldots,a+\frac{b-2}{2}$ from 2.29. And the other equations are insignificant as discussed above. So the height of the position is $a+\frac{b}{2} = \lfloor\frac{2a+b}{2}\rfloor$.

Hence we have proved when $a \geq 2b$,

$$x = \begin{cases} \left\lfloor \frac{2a+b}{2} \right\rfloor & if\ a+b\ odd; \\ \left\lceil \frac{2a-b}{2} \right\rceil & if\ a+b\ even. \end{cases}$$

The rest of the proof, i.e., when $a = b$ and $b < a < 2b$, is similar to the above.

CHAPTER 3

THE SPRAGUE-GRUNDY FUNCTION FOR CHOMP

3.1 The Sprague-Grundy Function for Chomp Positions

Obviously Chomp is an impartial game, therefore each position has its own nim value as discussed in Chapter 1. In Chapter 2 we developed computer programs to find patterns for the P-positions with the top rows fixed. For the results we got, for n-rowed Chomp positions, if the top $n-2$ rows are fixed to certain values, then either there are only finitely many P-positions, or when the $(n-1)$-th rows are large enough, the differences between the $(n-1)$-th rows and the n-th rows are constants. While finding P-positions is a big step forward for analyzing Chomp, it leaves a lot of questions unanswered, and the biggest one is how about the nim values. Since the P-positions have developed into such "gracious" patterns, the next natural question will be: do the nim values bear similar patterns. The answer is yes.

Again as in Chapter 2, we adopt the notation $[x_1, \cdots, x_k]$ for a Chomp position with k rows, and $x_1 + \cdots + x_k$ squares in the first row, $x_1 + \cdots + x_{k-1}$

the second, ..., x_1 the k-th row. We will also use $\{x_1, \cdots, x_k, y\}$ to represent a Chomp position $[x_1, \cdots, x_k]$ and its nim value y.

Theorem 3.1 *If we use the above notation for* $\{x_1, \cdots, x_k,\ y\}$*, and*

$$\begin{cases} k = 2 \quad \textit{and } x_1 < 121, \ \textit{or} \\ k = 3 \quad \textit{and } x_1 \leq 3, x_1 + x_2 \leq 5, \ \textit{or} \\ k = 4 \quad \textit{and } x_1 = 1, x_1 + x_2 + x_3 \leq 3, \end{cases}$$

then $y - x_k$ *is periodic for fixed* $[x_1, \cdots, x_{k-1}]$ *when* x_k *is large enough.*

Actually the Grundy function for two-rowed Chomp has a wonderful format:

Theorem 3.2 *For two-rowed Chomp positions,*

$$y = \begin{cases} x_1 + x_2 + \lfloor \frac{x_1 - 1}{2} \rfloor & \textit{if } x_2 \textit{ even;} \\ \frac{3x_2 - 3}{2} & \textit{if } x_2 \textit{ odd and } x_2 \leq x_1; \\ x_2 + \lfloor \frac{x_1}{2} \rfloor - 1 & \textit{if } x_2 \textit{ odd and } x_2 \geq x_1. \end{cases}$$

3.2 Proofs

Similar to [36] and Chapter 2, a new data structure is defined to calculate the nim values using Maple. Every Chomp position together with its nim value can be represented as a monomial in the formal power series

$$\frac{1}{(1 - x_1) \cdots (1 - x_k)(1 - y)}.$$

For example, $\{1, 0, 6, 4\}$ is written as $x_1 x_3^6 y^4$. Since we are fixing all the rows except one for the simplicity of calculation, we have a generating function with only two variables

$$\frac{1}{(1 - x_k)(1 - y)} - 1 - \sum_{[z_1, z_2, \ldots, z_{k-1}, w_1, w_2] \in \mathcal{P}} x_k^{w_1} y^{w_2},$$

where $[z_1, z_2, \ldots, z_{k-1}]$ are the fixed top rows, $\mathcal{P}$ is the set of all positions and nim values generated by the known positions and their nim values. For instance, once we have $\{1, 1, 3, 7\}$, $\{1, 1, 3+\alpha, 7\}$ and $\{1, 1, 3, 7+\alpha\}$, $\alpha \geq 0$ shall also be removed from the data structure because no other position with [1, 1] as the top two rows can have nim value 7, and the Chomp position [1, 1, 3] will not be needed for any further consideration. Hence

$$x_3^3 y^7 \left(\frac{1}{1-x_3} + \frac{1}{1-y}\right)$$

will be removed from the formal power series. The next position to be found will be the monomial with a positive coefficient and the least degree for x and then y. Once it is found, we update the generating function by removing the monomials generated by the newly found result from the power series. Positions and nim values can be subtracted from the formal power series multiple times during the calculation, but all what it does is to change the coefficient from 1 to a negative number instead of 0, which does not have any effect on our result.

The program then tries to find linear relationships within the n-th rows and the nim values. For example, results like $\{1, 3, 6, 6\}$, $\{1, 3, 22, 22\}$ and $\{1, 3, 38, 38\}$ suggest the formula $\{1, 3, 16 + 6x, 16 + 6x\}$. The newly generated formula will be double-checked so that it will not conflict with previous results. It can be proved that if the formula is still erroneous, it can be identified and corrected, and once no more individual positions can be found, the set of formulae found will complete the search of nim values for the positions with the specified fixed top rows. The proof is presented in the Maple package.

Therefore the proof of Theorem 3.1 is automatically done by the Maple package. Unfortunately, since we are fixing the top rows of the positions to be calculated, the Maple package was unable to find the generalized formula in Theorem 3.2, although it is easy to obtain by human eyes.

To prove Theorem 3.2 we only need to apply induction on both x_1 and x_2. Meanwhile, we give another proof from a different approach.

If we again adopt another set of notations used in [3] and Chapter 2, namely for two-rowed Chomp positions, we denote a as the number squares of the first row and b the second, we can rewrite Theorem 3.2 as

$$y = \begin{cases} \lfloor \frac{2a+b-1}{2} \rfloor & if\ a+b\ even; \\ min\left\{ \frac{3(a-b)-3}{2}, \lfloor \frac{2a-b}{2} \rfloor - 1 \right\} & if\ a+b\ odd. \end{cases} \tag{3.1}$$

It definitely has similarity to the result given in [3] of finding the number of squares x in the first column to make the position a $\mathcal{P}$-position when at most two rows can have more than one square:

$$x = \begin{cases} \lfloor \frac{2a+b}{2} \rfloor & if\ a+b\ even; \\ min\{\lceil \frac{2a-b}{2} \rceil, \lceil \frac{3(a-b)}{2} \rceil\} & if\ a+b\ odd. \end{cases} \tag{3.2}$$

In fact, there is intrinsic relationship between the two. Note that chomping off all the rows except the first row is not winning unless $a = 1$, and chomping off all the columns except the first is not winning unless $x = 1$. Without those extreme conditions, we can treat the Chomp position in equation 3.2 as two independent games: a nim game P_1 that consists of the bottom $x - 2$ squares, and a Chomp game P_2 $[b-1,\ a-b]$, the position acquired by cutting off the first column from the two-rowed base $[b,\ a-b]$ of the original game. Since no player dare to chomp to have only one column or one row left, and $[1,\ 1]$, the Chomp position to which the original game becomes when P_1 and P_2 are played to the end, is a $\mathcal{P}$-position, the players practically have to agree on playing the two independent games P_1 and P_2, and the winning strategy is the same as the sum of two nim games: keep the nim values of the two separate games the same. Therefore, to make a position in equation 3.2 a $\mathcal{P}$-position, the nim value y' for $[b-1,\ a-b]$ has to be $x - 2$. And it is not hard to verify that:

When $a + b$ is even, $y' = \lfloor \frac{2(a-1)+(b-1)-1}{2} \rfloor = \lfloor \frac{2a+b}{2} \rfloor - 2 = x - 2$;
When $a + b$ is odd and $a < 2b$, $y' = \frac{3((a-1)-(b-1))-3}{2} = \lceil \frac{3(a-b)-3-1}{2} \rceil = x - 2$;
When $a+b$ is odd and $a \geq 2b$, $y' = \lfloor \frac{2(a-1)-(b-1)}{2} \rfloor - 1 = \lceil \frac{2a-b-2}{2} \rceil - 1 = x - 2$.
Thus the proof is completed.

The Maple package can be downloaded at *http://www.math.temple.edu /~xysun* along with some pre-calculated results. Type in *Help()* for help.

3.3 Additional Comments

Due to the limitation of Maple and computer capacity, I was unable to get more results for three-rowed and four-rowed Chomp positions. However, a separate C++ program calculated three-rowed positions whose total number of pieces of the rows were up to 500 and suggests similar periodic relationship between the last rows and the nim values without the benefit of the automatic proof. Hence it is reasonable to conjecture that the periodic relationship exists for any given fixed top rows, which was later proved by Steven Byrnes [8]. Those results are also available from the website. The limited results also failed to show any similarity between the patterns for the P-positions and the Sprague-Grundy function, which should exist.

It will also be interesting to define a data structure to automatically prove Theorem 3.2 using computers.

CHAPTER 4

FRAENKEL'S N-HEAP WYTHOFF'S CONJECTURES

4.1 Wythoff's Game and Its Generalizations

Wythoff's Game [34] is an impartial game consisting of two piles of tokens. Players can remove any number of tokens from a single pile, or the same number of tokens from both piles. The P-positions are well-known and well explained by Fraenkel [15]: they are a sequences of pairs of integers $\{(A_n, B_n)\}_{n\geq 0}$, such that $A_n = \text{mex}\{A_m,\ B_m \mid 0 \leq m < n\}$ and $B_n = A_n + n$ with $A_0 = B_0 = 0$, where mex is the *M*inimal *EX*clusive value as discussed in Chapter 1. They can also be written as $A_n = \lfloor n\phi \rfloor$ and $B_n = \lfloor n\phi^2 \rfloor$, where $\phi = (1 + \sqrt{5})/2$ (the golden section).

Various generalizations and analysis on this game were done by Blass, Fraenkel and Guelman [5], WW [3], Coxeter [10], Fraenkel and Borosh [17], Fraenkel and Ozery [18], Fraenkel and Zusman [19], Yaglom and Yaglom [35]. Blass and Fraenkel [4], Dress [12], and Landman [27] also discussed the properties of the Sprague-Grundy function of the game.

Another generalization of Wythoff's game, involving more than two piles, was proposed by Fraenkel [16], which is listed in the survey article by Guy and

Nowakowski [23] as one of the "unsolved problems in combinatorial games". We are given N piles of tokens, whose sizes are $A^1, \ldots, A^N$, $A^1 \leq \cdots \leq A^N$. A player can remove any number of tokens from a single pile, or, for any non-zero vector of non-negative integers $(a_1, \ldots, a_N)$ whose *nim-sum* is 0, remove a_i tokens from the i-th pile (for $1 \leq i \leq N$). [Recall that the *nim-sum* (denoted by $\oplus$) is binary addition without carry. For example $3 \oplus 5$ equals $011 \oplus 101 = 110 = 6$.]

Denote all the P-positions by

$$(A^1, \ldots, A^{N-2}, A_n^{N-1}, A_n^N), A^{N-2} \leq A_n^{N-1} \leq A_n^N$$

and

$$A_n^{N-1} < A_{n+1}^{N-1} \quad for \quad all \quad n \geq 0 \quad .$$

Fraenkel's conjectures are as follows. Fix $A^1, \ldots, A^{N-2}$, then

Conjecture 1: There exists an integer N_1 (depending only on $A^1, \ldots, A^{N-2}$), such that when $n > N_1$, $A_n^N = A_n^{N-1} + n$.

Conjecture 2: There exist integers N_2 and α_2 such that when $n > N_2$, $A_n^{N-1} = \lfloor n\phi \rfloor + \epsilon_n + \alpha_2$ and $A_n^N = A_n^{N-1} + n$, where $\phi = (1+\sqrt{5})/2$ and $-1 \leq \epsilon_n \leq 1$.

Furthermore, $A_n^{N-1} = \text{mex}(\{A_i^{N-1}, A_i^N : 0 \leq i < n\} \cup T)$, where T is a (small) set of integers.

By preserving the moves related to the nim addition, the winning strategy of the multiple-heap Wythoff's game seems to have retained the golden section, just as the original game did.

In this chapter, a partial proof of the conjectures is presented for the three-heap Wythoff's game when $A^1 \leq 10$, which is a joint work with Prof. Doron Zeilberger.

4.2 Prerequisites

Throughout this chapter, we use the notation $\phi = (1+\sqrt{5})/2$, the golden section.

Definition 4.1 *We call a sequence of pairs of integers* $\{(A_n, B_n)\}_{n\geq n_0}$ *a* Wythoff's sequence *if there exist a finite set of integers* T *such that* $A_n = \text{mex}(\{A_i, B_i : n_0 \leq i < n\} \cup T)$, $B_n = A_n + n$ *and* $\{B_n\} \cap T = \emptyset$.

Definition 4.2 *A* special Wythoff's sequence *is a Wythoff's sequence such that there exist integers* N *and* α *such that when* $n > N$, $A_n = \lfloor n\phi \rfloor + \alpha + \epsilon_n$, *where* $\epsilon_n \in \{0, \pm 1\}$.

Lemma 4.1 *Given a Wythoff's sequence* $\{(A_n, B_n)\}_{n\geq n_0}$ *and* $N \geq n_0$ *such that* $A_n > \max(T)$ *for all* $n \geq N$*, then*

1. $1 \leq A_{n+1} - A_n \leq 2$,
2. $2 \leq B_{n+1} - B_n \leq 3$, *and*
3. *if* $A_n > B_N$, $A_{n+2} - A_n \geq 3$.

Proof: By definition, $A_n - A_{n-1} \geq 1$, $B_n - B_{n-1} = A_n + n - A_{n-1} - (n-1) \geq 2$. Also, since $\{A_i\}_{i\geq n_0} \cup \{B_i\}_{i\geq n_0} = \mathbb{Z} - T$ and $A_n > \max(T)$, the only numbers between A_n and A_{n+1} are in $\{B_i\}_{i\geq n_0}$, which are not pair-wise sequential, therefore $A_{n+1} - A_n \leq 2$ and $B_n - B_{n-1} = A_n - A_{n-1} + 1 \leq 3$. Furthermore, if $A_{n+2} - A_n = 2$, let $B_m = \min\{B_i : B_i > A_{n+2}\}$, then $m > N$, $B_{m-1} < A_n$, so $B_m - B_{m-1} > 3$, which is contradictory to what we just proved.

Lemma 4.2 $\{(\lfloor n\phi \rfloor, \lfloor n\phi \rfloor + n)\}_{n\geq 1}$*, the Wythoff's pairs as a sequence, is a special Wythoff's sequence. In addition to all the properties in Lemma 4.1, it also satisfies the following:*

1. $A = \{\lfloor n\phi \rfloor\}_{n\geq 1}$ *and* $B = \{n + \lfloor n\phi \rfloor\}_{n\geq 1}$ *are complementary, i.e.,* $A \cup B = \mathbb{Z}_{>0}$ *and* $A \cap B = \emptyset$,
2. $1 \leq \lfloor n\phi \rfloor - \lfloor (n-1)\phi \rfloor \leq 2$,
3. $|\lfloor n_1\phi \rfloor - \lfloor n_2\phi \rfloor - (n_1 - n_2)\phi| < 1$,
4. *if* $\lfloor n\phi \rfloor - \lfloor (n-1)\phi \rfloor = 1$*, then* $\lfloor (n+1)\phi \rfloor - \lfloor n\phi \rfloor = \lfloor (n-1)\phi \rfloor - \lfloor (n-2)\phi \rfloor = 2$,

5. *if* $\lfloor n\phi \rfloor - \lfloor (n-1)\phi \rfloor = \lfloor (n-1)\phi \rfloor - \lfloor (n-2)\phi \rfloor = 2$, *then* $\lfloor (n+1)\phi \rfloor - \lfloor n\phi \rfloor = \lfloor (n-2)\phi \rfloor - \lfloor (n-3)\phi \rfloor = 1$.

Proof: It is well explained by Fraenkel [15] that the sequence satisfies all the requirements for special Wythoff's sequence.

1. See in [15].

2. is a direct corollary of the previous lemma. The first two items are highlighted again because they are of special interest to us in the coming sections.

3. Since ϕ is irrational, $n_1\phi - 1 - n_2\phi < \lfloor n_1\phi \rfloor - \lfloor n_2\phi \rfloor < n_1\phi - n_2\phi + 1$.

4. Since $\phi \approx 1.618$, for any n, $\lfloor n\phi \rfloor - \lfloor (n-2)\phi \rfloor > n\phi - 1 - (n-2)\phi > 2.2$. Since the left-hand side of the inequality is an integer, it is at least 3.

5. Similarly for any n, $\lfloor n\phi \rfloor - \lfloor (n-3)\phi \rfloor < n\phi - (n-3)\phi + 1 < 5.9$. So the left-hand side can be at most 5.

Lemma 4.3 *Given a Wythoff's sequence* $\{(A_n, B_n)\}_{n \geq n_0}$ *and* $N \geq n_0$ *such that* $A_n > \max(T)$ *for all* $n \geq N$, *if we write* $A_n = \lfloor n\phi \rfloor + \alpha_n$ *and* $B_n = A_n + n + \alpha_n$, *then* $-1 \leq \alpha_{n+1} - \alpha_n \leq 1$ *for all* $n \geq N$.

Proof: $\alpha_{n+1} - \alpha_n = (A_{n+1} - A_n) - (\lfloor (n+1)\phi \rfloor - \lfloor n\phi \rfloor)$, so the inequality holds because of Lemma 4.1 and Lemma 4.2.

Lemma 4.4 *Given a Wythoff's sequence* $\{(A_n, B_n)\}_{n \geq n_0}$, *we assume that there exist integers* N, α, *and* $m_2 > m_1 > N + 2$, *such that*

- $A_n > \max(T)$, *when* $n > N$,
- $A_{m_2} > B_{m_1}$,
- $A_n = \lfloor n\phi \rfloor + \alpha + \epsilon_n$, *where* $m_1 - 2 \leq n \leq m_2$ *and* $-1 \leq \epsilon_n \leq 1$, *and*
- $\epsilon_{i-2}\epsilon_{i-1}\epsilon_i = 0$, *when* $m_1 - 2 \leq i \leq m_2$,

then $\epsilon_{i-1}\epsilon_i = 0$ *for any* $i > m_2$.

Proof: The last assumption is equivalent to the statement that there do not exist three consecutive non-zero ϵ's by Lemma 4.3 when $m_1 - 2 \leq i \leq m_2$, and we are to prove 1) $-1 \leq \epsilon_i \leq 1$ when $i > m_2$; and 2) there are no two consecutive non-zero ϵ's when $i > m_2$. Once proved, this lemma provides a way to evaluate the behavior of ϵ_n, and *when* the behavior starts.

We are going to prove the lemma in two steps:

First, if $\exists n > m_2$ such that $\epsilon_n \notin \{0, \pm 1\}$, let n be the smallest of such numbers, then $\epsilon_n = \pm 2$ by Lemma 4.3. There are four cases:

a) $A_n = A_{n-1} + 2$ and $\epsilon_n = -2$: $\epsilon_{n-1} = -1$ by Lemma 4.3, and $A_n = \lfloor n\phi \rfloor + \alpha - 2 \leq \lfloor (n-1)\phi \rfloor + \alpha$ by Lemma 4.2. However, $A_n = A_{n-1} + 2 = \lfloor (n-1)\phi \rfloor + \alpha + 1$, which is impossible.

b) $A_n = A_{n-1} + 2$ and $\epsilon_n = 2$: $\epsilon_{n-1} = 1$ by Lemma 4.3, so $\lfloor n\phi \rfloor - \lfloor (n-1)\phi \rfloor = A_n - A_{n-1} - (\epsilon_n - \epsilon_{n-1}) = 1$. By Lemma 4.2, $\lfloor (n+1)\phi \rfloor - \lfloor n\phi \rfloor = \lfloor (n-1)\phi \rfloor - \lfloor (n-2)\phi \rfloor = 2$. Since $A_n = A_{n-1} + 2$, there exists $m < n$ such that $B_m = A_n - 1 = A_{n-1} + 1$, i.e., $\lfloor m\phi \rfloor + m + \epsilon_m = \lfloor n\phi \rfloor + 1$. Since $\{\lfloor n\phi \rfloor + n\}$ and $\{\lfloor n\phi \rfloor\}$ are complementary and $\lfloor n\phi \rfloor = \lfloor (n+1)\phi \rfloor - 2$, we must have $\epsilon_m = 0$ and $\lfloor m\phi \rfloor + m = \lfloor n\phi \rfloor + 1$. Similarly since $\lfloor n\phi \rfloor - \lfloor (n-1)\phi \rfloor = 1$, $\lfloor (m-1)\phi \rfloor + m - 1 = \lfloor (n-1)\phi \rfloor - 1 = \lfloor m\phi \rfloor + m - 3$, thus $\lfloor m\phi \rfloor = \lfloor (m-1)\phi \rfloor + 2$. Also, since $2 \geq A_{n-1} - A_{n-2} = \lfloor (n-1)\phi \rfloor - \lfloor (n-2)\phi \rfloor + \epsilon_{n-1} - \epsilon_{n-2} = 3 - \epsilon_{n-2}$, $\epsilon_{n-2} = 1$ and $A_{n-1} - A_{n-2} = 2$. By our assumption of $\epsilon_{n-1}\epsilon_{n-2}\epsilon_{n-3} = 0$, we know $\epsilon_{n-3} = 0$, and by the definitions of A_n and B_n, $B_{m-1} = A_{n-1} - 1$, which is to say $\lfloor (m-1)\phi \rfloor + m - 1 + \epsilon_{m-1} = \lfloor (n-1)\phi \rfloor$. Again, since $\{\lfloor n\phi \rfloor + n\}$ and $\{\lfloor n\phi \rfloor\}$ are complementary and $\lfloor n\phi \rfloor = \lfloor (n-1)\phi \rfloor + 1$, ϵ_{m-1} has to be 1.

Now consider $A_{n-2} - A_{n-3}$, which is 1 or 2 by Lemma 4.1. If $A_{n-2} - A_{n-3} = 1$, we have $1 = A_{n-2} - A_{n-3} = \lfloor (n-2)\phi \rfloor - \lfloor (n-3)\phi \rfloor + 1 - \epsilon_{n-3}$, which means $\lfloor (n-2)\phi \rfloor - \lfloor (n-3)\phi \rfloor = 1$ and $\epsilon_{n-3} = 1$. But then $\epsilon_{n-1}\epsilon_{n-2}\epsilon_{n-3} = 1$, which is contradictory to our assumption. If $A_{n-2} - A_{n-3} = 2$, $B_{m-2} = A_{n-2} - 1$, which means $\lfloor (m-2)\phi \rfloor + m - 2 + \epsilon_{m-2} = \lfloor (n-2)\phi \rfloor$. Again, since $\{\lfloor n\phi \rfloor + n\}$ and $\{\lfloor n\phi \rfloor\}$ are complementary and $\lfloor (n-2)\phi \rfloor - \lfloor (n-3)\phi \rfloor = A_{n-2} - A_{n-3} - (\epsilon_{n-2} -$

$\epsilon_{n-3}) = 1$, ϵ_{m-2} has to be -1. But $\epsilon_{m-1} = 1$, contradictory to Lemma 4.3.

c) $A_n = A_{n-1} + 1$ and $\epsilon_n = 2$: By Lemma 4.3, $\epsilon_{n-1} = 1$, thus $\lfloor n\phi \rfloor - \lfloor (n-1)\phi \rfloor = A_n - A_{n-1} - \epsilon_n + \epsilon_{n-1} = 0$, which is impossible.

d) $A_n = A_{n-1} + 1$ and $\epsilon_n = -2$: By Lemma 4.1, $A_{n-1} = A_{n-2} + 2$, and by Lemma 4.3, $\epsilon_{n-1} = -1$. Therefore $-1 \le \epsilon_{n-2} = \lfloor (n-1)\phi \rfloor - \lfloor (n-2)\phi \rfloor - 3$, so $\epsilon_{n-2} = -1$, $\lfloor (n-1)\phi \rfloor - \lfloor (n-2)\phi \rfloor = 2$, and $\lfloor n\phi \rfloor - \lfloor (n-1)\phi \rfloor = A_n - A_{n-1} - (\epsilon_n - \epsilon_{n-1}) = 2$ thus $\lfloor (n-2)\phi \rfloor - \lfloor (n-3)\phi \rfloor = 1$ by Lemma 4.2. However $1 \le A_{n-2} - A_{n-3} = \lfloor (n-2)\phi \rfloor - \lfloor (n-3)\phi \rfloor - 1 - \epsilon_{n-3}$, which means $\epsilon_{n-3} = -1$ and $\epsilon_{n-1}\epsilon_{n-2}\epsilon_{n-3} = -1$, contradictory to our assumption.

Secondly, if there exists $n > m_2$ such that $\epsilon_n \epsilon_{n-1} \neq 0$, i.e., $\epsilon_n = \epsilon_{n-1} = \pm 1$, let n be the smallest of such numbers, then by Lemma 4.1 and by what we have just proved, $\epsilon_{n-2} = 0$. There are two cases:

e) $\epsilon_n = \epsilon_{n-1} = 1$: $2 \ge A_{n-1} - A_{n-2} = \lfloor (n-1)\phi \rfloor - \lfloor (n-2)\phi \rfloor + 1$, so $\lfloor (n-1)\phi \rfloor - \lfloor (n-2)\phi \rfloor = 1$ and $A_{n-1} - A_{n-2} = 2$. Also $A_n - A_{n-1} = \lfloor n\phi \rfloor - \lfloor (n-1)\phi \rfloor = 2$ by Lemma 4.2. So there exists $m < n$ such that $B_m = A_n - 1$ and $B_{m-1} = A_{n-1} - 1$, which means $\lfloor m\phi \rfloor + m + \epsilon_m = \lfloor n\phi \rfloor$ and $\epsilon_m = \pm 1$; $\lfloor (m-1)\phi \rfloor + m - 1 + \epsilon_{m-1} = \lfloor (n-1)\phi \rfloor$ and $\epsilon_{m-1} = \pm 1$. Since $\{\lfloor n\phi \rfloor + n\}$ and $\{\lfloor n\phi \rfloor\}$ are complementary and $\lfloor (n-1)\phi \rfloor - \lfloor (n-2)\phi \rfloor = 1$, $\epsilon_{m-1} = -1$, therefore $\epsilon_m = \epsilon_{m-1} = -1$ by Lemma 4.3, and $\lfloor m\phi \rfloor + m = \lfloor n\phi \rfloor + 1$. So $\lfloor (n+1)\phi \rfloor = \lfloor n\phi \rfloor + 2$, and $\lfloor (n+2)\phi \rfloor = \lfloor (n+1)\phi \rfloor + 1$ by Lemma 4.2. Thus $\lfloor (m+1)\phi \rfloor + m + 1 = \lfloor (n+2)\phi \rfloor + 1 = \lfloor m\phi \rfloor + m + 3$, and $3 \ge B_{m+1} - B_m = \lfloor (m+1)\phi \rfloor + m + 1 + \epsilon_{m+1} - (\lfloor m\phi \rfloor + m - 1) = 4 + \epsilon_{m+1}$. Therefore $\epsilon_{m+1} = -1$ and $\epsilon_{m+1}\epsilon_m\epsilon_{m-1} = -1$, contradictory to our assumption.

f) $\epsilon_n = \epsilon_{n-1} = -1$: $1 \le A_{n-1} - A_{n-2} = \lfloor (n-1)\phi \rfloor - \lfloor (n-2)\phi \rfloor - 1$, so $\lfloor (n-1)\phi \rfloor - \lfloor (n-2)\phi \rfloor = 2$ and $A_{n-1} - A_{n-2} = 1$. Thus $A_n - A_{n-1} = A_{n-2} - A_{n-3} = 2$ by Lemma 4.1, and $\lfloor n\phi \rfloor - \lfloor (n-1)\phi \rfloor = A_n - A_{n-1} = 2$. Hence $\lfloor (n+1)\phi \rfloor - \lfloor n\phi \rfloor = \lfloor (n-2)\phi \rfloor - \lfloor (n-3)\phi \rfloor = 1$ by Lemma 4.2. Therefore there exists $m < n$ such that $B_m = A_n - 1$, $B_{m-1} = A_{n-2} - 1$, that is $\lfloor n\phi \rfloor - 2 = \lfloor m\phi \rfloor + m + \epsilon_m$, and $\lfloor (m-1)\phi \rfloor + m - 1 + \epsilon_{m-1} = \lfloor (n-2)\phi \rfloor - 1$. Since $\{\lfloor n\phi \rfloor + n\}$ and $\{\lfloor n\phi \rfloor\}$ are complementary and $\lfloor n\phi \rfloor = \lfloor (n-1)\phi \rfloor + 2$, $\epsilon_m = \pm 1$. Similarly, since $\lfloor (n-2)\phi \rfloor - \lfloor (n-3)\phi \rfloor = 1$, $\epsilon_{m-1} = 1$, which means

$\epsilon_m = \epsilon_{m-1} = 1$ by Lemma 4.3. So $\lfloor m\phi \rfloor + m = B_m - 1 = A_n - 2 = \lfloor n\phi \rfloor - 3$, and $\lfloor (m+1)\phi \rfloor + m + 1 = \lfloor n\phi \rfloor - 1$, thus $\lfloor (m+1)\phi \rfloor = \lfloor m\phi \rfloor + 1$. Since $2 \leq B_{m+1} - B_m = 1 + \epsilon_{m+1}$, $\epsilon_{m+1} = 1$ and $\epsilon_{m+1}\epsilon_m\epsilon_{m-1} = 1$, contradictory to our assumption.

4.3 Partial Proof of the Conjectures

In this section, we adopt the following notation:

- $[a, b, c]$, $a \geq 0$, $b \geq 0$, $c \geq 0$, is a three-heap Wythoff's position having a, $a+b$ and $a+b+c$ tokens in the piles;
- $[m, A_n^m, B_n^m]$, where $n \geq 1$ and $A_n^m < A_{n+1}^m$, are all the P-positions whose first piles have m tokens;
- P^m is the set of P-positions whose first piles have m tokens;
- $T^m = \mathbb{Z}_{\geq 0} - \{A_i^m, A_i^m + B_i^m : i \geq 1\} - \{i : 0 \leq i < m\}$;
- $S^m = \mathbb{Z}_{\geq 0} - \{B_i^m : i \geq 1\}$;
- $\alpha_n^m = m + A_n^m - \lfloor B_n^m \phi \rfloor$;
- N_1^m is the integer such that when $n > N_1^m$, $A_n^m = \text{mex}\{A_i^m, B_i^m : 1 \leq i < n\}$ and $B_{n+1}^m = B_n^m + 1$;
- N_2^m, α^m, and ϵ_n^m are the integers such that when $n > N_2^m$; $A_n^m = \text{mex}\{A_i^m, B_i^m : 1 \leq i < n\}$, $B_{n+1}^m = B_n^m + 1$; and $\epsilon_n^m = m + A_n^m - \lfloor B_n^m \phi \rfloor - \alpha^m \in \{0, \pm 1\}$;
- N_3^m is the integer such that if N_2^m exists and when $n > N_3^m$, $\epsilon_n^m \epsilon_{n+1}^m = 0$;
- $p(m) = 2^{\lfloor \log_2(m) \rfloor + 1}$.

With the notation above, each list of three numbers *uniquely* identifies a three-heap position, and vice versa. For our convenience and without any confusion, we also use $[0, b, c]$ to denote two-heap positions.

We will also abuse the definition of (special) Wythoff's sequence by replacing the requirement of $B_n = A_n + n$ to $B_{n+1} - A_{n+1} = B_n - A_n + 1$ when n is large enough, because we can obtain a Wythoff's sequence by chopping off a number of pairs from the sequence and reorganizing the indices.

The conjectures can now be rephrased as follows. For any given $m \geq 0$, $[A_n^m, A_n^m + B_n^m]$ is a special Wythoff's sequence. In other words, N_1^m, N_2^m and α^m exist.

Claim 4.1 *$T^m = \{b : \exists a < m,$ such that $[a, m-a, b] \in P^a\}$, thus it is finite.*

Proof: Denote the right-hand side of the equation as T_1. If $b \in T_1$, $[m, b, c]$, is an N-position for any $c \geq 0$, because we can simply remove an appropriate number of tokens from the third pile to create a P-position. Similarly, $[m, c, b-c]$ is also an N-position for any $c \leq b$, because we can remove tokens from the second pile. Hence $T_1 \subset T^m$. On the other hand, if $b \in T^m$, $[m, b, c]$ is an N-position for any $c \geq 0$. By the rules, to find the P-position corresponding to $[m, b, c]$ for each c, we can

I) remove b_1, b_2, b_3 tokens from the three piles correspondingly, providing $b_1 \oplus b_2 \oplus b_3 = 0$ and $b_1 + b_2 + b_3 > 0$;

II) remove a_1 tokens from the first pile where $0 < a_1 \leq m$;

III) remove b_1 tokens from the second pile where $0 < b_1 \leq b + m$;

IV) remove c_1 tokens from the third pile where $0 < c_1 \leq m + b + c$.

There are only finitely many choices involving moves I), II) and III), but infinitely many choices of c, so there must exist $0 \leq c' < b + m$, such that a P-position has m, c' and $b + m$ tokens in the piles. Since $b \in T^m$, $c' < m$. Thus $b \in T_1$ and $T^m \subset T_1$.

It is easy to see that $T^1 = \{1\}$ because of $[0, 1, 1]$; $T^2 = \emptyset$; and $T^3 = \{1, 2, 3\}$ because of $[1, 2, 1]$, $[0, 3, 2]$ and $[2, 1, 3]$.

We implement the following steps in order to prove the conjectures on the Wythoff's game for any specific m.

Claim 4.2 *We can represent positions in the three-heap Wythoff's game symbolically, and therefore can create a generating function to find P-positions.*

Proof: Given a P-position $[a, b, c]$, $a \leq m$, the following positions whose first piles have m tokens can reach this position with one move:

$$[m, b + b' - (m - a), c + ((m - a) \oplus b') - b'],\ b' \geq 0,\ if\ a < m, \tag{4.1}$$

$$[m, a + a' - m, b + c + ((m - a - b) \oplus a') - a'],\ a' \geq m - a,\ if\ a + b < m, \tag{4.2}$$

$$[m, a + a' - m, b + ((m - a - b - c) \oplus a') - a'],\ a' \geq m - a,\ if\ a + b + c < m, \tag{4.3}$$

$$[m, b + b', c - b'],\ b' \leq c,\ if\ a = m, \tag{4.4}$$

$$[m, b + c, b'],\ b' \geq 0,\ if\ a = m, \tag{4.5}$$

$$[m, b, c + c'],\ c' \geq 0,\ if\ a = m, \tag{4.6}$$

$$[m, b', c - b'],\ 0 \leq b' \leq c,\ if\ a + b = m, \tag{4.7}$$

$$[m, c, c'],\ c' \geq 0,\ if\ a + b = m, \tag{4.8}$$

$$[m, b', b + c],\ b' \geq 0,\ if\ a + b = m, \tag{4.9}$$

$$[m, b', b],\ b' \geq 0,\ if\ a + b + c = m. \tag{4.10}$$

The ten sets of positions above correspond to the following moves. Add tokens to the original position to:

- all three rows, with $m-a$ tokens to the first pile, b' second and $(m-a) \oplus b'$ third,
- all three rows, with a' first, $m - a - b$ second and $(m - a - b) \oplus a'$ third,
- all three rows, with a' first, $(m - a - b - c) \oplus a'$ second and $m - a - b - c$ third,
- the second row, but not enough to exceed the third,
- the second row, and exceeding the third,
- the third row only,

- the first row, but not enough to exceed the third,
- the first row, and exceeding the third,
- the first and the third rows,
- the first and second rows, and both exceeding the third.

Also in cases of 4.1, 4.2 and 4.3, we may need to increase the second and third numbers to avoid possible negative values: if $[m, b', c']$ is the resulting N-position and $c' < 0$, we will change it to $[m, b' + c', -c']$; and if b' or $b' + c'$ is less than 0, we simply replace it with 0.

Therefore, if $[a, b, c]$ is a P-position, the positions listed above are all the N-position deduced from it. So for each position (A^1, A^2_n, A^3_n) in the game, N or P, by fixing the first pile, we can use $x_1^{A^2_n} x_2^{A^3_n}$ to represent it symbolically. By observing that for any given b, $(b \oplus c) - c$ is periodic as a function of c with period $p(b)$. we know that for each P-position $[a, b, c]$, all the N-positions deduced from it, possibly including $[a, b, c]$ itself, are:

$$\sum_{k=0}^{p(m-a)-1} \frac{x_1^b x_2^c}{1 - x_2^{((m-a)\oplus k)-k}}, \ if \ a < m, \tag{4.11}$$

$$\sum_{k=0}^{p(m-a-b)-1} \frac{x_2^{b+c}}{1 - x_2^{((m-a-b)\oplus(m-a+k))-(m-a+k)}}, \ if \ a < m, \tag{4.12}$$

$$\sum_{k=0}^{p(m-a-b-c)-1} \frac{x_2^{b+c}}{1 - x_2^{((m-a-b-c)\oplus(m-a+k))-(m-a+k)}}, \ if \ a < m, \tag{4.13}$$

$$\sum_{k=0}^{c} (x_1^{b+k} x_2^{c-k}), \ if \ a = m, \tag{4.14}$$

$$\frac{x_1^{b+c}}{1 - x_2}, \ if \ a = m, \tag{4.15}$$

$$\frac{x_1^b x_2^c}{1 - x_2}, \ if \ a = m, \tag{4.16}$$

$$\sum_{k=0}^{c} (x_1^k x_2^{c-k}), \ if \ a + b = m, \tag{4.17}$$

$$\frac{x_1^c}{1-x_2},\ if\ a+b=m, \tag{4.18}$$

$$\frac{x_2^{b+c}}{1-x_1},\ if\ a+b=m, \tag{4.19}$$

$$\frac{x_2^b}{1-x_1},\ if\ a+b+c=m. \tag{4.20}$$

Given a position $[a, b, c]$, let $N([a, b, c])$ be the set of all positions that can reach $[a, b, c]$ with one move, and denote by $f([a, b, c])$ the sum of the formal series 4.11–4.20, whenever applicable. Then f is the sum of the symbolic representations of $[a, b, c]$ and $N([a, b, c])$. We now define:

$$F_1(x_1, x_2) = \sum_{[a,b,c]\in P^a,\ all\ a<m} f([a, b, c]),$$

$$F_2(x_1, x_2) = \sum_{[a,b,c]\in P^m} f([a, b, c]),\ and$$

$$F(x_1, x_2) = \frac{1}{(1-x_1)(1-x_2)} - F_1(x_1, x_2) - F_2(x_1, x_2).$$

The sum for $F_2(x_1, x_2)$ is over the set of all known P-positions in P^m; and the sum for $F_1(x_1, x_2)$ is over the set of all P-positions whose first pile have less than m tokens with b and c large enough. We consider the Taylor expansion of $F(x_1, x_2)$ and find the lexicographically first monomial with strictly positive coefficient. The next P-position will be $[m, b, c]$, which always exists based on the rules of the game. In practice, we will use a faster approach to find A_n^m, namely $A_n^m = \text{mex}\{A_i^m, A_i^m + B_i^m : 0 \le i < n\}$, and still use the generating function to find B_i^m. The reason that we can use mex here is because of our assumption $A_{n-1}^m < A_n^m$, which indicates that any integer between the two must be in $\{A_i^m + B_i^m\}_{0\le i<n} \cup T$.

The game can also be visualized as follows. When the first pile has a fixed amount, m, of tokens, consider all the positions as points in the first quadrant with integral coordinates. For example, a position $[m, b, c]$ will be represented by the point (b, c) in our coordinate system. We call *instant winners* the positions at which a player can declare himself a winner immediately. In our game, they are the positions $[m, b, c]$ such that they can reach a certain $[a', b', c'] \in P^a$ with $a' < m$. Cross these points out of our coordinate system, and find the first point $[b, c]$ that has not been erased with the smallest possible x, and for that x, the smallest y coordinate. By the rules of the game, $[m, b, c]$ is a P-position. After finding each P-position$[m, b, c]$, we draw the following lines starting from (b, c): an upward vertical line, a leftward horizontal line, a 45° south-eastern slant line and an upward vertical line starting from the x-intercept of the slant line. Remove all the points on the lines, because they are the N-positions that can reach the newly found P-position with one move. Repeat the process to find the next P-position.

In Figure 4.1, $m = 1$; each (small) cross is an instant winner; and each "X" is a P-position.

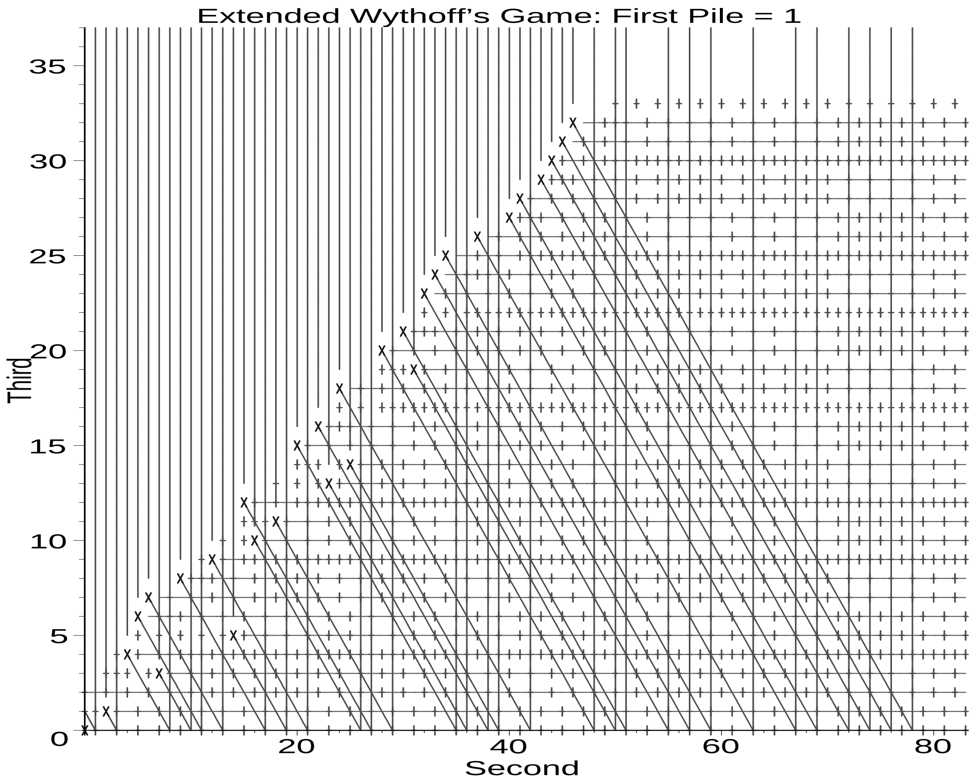

Figure 4.1: P-positions and Instant Winners when $m = 1$ and $n \leq 28$

Claim 4.3 *Given all* $[m, A_i^m, B_i^m] \in P^m$*, with* $i \leq N$*, We can decide whether a given integer* $c < B_N^m$ *is in* S^m *by the following rules.*

- *if there exists* $[a, b, c] \in P^a$ *and* $a + b = m$*, then* $b + c \in S^m$*;*
- *if there exists* $[a, b, c] \in P^a$ *and* $a + b + c = m$*, then* $b \in S^m$*;*
- *if there exists* n *such that* $B_i \neq c$ *when* $i < n$*;* $B_n > c$*; and* *coeff*$(F_{2,n}(x_1), x_1, i) \leq 0$ *with* $A_n \leq i \leq A_n + p(m)$*, then* $c \in S^m$*.*

Proof: Here we can see the advantage of symbolic over numeric computing, even though the latter would have been faster if we were *only* looking for the next P-positions.

Consider the generating function $F_1(x_1, x_2)$ generated by all the P-positions, whose first piles have fewer than m pieces, and their induced N-positions, namely, the sum of the formal power series 4.11, 4.12, 4.13, 4.17, 4.18, 4.19 and 4.20 over all the P-positions described above. Let $F_{1,n}(x_1, x_2)$ be the Taylor expansion of $F_1(x_1, x_2)$ of degree $\max\{A_i^m + B_i^m : i \leq n\} + p(m)$. Denote coeff$(f(x), x, n)$ as the coefficient of x^n of the Taylor expansion of $f(x)$, and let $F_{2,n}(x_1)$ be coeff$(F_{1,n}(x_1, x_2), x_2, c)$. So from the proof of the previous claim, $c \in S^m$ if we can show that there exists N such that $B_n^m \neq c$ when $A_n^m \leq N$, and coeff$(F_{2,n}(x_1), n)$ are all positive when $n > N$.

The first two cases are obvious because we can remove the same number of tokens from two different piles, or symbolically we can use 4.19 and 4.20, and work with the coefficients of the corresponding formal series. In the third case, we only consider the moves that remove tokens from all three piles, or equivalently, cases 4.1, 4.2 and 4.3, which generate formal series 4.11, 4.12 and 4.13. By our notation, each monomial $x_1^a x_2^b$ with positive coefficient in the Taylor expansions of the fractional expressions represents either the P-position that generates the terms or an N-position $[m, a, b]$ deduced from the P-position. By the previously mentioned fact that $(a \oplus b) - b$ is periodic as a function of b, with period $p(a)$, which divides $p(m)$ if $a \leq m$. Therefore if such an n exists as described above, coeff$(F_{2,n}(x_1), x_1, i) \geq 0$ for any $i \geq A_n$, and this finishes the proof of our claim. We denote all such numbers as $S_1^m(n)$, which is a subset of S^m.

For example when $m = 1$, $2 \in S^1$ because $[0,1,1] \in P^0$; $17 \in S^1$ because $[1,24,18] \in P^1$ and checking the generating function at that point confirms the result. In fact, manual check indicates that when $n < 29$, there is no P-position of the form $[1,n,17]$; when $n \geq 15$, $[1,2n-1,17]$ are all N-positions because $[0,29,18] \in P^0$, and $[1,2n,17]$ are all N-positions because $[0,25,16] \in P^0$. Such manual checks become impractical as m grows larger. Interested readers can try to check another simple one: $22 \in S^1$.

In the language of instant winners, $c \in S_1^m(n)$ means the instant winners will fill the horizontal line $y = c$ for $x > A_n^m$, e.g. check $c = 2, 17$ in Figure 4.1.

Claim 4.4 *There exists an integer N such that when $n > N$, $A_n^m > \max\{T^i : i \leq m\}$ and $B_n^m > m + \max\{T^i : i \leq m\}$. If for a given $n > N$, $B = \max(\{B_i^m : i \leq n\})$, $\operatorname{mex}(\{B_i^m : i \leq n\} \cup S_1^m(n)) = B + 1$, and $B + 1 - B_{n'}^i > m - a$ whenever $i + A_{n'}^i \leq m + A_{n+1}^m$ with $0 \leq i < m$, then $B_{n+1}^m = B + 1$.*

Proof: Since T^i, $i < m$, are all finite, there must exist an integer N as specified. If there is an $n > N$ as described in the assumption, let us consider the position $[m, A_{n+1}^m, B+1]$. To prove it is a P-position, we only need to show that it cannot reach another P-position with one move. For any $[a,b,c] \in P^a$ with $a < m$ and $a + b \leq m + A_{n+1}^m$, the moves like 4.7, 4.8, 4.9 and 4.10 require $b \leq m$ and $c \in T^i$ for some $i < m$, so the moves can change the second number to at most $\max\{T^i : i \leq m\}$ or the third number to $m + \max\{T^i : i \leq m\}$. So these moves cannot affect $[m, A_{n+1}^m, B+1]$ being a P-position or N-position. Since $B + 1 - c > m - a$ and $(a_1 \oplus a_2) - a_2 \leq a_1$ for any a_1 and a_2, when the moves like 4.1, 4.2, 4.3 change the first and second numbers of $[a,b,c]$ to m and A_{n+1}^m, they can change the third number from c to at most $m - a + c$, i.e., the third pile has at most $m + A_{n+1}^m + m + c - a < m + A_{n+1}^m + B + 1$ tokens, which means these moves are all irrelevant too. Thus we have been freed from the possible moves that involve the first pile. On the other hand, all the moves that involve only the second and third piles, namely 4.4, 4.5 and

4.6, will not increase both of the second and the third numbers at the same time, so $B^m_{n+1} = \text{mex}(\{B^m_i : i \leq n\} \cup S^m) = B + 1$.

Using the visual interpretation of game, we can view the result as: $B^m_{n+1} = \text{mex}(\{B^m_i : i \leq n\} \cup S^m) + 1$, if the instant winners are not involved in the decision-making. For example, check the P-positions when $c \geq 23$ in Figure 4.1. This claim provides us a sufficient condition to verify *when* $B^m_{n+1} = B^m_n + 1$.

Claim 4.5 *For a given m, if N^i_1, N^i_2 and N^i_3 exist for $i < m$, the following conditions imply both conjectures for m: given an integer N as in Claim 4.4, if there exist $n_1 > n_2 > N$ such that*

- $A^m_{n_2+3} + B^m_{n_2+3} < A^m_{n_1}$;
- $B^m_{j+1} = B^m_j + 1$ *for* $n_2 \leq j \leq n_1$;
- $B^m_{n_1} = \max(\{B^m_i : i \leq n_1\}$;
- $\text{mex}(\{B^m_i : i \leq n_1\} \cup S^m_1(n_1)) = B^m_{n_1} + 1$;
- $\max(\alpha^m_j : n_2 \leq j \leq n_1) - \min(\alpha^m_j : n_2 \leq j \leq n_1) \leq 2$;
- $B^m_{n_1} > B^i_{N^i_3}$, $i < m$;

Furthermore if we denote $\alpha' = \lfloor(\max(\alpha^m : n_2 \leq j \leq n_1) - \min(\alpha^m : n_2 \leq j \leq n_1))/2\rfloor$ *and* $\epsilon^m_i = m + A^m_i - \lfloor B^m_i \phi \rfloor - \alpha'$, $i \geq 1$, *we also assume:*

- $\alpha^i - \alpha' \geq 4(m - i)$, $0 \leq i < m$;
- $\epsilon^m_j \epsilon^m_{j-1} \epsilon^m_{j-2} = 0$, $n_2 < j < n_1$.

Proof: Note that although there are eight conditions in the assumption, the first six are in fact necessary conditions for the conjectures.

We prove this Claim by induction and assume all the conditions are true for $n \geq n_1$, i.e., $\text{mex}(\{B^m_i : i \leq n\} \cup S^m_1(n)) = B^m_n + 1$, $B^m_n = B^m_{n-1} + 1$, and $|\epsilon_n| \leq 1$.

To prove $B^m_{n+1} = B^m_n + 1$, by Claim 4.4, we need to show that if $i + A^i_{n'} \leq m + A^m_{n+1}$ with $i < m$, then $B^m_n + 1 > B^i_{n'} + m - i$. Since $A^m_{n+1} \leq A^m_n + 2$, $\lfloor B^m_n \phi \rfloor + \alpha' + \epsilon^m_n + 2 - \lfloor B^i_{n'} \phi \rfloor - \alpha^i - \epsilon^i_{n'} \geq 0$, therefore $\lfloor B^m_n \phi \rfloor - \lfloor B^i_{n'} \phi \rfloor \geq \alpha^i - \alpha' - 4$; $(B^m_n - B^i_{n'})\phi + 1 > \alpha^i - \alpha' - 4$; $B^m_n + 1 - B^i_{n'} > (\alpha^i - \alpha' - 5)/\phi + 1$; so $B^m_n + 1 - B^i_{n'} \geq m - i$. The only time that the equal sign may hold is when $m = i + 1$, $\alpha^i - \alpha' = 4$, $\epsilon^i_{n'} = -1$, $\epsilon^m_n = 1$, and $B^m_n = B^i_{n'} = B$. By Lemma 4.4 and the assumption $B^i_{n'} = B^m_n > B^i_{N^i_3}$, $\epsilon^i_{n'-1} = \epsilon^m_{n-1} = 0$. Thus $A^m_n - A^m_{n-1} = \lfloor B\phi \rfloor - \lfloor (B-1)\phi \rfloor + 1$ and $A^i_{n'} - A^i_{n'-1} = \lfloor B\phi \rfloor - \lfloor (B-1)\phi \rfloor - 1$. If $\lfloor B\phi \rfloor - \lfloor (B-1)\phi \rfloor = 1$, $A^i_{n'} - A^i_{n'-1} = 0$; and if $\lfloor B\phi \rfloor - \lfloor (B-1)\phi \rfloor = 2$, $A^m_n - A^m_{n-1} = 3$. Neither of the two cases is possible. So $B^m_{n+1} = B^m_n + 1$ and $|\epsilon^m_{n+1}| \leq 1$ by Lemma 4.4. It is also obvious now that $S^m_1(n) = S^m_1(n+1)$, $B^m_{n+1} = \max(\{B^m_i : i \leq n+1\}$ and $\text{mex}(\{B^m_i : i \leq n+1\} \cup S^m_1(n+1)) = B^m_{n+1} + 1$, therefore we have completed the induction. In this case, $\alpha^m = \alpha'$ and $S^m = S^m_1(n_1)$.

Claim 4.6 *When $m < 10$, $\{(A^m_n, A^m_n + B^m_n)\}_{n \geq 0}$ are special Wythoff's sequences, and thus we have proved the conjectures.*

Proof: By using Claim 4.5, we only need to show that the two integers n_1 and n_2 do exist. Table 4.1 lists results for $m \leq 10$, in which we still use the same notations T^m, S^m, α^m, N^m_1 and N^m_2 as described at the beginning of the section. Complete results and the associated Maple package are available at *http://math.temple.edu/∼xysun/wythoff/wythoff.htm*.

As we can see, the results for $m = 1$ are consistent with the ones predicted by Fraenkel [16], that also appear in Guy and Nowakowski [23], since the 21st and 28th P-positions are $[1, 32, 23]$ and $[1, 44, 30]$ respectively. (Note that our notation differs slightly from that of [16], so some of the signs are reversed).

The method discussed here should be able to be extended to prove the conjectures for Wythoff's games with more than three heaps. A numerical method, instead of the symbolic one presented here, may also be developed to improve the performance, provided Claim 4.3 can be proved without using the generating functions.

m	T^m	S^m	α^m	N_1^m	N_2^m
0			0	1	1
1	1	2, 17, 22	-4	21	28
2		1, 5, 8, 24, 26, 32	-10	28	58
3	1, 2, 3	2, 3, 4, 5, 8, 10, 11, 12, 28, 41, 57	-16	48	73
4	3, 6	2, 5, 6, 7, 8, 9, 11, 12, 14, 17, 46,	-20	126	208
		48, 59	-20	126	208
5	4, 7, 10	1, 3, 5, 6, 8, 10, 11, 12, 15, 16, 17, 18, 19, 28, 56, 77, 83	-26	71	123
6	2, 3, 4, 6	1, 2, 3, 4, 5, 7, 8, 10, 11, 12, 13, 14, 15, 16, 18, 21, 44, 58, 95, 96, 132	-32	113	232
7	1, 4, 6, 7, 9	0, 1, 2, 4, 5, 6, 7, 8, 10, 12, 13, 14, 15, 17, 18, 19, 21, 22, 23, 24, 28, 30, 86, 88, 232, 251	-39	227	343
8	3, 5, 10, 13	1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 13, 14, 16, 17, 19, 20, 21, 22, 24, 26, 27, 28, 33, 34, 46, 155, 257, 390, 415	-46	388	648
9	6, 4, 11, 15	0, 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 18, 19, 20, 24, 25, 26, 27, 30, 36, 37, 44, 48, 62, 254, 388, 421, 676	-52	645	645
10	6, 7, 8, 9, 13	0, 1, 2, 3, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 25, 27, 28, 29, 30, 31, 32, 34, 39, 50, 53, 103, 391, 424, 690	-56	656	656

Table 4.1: Results on Three-Heap Wythoff's Games

CHAPTER 5

WYTHOFF'S SEQUENCE

5.1 Overview

The P-positions of the Wythoff's game are called the Wythoff's pairs, which are pairs of integers $\{(\lfloor n\phi \rfloor, \lfloor n\phi^2 \rfloor)\}_{n\geq 0}$, where $\phi = (1+\sqrt{5})/2$, the golden section, which notation we adopt throughout this chapter. The first few pairs are listed in the following table:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$A_n = \lfloor n\phi \rfloor$	0	1	3	4	6	8	9	11	12	14	16	17	19	21
$B_n = \lfloor n\phi^2 \rfloor$	0	2	5	7	10	13	15	18	20	23	26	28	31	34

Table 5.1: Wythoff's Pairs

Wythoff's pairs have close relationships with the Fibonacci numbers. For example, let us consider the sequence A_1, B_1, A_{B_1}, B_{B_1}, $A_{B_{B_1}}$, $B_{B_{B_1}}$, $\ldots$, which is 1, 2, 3, 5, 8, 13, 21, 34, $\ldots$, which in turn is the Fibonacci sequence without the first number. In fact, any such sequence starting from A_n and B_n is a Fibonacci sequence generated by those two integers, as proved by Hoggatt and Hillman [25], Horadam [26], and Silber [32]. Other properties, relationships and applications were investigated extensively by numerous people, whom we are not going to list here.

In this chapter, we are going to discuss similar sequences and their relationships with Fraenkel's N-heap Wythoff's conjectures. We start with the definition of Wythoff's sequence and its construction in section 5.2, the basic properties of Wythoff's sequence in section 5.3, and special Wythoff's sequence and the equivalency of the two conjectures on N-heap Wythoff's game in section 5.4.

5.2 Wythoff's Sequence

In Chapter 4 we defined the (special) Wythoff's sequence, which we repeat here:

Definition 5.1 *We call a sequence of pairs of integers* $\{(A_n, B_n)\}_{n \geq n_0 > 0}$ *a* Wythoff's sequence *if there exist a finite set of integers* T *such that* $A_n = \text{mex}(\{A_i, B_i : n_0 \leq i < n\} \cup T)$, $B_n = A_n + n$ *and* $\{B_n\} \cap T = \emptyset$.

Definition 5.2 *A* special Wythoff's sequence *is a Wythoff's sequence such that there exist integers* N *and* α *such that when* $n > N$, $A_n = \lfloor n\phi \rfloor + \alpha + \epsilon_n$, *where* $\epsilon_n \in \{0, \pm 1\}$.

When it is not confusing, we will abuse the definition of (special) Wythoff's sequence by replacing the requirement of $B_n = A_n + n$ to $B_{n+1} - A_{n+1} = B_n - A_n + 1$ when $n > 0$ is large enough, because we can easily obtain a Wythoff's sequence by chopping off a number of pairs at the beginning of the sequence and reorganizing the remaining indices.

The following theorem provides another way to create a Wythoff's sequence.

Theorem 5.1 $\{(A_n, B_n)\}$ *is a Wythoff's sequence if and only if there exist two finite sets of integers* S_1 *and* $S_2 \subset \mathbb{Z}_{\geq 0}$, *such that* $A_n = \text{mex}(\{A_i, B_i : i <$

$n\} \cup S_1)$, $B_n = \text{mex}\{\{A_n + B_i - A_i : i < n\} \cup \{A_n + t : t \in S_2\} \cup S_1\}$. In such a case, $S_2 = \mathbb{Z}_{\geq 0} - \{B_i - A_i : i > 0\}$.

Proof: Given S_1 and S_2 are as described above, there exists N_0 such that when $n \geq N_0$, $A_n > \max(S_1)$ and $B_n - A_n > \max(S_2)$. If we write $\alpha_n = \max\{B_i - A_i : i < n\} + 1$ and $D_n = \{i : 0 \leq i < \alpha_n\} - S_2 - \{B_i - A_i : i < n\}$, then for any $n \geq N_0$, it is obvious that $B_n - A_n \leq \alpha_n$ and $\alpha_n \leq \alpha_{n+1}$. Now $B_n - A_n < \alpha_n$ iff $B_n - A_n \in D_n$; iff $\alpha_{n+1} = \alpha_n$; iff $D_n = D_{n+1} \cup \{B_n - A_n\}$. Also, $B_n - A_n = \alpha_n$ iff $\alpha_{n+1} = \alpha_n + 1$; iff $D_{n+1} = D_n$. Note that $D_{n+1} \subset D_n \subset D_{n_0}$ are all finite, so there exists $N \geq N_0$ such that for any $n \geq N$, $D_n = D_{n+1}$, and $B_{n+1} - A_{n+1} = \alpha_{n+1} = \alpha_n + 1 = B_n - A_n + 1$.

Conversely, if $\{(A_n, B_n)\}$ is a Wythoff's sequence, we can define $S_1 = \mathbb{Z}_{\geq 0} - \{A_i, B_i : i > 0\}$ and $S_2 = \mathbb{Z}_{\geq 0} - \{B_i - A_i : i > 0\}$, which are both finite by the definition of the Wythoff's sequence.

For the last part of the theorem, observe that $S_2 \subset \mathbb{Z}_{\geq 0} - \{B_i - A_i : i > 0\}$ by the definition of B_n. If there exists $d \in \mathbb{Z}_{\geq 0} - \{B_i - A_i : i > 0\} - S_2$, $B_n \neq A_n + d$ for all n. When n is large enough such that $B_i - A_i > d$ for all $i \geq n$, there exists $m < n$ such that $A_n + d = B_m$ for each n, since ever large integer has to be an A or B, and $\{A_i\}$ is an ascending sequence when i is large enough by the definitions. Therefore for any n large enough, there exist m_1 and m_2 such that $A_{n+1} - A_n = B_{m_1} - B_{m_2}$. By Lemma 5.1 in the following section, $A_{n+1} - A_n = 2$ and $B_{n+1} - B_n = A_{n+1} - A_n + 1 = 3$. Given any such n, $A_{3n} = 2(3n - n) + A_n = 4n + A_n = 3n + B_n = 3(2n - n) + B_n = B_{2n}$, which is contradictory to Definition 5.1.

So even though we can start with two random finite sets of integers, S_1 and S_2, such that $\{A_n, B_n\} \cap S_1 = \emptyset$ and $\{B_n - A_n\} \cap S_2 = \emptyset$, after some chaotic data at the beginning, the sequence of pairs of integers $\{(A_n, B_n)\}$ defined using mex in the theorem will eventually grow in an orderly manner, and become a Wythoff's sequence.

5.3 Properties of Wythoff's Sequence

From this section and on, for any Wythoff's sequence $\{(A_n, B_n)\}_{n\geq n_0}$, we always assume that when $n \geq n_0$, $A_{n_0} > max(T)$ as in Definition 5.1; or equivalently, $A_{n_0} > \max(S_1)$, $B_{n_0} > A_{n_0} + \max(S_2) + 1$ and $B_{n+1} - A_{n+1} = B_n - A_n + 1$ as in Theorem 5.1. Otherwise, we can always increase the value of n_0 and the sizes of T, S_1 and S_2 by eliminating the early entries of the sequence.

Lemma 5.1 *Given a Wythoff's sequence* $\{(A_n, B_n)\}_{n\geq n_0}$*,*

1. $1 \leq A_{n+1} - A_n \leq 2$*,*
2. $2 \leq B_{n+1} - B_n \leq 3$*, and*
3. $||\lfloor n_1\phi\rfloor - \lfloor n_2\phi\rfloor - (n_1 - n_2)\phi| < 1$.

Proof: See Chapter 4.

Theorem 5.2 *Given a Wythoff's sequence* $\{(A_n, B_n)\}_{n\geq n_0}$*, there exists a constant c, such that* $A_{A_n+c} = B_n - 1$ *and* $A_{B_n+c} = B_{A_n+c} + 1 = A_n + B_n + c$*.*

Proof: By lemma 5.1, there exists k_0 such that $A_{k_0} = B_{n_0} - 1$. Consider all the integers from 1 to $B_{n_0} - 1$, there are $(k_0 - n_0 + 1)$ A's, no B's, and $|T|$ T's, which means $B_{n_0} - 1 = |T| + k_0 - n_0 + 1$. Let $c = k_0 - A_{n_0}$. We now have $|T| = B_{n_0} - 1 - k_0 + n_0 - 1 = A_{n_0} - k_0 + 2n_0 - 2 = 2n_0 - 2 - c$.

Now for any $n \geq n_0$, there exists $A_k = B_n - 1$. Consider all the integers from 1 to B_n, there are $(k - n_0 + 1)$ A's, $(n - n_0 + 1)$ B's, and $|T|$ T's, so $B_n = k - n_0 + 1 + n - n_0 + 1 + |T| = k + n - c$, hence $k = B_n - n + c = A_n + c$. Therefore $B_n = A_k + 1 = A_{A_n+c} + 1$.

Consider all the integers from 1 to B_{A_n+c}, there are (A_n+c-n_0+1) B's and $|T|$ T's, so there are $(B_{A_n+c} - A_n - c + n_0 - 1 - |T|) = (A_{A_n+c} + n_0 - 1 - |T|)$ A's, the largest of which is $A_{k'} = B_{A_n+c} - 1$. So k' must be $A_{A_n+c} + n_0 - 1 - |T| + n_0 - 1 = B_n - 1 + 2n_0 - 2 - (2n_0 - 2 - c) = B_n + c - 1$. By Lemma 5.1 and the previous result, $A_{B_n+c} = A_{k'+1} = B_{A_n+c} + 1 = A_{A_n+c} + A_n + c + 1 = A_n + B_n + c$.

Corollary 5.1 *Given a Wythoff's sequence* $\{(A_n, B_n)\}_{n\geq n_0}$ *and* c *as in Theorem 5.2,* $A_{A_n+c+1} - A_{A_n+c} = 2$*;* $A_{B_n+c+1} - A_{B_n+c} = 1$*;* $B_{A_n+c+1} - B_{A_n+c} = 3$*;* $B_{B_n+c+1} - B_{B_n+c} = 2$.

Proof: $A_{m+c+1} - A_{m+c} = 2$ iff there exists n such that $A_{m+c+1} - 1 = B_n = A_{m+c} + 1$; iff $A_{m+c} = A_{A_n+c}$; iff $m = A_n$. The rest of the equations are obvious from the preceding fact.

Notice that if there exist $m_1 > m_2 > n_0$ such that $A_{m_1} \geq B_{m_2}$ and we know $\{(A_n, B_n) : m_2 \leq n \leq m_1\}$, we can construct the sequence for $m > m_1$ without using the definition of the Wythoff's sequence, i.e., mex. There are two ways of doing so recursively:

1. For any $m > m_1$, by Theorem 5.2, if $m - c$ is of the form $A_{m'}$, $A_m = A_{m'} + m' - 1$ and $B_m = m + B_{m'} - 1$; otherwise, $m - c = B_{m'}$, $A_m = A_{m'} + m$ and $B_m = B_{m'} + 2m - m'$.

2. If A_m is known and if $m - c$ is in the A's, by Corollary 5.1, $A_{m+1} = A_m + 2$ and $B_{m+1} = B_m + 3$; otherwise, $A_{m+1} = A_m + 1$ and $B_{m+1} = B_m + 2$. Here we can see that the two sequences are self-generating, i.e., we can construct the sequence of $\{A_n\}_{n\geq m_2}$ or $\{B_n\}_{n\geq m_2}$ without any knowledge of the other.

Corollary 5.2 *Given a Wythoff's sequence* $\{(A_n, B_n)\}_{n\geq n_0}$*, then for any* $n \geq A_{n_0}$*, the number of A's less than* n *is* $A_{n+c} - n - n_0 + 1$*; for any* $n \geq B_{n_0}$*, the number of B's less than* n *is* $2n - A_{n+c} + c - n_0$.

Proof: Let $f(n) = A_{n+c} - n - n_0 + 1$. We claim that $f(n)$ is the number of A's less than n. First, $f(A_{n_0}) = A_{A_{n_0}+c} - A_{n_0} - n_0 + 1 = B_{n_0} - A_{n_0} - n_0 = 0$, which is the number of A's less than A_{n_0}. By induction, if the claim is true for $n - 1$, there are two cases: if $n - 1 = B_m$, by Corollary 5.1, $f(n) = A_{B_m+1+c} - (B_m + 1) - n_0 + 1 = A_{B_m+c} - B_m - n_0 + 1 = f(n-1)$; if $n - 1 = A_m$, $f(n) = A_{A_m+1+c} - (A_m + 1) - n_0 + 1 = A_{A_m+c} + 2 - A_m - n_0 = f(n-1) + 1$. So the claim is proved. On the other hand, if we write $g(n) = 2n - A_{n+c} + c - n_0$,

$g(B_{n_0}) = 2B_{n_0} - A_{B_{n_0}+c} + c - n_0 = 2B_{n_0} - A_{n_0} - B_{n_0} - c + c - n_0 = 0$. If $n > B_{n_0}$ and if $n-1 = B_m$, $g(n) = 2n - A_{B_m+1+c} - n_0 = 2n - A_{B_m+c} - 1 - n_0 = g(n-1)+1$; if $n - 1 = A_m$, $g(n) = 2n - A_{A_m+1+c} - n_0 = 2n - A_{A_m+c} - 2 - n_0 = g(n-1)$. So $g(n)$ is the number of B's less than n.

A special case of the theorem and corollaries is when the Wythoff's sequence is the original Wythoff's pairs. In such an occasion, $n_0 = 0$ and $c = 0$, which were proved by Hoggatt and Hillman [25], Hoggatt and Bicknell-Johnson [24], and Silber [32].

5.4 Special Wythoff's Sequence and N-heap Wythoff's Conjectures

Throughout this section we use Wythoff's sequence $\{(A_n, B_n)\}_{n \geq n_0}$ and c as in Theorem 5.2. Note that when n is large enough, it must be of the form A_{A_m}, A_{B_m}, B_{A_m}, or B_{B_m}. Since for any m, there exist m_1 and m_2 such that $A_m = B_{m_1} \pm 1$ and $B_m = A_{m_2} + 1$, so n must be of the form $B_{A_{m+c+\epsilon_2}} + c + \epsilon_1$, where $\epsilon_1 \in \{-1, 0, 1\}$ and $\epsilon_2 \in \{0, 1\}$.

Theorem 5.3 *Every Wythoff's sequence is special.*

Proof: Let $\alpha_n = A_n - \lfloor n\phi \rfloor$. We only need to prove that as m and n grow, $|\alpha_m - \alpha_n|$ eventually decreases to at most 2.

By Corollary 5.1, $A_{B_n+c+1} - A_{B_n+c} - \phi = 1 - \phi$ and $A_{B_n+c-1} - A_{B_n+c} + \phi = -2 + \phi$, so $A_{B_n+c+\epsilon} - A_{B_n+c} - \phi\epsilon = (3\epsilon - 2\phi\epsilon - \epsilon^2)/2$ when $|\epsilon| \leq 1$. Therefore if we write $\gamma = (A_{B_m+c+\epsilon_m} - A_{B_m+c} - \phi\epsilon_m) - (A_{B_n+c+\epsilon_n} - A_{B_n+c} - \phi\epsilon_n)$, $|\gamma| = |(\epsilon_m - \epsilon_n)(3 - 2\phi - \epsilon_m - \epsilon_n)/2| \leq \phi - 1$, when $|\epsilon_m|$, $|\epsilon_n| \leq 1$. Also note that $A_{A_n+c+\epsilon} - A_{A_n+c} = 2\epsilon$, when $\epsilon \in \{0, 1\}$.

We also adopt the following notation: $\beta_1 = \lfloor (B_{A_{m+c+\epsilon_{2m}}} + c + \epsilon_{1m})\phi \rfloor - \lfloor (B_{A_{n+c+\epsilon_{2n}}} + c + \epsilon_{1n})\phi \rfloor - ((B_{A_{m+c+\epsilon_{2m}}} + c + \epsilon_{1m})\phi - (B_{A_{n+c+\epsilon_{2n}}} + c + \epsilon_{1n})\phi)$ and $\beta_2 = \lfloor m\phi \rfloor - \lfloor n\phi \rfloor - (m - n)\phi$.

Now if $\epsilon_{1m}, \epsilon_{1n} \in \{-1, 0, 1\}$ and $\epsilon_{2m}, \epsilon_{2n} \in \{0, 1\}$,

$$
\begin{aligned}
& \alpha_{B_{A_m+c+\epsilon_{2m}}+c+\epsilon_{1m}} - \alpha_{B_{A_n+c+\epsilon_{2n}}+c+\epsilon_{1n}} \\
= \ & A_{B_{A_m+c+\epsilon_{2m}}+c+\epsilon_{1m}} - A_{B_{A_n+c+\epsilon_{2n}}+c+\epsilon_{1n}} \\
& -(\lfloor (B_{A_m+c+\epsilon_{2m}} + c + \epsilon_{1m})\phi \rfloor - \lfloor (B_{A_n+c+\epsilon_{2n}} + c + \epsilon_{1n})\phi \rfloor) \\
= \ & A_{B_{A_m+c+\epsilon_{2m}}+c+\epsilon_{1m}} - A_{B_{A_n+c+\epsilon_{2n}}+c+\epsilon_{1n}} \\
& -((B_{A_m+c+\epsilon_{2m}} - B_{A_n+c+\epsilon_{2n}})\phi + (\epsilon_{1m} - \epsilon_{1n})\phi + \beta_1) \\
= \ & A_{B_{A_m+c+\epsilon_{2m}}+c} - A_{B_{A_n+c+\epsilon_{2n}}+c} + \gamma - (B_{A_m+c+\epsilon_{2m}} - B_{A_n+c+\epsilon_{2n}})\phi - \beta_1 \\
= \ & A_{A_m+c+\epsilon_{2m}} - A_{A_n+c+\epsilon_{2n}} + (B_{A_m+c+\epsilon_{2m}} - B_{A_n+c+\epsilon_{2n}})(1-\phi) + \gamma - \beta_1 \\
= \ & (A_{A_m+c+\epsilon_{2m}} - A_{A_n+c+\epsilon_{2n}})(2-\phi) \\
& +(A_m - A_n + \epsilon_{2m} - \epsilon_{2n})(1-\phi) + \gamma - \beta_1 \\
= \ & (A_{A_m+c} - A_{A_n+c} + 2(\epsilon_{2m} - \epsilon_{2n}))(2-\phi) \\
& +(A_m - A_n + \epsilon_{2m} - \epsilon_{2n})(1-\phi) + \gamma - \beta_1 \\
= \ & (A_{A_m+c} - A_{A_n+c})(2-\phi) + (A_m - A_n)(1-\phi) + (\epsilon_{2m} - \epsilon_{2n})(5 - 3\phi) \\
& +\gamma - \beta_1 \\
= \ & (B_m - B_n)(2-\phi) + (A_m - A_n)(1-\phi) + (\epsilon_{2m} - \epsilon_{2n})(5-3\phi) \\
& +\gamma - \beta_1 \\
= \ & (A_m - A_n)(3 - 2\phi) + (m-n)(2-\phi) \\
& +(\epsilon_{2m} - \epsilon_{2n})(5-3\phi) + \gamma - \beta_1 \\
= \ & (\lfloor m\phi \rfloor + \alpha_m - \lfloor n\phi \rfloor - \alpha_n)(3-2\phi) + (m-n)(2-\phi) \\
& +(\epsilon_{2m} - \epsilon_{2n})(5-3\phi) + \gamma - \beta_1 \\
= \ & ((m-n)\phi + \beta_2 + (\alpha_m - \alpha_n))(3-2\phi) + (m-n)(2-\phi) \\
& +(\epsilon_{2m} - \epsilon_{2n})(5-3\phi) + \gamma - \beta_1 \\
= \ & -(\alpha_m - \alpha_n)(2\phi - 3) - \beta_2(2\phi - 3) + (\epsilon_{2m} - \epsilon_{2n})(5 - 3\phi) + \gamma - \beta_1.
\end{aligned}
$$

$$
\begin{aligned}
\text{So} \quad & |\alpha_{B_{A_m+c+\epsilon_{2m}}+c+\epsilon_{1m}} - \alpha_{B_{A_n+c+\epsilon_{2n}}+c+\epsilon_{1n}}| \\
\leq \quad & |\alpha_m - \alpha_n|(2\phi - 3) + |\beta_2|(2\phi - 3) + |\epsilon_{2m} - \epsilon_{2n}|(5 - 3\phi) + |\gamma| + |\beta_1| \\
< \quad & |\alpha_m - \alpha_n|(2\phi - 3) + (2\phi - 3) + (5 - 3\phi) + \phi - 1 + 1 \\
= \quad & |\alpha_m - \alpha_n|(2\phi - 3) + 2.
\end{aligned}
$$

Since it is an integer, $|\alpha_{B_{A_m+c+\epsilon_{2m}}+c+\epsilon_{1m}} - \alpha_{B_{A_n+c+\epsilon_{2n}}+c+\epsilon_{1n}}| \leq \max(|\alpha_m - \alpha_n| - 1, 2)$.

Now for any integers m and n, we can construct two sequences $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$, such that $a_k = m$, $b_k = n$, $A_{n_0} \leq \min(a_1, b_1) < B_{A_{n_0}+c+1} + 1$, and $a_i = B_{A_{a_{i-1}}+c+\epsilon_{a2i}} + \epsilon_{a1i}$, $b_i = B_{A_{b_{i-1}}+c+\epsilon_{b2i}} + \epsilon_{b1i}$, where $1 < i \leq k$, $\epsilon_{a1i}, \epsilon_{b1i} \in \{0, \pm 1\}$, and $\epsilon_{a2i}, \epsilon_{b2i} \in \{0, 1\}$. Hence if $\max(|\alpha_i - \alpha_j| : i, j \geq 1) = N$ is finite, then when $k \geq N$, or equivalently when m and n are large enough, $|\alpha_m - \alpha_n| = |\alpha_{a_k} - \alpha_{b_k}|$ decreases to at most 2. The assumption is proved in the following lemma, which completes our proof.

Lemma 5.2 *α_n is bounded for all n.*

Proof:

Let $\beta_3 = (\lfloor A_m\phi \rfloor - \lfloor A_n\phi \rfloor) - (A_m - A_n)\phi$, and $\beta_4 = (\lfloor m\phi \rfloor - \lfloor n\phi \rfloor) - (m-n)\phi$.

$$\begin{aligned}
& \alpha_{A_m} - \alpha_{A_n} \\
= \; & A_{A_m} - A_{A_n} - (\lfloor A_m\phi \rfloor - \lfloor A_n\phi \rfloor) \\
= \; & B_m - B_n - (A_m - A_n)\phi - \beta_3 \\
= \; & (A_m - A_n)(1-\phi) + (m-n) - \beta_3 \\
= \; & (\lfloor m\phi \rfloor - \lfloor n\phi \rfloor + \alpha_m - \alpha_n)(1-\phi) + (m-n) - \beta_3 \\
= \; & ((m-n)\phi + \beta_4 + \alpha_m - \alpha_n)(1-\phi) + (m-n) - \beta_3 \\
= \; & -(\alpha_m - \alpha_n)(\phi - 1) - \beta_3 - \beta_4(\phi - 1).
\end{aligned}$$

Define $\delta_0 = 1$ and $\delta_i = 1 - (\phi - 1)\delta_{i-1}$ recursively for $i \geq 1$. If we write $\delta_n = (1-\phi)^n g_n$, $g_n = g_{n-1} + 1/(1-\phi)^n$, i.e., $g_n = \sum_{i=1}^{n} 1/(1-\phi)^i = ((1-\phi)^{n+1} - 1)/(-\phi(1-\phi)^n)$. So

$$\delta_n = (1-\phi)^n g_n = \phi - 1 + (1-\phi)^{n+2}.$$

Hence $\delta_n \to \phi - 1$, as $n \to \infty$, and $|\delta_n| \leq 1$.

Note that for any integer m, we can construct a sequence $a_1, \ldots, a_k$, such that $a_k = m$, $A_{n_0} \leq a_1 < A_{A_{n_0}}$, $a_i = A_{a_{i-1}}$, where $1 < i \leq k$.

Let $\beta_3^i = (\lfloor a_i\phi \rfloor - \lfloor a_{i-1}\phi \rfloor) - (a_i - a_{i-1})\phi$, and $\beta_4^i = (\lfloor a_{i-1}\phi \rfloor - \lfloor a_{i-2}\phi \rfloor) - (a_{i-1} - a_{i-2})\phi = \beta_3^{i-1}$, then

$$\alpha_{a_i} - \alpha_{a_{i-1}} = -(\alpha_{a_{i-1}} - \alpha_{a_{i-2}})(\phi - 1) - \beta_3^i - \beta_4^i, \; 1 < i \leq k$$

by the previous result.

$$
\begin{aligned}
\text{Now} \quad & \alpha_m \\
= \quad & \alpha_{a_1} + \textstyle\sum_{i=2}^{k}(\alpha_{a_i} - \alpha_{a_{i-1}}) \\
= \quad & \alpha_{a_1} - (\alpha_{a_{k-1}} - \alpha_{a_{k-2}})(\phi - 1) - \beta_3^k - \beta_4^k(\phi - 1) \\
& + \textstyle\sum_{i=2}^{k-1}(\alpha_{a_i} - \alpha_{a_{i-1}}) \\
= \quad & \alpha_{a_1} - (\beta_3^k + \beta_4^k(\phi - 1))\delta_0 + (\alpha_{a_{k-1}} - \alpha_{a_{k-2}})\delta_1 \\
& + \textstyle\sum_{i=2}^{k-2}(\alpha_{a_i} - \alpha_{a_{i-1}}) \\
= \quad & \alpha_{a_1} - (\beta_3^k + \beta_4^k(\phi - 1))\delta_0 - (\beta_3^{k-1} + \beta_4^{k-1}(\phi - 1))\delta_1 \\
& + (\alpha_{a_{k-2}} - \alpha_{a_{k-3}})\delta_2 + \textstyle\sum_{i=2}^{k-3}(\alpha_{a_i} - \alpha_{a_{i-1}}) \\
= \quad & \cdots \\
= \quad & \alpha_{a_1} - \textstyle\sum_{i=3}^{k}((\beta_3^i + \beta_4^i(\phi - 1))\delta_{k-i}) + (\alpha_{a_2} - \alpha_{a_1})(\delta_{k-2} + 1).
\end{aligned}
$$

We also have

$$
\begin{aligned}
& |\textstyle\sum_{i=3}^{k}(\beta_3^i \delta_{k-i})| \\
= \quad & |\textstyle\sum_{i=3}^{k}((\lfloor a_i\phi \rfloor - a_i\phi)\delta_{k-i}) - \textstyle\sum_{i=3}^{k}((\lfloor a_{i-1}\phi \rfloor - a_{i-1}\phi))\delta_{k-i})| \\
= \quad & |\textstyle\sum_{i=3}^{k}((\lfloor a_i\phi \rfloor - a_i\phi)\delta_{k-i}) - \textstyle\sum_{i=2}^{k-1}((\lfloor a_i\phi \rfloor - a_i\phi))\delta_{k-i-1})| \\
= \quad & |\textstyle\sum_{i=3}^{k-1}((\lfloor a_i\phi \rfloor - a_i\phi)(\delta_{k-i} - \delta_{k-i-1})) + (\lfloor a_k\phi \rfloor - a_k\phi)\delta_0) \\
& -(\lfloor a_2\phi \rfloor - a_2\phi))\delta_{k-3})| \\
\leq \quad & \textstyle\sum_{i=3}^{k-1}(|\lfloor a_i\phi \rfloor - a_i\phi||\delta_{k-i} - \delta_{k-i-1}|) + |(\lfloor a_k\phi \rfloor - a_k\phi)\delta_0| \\
& +|(\lfloor a_2\phi \rfloor - a_2\phi)\delta_{k-3}| \\
< \quad & \textstyle\sum_{i=3}^{k-1} |\delta_{k-i} - \delta_{k-i-1}| + \delta_0 + \delta_{k-3} \\
= \quad & \textstyle\sum_{i=3}^{k-1} |(1-\phi)^{k-i+2} - (1-\phi)^{k-i+1}| + \delta_0 + \delta_{k-3} \\
< \quad & 2\textstyle\sum_{i=0}^{\infty}(\phi - 1)^i + 2 \\
= \quad & 2\phi + 4.
\end{aligned}
$$

Similarly, $|\sum_{i=3}^{k}(\beta_4^i(\phi - 1)\delta_{k-i})|$ has a constant upper bound too, which means α_m is bounded by a value determined only by the values of α_{a_1} and α_{a_2}, regardless of the values of m (or k). Since there are only finitely many choices of a_1 and a_2, α_m is bounded for all m.

From the proof of Lemma 5.2, we can see that when $|\alpha_m - \alpha_n| \geq 3$, $(\alpha_{A_m} - \alpha_{A_n})$ and $(\alpha_m - \alpha_n)$ always have different signs. $(\lfloor (m+1)\phi \rfloor - \lfloor m\phi \rfloor)| \leq 1$, Let us consider $\alpha(m) = \alpha_m$ as a function. The graph of the function is a set of discrete points that oscillate. The amplitude of graph, if we are allowed to abuse the word, decreases slowly but persistently as m grows. By Theorem 5.3, the amplitude eventually decreases to 1, when the oscillation of the graph becomes somewhat unpredictable.

Lemma 5.3 *In the two conjectures on the N-heap Wythoff's game, $A_n^{N-1} = \operatorname{mex}(\{A_i^{N-1}, A_i^N : 0 \leq i < n\} \cup T)$, where T is a finite set depending only on $A^1, \ldots, A^{N-2}$. In fact, $T = \{a : \exists b \leq A^{N-2}$ and $k \leq N-2$, such that $A^{k-1} \leq b \leq A^k$ and $(A^1, \ldots, A^{k-1}, b, A^k, \ldots, A^{N-2}, a)$ is a P-position$\}$.*

Proof: By definition, $T = \mathbb{Z}_{\geq 0} - \{A_i^{N-1}, A_i^N : i > 0\}$. Write T' as the last set in the lemma. We claim $T = T'$. First, $T' \subset T$ because for any $b \geq a$, $(A^1, \ldots, A^{N-2}, a, b)$ is an N-position since we can remove tokens from the last pile to create a P-position; and similar argument can be applied when $A^{N-2} < b < a$.

For any $a \in T$ and $b \geq a$, $(A^1, \ldots, A^{N-2}, a, b)$ is an N-position by the definition of T. There are several kind of moves from this position to create a P-position:

1. Remove $a_1, \ldots, a_N$ tokens from all corresponding piles, where $a_1 \oplus \cdots \oplus a_N = 0$, so that $(A^1 - a_1, \ldots, A^{N-2} - a_{N-2}, a - a_{N-1}, b - a_N)$ is a P-position.

2. Remove $a_k \leq A^k$ tokens from the k-th pile, so that $(A^1, \ldots, A^{k-1}, A^k - a_k, A^{k+1}, \ldots, A^{N-2}, a, b)$ is a P-position;

3. Remove $a_{N-1} \leq a$ tokens from the $(N-1)$-th pile, so that $(A^1, \ldots, A^{N-2}, a - a_{N-1}, b)$ is a P-position;

4. Remove $a_N \leq b$ tokens from the N-th pile, so that $(A^1, \ldots, A^{N-2}, a, b - a_N)$ is a P-position;

There are only finitely many possible moves using the first three kinds of moves, but there are infinitely many choices of b, so there exists an integer b' such that $(A^1, \ldots, A^{N-2}, a, b-b')$ is a P-position. Again by the definition of T and by the convention that we adopted: $A^1 \leq \cdots \leq A^{N-2} \leq a \leq b$, we must have $b - b' \leq A^{N-2}$, which shows $T \subset T'$. T' is obviously finite, since $A^1 \leq \cdots \leq A^{N-2}$ are finite.

Now let $a = \text{mex}(\{A_i^{N-1}, A_i^N : 0 \leq i < n\} \cup T)$. It is obvious that $A_n^{N-1} \geq a$. Also, by the definition of T and mex, $(A^1, \ldots, A^{N-2}, c, a)$ is an N-positionfor any $A^{N-2} \leq c < a$, but there must exist an integer $b \geq a$ such that $(A^1, \ldots, A^{N-2}, a, b)$ is a P-position. Finally, since we assume $A_{n-1}^{N-1} < A_n^{N-1}$, if there exists b such that $A_{n-1}^{N-1} < b < A_n^{N-1}$, we must have either $b \in T$ or $b \in \{A_i^N\}_{i\geq 1}$. Combining all the arguments above, $a = A_n^{N-1}$.

Corollary 5.3 *The two conjectures on the N-heap Wythoff's game are equivalent.*

Proof: Conjecture 1, together with the previous lemma, states that the P-positions for any given m form a Wythoff's sequence, while Conjecture 2 states further that it is a special Wythoff's sequence. The result follows from Theorem 5.3.

Theorem 5.4 *Given a Wythoff's sequence $\{(A_n, B_n)\}_{n\geq n_0}$ and α are as in Definition 5.2, $\alpha = -c$.*

Proof: Let $\beta_5 = \lfloor (A_n + c)\phi \rfloor - (A_n + c)\phi$ and $\beta_6 = \lfloor n\phi \rfloor - n\phi$. Then

$$A_n + n - 1 = B_n - 1 = A_{A_n+c} = \lfloor (A_n+c)\phi \rfloor + \alpha + \epsilon_{A_n+c} = A_n\phi + c\phi + \alpha + \epsilon_{A_n+c} + \beta_5.$$

So

$$\begin{aligned} \epsilon_{A_n+c} + \beta_5 &= A_n(1-\phi) - 1 + n - c\phi - \alpha \\ &= (n\phi + \alpha + \epsilon_n + \beta_6)(1-\phi) + n - c\phi - \alpha - 1 \\ &= -(c+\alpha)\phi + (\epsilon_n + \beta_6)(1-\phi) - 1, \end{aligned}$$

hence

$$-(c+\alpha)\phi = \epsilon_{A_n+c} + \beta_5 + (\epsilon_n + \beta_6)(\phi - 1) + 1.$$

Note that the left-hand side of the last equation does not depend on the choice of n, while the right-hand side does. The theorem is proved if we can make the right choice of n so that the absolute value of the right-hand side is less than ϕ. Note that $-1 < \beta_5, \beta_6 < 0$, and $\epsilon_{A_n+c}, \epsilon_n \in \{0, \pm 1\}$, therefore the proof is completed if we can find an integer N such that

$$\epsilon_{A_N+c} = 0; \quad or \tag{5.1}$$

$$\epsilon_N = 0 \; and \; \epsilon_{A_N+c} \in \{0, -1\}; \quad or \tag{5.2}$$

$$\epsilon_N = -1 \; and \; \epsilon_{A_N+c} \in \{0, 1\}. \tag{5.3}$$

First we can assume ϵ_n is not a constant, otherwise we can adjust the value of α so that ϵ_n is always 0. Secondly, since $|\epsilon_n| \leq 1$ and $|\epsilon_n - \epsilon_{n-1}| \leq |(A_n - A_{n-1}) - (\lfloor n\phi \rfloor - \lfloor (n-1)\phi \rfloor)| \leq 1$, there always exists an n large enough such that $\epsilon_n = 0$. By the condition 5.2 above, we only have to consider the case when $\epsilon_n = 0$ and $\epsilon_{A_n+c} = 1$. From now on, we always assume n is large enough.

If $A_n = A_{n-1} + 1$, by Corollary 5.1, there exist m such that $n = B_m + 1 + c$; and by Lemma 5.1, there exists m' such that $B_m + 1 = A_{m'}$. Therefore, $\epsilon_{A_{m'}+c} = \epsilon_n = 0$, which proves the theorem by choosing $N = m'$ and using condition 5.1.

If $A_n = A_{n-1} + 2$, $3 \leq \lfloor (A_n + c)\phi \rfloor - \lfloor (A_{n-1} + c)\phi \rfloor \leq 4$. There also exists m such that $A_{n-1} + 1 = B_m = A_{A_m+c} + 1$, therefore $n - 1 = A_m + c$. Furthermore $A_{A_n+c} - A_{A_{n-1}+c} = (A_{B_m+1+c} - A_{B_m+c}) + (A_{B_m+c} - A_{B_m-1+c}) = 3$ by Corollary 5.1, which means $\lfloor (A_n + c)\phi \rfloor - \lfloor (A_{n-1} + c)\phi \rfloor + \epsilon_{A_n+c} - \epsilon_{A_{n-1}+c} = 3$. Therefore $\lfloor (A_n + c)\phi \rfloor - \lfloor (A_{n-1} + c)\phi \rfloor = 3$ and $\epsilon_{A_{n-1}+c} = 1$, because of our assumption of $\epsilon_{A_n+c} = 1$. Since $2 = A_n - A_{n-1} = \lfloor n\phi \rfloor - \lfloor (n-1)\phi \rfloor - \epsilon_{n-1}$, we have either $\lfloor n\phi \rfloor - \lfloor (n-1)\phi \rfloor = 1$ and $\epsilon_{n-1} = -1$, or $\lfloor n\phi \rfloor - \lfloor (n-1)\phi \rfloor = 2$ and $\epsilon_{n-1} = 0$. In the former case we can prove the theorem by choosing $N = n - 1$ and using condition 5.3 because $\epsilon_{n-1} = -1$ and $\epsilon_{A_{n-1}+c} = 1$; while in the latter case $\epsilon_{A_m+c} = \epsilon_{n-1} = 0$, so we can choose $N = m$ and use condition 5.1.

Theorem 5.2, together with the comments at the end of the section 5.3, indicates that any Wythoff's sequence is "shifted" Wythoff pairs. It also maintains the relationship with the golden section with another "shift" α and some "controlled error" ϵ. Theorem 5.4 tells us the values of the two shifts are in fact the same. The fact can be seen from the following example: Given any integer a, consider the sequence $\{(A_n = \lfloor n\phi \rfloor + a, B_n = \lfloor n\phi \rfloor + n + a)\}$, with n large enough. The sequence obviously is a special Wythoff's sequence with $\alpha = a$, because it is generated from the Wythoff's pairs. At the mean time, $A_{A_n-a} = A_{\lfloor n\phi \rfloor} = \lfloor \lfloor n\phi \rfloor \phi \rfloor + a = \lfloor n\phi \rfloor + n - 1 + a = B_n - 1$, where the equation in the middle can be derived from the fact that the constant c for the Wythoff's pairs is 0, or from [2]. Similarly, $A_{B_n-a} = A_{\lfloor n\phi \rfloor + n} = \lfloor (\lfloor n\phi \rfloor + n)\phi \rfloor + a = 2\lfloor n\phi \rfloor + n + a = A_n + B_n - a$. So the constant c for the sequence is $-a = -\alpha$.

To determine the value of α for any Wythoff's sequence, instead of calculating a large number of pairs of integers as the definition requires, we only need the pairs at the beginning of the sequence. As shown in the proof of Theorem 5.2, all we need to know is the integer k such that $A_k = B_{n_0} - 1$, which is to find all the A's less than B_{n_0}. So by using the notation in the proof of Corollary 5.2, $f(B_{n_0}) = A_{B_{n_0}+c} - B_{n_0} - n_0 + 1 = A_{n_0} - n_0 + 1 + c = B_{n_0} - 2n_0 + 1 + c$, therefore it only requires the values of roughly $B_{n_0} - 2n_0 + 1$ pairs of integers.

5.5 Remarks

Lemma 5.1 and Corollary 5.3, and maybe some others, were also independently discovered by Aviezri Fraenkel and Dalia Krieger. This chapter benefited tremendously from their comments and suggestions on the draft copy, which have made it much clearer and much more readable.

CHAPTER 6

TERNARY SQUARE-FREE WORDS

6.1 Introduction to Words

A word w is a finite sequence of letters from a certain alphabet Σ. The length of a word is the number of letters of the word. English as a language is a set of words from an alphabet of twenty-seven letters, i.e., {a, b, ..., y, z}. But our words are not limited to the words we encounter in English every day. For example, a DNA sequence is a word of four letters: {A, C, G, T}. A function in a Maple program can also be considered as a word whose alphabet consists of the keywords such as {if, then, else, while, do, ...} and user defined variables.

Binary words are the words from a two-letter alphabet {0, 1}, whereas ternary words are from a three-letter alphabet {0, 1, 2}. A word is square-free if it does not contain two identical consecutive subwords (factors), i.e., w cannot be written as $axxb$ where a, b, x are words with x non-empty.

It is easy to see that there are only finitely many binary square-free words. However, there are infinitely many ternary square-free words. The fact was proved by utilizing what is now called the Prouhet-Thue-Morse sequence (see in [29]). Brinkhuis [7], Brandenburg [6] (also in [1]), Zeilberger [13] and Grimm [21] showed that the numbers of such words of length n are greater than $2^{n/24}$, $2^{n/21}$, $2^{n/17}$, and $65^{n/40}$ respectively. Details on words and related topics can be found at [14] and [33].

While the best available upper bound has been very close to the estimate as described later, the available lower bounds still have much room for improvement. Finding better lower bounds has posed as a algorithmic challenge, as well as a theoretic one. As explained later, the complexity of the algorithm used here is likely (very) exponential.

6.2 Brinkhuis Triples

We denote $a(n)$ to be the number of ternary square-free words of length n. It is easy to see that

$$a(m+n) \leq a(m)a(n) \tag{6.1}$$

for all $m,\ n \geq 0$, which implies (see in [1]) the existence of the limit

$$s := \lim_{n\to\infty} a(n)^{1/n}, \tag{6.2}$$

which is also called the growth rate or "connective constant" of ternary square-free words.

It is widely believed that the available upper bounds are very close to the actual value of s. In fact, it has been estimated by Noonan and Zeilberger in [30] that $s \approx 1.302$ using Zinn-Justin's method and they have also proved that $s \leq 1.30201064$ by implementing the Golden-Jackson method.

Definition 6.1 *An n-Brinkhuis k-triple is three sets of words $\mathcal{B} = \{\mathcal{B}^0,\ \mathcal{B}^1,\ \mathcal{B}^2\}$, $\mathcal{B}^i = \{w^i_j | 1 \leq j \leq k\}$, where w^i_j are square-free words of length n, such that for any square-free word $i_1i_2i_3$, $0 \leq i_1, i_2, i_3 \leq 2$, and any $1 \leq j_1, j_2, j_3 \leq k$, the word $w^{i_1}_{j_1}w^{i_2}_{j_2}w^{i_3}_{j_3}$ of length $3n$ is also square-free.*

Based on an n-Brinkhuis k-triple, we can define the following set of uniformly growing morphisms:

$$\rho = \begin{cases} 0 \to w^0_{j_0},\ 1 \leq j_0 \leq k; \\ 1 \to w^1_{j_1},\ 1 \leq j_1 \leq k; \\ 2 \to w^2_{j_2},\ 1 \leq j_2 \leq k. \end{cases} \tag{6.3}$$

As proven in [6], [11] and [28], ρ are square-free morphisms, i.e., they map each square-free word of length m onto k^m different images of square-free words of length nm.

Therefore, the existence of an n-Brinkhuis k-triple indicates that

$$\frac{a(mn)}{a(m)} \geq k^m \tag{6.4}$$

for any $m \geq 1$, which implies

$$s^{n-1} = \lim_{n\to\infty} \left(\frac{a(mn)}{a(m)}\right)^{1/m} \geq k, \tag{6.5}$$

and thus yields the lower bound of $s \geq k^{1/(n-1)}$.

Given the permutation $\tau = (0,\, 1,\, 2)$, we can have

Definition 6.2 *A quasi-special n-Brinkhuis k-triple is an n-Brinkhuis k-triple such that $\mathcal{B}^1 = \tau(\mathcal{B}^0)$, $\mathcal{B}^2 = \tau(\mathcal{B}^1)$.*

Definition 6.3 *A special n-Brinkhuis k-triple is a quasi-special n-Brinkhuis k-triple such that $w \in \mathcal{B}^0$ implies $\overline{w} \in \mathcal{B}^0$, where $\overline{w}$ is the reversion of w.*

Grimm [21] was able to construct a special 41-Brinkhuis 65-triple, hence proved $s \geq 65^{1/40}$.

6.3 Lower Bound of the Connective Constant

Definition 6.4 *A word w is admissible if $(w, \tau(w), \tau^2(w))$ is a quasi-special Brinkhuis 1-triple by itself.*

Definition 6.5 *An optimal quasi-special (special) n-Brinkhuis k-triple is a quasi-special (special) n-Brinkhuis k-triple such that any quasi-special (special) n-Brinkhuis l-triple has $l \leq k$.*

To find the optimal quasi-special n-Brinkhuis triples, we only need to find the set of all admissible words of length n, and its largest subset in which any three words w_1, w_2, w_3 can form a quasi-special n-Brinkhuis 3-triple, i.e., $\{\{w_1, w_2, w_3\}, \{\tau(w_1), \tau(w_2), \tau(w_3)\}, \{\tau^2(w_1), \tau^2(w_2), \tau^2(w_3)\}\}$ is a quasi-special n-Brinkhuis 3-triple. A Maple package was written to calculate such words and sets. The results are listed below.

Proposition 6.1 *Special n-Brinkhuis triples yield the best possible results for each $13 \leq n \leq 20$, and quasi-special Brinkhuis triples do not yield better results than special n-Brinkhuis triples for each $13 \leq n \leq 39$, except 37.*

In the Table 6.1, n is the length of the words; b_1 and k_1 are the numbers of all available optimal quasi-special Brinkhuis triples and the numbers of elements in the triples; b_2 and k_2 are those of the special Brinkhuis triples. Notice the numbers of the triples and their sizes do not always grow as n does, and occasionally there are extraordinary amount of the triples for certain word lengths, i.e., 34 and 37.

Although there are often more choices for the regular and quasi-special Brinkhuis triples than the special Brinkhuis triples as listed above, none of them can be combined to form larger triples. And the exception of $n = 37$ has hardly any significance because the results are superceded by the 36-Brinkhuis 32-triples already. These results strongly suggest that the special Brinkhuis triples will generally yield the best results regardless of n.

It is reasonable to believe that there exist n-Brinkhuis triples that are not quasi-special when $n > 20$, or quasi-special n-Brinkhuis triples that are not special when $n > 39$. However, as explained in the proof of the following proposition, it is vary hard to find such triples due to the complexity.

n	b_1	k_1	b_2	k_2
13	1	1	1	1
14	0	0	0	0
15	0	0	0	0
16	0	0	0	0
17	1	1	1	1
18	1	2	1	2
19	1	1	1	1
20	0	0	0	0
21	0	0	0	0
22	0	0	0	0
23	1	3	1	3
24	5	2	3	2
25	1	5	1	5
26	2	2	2	2
27	1	3	1	3
28	4	4	2	4
29	2	6	2	6
30	1	8	1	8
31	4	7	2	7
32	1	8	1	8
33	1	12	1	12
34	33	10	5	10
35	2	18	2	18
36	1	32	1	32
37	66	32	24	31
38	9	28	3	28
39	1	32	1	32
40			2	48
41			8	65
42			4	76
43			2	110

Table 6.1: Results of Optimal Quasi-special and Special Brinkhuis Triples with n up to 43

Proposition 6.2 *The following 43-Brinkhuis 110-triple exists, and thus shows* $s \geq 110^{1/42} = 1.118419\ldots > 65^{1/40} = 1.109999\ldots$*:*

{ *0120212012102120102012102010212012102120210,*
0120212010210120102012102010210120102120210,
0120212010201210212021020120212012102120210,
0120212012102120210201202120121020102120210,
0120210201210120210121020120212012102120210,
0120212012102120210201210120210121020120210,
0120210201210120102120210120212012102120210,
0120212012102120210120212010210121020120210,
0120210201210120212010210120212012102120210,
0120212012102120210120102120210121020120210,
0120212010201210212010210120212012102120210,
0120212012102120210120102120121020102120210,
0120212010210121021201210120212012102120210,
0120212012102120210121021201210120102120210,
0120212012101201021201210120212012102120210,
0120212012102120210121021201021012102120210,
0120212010210121021202102010212012102120210,
0120212012102120102012021201210120102120210,
0120212012101201021202102010212012102120210,
0120212012102120102012021201021012102120210,
0120212010210120102012102010212012102120210,
0120212012102120102012102010210120102120210,
0120210201210120102012102010212012102120210,
0120212012102120102012102010210121020120210,
0120210201210120212012102010212012102120210,
0120212012102120102012102120210121020120210,
0120212010201210212012102010212012102120210,
0120212012102120102012102120121020102120210,
0120210201210212021012102010212012102120210,
0120212012102120102012101202120121020120210,
0120212012101201021012102010212012102120210,
0120212012102120102012101201021012102120210,
0120212010201210201021012010212012102120210,
0120212012102120102101201020121020102120210,
0120212010212021020121012010212012102120210,
0120212012102120102101210201202120102120210,
0120212010210121020121012010212012102120210,
0120212012102120102101210201210120102120210,

0120212012101201021202102012021012102120210,
0120212012101202102012021201021012102120210,
0120210201210120212012102012021012102120210,
0120212012101202102012102120210121020120210,
0120212010201210212012102012021012102120210,
0120212012101202102012102120121020102120210,
0120212010210120102120121012021012102120210,
0120212012101202101210212010210120102120210,
0120210201210120102120121012021012102120210,
0120212012101202101210212010210121020120210,
0120212010210120102120210201021012102120210,
0120212012101201020120212010210120102120210,
0120210201210120102120210201021012102120210,
0120212012101201020120212010210121020120210,
0120212010210121021201210201021012102120210,
0120212012101201020121021201210120102120210,
0120212010210120102012021201021012102120210,
0120212012101201021202102010210120102120210,
0120210201210120102012021201021012102120210,
0120212012101201021202102010210121020120210,
0120210201210212021012021201021012102120210,
0120212012101201021202101202120121020120210,
0120210201210120210121021201021012102120210,
0120212012101201021201210120210121020120210,
0120212010210120102012101201021012102120210,
0120212012101201021012102010210120102120210,
0120210201210120212012101201021012102120210,
0120212012101201021012102120210121020120210,
0120212010201210212012101201021012102120210,
0120212012101201021012102120121020102120210,
0120210201210120212012102012021020102120210,
0120212010201202102012102120210121020120210,
0120212010201210212012102012021020102120210,
0120212010201202102012102120121020102120210,
0120212010210120102120121012021020102120210,
0120212010201202101210212010210120102120210,
0120210201210120102120121012021020102120210,
0120212010201202101210212010210121020120210,
0120210201210212012101201020121020102120210,
0120212010201210201021012102120121020120210,

0120210201210212012101202120121020102120210,
0120212010201210212021012102120121020120210,
0120212010210120102012102120121020102120210,
0120212010201210212012102010210120102120210,
0120210201210212021012102120121020102120210,
0120212010201210212012101202120121020120210,
0120210201210212021020102120121020102120210,
0120212010201210212010201202120121020120210,
0120212010201210120102120210121020102120210,
0120212010201210120212010210121020102120210,
0120210201210212021020120210121020102120210,
0120212010201210120210201202120121020120210,
0120212010210120102120210201202120102120210,
0120212010212021020120212010210120102120210,
0120210201210120102120210201202120102120210,
0120212010212021020120212010210121020120210,
0120212010210121021201210201202120102120210,
0120212010212021020121021201210120102120210,
0120212010210121021202102010210120102120210,
0120212010210120102012021201210120102120210,
0120210201210120102012102010210120102120210,
0120212010210120102012102010210121020120210,
0120210201210120212012102010210120102120210,
0120212010210120102012102120210121020120210,
0120210201210212021012102010210120102120210,
0120212010210120102012101202120121020120210,
0120210201210120102012021201210120102120210,
0120212010210121021202102010210121020120210,
0120210201210212021012021201210120102120210,
0120212010210121021202101202120121020120210,
0120210201210120210121021201210120102120210,
0120212010210121021201210120210121020120210 }

Proof: Each admissible word is of length at least 13 and of the forms of either $012021 \cdots 120210$ or $012102 \cdots 201210$ as proved by Grimm [21]. So we first find all the square-free words of length $n-12$, attach the two pairs of prefixes and suffixes to these words, then determine if the results are square-free and admissible words, and label them from 1 to m, where m is the total number of such words. The next step is to find all quasi-special (special) Brinkhuis 3-triples and replace the words with the labels we just assigned to them. Thus each triple correspond to a unique ordered list of three different integers, and we have created a set of lists of integers S. Note that if the square-free words of length $n-12$ are known, the rest of the process above only take polynomial time. Now the problem is reduced to find the largest subset T of $\{1,\dots,m\}$ so that the list of any three elements of T is an element of S. Such a question is obviously NP, because the certificate will be the solution itself, and the time required to verify the certificate will be $O(\binom{n}{3})$, thus polynomial. Fortunately, we are not obliged to tell how long it takes to get the certificate.

We now create a graph G so that each element in S is a vertex of G, and any two vertices are connected if and only if any combination of three different numbers from the two lists can form a quasi-special (special) Brinkhuis 3-triple. For example, if $[1,2,3]$ and $[1,2,4]$ are vertices of the graph, they can be connected if and only if $[1,3,4]$ and $[2,3,4]$ are vertices of the graph too. And in this case, the four vertices will form a complete graph. Now we have reduced the problem into finding the largest complete subgraph of a graph, which is known to be NP-complete, in polynomial time. Although what we did does not imply the original problem to be NP-complete, it does shed some light on how to solve the problem: we will use the backtracking method to find the largest Brinkhuis triple.

We say a number i is compatible with a list of numbers $i_1,\dots,i_n$ if any three words chosen from the corresponding words $w_i, w_{i_1},\dots,w_{i_n}$ can form a quasi-special (special) Brinkhuis 3-triple.

Assuming all the numbers in the vertices are ordered increasingly, we try to construct the largest quasi-special (special) Brinkhuis triples recursively: We start with the pair of numbers, a_1 and a_2, who has the largest set of compatible numbers of all pairs of numbers in $\{1,\dots,m\}$. After we have a list $a_1,\dots,a_{n-1}$ such that every three numbers in the list can form a quasi-special (special) Brinkhuis 3-triple, we try to find a_n as the number such that a_n is compatible with $a_1,\dots,a_{n-1}$, and $a_1,\dots,a_n$ has the largest possible set of compatible numbers. If there is a tie, we choose the smallest possible number. Once we cannot add another number to the current list of $a_1,\dots,a_n$, we have found a "locally optimal" Brinkhuis n-triple. We then backtrack to a_{n-1} and search for the next best choice of a_n. When all such choices are analyzed, we backtrack to a_{n-2}. We repeat the process until we backtrack to a_1 and a_2, when we try the pair of numbers who has the next largest set of compatible numbers. We will continue until all the possibilities are considered. Of course, we can always break out of the search if the size of the list of numbers found plus the number of compatible numbers available is less than the best known size of the triples at the time.

The complexity of searching the largest complete subgraph of n vertices is equivalent to searching the largest independent set of vertices of the complement of the graph, whose *average* rate of growth is subexponential, i.e., $O(n^{\log n})$. However, the exact amount of labor required for a specific kind of graphs can be very exponential. Theoretically, we can take advantage of the special structure of the graphs to increased the performance: if vertices $[1, 2, 3]$ and $[4, 5, 6]$ are connected, there is automatically a complete subgraph of 20 vertices, namely any combinations of three numbers from 1 to 6. But such an approach will use recursive programming, which would have required exponential space and thus is impractical. Unless we can find other methods to find the lower bound, using Brinkhuis triples cannot provide must better results, even with more powerful (multi-processor) computers. Unfortunately, this is the best method known yet, if not the only one.

The Maple package and the results on optimal Brinkhuis triples are all available at *http://www.math.temple.edu/~xysun/ternarysf /ternary_square_free.htm.*

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