EXTENDING ACTIONS OF HOPF ALGEBRAS TO ACTIONS OF THE DRINFEL'D DOUBLE

A Dissertation
Submitted to
the Temple University Graduate Board

in Partial Fulfillment
of the Requirements for the Degree of
DOCTOR OF PHILOSOPHY

by
Zachary Cline
May, 2019

Examining Committee Members:

Martin Lorenz, Advisory Chair, Mathematics
Chelsea Walton, University of Illinois Urbana-Champaign, Mathematics
Edward Letzter, Mathematics
David Futer, Mathematics
Peter Riseborough, External Member, Physics

ABSTRACT

EXTENDING ACTIONS OF HOPF ALGEBRAS TO
ACTIONS OF THE DRINFEL'D
DOUBLE

Zachary Cline
DOCTOR OF PHILOSOPHY

Temple University, May, 2019

Dr. Martin Lorenz, Chair

Mathematicians have long thought of symmetry in terms of actions of groups, but group actions have proven too restrictive in some cases to give an interesting picture of the symmetry of some mathematical objects, e.g. some noncommutative algebras. It is generally agreed that the right generalizations of group actions to solve this problem are actions of Hopf algebras, the study of which has exploded in the years since the publication of Sweedler's *Hopf algebras* in 1969.

Different varieties of Hopf algebras have been useful in many fields of mathematics. For instance, in his "Quantum Groups" paper, Vladimir Drinfel'd introduced quasitriangular Hopf algebras, a class of Hopf algebras whose modules each provide a solution to the quantum Yang-Baxter equation. Solutions of this equation are a source of knot and link invariants and in physics, determine if a one-dimensional quantum system is integrable. Drinfel'd also introduced the Drinfel'd double construction, which produces for each finite-dimensional Hopf algebra a quasitriangular one in which the original embeds.

This thesis is motivated by work of Susan Montgomery and Hans-Jürgen Schneider on actions of the Taft (Hopf) algebras $T_n(q)$ and extending such actions to the Drinfel'd double $D(T_n(q))$. In 2001, Montgomery and Schneider classified all nontrivial actions of $T_n(q)$ on an n-dimensional associative algebra A. It turns out that A must be isomorphic to the group algebra of grouplike elements $\Bbbk G(T_n(q))$.

They further determined that each such action extends uniquely to an action of the Drinfel'd double $D(T_n(q))$ on A, effectively showing that each action has a unique compatible coaction. We generalize Montgomery and Schneider's results to Hopf algebras related to the Taft algebras: the Sweedler (Hopf) algebra, bosonizations of 1-dimensional quantum linear spaces, generalized Taft algebras, and the Frobenius-Lusztig kernel $u_q(\mathfrak{sl}_2)$. For each Hopf algebra H, we determine

1. whether there are non-trivial actions of H on A,
2. the possible H-actions on A, and
3. the possible $D(H)$-actions on A extending an H-action and how many there are.

ACKNOWLEDGEMENTS

I would first like to thank God, the Creator, Redeemer, and Sustainer, for his faithfulness, even in the lack of my own. Without Him, I am nothing.

I am eternally grateful to my advisors, Dr. Chelsea Walton and Dr. Martin Lorenz. Dr. Walton guided me throughout my research with the patience of a saint, often introducing new topics to me and asking questions that I had not thought of. If I came to her with an idea, she was always happy to toss it around with me; and if I came to her with nothing, she always had good insight for what to try next. Dr. Walton has been a mentor not just with regard to research, but also with regard to the profession, through teaching me about submitting papers to journals, giving conference talks, and searching for jobs. I would also like to thank Dr. Lorenz for piquing my interest in noncommutative algebra and introducing me to Hopf algebras in the first place.

I would like to express my sincere gratitude to the faculty in Temple's math department. I have learned more mathematics in the last six years than I could have possibly imagined at the start of my time as a graduate student. Additionally, I learned how to teach math well during my time here. I would especially like to thank Dr. Maria Lorenz, Dr. Ed Letzter, Dr. Vasily Dolgushev, Dr. David Futer, and Dr. Matthew Stover.

My time as a graduate student at Temple would have been much more difficult (and boring) without the company of fellow graduate students. Thank you to all the graduate students in the math department, both past and present, for your company, advice, and support. I owe special thanks to Tim Morris, for the many conversations we have had: studying together, teaching each other something new, encouraging each other, commiserating together, or just shooting the breeze to take a break. I am also thankful to former algebra students Adam Jacoby and Geoff Schneider, who always provided good advice and insight.

Finally, I would like to thank all my family and friends who have supported me throughout the past six years. Finishing this degree might have just killed me without my supportive and patient wife, Holly. Thank you for making sure I ate and slept (semi)regularly, and for encouraging me daily. I truly appreciate all the sacrifices you have made for me over these years. None of this would have been worth it if I didn't have you to come home to at the end of the day. Thank you to my parents for everything you have done for me not just during my time at Temple, but before that as well. Thank you for believing in my potential and making me believe in myself. There is not enough space here to thank all the people I ought to, but to every person that has encouraged and helped me in the past six years, thank you. I couldn't have done this without you.

TABLE OF CONTENTS

ABSTRACT **ii**

ACKNOWLEDGEMENTS **iv**

LIST OF TABLES **viii**

1 INTRODUCTION **1**
1.1 Symmetry 1
1.2 Hopf algebras 3
1.3 Motivation from Montgomery–Schneider 6
1.4 Main results and related work 8

2 PRELIMINARIES **12**
2.1 Hopf algebras 12
2.1.1 (Co)algebras 13
2.1.2 (Co)modules 16
2.1.3 Bialgebras and Hopf algebras 17
2.1.4 q-Binomial symbols 20
2.1.5 Perfect dualities 22
2.2 (Co)Actions of Hopf algebras 22
2.2.1 Monoidal structure of representations 23
2.2.2 Yetter-Drinfel'd modules, bosonizations 25
2.3 The Drinfel'd double 28
2.4 Nichols algebras 30
2.5 Inner-faithful module algebras 32

3 THE TAFT ALGEBRAS **35**
3.1 Work of Montgomery–Schneider ($n \geq 3$) 35
3.2 The Sweedler algebra ($n = 2$) 36

4 $H_n(\zeta, m, t)$, A CORADICALLY GRADED GENERALIZATION OF TAFT ALGEBRAS 38

4.1 The Hopf Algebras $H_n(\zeta, m, t)$ 38

4.2 The dual and double of $H_n(\zeta, m, t)$ 43

4.2.1 The dual $H_n(\zeta, m, t)^*$ 43

4.2.2 The Drinfel'd double $D(H_n(\zeta, m, t))$ 46

4.3 The possible structures of $A(H_n(\zeta, m, t))$ 47

4.4 Extensions to $D(H_n(\zeta, m, t))$ 50

5 NON CORADICALLY-GRADED GENERALIZED TAFT ALGEBRAS 54

5.1 The possible structures of $A(T(n, N, 1))$ 55

5.2 The dual and double of $T(n, N, 1)$ 56

5.2.1 The dual $T(n, N, 1)^*$ 56

5.2.2 The double $D(T(n, N, 1))$ 63

5.3 Extensions to $D(T(n, N, 1))$ 64

6 THE FROBENIUS-LUSZTIG KERNEL $u_q(\mathfrak{sl}_2)$ 67

6.1 The dual and double of $u_q(\mathfrak{sl}_2)$ 68

6.2 The possible structures of $A(u_q(\mathfrak{sl}_2))$ and extensions to $D(u_q(\mathfrak{sl}_2))$. 73

7 FUTURE DIRECTIONS 77

REFERENCES 80

A EXTRA CALCULATIONS 84

A.1 Calculations from Chapter 5 84

A.1.1 Remainder of the proof of Proposition 5.2.7 84

A.2 Calculations from Chapter 6 87

A.2.1 Proof of (6.1) 87

A.2.2 Remainder of the proof of Theorem 6.1.7 91

LIST OF TABLES

1 Summary of Main Results 1 10

2 Summary of Main Results 2 11

CHAPTER 1

INTRODUCTION

Unless otherwise specified, throughout this work, $\Bbbk$ will denote an algebraically closed field of characteristic 0, algebraic structures will be over $\Bbbk$, and $\otimes$ will mean $\otimes_{\Bbbk}$. 'Dimension' will always mean $\Bbbk$-vector space dimension. Technical terms in the introduction and definitions throughout the manuscript are italicized. An index of notation and terminology is provided at the end to aid the reader.

1.1 Symmetry

One objective of mathematics in general is the discovery, creation, and interpretation of patterns, and in the physical world, one of the most observed and prevalent patterns is symmetry. From childhood, we are taught about symmetry in school; we observe it in the plants and animals of nature; it has been used in architecture and the arts since the beginning of civilization. Yet, only fairly recently in the scope of human history did mathematicians realize that they could study symmetry rigorously, with the discovery (or invention) of groups.

If we fix a certain property or structure of some mathematical object, the set of operations on the object which preserve that property form a group. The symmetric group S_n is the group of symmetries of arrangements of n objects, the dihedral group D_{2n} is the group of symmetries of an n-sided regular polygon, and $GL_n(\mathbb{R})$ comprises the linear symmetries of real n-space. We can think of symmetries more generally in terms of group actions: a group action of G on a set X is simply a group homomorphism $\phi : G \to S_X$, where S_X is the symmetric group on X. The act of specifying which property we want to preserve for our desired notion of 'symmetry' is to simply replace S_X with a subgroup.

For instance, Group Representation Theory is devoted to studying groups by their linear actions on vector spaces. A *representation* of G is a vector space V equipped with a group homomorphism $\phi : G \to GL(V)$. We will denote the category of representations of a group G by **Rep(**G**)**. (In fact, some purely group-theoretic facts have been proven using actions and representations, such as Burnside's $p^a q^b$ Theorem and results in the classification of finite simple groups [21, Section 3.6])

While group actions capture symmetry in the classical sense discussed above, they have proven too restrictive to yield interesting information about all mathematical objects, such as noncommutative algebras. Noncommutative algebras typically have small, uninteresting automorphism groups, and so are too rigid for group actions to reflect their rich structure. Thus, we need a different notion of symmetry to handle symmetries of $\Bbbk$-algebras in general, and a natural first place to look are actions of some other algebraic structure.

To see what sort of structure we should use to handle symmetries of $\Bbbk$-algebras in general, recall that through the diagonal action, the tensor product of two group representations is a representation as well: for $V, W \in$ **Rep(**G**)**

$$g \cdot (v \otimes w) = (g \cdot v) \otimes (g \cdot w) \qquad g \in G,\ v \in V,\ w \in W. \tag{1.1}$$

Moreover, the one-dimensional vector space $\Bbbk$ is a representation of any group by the trivial action: $g \cdot 1_{\Bbbk} = 1_{\Bbbk}$. These facts are used to define actions of groups on $\Bbbk$-algebras generated by V, giving the classical symmetry of $\Bbbk$-algebras. If one wants classical symmetries of algebras generated by the dual space V^*, then note that V^* can be made a representation via

$$(g \cdot f)(v) = f(g^{-1} \cdot v) \qquad g \in G,\ v \in V,\ f \in V^*. \tag{1.2}$$

Together, these properties make **Rep(**G**)** a *rigid monoidal category*. Thus, we should expect to need an algebraic object whose representations form a rigid monoidal category, and whose representation theory contains that of groups as a special case. It has generally been accepted that the right objects of study to generalize group representation theory to this end are Hopf algebras. We reserve the precise definition of Hopf algebra for Section 2.1, but define it briefly here.

Definition 1.1.1. A *Hopf algebra* is a $\Bbbk$-vector space H, equipped with

- an associative $\Bbbk$-algebra structure (H, m, η), where $m : H \otimes H \to H$ and $\eta : \Bbbk \to H$ are the *multiplication* and *unit* maps, respectively;
- a coassociative $\Bbbk$-coalgebra structure (H, Δ, ϵ), where $\Delta : H \to H \otimes H$ and $\epsilon : H \to \Bbbk$ are the *comultiplication* and *counit* maps, respectively;
- and an anti-automorphism $S : H \to H$, called the *antipode*,

satisfying compatibility conditions.

1.2 Hopf algebras

While Hopf algebras originated in the study of algebraic topology and algebraic groups [4], they are objects which appear in a variety of contexts and are now studied in their own right. They generalize two classical algebraic objects: group algebras and universal enveloping algebras of Lie algebras. The *group algebra* of a group G, denoted $\Bbbk G$, is the $\Bbbk$-vector space with basis G, and multiplication coming from extending the group operation linearly. The *universal enveloping algebra* of a Lie algebra $\mathfrak{g}$, denoted $U(\mathfrak{g})$, is the quotient of the tensor algebra $T(\mathfrak{g})$ by the ideal generated by elements of the form $x \otimes y - y \otimes x - [x, y]$ for $x, y \in \mathfrak{g}$. For a group G (or a Lie algebra $\mathfrak{g}$), the coalgebra structure and antipode of $\Bbbk G$ (or $U(\mathfrak{g})$) are given respectively by

$$\begin{array}{llll} \Delta(g) = g \otimes g & \epsilon(g) = 1 & S(g) = g^{-1} & (g \in G); \\ \Delta(x) = 1 \otimes x + x \otimes 1 & \epsilon(x) = 0 & S(x) = -x & (x \in \mathfrak{g}). \end{array} \tag{1.3}$$

Many classical results about Hopf algebras involve proving or disproving analogous versions (or generalizations) of theorems from group theory, e.g. the Nichols-Zoeller Theorem [26] which generalizes Lagrange's Theorem (that the number of elements of any subgroup must divide the number of elements of the group). While Hopf algebras were introduced in the 1940's, the theory is still developing today, with most results pertaining to particular classes or examples. The classification of these objects is far from complete, with most results limited to Hopf algebras whose dimensions are products of a few prime integers [2, 10].

Since a Hopf algebra H is in particular an algebra, we will define a representation of H to be an H-module, and will denote the category of representations of H by ${}_H\mathcal{M}$ (the reason for which is more clear in Section 2.1.2). As desired, ${}_H\mathcal{M}$ is indeed a rigid monoidal category. (For more explanation of this, see Section 2.2.1.) The tensor product, trivial, and dual actions are given as follows: For a Hopf algebra H and $V, W \in {}_H\mathcal{M}$, we have that $V \otimes W$, $\Bbbk$, and V^* are also representations via, respectively,

$$\begin{gathered} h \cdot (v \otimes w) = (h_{(1)} \cdot v) \otimes (h_{(2)} \cdot w), \qquad h \cdot 1_{\Bbbk} = \epsilon(h) 1_{\Bbbk}, \\ (h \cdot f)(v) = f(S(h) \cdot v) \qquad \text{(for } h \in H,\ v \in V,\ w \in W, \text{ and } f \in V^*). \end{gathered} \tag{1.4}$$

(Here, we are using Sweedler's summation-less notation, $\Delta(h) = h_{(1)} \otimes h_{(2)}$.) Also, recalling that a representation of G is the same thing as a $\Bbbk G$-module, by comparing (1.1), (1.2), (1.3), and (1.4), one sees that representations of Hopf algebras generalize group representations. Motivated by the group case, we call a nonzero element g of a Hopf algebra H *grouplike* if $\Delta(g) = g \otimes g$. As we will see in Remark 2.1.8, the set of grouplike elements of H forms a group, denoted $G(H)$.

Moreover, since enveloping algebras are also Hopf algebras, with the structure given in (1.3), we see how the tensor product, trivial, and dual representations in **Rep($\mathfrak{g}$)** arise for a Lie algebra $\mathfrak{g}$, by (1.4):

$$\begin{gathered} x \cdot (v \otimes w) = v \otimes (x \cdot w) + (x \cdot v) \otimes w, \qquad x \cdot 1_{\Bbbk} = 0, \\ (x \cdot f)(v) = f(-x \cdot v) \qquad \text{(for } x \in \mathfrak{g},\ v \in V,\ w \in W, \text{ and } f \in V^*). \end{gathered}$$

Like for groups and Lie algebras, for any Hopf algebra H, we can use (1.4) to define an action of H on an algebra A.

Definition 1.2.1. Let H be a Hopf algebra. An *H-module algebra* A is an algebra (or monoid) in ${}_H\mathcal{M}$. Put another way, A is simultaneously an H-module and $\Bbbk$-algebra such that

$$h \cdot (ab) = (h_{(1)} \cdot a)(h_{(2)} \cdot b), \qquad h \cdot 1_A = \epsilon(h) 1_A \quad \text{(for } h \in H \text{ and } a, b \in A\text{)}.$$

We also say that H *acts on* A and call this a *Hopf action on* A.

This definition of an action of a Hopf algebra on an algebra gives us a way to define a new type of symmetry. Now group algebras and enveloping algebras are examples of *cocommutative* Hopf algebras, meaning $\Delta = \tau \circ \Delta$ for the twist map $\tau : H \otimes H \to H \otimes H$. In fact, a theorem of Cartier, Kostant, Milnor, and Moore states that all cocommutative Hopf algebras are smash products of the two previously mentioned types [1, Theorem 1.1]. Thus, symmetry coming from cocommutative Hopf algebras are considered classical. On the other hand, symmetry from a Hopf action that does not factor through that of a cocommutative Hopf algebra is considered *quantum symmetry*.

This work is focused on classifying actions of *pointed* Hopf algebras H on the group algebra of grouplike elements $\Bbbk G(H)$, essentially studying quantum symmetries of classical objects. (See Section 2.1.1 for a definition of a 'pointed' Hopf algebra.) Actions of such Hopf algebras H are then extended to actions of the *Drinfel'd double* $D(H)$.

The purpose of looking for actions of $D(H)$ is to find solutions to the quantum Yang-Baxter equation, which provide a source of link/knot invariants and play a role in the theory of quantum integrable systems [17, 18]. For a vector space V, a map $c \in \mathrm{Aut}_\Bbbk(V \otimes V)$ is called a solution of the *quantum Yang-Baxter equation* if the identity

$$(c \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c)(c \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes c)(c \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c) \tag{1.5}$$

holds in $\mathrm{Aut}_\Bbbk(V \otimes V \otimes V)$. In [14], Drinfel'd introduced the notion of *quasitriangular* Hopf algebras, whose modules each lead to a solution of the quantum Yang-Baxter equation. He also introduced the *quantum double* of a finite-dimensional

Hopf algebra H (now called the *Drinfel'd double* of H), denoted $D(H)$, which is a canonical quasitriangular Hopf algebra in which H embeds. Thus, modules of a finite-dimensional Hopf algebra H which admit an extension to the structure of a $D(H)$-module give solutions of the quantum Yang-Baxter equation.

Thus, for the sake of both studying symmetries of associative algebras and for finding solutions of the quantum Yang-Baxter equation, we are interested in the question of when actions of a finite-dimensional Hopf algebra H on A leads to a non-trivial action of $D(H)$ on A. In particular, we explore the question of when a group (G-)action on A by algebra automorphisms can extend non-trivially to an action of a Hopf algebra H on A, and when this action can then extend non-trivially to an action of $D(H)$ on A (see Question 1.3.3).

1.3 Motivation from Montgomery–Schneider

The scope of this thesis is based on the work of Susan Montgomery and Hans-Jürgen Schneider on actions of the n^2-dimensional *Taft Hopf algebra*, $T_n(q)$. For $n \in \mathbb{N}$, $n \geq 2$, and a primitive n^{th} root of unity $q \in \Bbbk$, this Hopf algebra is generated as an algebra by elements g and x, with relations

$$g^n = 1, \quad x^n = 0, \quad gx = qxg.$$

The rest of the Hopf algebra structure is given in Example 2.1.11. For now, note that the group of grouplike elements is the cyclic group of order n: $G(T_n(q)) = \langle g \rangle$. In [24], Montgomery and Schneider classified the n-dimensional $T_n(q)$-module algebras with no nonzero nilpotent elements, for which x does not act by zero. In fact, x acting by nonzero is exactly the condition that this module structure is *inner-faithful*, i.e., that the action does not factor through any proper Hopf quotient of $T_n(q)$ (see, e.g., Corollary 2.5.3). Moreover, by Proposition 2.5.5 below, the value n is the smallest possible dimension of an inner-faithful $T_n(q)$-module algebra with no nonzero nilpotent elements. Their classification was the following.

Theorem 1.3.1. *[24, Theorem 2.5] Take $n \geq 2$. Let A be an n-dimensional inner-faithful $T_n(q)$-module algebra with no nonzero nilpotent elements. Then there exists* *an element* $u \in A$ *and nonzero scalars* $\beta, \gamma \in \Bbbk$ *such that* $A = \Bbbk[u]/(u^n - \beta)$, *where* $g \cdot u = q^{-1}u$, *and* $x \cdot u = \gamma 1_A$. $\square$

By scaling u, we can assume without loss of generality that $u^n = 1_A$ in A above. Thus, A is in fact isomorphic as an algebra to the group algebra $\Bbbk G$, where $G = G(T_n(q)) \cong \mathbb{Z}/n\mathbb{Z}$ is the group of grouplike elements of $T_n(q)$. Moreover, note that since G is abelian, $G \cong \widehat{G}$, the character group of G. (This isomorphism is in general not unique.) The action of the Hopf subalgebra $\Bbbk G \subseteq T_n(q)$ on $A \cong \Bbbk G$ is induced by the character group: Fix generators $g \in \widehat{G}$ and $u \in G$ so that $\langle g, u \rangle = q^{-1}$; then, in $A \cong \Bbbk G$, we get that $g \cdot u^m = q^{-m}u^m = \langle g, u^m \rangle u^m$. In general, for G abelian, there is always an action of $\Bbbk\widehat{G}$ on $\Bbbk G$ given by

$$g \cdot u = \langle g, u \rangle u \quad g \in \widehat{G},\ u \in G. \tag{1.6}$$

Thus, Montgomery and Schneider classified all the inner-faithful actions of $T_n(q)$ on the group algebra of its grouplike elements $\Bbbk G(T_n(q))$, extending the action of $\Bbbk G(T_n(q))$ on itself as just described. We set the following notation.

Notation 1.3.2 ($A(H)$)**.** For a Hopf algebra H with a finite abelian group of grouplike elements $G := G(H)$, let $A(H)$ denote an inner-faithful H-module algebra that is isomorphic to $\Bbbk G$ as an algebra so that $\Bbbk G \subset H$ acts on $A(H) \cong \Bbbk G$ as $\Bbbk\widehat{G}$ does in (1.6).

Montgomery and Schneider showed further that for $n \geq 3$, each such action of $T_n(q)$ on $A(T_n(q))$ can be extended uniquely to an action of the Drinfel'd double $D(T_n(q))$ on $A(T_n(q))$; we recall the details of their result in Theorem 3.1.2. Therefore, each module algebra $A(T_n(q))$ gives a solution to the quantum Yang-Baxter equation, and the symmetries of $A(T_n(q))$ coming from the action of $D(T_n(q))$ are, in a sense, determined uniquely by the symmetries coming from the action of $T_n(q)$. Motivated by their work, we investigate the following questions.

Question 1.3.3. Let H be a finite-dimensional Hopf algebra with an abelian group of grouplike elements.

(a) Do the module algebra structures $A(H)$ as described in Notation 1.3.2 exist?

If (a) is affirmative, then:

(b) What are the possible H-module structures on $A(H)$?

(c) What are the possible $D(H)$-module algebra structures on $A(H)$ extending that in (b)? How many extensions are there? In particular, is there a unique extension as in the case of the Taft algebras ($T_n(q)$ with $n \geq 3$)?

Remark 1.3.4. The first case to consider is, naturally, the case $H = \Bbbk G$ for G a finite abelian group. Here, $A(H) = H$ with the action determined by (1.6), which addresses Question 1.3.3(a,b). Note that $D(\Bbbk G) \cong \Bbbk G \otimes \Bbbk G$ as Hopf algebras with the tensor product Hopf algebra structure. The second copy of $\Bbbk G$ corresponds to the original H, and the first copy corresponds to the dual $(\Bbbk G)^* \cong \Bbbk G$. Thus, any extension of an action of $\Bbbk G$ on $A(\Bbbk G)$ to one of $D(\Bbbk G)$ on $A(\Bbbk G)$ is given by any other action (not necessarily faithful) of $\widehat{G} \cong G$ on $\Bbbk G$ by algebra automorphisms.

1.4 Main results and related work

Because the answers to Question 1.3.3 are interesting for the Taft algebras $T_n(q)$, we will answer these questions for some pointed, finite-dimensional Hopf algebras related to Taft algebras. In Chapter 2, we provide background information pertaining to actions of pointed Hopf algebras and their Drinfel'd doubles that will be used throughout. Chapter 3 goes over the case of the Taft algebras in more detail, and gives an answer to Question 1.3.3(c) for the Sweedler algebra $T_2(-1)$. Chapter 4 is dedicated to a family of coradically graded Hopf algebras, $H_n(\zeta, m, t)$, for which the Taft algebras are a subclass; these Hopf algebras arise as bosonizations of quantum linear spaces from Andruskiewitsch and Schneider's work [5]. Explicit computations are given for the dual $H_n(\zeta, m, t)^*$, with the dual pairing given, and for $D(H_n(\zeta, m, t))$ before addressing Question 1.3.3. Non-trivial liftings of $H_n(\zeta, m, t)$, namely the generalized Taft algebras $T(n, N, 1)$, are the subject of Chapter 5. Again, explicit computations of the dual and double are given for $T(n, N, 1)$. It is known that a Taft algebra can be considered as the positive Borel part of the Frobenius-Lusztig kernel $u_q(\mathfrak{sl}_2)$, and Chapter 6 answers Question 1.3.3 for the full small quantum group $u_q(\mathfrak{sl}_2)$. Directions for future research are discussed in Chapter 7, while some computations omitted in the body for the sake of brevity are included in Appendix A.

Our main results are summarized as follows.

Theorem 1.4.1. *Consider the finite-dimensional pointed Hopf algebras $T_2(-1)$, $H_n(\zeta, m, t)$, $T(n, N, 1)$, and $u_q(\mathfrak{sl}_2)$ discussed above. Then, Question 1.3.3 is answered for these Hopf algebras, as detailed in Tables 1 and 2.*

The results of Montgomery and Schneider for $T_n(q)$ are included in Table 1 and 2 for comparison, and the proof of Theorem 1.4.1 is the main focus of most of this thesis.

It is worth mentioning that presentations for the dual and double of these Hopf algebras are computed, which may be of independent interest. As an example, we give a complete proof of the fact that $u_q(\mathfrak{sl}_2)^*$ is isomorphic to a quotient of the quantum group $\mathcal{O}_q(\mathrm{SL}_2)$, and give the dual pairing. While this fact is seemingly well-known (see, e.g., Brown-Goodearl's work in [11, III.7.10]), there did not seem to be a full proof in the literature.

We end this section by mentioning some related results in the literature that may be of interest. In [13], Cohen, Fischman, and Montgomery examine conditions on a Hopf algebra H and left H-module H-comodule algebra A under which A can be realized as a $D(H)$-module algebra. In particular, they show that if H has a bijective antipode and either (i) A is a faithful $A\#H$-module, or (ii) A/A^{coH} is H-Galois and A is H-commutative (i.e. $ab = (a_{(-1)} \cdot b)a_{(0)}$ for all $a, b \in A$), then A is a $D(H)$-module algebra. Chen and Zhang classified all $D(T_2(-1))$-module algebras of dimension 4 up to isomorphism as $D(T_2(-1))$-modules in [12], in particular giving all $D(T_2(-1))$-module algebra structures on $M_2(\Bbbk)$. In [19], Kinser and Walton examine actions of Taft algebras on path algebras of quivers, and extend such actions to $D(T_n(q))$.

H (gens. of H) (gens. of H^*)	**Actions of H on $A(H)$** ($\forall H$, $A(H)$ is gen. by u) on Question 1.3.3(a,b)	**Extension to actions of $D(H)$** on Question 1.3.3(c)	**# / Parametrization of extns. to actions of $D(H)$**
$T_n(q)$ (g, x) (G, X)	$g \cdot u = q^{-1}u, \ \ x \cdot u = \gamma 1$ $0 \neq \gamma \in \Bbbk$ [Theorem 1.3.1]	$G \cdot u = qu, \ \ X \cdot u = \lambda u^2$ $\lambda \in \Bbbk, \ \gamma\lambda = q - 1$ [Theorem 3.1.2]	1
$T_2(-1)$ (g, x) (G, X)	$g \cdot u = -u, \ \ x \cdot u = \gamma 1$ $0 \neq \gamma \in \Bbbk$ [Theorem 1.3.1]	$G \cdot u = -u, \ \ X \cdot u = \lambda 1$ $\lambda \in \Bbbk$ [Proposition 3.2.2]	$\Bbbk$
$H_n(\zeta, m, t)$ (y, x) (Y, X)	exists if gcd$(mt, n) = m$: $y \cdot u = \zeta u,$ $x \cdot u = \gamma u^{t+1}$ $0 \neq \gamma \in \Bbbk$ [Proposition 4.3.2]	$Y \cdot u = \zeta^d u,$ $X \cdot u = \delta u^{n+1-t},$ $m \equiv -dt \pmod{n},$ $\delta \in \Bbbk,$ $\gamma\delta = \frac{\zeta^{-m}-1}{(n-t)_{\zeta^m}}$ if $2m \neq n$ [Theorem 4.4.1]	t, if $2m \neq n$ $t \times \Bbbk$, if $2m = n$

Table 1: Summary of Main Results 1

H (gens. of H) (gens. of H^*)	**Actions of H on $A(H)$** for Question 1.3.3(a,b) ($\forall H$, $A(H)$ is gen. by u)	**Extension to actions of $D(H)$** for Question 1.3.3(c)	**# / Parametrization of extns. to actions of $D(H)$**
$T_n(q)$ (g, x) (G, X)	$g \cdot u = q^{-1}u, \;\; x \cdot u = \gamma 1$ $0 \neq \gamma \in \Bbbk$ [Theorem 1.3.1]	$G \cdot u = qu, \;\; X \cdot u = \lambda u^2$ $\lambda \in \Bbbk, \; \gamma\lambda = q - 1$ [Theorem 3.1.2]	1
$T(n, N, 1)$ (g, x) (G, X)	$g \cdot u = \zeta u,$ $x \cdot u = \gamma u^{1+\frac{n}{N}}$ $\gamma \in \Bbbk, \; \gamma^N = -(1-\zeta)^N$ [Proposition 5.1.1]	exists if $N = 2$ and $\frac{n}{2}$ odd: $G \cdot u = \zeta^a u$ ($a \in \mathbb{N}$ odd), $X \cdot u = \delta u^{1-\frac{n}{N}}$ $\delta \in \Bbbk, \gamma\delta = \zeta - 1$ [Theorem 5.3.1]	$\frac{n}{2}$, if $N = 2$ and $\frac{n}{2}$ odd 0, otherwise
$u_q(\mathfrak{sl}_2)$ (K, E, F) (a, b, c, d)	$K \cdot u = q^2 u,$ $F \cdot u = \gamma 1,$ $E \cdot u = \delta u^2$ $\gamma, \delta \in \Bbbk, \; \gamma\delta = -q$ [Proposition 6.2.3]	$a \cdot u = qu, \;\; b \cdot u = (q - q^{-1})1,$ $c \cdot u = 0, \;\; d \cdot u = q^{-1}u,$ — or — $a \cdot u = q^{-1}u, \;\; b \cdot u = 0,$ $c \cdot u = (q - q^{-1})1, \;\; d \cdot u = qu$ [Theorem 6.2.4]	2

Table 2: Summary of Main Results 2

CHAPTER 2

PRELIMINARIES

The main objects of study are Hopf algebras. The relevant general theory is developed in Section 2.1. In Section 2.2, we consider actions of Hopf algebras on associative algebras, and develop the theory of Yetter-Drinfel'd modules and bosonizations. Many of the pointed Hopf algebras we consider later are (liftings of) bosonizations. The Drinfel'd double is the subject of Section 2.3. In Section 2.4, we introduce an important class of Hopf algebras in the category of Yetter-Drinfel'd modules, namely Nichols algebras. These have proven useful in the classification program of pointed Hopf algebras. Finally, we discuss inner-faithful module algebras in Section 2.5 and their structure for pointed Hopf algebras H with $G(H)$ finite cyclic.

2.1 Hopf algebras

Even rigorously defining what a Hopf algebra is requires a bit of background. A Hopf algebra is a $\Bbbk$-vector space with a compatible algebra and coalgebra structure, along with a special map called the antipode. Each of these terms, besides 'algebra', needs a proper explanation. We first discuss the dual notion of an associative algebra: a coassociative coalgebra.

2.1.1 (Co)algebras

It is easiest to understand coassociative coalgebras as the dual notion to associative algebras. One typically learns that an associative $\Bbbk$-algebra is a unital ring A together with a unital ring homomorphism $f : \Bbbk \to A$ so that $f(\Bbbk) \subseteq Z(A)$. This map gives the ring A the additional structure of a $\Bbbk$-vector space, and so a $\Bbbk$-algebra is simultaneously a ring and a vector space with compatible structures. Of course, this definition prioritizes the ring structure. Alternatively, we could prioritize the vector space structure, and define an *associative* $\Bbbk$*-algebra* as a $\Bbbk$-vector space A equipped with two $\Bbbk$-linear maps, $m : A \otimes A \to A$ and $\eta : \Bbbk \to A$ for which the diagrams

$$
\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\mathrm{id}_A \otimes m"] \arrow[d, "m \otimes \mathrm{id}_A"'] & A \otimes A \arrow[d, "m"] \\
A \otimes A \arrow[r, "m"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \otimes A \arrow[dd, "m"] & \\
\Bbbk \otimes A \arrow[ur, "\eta \otimes \mathrm{id}_A"] \arrow[dr, equal] & & A \otimes \Bbbk \arrow[ul, "\mathrm{id}_A \otimes \eta"'] \arrow[dl, equal] \\
 & A &
\end{tikzcd}
\tag{2.1}
$$

commute. The first diagram expresses associativity of the multiplication and the second gives that $\eta(1_\Bbbk)$ is the unit of A. (Here, η and f above are the same map.) We call m and η the *multiplication* (or *product*) and *unit* maps, respectively, for the algebra A.

One benefit of the second definition of an associative $\Bbbk$-algebra is that we can generalize the notion of "algebra" to arbitrary monoidal categories (see [28, Chapter 11]). We will see some examples in Section 2.2.1. The more pertinent benefit is that we can now define a coalgebra as the dual notion of an algebra, with the axioms given by "reversing all arrows". More precisely, a *coassociative* $\Bbbk$*-coalgebra* is a $\Bbbk$-vector space C equipped with two $\Bbbk$-linear maps, $\Delta : C \to C \otimes C$ and $\epsilon : C \to \Bbbk$

for which the diagrams

$$
\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\mathrm{id}_C \otimes \Delta"] \\
C \otimes C \arrow[r, "\Delta \otimes \mathrm{id}_C"'] & C \otimes C \otimes C
\end{tikzcd}
\qquad
\begin{tikzcd}
& C \arrow[dl, equal] \arrow[dr, equal] \arrow[dd, "\Delta"] & \\
\Bbbk \otimes C & & C \otimes \Bbbk \\
& C \otimes C \arrow[ul, "\epsilon \otimes \mathrm{id}_C"] \arrow[ur, "\mathrm{id}_C \otimes \epsilon"'] &
\end{tikzcd}
\tag{2.2}
$$

commute. We call Δ and ϵ the *comultiplication* (or *coproduct*) and *counit* maps, respectively, for C.

For any $c \in C$, we have $\Delta(c) = \sum_{i=1}^{n} a_i \otimes b_i$ for some $n \in \mathbb{N}$ and $a_i, b_i \in C$. If we start performing calculations with many elements, the introduced notation and plethora of summations becomes bulky and hard to keep track of.

Notation 2.1.1. To make such calculations easier, when all maps involved are $\Bbbk$-linear, Sweedler introduced the notation $\Delta(c) = c_{(1)} \otimes c_{(2)}$, which is now called *Sweedler notation*.

As an example, the commutativity of the diagrams in (2.2) is expressed

$$c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)} = c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)}, \tag{2.3}$$

$$\epsilon(c_{(1)})c_{(2)} = c = c_{(1)}\epsilon(c_{(2)}). \tag{2.4}$$

By virtue of (2.3), there is no ambiguity in writing $c_{(1)} \otimes c_{(2)} \otimes c_{(3)}$, which is sometimes written $\Delta^{(2)}(c)$. Just as associativity of multiplication leads to a generalized associativity (that any placement of parentheses results in the same product) so too the coassociativity leads to a generalized coassociativity (that applying Δ successively n times always results in the same coproduct, regardless of which slot we apply Δ to at each step.) Thus, more generally, we write $\Delta^{(n-1)}(c) = c_{(1)} \otimes c_{(2)} \otimes \cdots \otimes c_{(n)}$.

Example 2.1.2. Let X be any set and let $\Bbbk X$ denote the $\Bbbk$-vector space with basis X. We can give $\Bbbk X$ a coalgebra structure by defining

$$\Delta(x) = x \otimes x \qquad \epsilon(x) = 1 \tag{2.5}$$

for any $x \in X$.

For any coalgebra C, we call an element *grouplike* if it satisfies (2.5). The set of all grouplike elements is linearly independent and denoted $G(C)$. The motivation for the term will become clear later. (See Example 2.1.9.) An element $c \in C$ will be called (g,h)-*skew primitive* if it satisfies $\Delta(c) = g \otimes c + c \otimes h$ for some $g, h \in G(C)$. The space of all such elements is denoted $P_{g,h}(C)$. Note that the axioms of a coalgebra force $\epsilon(c) = 0$ for any skew primitive c. As an example, if $g, h \in G(C)$, then $g - h \in P_{g,h}(C)$. We will see more interesting examples of skew primitive elements in Examples 2.1.10 and 2.1.11.

We will make regular use of the following standard terminology surrounding coalgebras. Let C and D be coalgebras.

- *coalgebra homomorphism*: a map $\phi : C \to D$ such that for any $c \in C$, $\Delta(f(c)) = f(c_{(1)}) \otimes f(c_{(2)})$ and $\epsilon(f(c)) = \epsilon(c)$.

- *coideal*: kernel of a coalgebra homomorphism. A subspace $I \subseteq C$ is a coideal if and only if $\Delta(I) \subseteq C \otimes I + I \otimes C$ and $\epsilon(I) = 0$.

- *subcoalgebra* of C: a subspace V of C such that $\Delta(V) \subseteq V \otimes V$. Of course, as for most algebraic objects, there are obvious versions of the isomorphism theorems for coalgebras.

- *simple* coalgebra: a coalgebra which has only two subcoalgebras, (0) and itself.

- *coradical* of C: the (direct) sum of the simple subcoalgebras of C. It is denoted C_0.

- *pointed* coalgebra: a coalgebra C whose simple subcoalgebras are all 1-dimensional, i.e. $C_0 = \Bbbk G(C)$.

- *coalgebra filtration* of C: an increasing (with respect to $\subseteq$) and exhaustive family of subspaces $\{V_i\}_{i \geq 0}$ satisfying $\Delta(V_i) \subseteq \sum_{j=0}^{i} V_j \otimes V_{i-j}$ for all $i \geq 0$.

- *coradical filtration* of C: the coalgebra filtration defined inductively by C_0 being the coradical and $C_i = \Delta^{-1}(C_{i-1} \otimes C + C \otimes C_0)$ for all $i > 0$.

- *coalgebra grading* of C: a vector space decomposition $C = \bigoplus_{i\geq 0} C(i)$ such that $\epsilon(C(0)) = 0$ and $\Delta(C(i)) \subseteq \sum_{j=0}^{i} C(j) \otimes C(i-j)$. The associated graded coalgebra for the coradical filtration will be denoted $\mathrm{gr}(C)$.

- *coradically graded* coalgebra: a graded coalgebra C such that $C \cong \mathrm{gr}(C)$ as graded coalgebras, i.e if $C_i = \bigoplus_{j=0}^{i} C(j)$ for all $i \in \mathbb{N}$.

Let (C, Δ, ϵ) be any coalgebra and let $\tau : C \otimes C \to C \otimes C$ denote the typical twist map: $c \otimes d \mapsto d \otimes c$. Then we can define a new coalgebra structure on C by replacing Δ with $\tau \circ \Delta$. We call this the *coopposite coalgebra* and denote it C^{cop}. If $\Delta = \tau \circ \Delta$, we call C *cocommutative*. If $(D, \widehat{\Delta}, \widehat{\epsilon})$ is another coalgeba, then $C \otimes D$ can be given a coalgebra structure with coproduct $\Delta \otimes \widehat{\Delta}$ and counit $\epsilon \otimes \widehat{\epsilon}$. This is called the *tensor product coalgebra structure* on $C \otimes D$.

Now, since the definition of coalgebra is dual to that of an algebra, we have that the vector space dual C^* with multiplication Δ^* and unit ϵ^* is an associative $\mathbb{k}$-algebra. On the other hand, if A is a $\mathbb{k}$-algebra, then $A^\circ := m^{*-1}(A^* \otimes A^*)$ with comultiplication m^* and counit η^* is a $\mathbb{k}$-coalgebra.[1] Note that when A is finite-dimensional, $A^\circ = A^*$. This leads to the following example.

Example 2.1.3. Let $n \in \mathbb{N}$, $n \geq 1$, and let $\mathrm{M}_n(\mathbb{k})$ denote the $\mathbb{k}$-algebra of $n \times n$ matrices with entries in $\mathbb{k}$. Let $E_{i,j}$ denote the matrix with a single 1 in the i^{th} row and j^{th} column and 0 elsewhere. Then $\{E_{i,j}\}$ is the standard basis of $\mathrm{M}_n(\mathbb{k})$. The dual space, $\mathrm{C}_n(\mathbb{k}) := \mathrm{M}_n(\mathbb{k})^*$ is a coalgebra called a *comatrix coalgebra over* $\mathbb{k}$. Denoting the dual basis to $\{E_{i,j}\}$ by $\{e_{i,j}\}$, the comultiplication and counit are given by

$$\Delta(e_{i,j}) = \sum_{\ell=1}^{n} e_{i,\ell} \otimes e_{\ell,j} \qquad \epsilon(e_{i,j}) = \delta_{i,j}.$$

2.1.2 (Co)modules

Dual to the notion of a module for an algebra is the notion of a comodule for a coalgebra. For an associative $\mathbb{k}$-algebra A, a *right A-module* is a $\mathbb{k}$-vector space M

[1] We must consider A° and not simply A^* because, while $A^* \otimes A^* \subseteq (A \otimes A)^*$, equality does not hold in general.

equipped with a linear map $\mu : M \otimes A \to M$ so that the diagrams

$$\begin{tikzcd} M \otimes A \otimes A \arrow[r, "\mu\otimes\mathrm{id}_A"] \arrow[d, "\mathrm{id}_M\otimes m"'] & M \otimes A \arrow[d, "\mu"] \\ M \otimes A \arrow[r, "\mu"'] & M \end{tikzcd} \qquad \begin{tikzcd} M \otimes \Bbbk \arrow[rr, equal] \arrow[dr, "\mathrm{id}_M\otimes\eta"'] & & M \\ & M \otimes A \arrow[ur, "\mu"'] & \end{tikzcd}$$

commute. Thus, for a coassociative $\Bbbk$-coalgebra C, a *right C-comodule* is a $\Bbbk$-vector space M equipped with a linear map $\rho : M \to M \otimes C$ so that the diagrams

$$\begin{tikzcd} M \arrow[r, "\rho"] \arrow[d, "\rho"'] & M \otimes C \arrow[d, "\rho\otimes\mathrm{id}_C"] \\ M \otimes C \arrow[r, "\mathrm{id}_M\otimes\Delta"'] & M \otimes C \otimes C \end{tikzcd} \qquad \begin{tikzcd} M \arrow[rr, equal] \arrow[dr, "\rho"'] & & M \otimes \Bbbk \\ & M \otimes C \arrow[ur, "\mathrm{id}_M\otimes\epsilon"'] & \end{tikzcd} \tag{2.6}$$

commute. We will use the modified Sweedler notation $\rho(m) = m_{(0)} \otimes m_{(1)}$. In this notation, (2.6) states that for all $m \in M$,

$$m_{(0)(0)} \otimes m_{(0)(1)} \otimes m_{(1)} = m_{(0)} \otimes m_{(1)(1)} \otimes m_{(1)(2)}, \tag{2.7}$$

$$m_{(0)}\epsilon(m_{(1)}) = m. \tag{2.8}$$

In light of (2.7), there is no ambiguity in writing $m_{(0)} \otimes m_{(1)} \otimes m_{(2)}$ and similarly for any number of applications of ρ. Of course, we can define *left C-comodules* analogously, in which case, we would write $\rho(m) = m_{(-1)} \otimes m_{(0)}$. (The convention is that the 0 subscript always corresponds to the elements of M.)

Notation 2.1.4. We will denote the category of right A-modules (resp. left A-modules, right C-comodules, left C-comodules) by $\mathcal{M}_A$ (resp. ${}_A\mathcal{M}$, $\mathcal{M}^C$, ${}^C\mathcal{M}$).

Example 2.1.5. Let $n \in \mathbb{N}$, $n \geq 1$, and let M be a $\Bbbk$-vector space with basis $v_1, \ldots, v_n$. Then M is a right $\mathrm{C}_n(\Bbbk)$-comodule via $\rho(v_j) = \sum_{i=1}^n v_i \otimes e_{i,j}$. This is dual to the action of $\mathrm{M}_n(\Bbbk)$ on an n-dimensional vector space.

2.1.3 Bialgebras and Hopf algebras

With a bit of background on coalgebras and comodules, we can now discuss bialgebras and Hopf algebras.

A $\Bbbk$-*bialgebra* is a $\Bbbk$-vector space B equipped with $\Bbbk$-linear maps m, η, Δ, and ϵ, so that

1. (B, m, η) is an associative $\Bbbk$-algebra,
2. (B, Δ, ϵ) is a coassociative $\Bbbk$-coalgebra, and
3. Δ and ϵ are algebra homomorphisms (or equivalently, m and η are coalgebra homomorphisms).

For item 3, we are considering $B \otimes B$ as an algebra (or coalgera) with the tensor product structure. Note that we can twist the multiplication and/or the comultiplication with the twist map $\tau : a \otimes b \mapsto b \otimes a$ to get three other bialgebra structures on B: B^{op}, B^{cop} and $B^{op\,cop}$. For example, $B^{op\,cop}$ has multiplication $m \circ \tau$ and comultiplication $\tau \circ \Delta$. A *bi-ideal* of B is a subspace which is simultaneously an ideal and coideal of B. These are seen to be the kernels of *bialgebra homomorphisms*, maps between bialgebras which are simultaneously algebra and coalgebra homomorphisms.

Example 2.1.6. If the set in 2.1.2 is a multiplicative monoid M then $\Bbbk M$ is a bialgebra by extending the multiplication of M linearly with 1_M being the identity.

We can form tensor products of bialgebras, with both the tensor product algebra and coalgebra structure. Also, the dual coalgebra B° is a subalgebra of the dual coalgebra B^*, and so we have a bialgebra structure on B°, which we call the *dual bialgebra*.

For a coalgebra C and algebra A, the space $\mathrm{Hom}_\Bbbk(C, A)$ becomes an algebra under the *convolution product* $f * g = m \circ (f \otimes g) \circ \Delta$; the identity is $u \circ \epsilon$. Thus, for a bialgebra B, $\mathrm{End}_\Bbbk(B)$ is an algebra.

Definition 2.1.7. A *Hopf algebra* is a bialgebra H for which id_H has a convolution inverse in $\mathrm{End}_\Bbbk(H)$, called the *antipode* and denoted by S. In other words, a Hopf algebra is a bialgebra equipped with a linear map $S : H \to H$ so that $h_{(1)}S(h_{(2)}) = \epsilon(h)1_H = S(h_{(1)})h_{(2)}$ for all $h \in H$.

Remark 2.1.8. For a Hopf algebra H, and a grouplike element g, the above equation implies that $S(g)$ must be a two-sided inverse for g. In fact, for any Hopf algebra H, the set $G(H)$ forms a group. (For an arbitrary bialgebra B, the set $G(B)$ always forms a monoid.) Additionally, for $x \in P_{(g,h)}(H)$, the equation above implies that $S(x) = -g^{-1}xh^{-1}$.

By virtue of being a convolution inverse of an algebra and coalgebra map, S is both an anti-algebra map and anti-coalgebra map, i.e. $S(ab) = S(b)S(a)$ and $\Delta(S(a)) = S(a_{(2)}) \otimes S(a_{(1)})$ for all $a \in H$. The bialgebra $H^{op\ cop}$ is also a Hopf algebra with the same antipode, while H^{op} and H^{cop} are Hopf algebras if and only if S is bijective, in which case the antipode of these is S^{-1}. All finite-dimensional Hopf algebras have a bijective antipode.

The tensor product of Hopf algebras is a Hopf algebra with antipode given by the tensor product of antipodes. Also, the dual bialgebra H° of any Hopf algebra H is a Hopf algebra with antipode given by $S^\circ = S^*|_{H^\circ}$. A *Hopf ideal* is a bi-ideal I such that $S(I) \subseteq I$, and a *Hopf algebra homomorphism* is a bialgebra homomorphism between Hopf algebras. It turns out that the axioms for a bialgebra homomorphism $f : C \to D$ force $f \circ S_C = S_D \circ f$.

Example 2.1.9. If the monoid (set) in Example 2.1.6 (2.1.2) is a group G, then the resulting group algebra $\Bbbk G$ is a Hopf algebra. The comultiplication and counit are again given by $\Delta(g) = g \otimes g$ and $\epsilon(g) = 1$ and the antipode is given by $S(g) = g^{-1}$.

Example 2.1.10. Let $\mathfrak{g}$ be a Lie algebra over $\Bbbk$ and let $U(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$. By defining $\mathfrak{g} \subseteq P_{1,1}(U(\mathfrak{g}))$, i.e. $\Delta(x) = 1 \otimes x + x \otimes 1$, $\epsilon(x) = 0$, and $S(x) = -x$ for all $x \in \mathfrak{g}$, we get a Hopf algebra structure on $U(\mathfrak{g})$. (One should check that Δ and S are well-defined.)

Example 2.1.11. Let $n \in \mathbb{N}$, $n \geq 2$, and suppose $\Bbbk$ contains a primitive n^{th} root of unity q. The *Taft algebra* $T_n(q)$ is the algebra generated by g and x with relations

$$g^n = 1, \qquad x^n = 0, \qquad gx = qxg.$$

$T_n(q)$ is a Hopf algebra with $g \in G(T_n(q))$ and $x \in P_{g,1}(T_n(q))$. (Thus, we have $\epsilon(g) = 1,\ S(g) = g^{n-1},\ \epsilon(x) = 0,$ and $S(x) = -g^{-1}x$.) The Taft algebras are neither commutative nor cocommutative. The case $n = 2$, namely the Hopf algebra $T_2(-1)$, is called *the Sweedler Hopf algebra*.

A bialgebra or Hopf algebra is called *pointed* if its underlying coalgebra structure is so. It is well-known that any bialgebra generated by grouplike and skew primitive elements is pointed. Each of the Hopf algebras in Examples 2.1.9-2.1.11 are pointed. Andruskiewitsch and Schneider conjecture that, conversely, all finite-dimensional pointed Hopf algebras, H, over an algebraically closed field of characteristic 0 are generated by grouplike and skew primitive elements, [6, Conjecture 5.7]; Angiono verified this conjecture in the case when $G(H)$ is abelian, [8, Theorem 2].

Next, we discuss when a Hopf algebra's coradical filtration gives a filtration of the Hopf algebra. The coradical filtration of Hopf algebras has been useful in the work of Andruskiewitsch and Schneider in classifying pointed Hopf algebras [2, 6]. A coalgebra filtration $\{V_i\}_{i\geq 0}$ of a bialgebra B is called a *bialgebra filtration* if it is also an algebra filtration, i.e. $V_iV_j \subseteq V_{i+j}$ for all i, j. If B is a Hopf algebra, a bialgebra filtration is a *Hopf algebra filtration* if in addition, $S(V_i) \subseteq V_i$ for all i. A *bialgebra grading* is a coalgebra grading $B = \bigoplus_{i\geq 0} B(i)$ of a bialgebra B which is also an algebra grading ($1 \in B(0)$ and $B(i)B(j) \subseteq B(i+j)$). If B is a Hopf algebra, a bialgebra grading is a *Hopf algebra grading* if in addition, $S(B(i)) \subseteq B(i)$ for all $i \geq 0$. A *coradically graded bialgebra (or Hopf algebra)* is a graded bialgebra (or Hopf algebra) whose underlying coalgebra is coradically graded.

The coradical filtration of a bialgebra (Hopf algebra) B is a bialgebra (Hopf algebra) filtration if and only if B_0 is a subalgebra (Hopf subalgebra) of B. In this case, gr(B) is coradically graded, and B is called a *lifting* of gr(B). In particular, every pointed Hopf algebra H is a lifting of gr(H). The Hopf algebras in Examples 2.1.9 and 2.1.11 are coradically graded and the universal enveloping algebra in Example 2.1.10 is a lifting of the symmetric algebra $S(\mathfrak{g})$.

2.1.4 q-Binomial symbols

In many Hopf algebras, there will be a relation like that in the Taft algebras, of the form $yx = qxy$, for $q \in \Bbbk$. It is thus helpful to consider the *quantum binomial* *coefficients*, $\binom{n}{m}_q$, which are defined using any x, y such that $yx = qxy$ by

$$(x+y)^n = \sum_{m=0}^{n} \binom{n}{m}_q x^{n-m} y^m. \tag{2.9}$$

The q-binomial coefficients are polynomials in q and are related to the following symbols. For any integer $n \geq 0$, set

$$(n)_q := 1 + q + q^2 + \ldots + q^{n-1} = \frac{q^n - 1}{q - 1} \quad (\text{if } q \neq 1);$$
$$(n)_q! := (1)_q\,(2)_q \cdots (n)_q = \frac{(q-1)(q^2-1)\cdots(q^n-1)}{(q-1)^n} \quad (\text{if } q \neq 1).$$

By convention, we also define $(0)_q! = 1$.

The relationship between these symbols and q-binomial coefficients is given by [28, Proposition 7.2.1(a)]: If $(n-1)_q! \neq 0$, then one obtains that

$$\binom{n}{m}_q = \frac{(n)_q!}{(m)_q!(n-m)_q!}.$$

It is clear that $(n)_q = 0$ if and only if ord$(q)|n$. Thus, if q is an n^{th} root of unity and $yx = qxy$, we have

$$(x+y)^n = x^n + y^n. \tag{2.10}$$

We also have the following variation, which will be useful for the computation of $D(u_q(\mathfrak{sl}_2))$ in Section 6.1 and Appendix A. Let $q \neq \pm 1 \in \Bbbk$. For any integer n, set

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \cdots + q^{-n+1}.$$

For a positive integer n, set $[n]_q! = [1]_q[2]_q \cdots [n]_q$. Also define

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

As a convention, we will set $[0]_q! = 1$ and $\begin{bmatrix} n \\ k \end{bmatrix}_q = 1$ if $n < k$. The relationship between these and the symbols $(k)_q$ defined above is given by

$$[n]_q = q^{-(n-1)}(n)_{q^2}, \quad [n]_q! = q^{-n(n-1)/2}(n)_{q^2}!, \quad \text{and}$$
$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^{-k(n-k)} \binom{n}{k}_{q^2}. \tag{2.11}$$

2.1.5 Perfect dualities

For computing presentations of Drinfel'd doubles, we will first need presentations of dual Hopf algebras, in such a way that we know the dual pairing. One helpful way for thinking about dual Hopf algebras is perfect dualities, which we recall from [18, Definition V.7.1]. Let H and K be Hopf algebras and $\langle\ ,\ \rangle$ a bilinear form on $H \times K$. We say H and K are *in duality*, or that the bilinear form induces a duality between them, if the following hold for any $u, v \in H$ and $x, y \in K$:

$$\begin{gathered}\langle uv, x\rangle = \langle u, x_{(1)}\rangle\langle v, x_{(2)}\rangle, \quad \langle u, xy\rangle = \langle u_{(1)}, x\rangle\langle u_{(2)}, y\rangle, \\ \langle 1, x\rangle = \epsilon_K(x), \quad \langle u, 1\rangle = \epsilon_H(u), \quad \langle S_H(u), x\rangle = \langle u, S_K(x)\rangle.\end{gathered} \tag{2.12}$$

With $\phi : H \to K^*$ and $\psi : K \to H^*$ defined by $\phi(u)(x) = \langle u, x\rangle = \psi(x)(u)$, we say the duality between H and K is *perfect* if ϕ and ψ are injective. Observe that a perfect duality between finite-dimensional Hopf algebras induces an isomorphism $K \cong H^*$.

2.2 (Co)Actions of Hopf algebras

As mentioned in Section 1.2, we are primarily interested here in actions of Hopf algebras on other algebras, which generalize actions of groups on algebras. The feature of group representations that allows us to define actions of groups on algebras, the base field, and duals, is that the category of representations of a group G is a rigid monoidal category. The representations of a Hopf algebra H, ${}_H\mathcal{M}$, also form a rigid monoidal category. Moreover, due to Example 2.1.9, the representation theory of Hopf algebras generalizes the representation theory of groups. We briefly describe the features of the monoidal structure here. We also introduce a special class of objects which are both modules and comodules over H in a compatible way and explain their significance to the classification of pointed Hopf algebras.

2.2.1 Monoidal structure of representations

One feature of bialgebras is that the categories of modules and comodules form monoidal categories. We will not give a fully rigorous definition of these sorts of categories here, but describe them nonetheless. A *monoidal category* is essentially a category with well-defined and well-behaved tensor products, $\otimes$, and an isomorphism class of objects which constitute an identity object, $\mathbb{1}$, for the tensor product. (See [32, Chapter 1] for more details.) Such a category is called *rigid* if there are well-defined duals, evaluation, and coevaluation maps. The prototype of a rigid monoidal category is $\Bbbk$-**Vect**, the category of vector spaces over $\Bbbk$, with $\otimes$ being the vector space tensor product, $\mathbb{1} = \Bbbk$, and duals being vector space duals. Another example of a monoidal category is **Set**, the category of sets, with $\otimes$ being direct product and $\mathbb{1}$ the set with one element.

Because of the extra structure of a monoidal category, we can define *algebras* (or *monoids*) as objects A equipped with morphisms $m : A \otimes A \to A$ and $\eta : \mathbb{1} \to A$ satisfying the diagrams in (2.1) (with $\mathbb{1}$ replacing $\Bbbk$). Similarly, we can define *coalgebra* (or *comonoids*) as objects C equipped with morphisms $\Delta : C \to C \otimes C$ and $\epsilon : C \to \mathbb{1}$ satisfying the diagrams in (2.2) (again with $\mathbb{1}$ replacing $\Bbbk$). In $\Bbbk$-**Vect**, algebras and coalgebras are associative $\Bbbk$-algebras and coassociative $\Bbbk$-coalgebras, respectively. In **Set**, algebras are traditional monoids, and coalgebras are simply sets where Δ is the diagonal map and ϵ is the unique map to the set with one element. The examples pertinent to this work, however, come from (co)modules.

Example 2.2.1. Let H be a bialgebra. Then the category of left H-modules ${}_H\mathcal{M}$ is a monoidal category with $\otimes$ being the tensor product of vector spaces and $\mathbb{1} = \Bbbk$. For $M, N \in {}_H\mathcal{M}$, the module structures on $M \otimes N$ and $\Bbbk$ are given respectively by

$$h \cdot (m \otimes n) = h_{(1)} \cdot m \otimes h_{(2)} \cdot n \quad (h \in H,\ m \in M,\ n \in N),$$
$$h \cdot 1_{\Bbbk} = \epsilon(h) \quad (h \in H).$$

If H is a Hopf algebra, then ${}_H\mathcal{M}$ is rigid, with the module structure on M^* defined by

$$(h \cdot p)(m) = p(S(h) \cdot m) \quad (h \in H,\ p \in M^*,\ m \in M).$$

Example 2.2.2. Similarly, the category ${}^H\mathcal{M}$ of left H-comodules is a monoidal category. For $M, N \in {}^H\mathcal{M}$, the comodule structures on $M \otimes N$ and $\Bbbk$ are given by

$$\rho(m \otimes n) = m_{(-1)}n_{(-1)} \otimes m_{(0)} \otimes n_{(0)} \quad (h \in H,\ m \in M,\ n \in N),$$
$$\rho(1_\Bbbk) = 1_H \otimes 1_\Bbbk.$$

We now consider algebras and coalgebras in these monoidal categories. Let H be a bialgebra over $\Bbbk$. Since ${}_H\mathcal{M}$ is a monoidal category, we define *left H-module algebras* as algebras in this category. In other words, a left H-module algebra is a left H-module A satisfying

$$h \cdot (ab) = (h_{(1)} \cdot a)(h_{(2)} \cdot b) \quad \text{and} \quad h \cdot 1_A = \epsilon(h)1_A.$$

We will also say that H *acts* on the algebra A. We could similarly define right H-module algebras.

If A is an H-module algebra, then there is an associative $\Bbbk$-algebra structure on $A \otimes H$. The identity is $1_A \otimes 1_H$ and multiplication is defined by

$$(a \otimes h)(b \otimes k) = a(h_{(1)} \cdot b) \otimes h_{(2)}k.$$

This algebra is called the *smash product* of A and H and is denoted $A\#H$.

Similarly, we define *left H-comodule coalgebras* as coalgebras in ${}^H\mathcal{M}$. In other words, a left H-comodule coalgebra is a left H-comodule C satisfying

$$c_{(1)(-1)}c_{(2)(-1)} \otimes c_{(1)(0)} \otimes c_{(2)(0)} = c_{(-1)} \otimes c_{(0)(1)} \otimes c_{(0)(2)} \quad \text{and}$$
$$c_{(-1)}\epsilon(c_{(0)}) = \epsilon(c)1_C.$$

We will also say that H *coacts* on the coalgebra C. If C is an H-comodule coalgebra, then there is a coassociative $\Bbbk$-coalgebra structure on $C \otimes H$. The counit and coproduct are given, respectively, by

$$\epsilon(c \otimes h) = \epsilon(c)\epsilon(h) \quad \text{and} \quad \Delta(c \otimes h) = (c_{(1)} \otimes c_{(2)(-1)}h_{(1)}) \otimes (c_{(2)(0)} \otimes h_{(2)}).$$

This coalgebra is called the *smash coproduct* of C and H and is denoted $C\natural H$.

2.2.2 Yetter-Drinfel'd modules, bosonizations

Because Hopf algebras have both an algebra and coalgebra structure, we can consider spaces which are simultaneously H-modules and H-comodules. If we want such a space to have any useful structure to it, the module and comodule structure should be compatible in some sense. The compatibility condition that seems most obvious for a left H-module and left H-comodule,

$$\rho(h \cdot m) = h_{(1)}m_{(-1)} \otimes h_{(2)} \cdot m_{(0)},$$

defines what are known as *left H-Hopf modules*. These are very well understood. (See, e.g., [28, Section 8.2].) A less obvious, but very useful, compatibility condition leads to Yetter-Drinfel'd modules. These allow us to combine the smash product and smash coproduct structures to form a new Hopf algebra in a process called bosonization.

Let H be a Hopf algebra. A *(left-left) Yetter-Drinfel'd module M* over H is simultaneously a left H-module and a left H-comodule, satisfying the compatibility condition

$$\rho(h \cdot m) = h_{(1)}m_{(-1)}S(h_{(3)}) \otimes h_{(2)} \cdot m_{(0)},$$

for all $h \in H$ and $m \in M$. We will denote the category of Yetter-Drinfel'd modules over H by ${}^H_H\mathcal{YD}$. If $H = \Bbbk\Gamma$ is the group algebra of a group Γ, we will write ${}^\Gamma_\Gamma\mathcal{YD}$ for ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$.

Remark 2.2.3. [6, Remark 1.5] If Γ is an abelian group, then a Yetter-Drinfel'd module over $\Bbbk\Gamma$ is the same as a Γ-graded $\Bbbk\Gamma$-module. If Γ is finite abelian, then the module structure is diagonalizable, and we have

$$V = \bigoplus_{g\in\Gamma,\chi\in\widehat{\Gamma}} V_g^\chi, \quad V_g^\chi = V^\chi \cap V_g = \{v \in V : \rho(v) = g \otimes v,\ \gamma v = \chi(\gamma)v\ \forall \gamma \in \Gamma\}.$$

With the H-module and H-comodule structures on tensor products defined as above, ${}^H_H\mathcal{YD}$ is a monoidal category. The main reason to consider ${}^H_H\mathcal{YD}$ is its use in the study of pointed Hopf algebras and classification of pointed Hopf algebras using bosonizations. (See e.g. [6, 7, 22]) Results in the classification program use the fact that ${}^H_H\mathcal{YD}$ is in fact a *braided* monoidal category, which we describe here. For motivation, recall that in $\Bbbk$**-Vect**, we can form tensor products of algebras and coalgebras, which is what allows us to define bialgebras over $\Bbbk$. (Recall that we need $\Delta : B \to B \otimes B$ to be an algebra map. In particular, this means we need $B \otimes B$ to be an algebra.) In arbitrary monoidal categories, however, we can only form tensor product algebra and coalgebra structures if the category is braided.

Again, without giving the long formal definition, a *braided monoidal category* is a monoidal category together with well-behaved natural isomorphisms

$$c_{A,B} : A \otimes B \to B \otimes A$$

for any pair of objects A, B. If A and B are two algebras in a braided monoidal category, then we can give $A \otimes B$ an algebra structure as well by defining $m_{A\otimes B} = (m_A \otimes m_B) \circ (\mathrm{id} \otimes c_{B,A} \otimes \mathrm{id})$.[2] We can similarly define tensor product coalgebras. With these tensor product algebra and coalgebra structures, *bialgebras* and *Hopf algebras* in such a category are defined analogously to the definition given in Section 2.1.3. This actually generalizes the previous definition, by noting that $\Bbbk$**-Vect** has a braiding $c_{V,W} : V \otimes W \to W \otimes V$ given by the usual twist map: $c_{V,W}(v \otimes w) = w \otimes v$. As another example, **Set** has braiding given by the twist map as well $c_{V,W}(v, w) = (w, v)$. Hopf algebras in **Set** with this braiding are simply groups.

The braiding of ${}^H_H\mathcal{YD}$ is just a bit more complicated, given by

$$c_{M,N}(m \otimes n) = m_{(-1)} \cdot n \otimes m_{(0)}. \tag{2.13}$$

The tensor product of algebras $A, B \in {}^H_H\mathcal{YD}$ is denoted $A\underline{\otimes}B$ to distinguish it from the usual tensor product of two $\Bbbk$-algebras. In $A\underline{\otimes}B$, we have

$$(a \otimes b)(a' \otimes b') = a(b_{(-1)} \cdot a') \otimes b_{(0)}b'.$$

A Hopf algebra in ${}^H_H\mathcal{YD}$ is typically called a *braided Hopf algebra*. It is a coalgebra and algebra $B \in {}^H_H\mathcal{YD}$ such that $\Delta : B \to B\underline{\otimes}B$ and $\epsilon : B \to \Bbbk$ are algebra maps.

[2]Here, we are ignoring the associativity isomorphisms from the monoidal category, which is okay due to Mac Lane's Coherence Theorem. For more information on braided monoidal categories, see [23, 28].

Of course, if B is a braided Hopf algebra in ${}^H_H\mathcal{YD}$, then in particular, B is a left H-module algebra and a left H-comodule coalgebra. Thus, $B \otimes H$ is a $\Bbbk$-algebra and a $\Bbbk$-coalgebra via the smash product and smash coproduct structures (Section 2.2.1), respectively. By combining these structures, and using the antipode of B and H, we get that $B \otimes H$ is in fact a Hopf algebra over $\Bbbk$, which we describe as follows.

Definition-Theorem 2.2.4 ([28, Theorems 11.6.7, 11.6.9])**.** Let H be a Hopf algebra over $\Bbbk$ and let B be a braided Hopf algebra in ${}^H_H\mathcal{YD}$. Then $B \otimes H$ is a Hopf algebra over $\Bbbk$ with

- unit $1_B \otimes 1_H$,
- multiplication $(a \otimes h)(b \otimes k) = a(h_{(1)} \cdot b) \otimes h_{(2)}k$,
- counit $\epsilon(b \otimes h) = \epsilon_B(b)\epsilon_H(h)$,
- comultiplication $\Delta(b \otimes h) = (b_{(1)} \otimes b_{(2)(-1)}h_{(1)}) \otimes (b_{(2)(0)} \otimes h_{(2)})$,
- and antipode $S(b \otimes h) = (1 \otimes S_H(b_{(-1)}h))(S_B(b_{(0)}) \otimes 1)$

for $a, b \in B$ and $h, k \in H$. This Hopf algebra is called the *bosonization* or *biproduct* of B and H, and is denoted by $B\#H$. □

Bosonizations have become an essential tool in the classification of pointed Hopf algebras, thanks to Radford's abstract characterization of those Hopf algebras that can be realized as bosonizations.

Theorem 2.2.5. *[27, Theorem 3] Let H be a Hopf algebra and L a bialgebra, and suppose we have bialgebra maps $L \underset{\pi}{\overset{j}{\leftrightarrows}} H$ satisfying $\pi \circ j = id_H$. Let $B = L_{coinv} = \{\ell \in L : \ell_{(1)} \otimes \pi(\ell_{(2)}) = \ell \otimes 1\}$. Then B is a braided bialgebra in ${}^H_H\mathcal{YD}$ and we have an isomorphism of bialgebras $f : B\#H \to L$ given by $f(b\#h) = bj(h)$.* □

Recall from Section 2.1.3 that the coradical filtration of a Hopf algebra L is a Hopf algebra filtration if and only if L_0 is a Hopf subalgebra. In particular, the coradical filtration of a pointed Hopf algebra L is a Hopf algebra filtration. Thus, in this case, the associated graded coalgebra $\mathrm{gr}(L)$ is a graded Hopf algebra, with

$\Bbbk G(L) = L_0 = \text{gr}(L)(0)$ a Hopf subalgebra. With $j : \Bbbk G(L) \to \text{gr}(L)$ the inclusion and $\pi : \text{gr}(L) \to \text{gr}(L)(0) = \Bbbk G(L)$ the typical projection map, $\text{gr}(L)$ and $\Bbbk G(L)$ satisfy the hypotheses of Theorem 2.2.5. Thus,

$$\text{gr}(L) \cong R\#\Bbbk G(L),$$

where $R = \{\ell \in L : \ell_{(1)} \otimes \pi(\ell_{(2)}) = \ell \otimes 1\}$. The braided Hopf algebra R is in fact graded: $R = \bigoplus_{n\geq 0} R(n)$ with $R(n) = \text{gr}(L)(n) \cap R$. Moreover, $R(0) = \Bbbk 1$ and $R(1) = P(R)$. The braided Hopf algebra R is called the *diagram* of L and the dimension of $R(1)$ is called the *rank* of L. As defined in Section 2.1.3, the Hopf algebra L is called a *lifting* of the coradically graded Hopf algebra $\text{gr}(L)$.

Andruskiewitsch and Schneider have used Radford's result to launch a very active program of classifying finite-dimensional pointed Hopf algebras [2, 6]. Their method is to determine all possible diagrams R when $L_0 = \Bbbk\Gamma$, Γ a group, and then determine all possible liftings of $R\#\Bbbk\Gamma$. A good source of diagrams are Nichols algebras, which we describe in Section 2.4.

2.3 The Drinfel'd double

The main appeal of the Drinfel'd double construction is that every finite dimensional Hopf algebra H embeds in its double $D(H)$, and every Drinfel'd double is quasitriangular. Without giving a rigorous definition, a *quasitriangular* Hopf algebra has an invertible element $R \in H \otimes H$, sometimes called an *R-matrix*, which satisfies

$$R\Delta(h) = \Delta^{cop}(h)R \quad \text{for all } h \in H.$$

Moreover, $\tau(R)$ satisfies the quantum Yang-Baxter equation (1.5), and equips each H-module with a solution as well. Therefore, $D(H)$-modules give solutions to the quantum Yang-Baxter equation, and from any finite-dimensional Hopf algebra, we can arrive at families of solutions through the Drinfel'd double construction.

Moreover, for a Hopf algebra H, the categories ${}_{D(H)}\mathcal{M}$ and ${}_{H}\mathcal{YD}^{H}$ are equivalent [28, Section 13.1]. Here, ${}_{H}\mathcal{YD}^{H}$ is a variant of ${}_{H}^{H}\mathcal{YD}$ which is equivalent as a category to ${}^{H^{cop}}_{H^{cop}}\mathcal{YD}$. That is, objects of ${}_H\mathcal{YD}^H$ are left H-modules and right H-comodules satisfying

$$\rho(h \cdot m) = h_{(2)} \cdot m_{(0)} \otimes h_{(3)} m_{(1)} S^{-1}(h_{(1)}).$$

Toward defining the Drinfel'd double, we first introduce the transpose actions of a Hopf algebra H on its dual. Here, for $p \in H^*$ and $a \in H$, we write $\langle p, a \rangle$ for $p(a) \in \Bbbk$. The transpose action is given by

$$\langle a \succ p, b \rangle := \langle p, ba \rangle, \quad \langle p \prec a, b \rangle := \langle p, ab \rangle, \quad \text{for } a, b \in H,\ p \in H^*.$$

Since $H^\circ \subseteq H^*$ is a Hopf algebra with comultiplication given by m^*, we have $\langle p, ab \rangle = \langle p, m(a \otimes b) \rangle = \langle m^*(p), a \otimes b \rangle = \langle p_{(1)}, a \rangle \langle p_{(2)}, b \rangle$ for $p \in H^\circ$. Therefore, $a \succ p = \langle p_{(2)}, a \rangle p_{(1)}$ and $p \prec a = \langle p_{(1)}, a \rangle p_{(2)}$. Combining these two facts gives

$$a \succ p \prec b = \langle p_{(1)}, b \rangle \langle p_{(3)}, a \rangle p_{(2)}. \tag{2.14}$$

Definition 2.3.1. Let H be a finite-dimensional Hopf algebra with antipode S. (Recall that the antipode S is then necessarily invertible.) The *Drinfel'd double*, $D(H)$, of H, is the Hopf algebra with coalgebra structure given by the tensor product coalgebra structure

$$D(H) = H^{*cop} \otimes H, \tag{2.15}$$

with multiplication given by

$$(p \otimes a)(q \otimes b) = p\left(a_{(1)} \succ q \prec S^{-1}(a_{(3)})\right) \otimes a_{(2)} b, \tag{2.16}$$

with unit $\epsilon \otimes 1$, and with antipode

$$S_{D(H)}(p \otimes a) = (\epsilon \otimes S(a))(p \circ S^{-1} \otimes 1) \quad \text{for } p \in H^*, a \in H.$$

Simple tensors in $D(H)$ are written as $p \bowtie a$.

Note that both H and H^{*cop} embed in $D(H)$, and we will think of elements of the former two as elements of the latter, by identifying $p \bowtie 1$ with p and $\epsilon \bowtie a$ with a. These identifications are justified by the following.

Lemma 2.3.2. *Let H be a finite-dimensional Hopf algebra. Then for $p, q \in H^*$ and $a, b \in H$, we have the following identities in $D(H)$:*

$$(p \bowtie 1)(\epsilon \bowtie a) = p \bowtie a, \quad (p \bowtie 1)(q \bowtie 1) = pq \bowtie 1,$$
$$(\epsilon \bowtie a)(\epsilon \bowtie b) = \epsilon \bowtie ab, \quad S_{D(H)}(p \bowtie 1) = p \circ S^{-1} \bowtie 1, \text{ and}$$
$$S_{D(H)}(\epsilon \bowtie a) = \epsilon \bowtie S(a).$$

As a consequence, if $\{a_i\}_{i=1}^n$ is a set of generators for H and $\{p_i\}_{i=1}^m$ is a set of generators for H^*, then $\{p_i \bowtie 1\}_{i=1}^m \cup \{\epsilon \bowtie a_i\}_{i=1}^n$ generates $D(H)$ as an algebra. □

From now on, we suppress the $\bowtie$ notation. It is clear that the relations between generators of H and H^* will also be relations in $D(H)$. Thus, to achieve an algebra presentation of $D(H)$, it remains to show how elements of H move past those of H^*. We will compute relations giving this "commutation" between elements of H and H^* using the following consequence of (2.16): For any $p \in H^*$ and $a \in H$, we have in $D(H)$:

$$ap \;=\; (a_{(1)} \succ p \prec S^{-1}(a_{(3)}))a_{(2)} \;=\; \langle p_{(1)}, S^{-1}(a_{(3)})\rangle\langle p_{(3)}, a_{(1)}\rangle p_{(2)}a_{(2)}. \quad (2.17)$$

The explicit computation of the double of many finite-dimensional, pointed Hopf algebras will be given later (see Sections 4.2.2, 5.2, 6.1).

2.4 Nichols algebras

Recall that an object of the category ${}^H_H\mathcal{YD}$ is simultaneously a left H-module and left H-comodule satisfying

$$\rho(h \cdot m) = h_{(1)}m_{(-1)}S(h_{(3)}) \otimes h_{(2)} \cdot m_{(0)}.$$

This is a braided monoidal category and we typically call Hopf algebras in this category *braided Hopf algebras*. (See Section 2.2.2.) Also recall that each braided Hopf algebra B in ${}^H_H\mathcal{YD}$ gives rise to a new Hopf algebra, called the bosonization of B and H, and denoted $B\#H$. Thus, it would be useful to have a way to produce braided Hopf algebras in ${}^H_H\mathcal{YD}$. For any $V \in {}^H_H\mathcal{YD}$, there is a canonical graded braided Hopf algebra $\mathfrak{B}(V) \in {}^H_H\mathcal{YD}$, called a *Nichols algebra*. These were first discovered by Warren D. Nichols and appeared in [25]. For a current survey of the Nichols algebras pertinent to the classification program of finite-dimensional pointed Hopf algebras, see the work of Andruskiewitsch and Angiono [3].

Definition 2.4.1. Let H be a Hopf algebra and let $V \in {}^H_H\mathcal{YD}$. A graded braided Hopf algebra $R = \bigoplus_{n\geq 0} R(n)$ in ${}^H_H\mathcal{YD}$ is called a *Nichols algebra* of V, denoted $\mathfrak{B}(V)$, if

- $\Bbbk = R(0)$ and $V \cong R(1)$ as Yetter-Drinfel'd modules,
- $R(1) = P_{1,1}(R)$, and
- R is generated as an algebra by $R(1)$.

The dimension of $V \cong R(1)$ will be called the *rank* of $\mathfrak{B}(V)$.

It turns out Nichols algebras of $V \in {}^H_H\mathcal{YD}$ always exist and are unique up to isomorphism. In fact, $\mathfrak{B}$ upgrades to a functor. We now wish to describe a special class of Nichols algebras introduced by Andruskiewitsch and Schneider: those coming from braided vector spaces of "Cartan type".

Definition 2.4.2. A *braided vector space* is a $\Bbbk$-vector space V equipped with a map $c \in \mathrm{End}_{\Bbbk}(V \otimes V)$ satisfying the *quantum Yang-Baxter equation* (or *braid equation*):

$$(c \otimes \mathrm{id}) \circ (\mathrm{id} \otimes c) \circ (c \otimes \mathrm{id}) = (\mathrm{id} \otimes c) \circ (c \otimes \mathrm{id}) \circ (\mathrm{id} \otimes c).$$

We say a braided vector space (V, c) is of

- *diagonal type* if there is a basis $(x_1, \dots, x_\theta)$ of V such that

  $$c(x_i \otimes x_j) = q_{i,j} x_j \otimes x_i$$

  for all i, j. Every braided vector space of diagonal type can be realized as an object of ${}^\Gamma_\Gamma\mathcal{YD}$ for some abelian group Γ. Conversely, every $M \in {}^H_H\mathcal{YD}$ is a braided vector space via the braiding map $c_{M,M}$.

- *Cartan type* if, further, $q_{i,i} \neq 1$ for all i and there are integers $a_{i,j}$ satisfying

  1. $a_{i,i} = 2$,
  2. $0 \leq -a_{i,j} \leq \mathrm{ord}(q_{i,i})$ if $i \neq j$, and
  3. $q_{i,j} q_{j,i} = q_{i,i}^{a_{i,j}}$.

  If the matrix (a_{ij}) is a Cartan matrix associated to a finite-dimensional semisimple Lie algebra, we can specify the type further.

If V is a braided vector space of Cartan type, viewed as an object in ${}^H_H\mathcal{YD}$ for some Hopf algebra H, we say $\mathfrak{B}(V)$ is a *Nichols algebra of Cartan type*. More specifically, if V is a braided vector space of type $(A_1)^{\times\theta}$, then the Nichols algebra $\mathfrak{B}(V) \in {}^H_H\mathcal{YD}$ is called a *quantum linear space over H*.

Quantum linear spaces are studied in detail in [5]. We study these more in Chapter 4.

2.5 Inner-faithful module algebras

Throughout this work, we consider module algebras over some pointed Hopf algebras that are faithful in the following sense.

Definition 2.5.1. Let H be a Hopf algebra and M a left H-module. We say that M is an *inner-faithful* H-module, or that the action of H on M is *inner-faithful* provided $I \cdot M \neq 0$ for any nonzero Hopf ideal I of H. In other words, the action of H on M is inner-faithful provided the action on M does not factor through any proper Hopf quotient of H.

If A is an H-module algebra such that the action of H on A is inner-faithful, we call A an *inner-faithful* H-module algebra.

Clearly, if the action of H on M is faithful, then it is inner-faithful. Since all of the Hopf algebras we will consider in this work are pointed, the following standard results will be useful.

Lemma 2.5.2. *Let H be a pointed Hopf algebra and I a nonzero Hopf ideal of H. Then I contains a nonzero element of $P_{g,1}(H)$ for some $g \in G(H)$.*

Proof. Consider the projection map $f : H \to H/I$. Since $I \neq 0$, f is not injective. Therefore, by [31, 6.1.1], we can fix some $g, h \in G(H)$, with $f \mid_{P_{g,h}(H)}$ not injective. Choose nonzero $x \in P_{g,h}(H)$ such that $f(x) = 0$ (i.e. $x \in I$), and take $x' = xh^{-1}$. Then $x' \in P_{gh^{-1},1}(H) \cap I$ and $x' \neq 0$, or else $x = x'h = 0$. □

Corollary 2.5.3. *Let H be a pointed Hopf algebra and A an H-module algebra. Then the action of H on A is inner-faithful if and only if for each $g \in G(H)$ and nonzero $x \in P_{g,1}(H)$ we have that $x \cdot A \neq 0$.* □

Since $g - 1 \in P_{g,1}(H)$, we have the following consequence.

Corollary 2.5.4. *Suppose that H acts on A inner-faithfully. Then the group of grouplike elements $G(H)$ acts faithfully on A by algebra automorphisms.* □

These results actually give us a lower bound on the $\Bbbk$-vector space dimension of inner-faithful module algebras with no nonzero nilpotent elements.

Proposition 2.5.5. *Suppose that a finite group G acts faithfully by algebra automorphisms on a finite-dimensional $\Bbbk$-algebra A with no nonzero nilpotent elements. Then*

$$\dim_{\Bbbk}(A) \geq \max\{\text{ord}(g) : g \in G\}.$$

Proof. Let $g \in G$ and $n = \text{ord}(g)$. Since $\langle g \rangle$ is finite abelian, the action of g on A is diagonalizable with

$$A = \bigoplus_{i=0}^{n-1} A_i, \qquad A_i = \{a \in A : g \cdot a = q^i a\},$$

where q is a fixed primitive n^{th} root of unity. Because $\text{ord}(g) = n$, and the action is faithful, there exists j such that $\gcd(j, n) = 1$ and $A_j \neq 0$. Without loss of generality, by choosing a different q, we can take $j = 1$. Choose nonzero $u \in A_1$. Since A has no nonzero nilpotent elements, $u^i \neq 0$ for all i. Also, $g \cdot u^i = q^i u^i$ for all i, showing that $u^i \in A_i$. Thus, $A_i \neq 0$ for all i. Therefore, $\dim_{\Bbbk}(A) \geq n$. □

Remark 2.5.6. For any Hopf algebra H, Proposition 2.5.5 shows that if $G(H)$ is cyclic of order n, then the smallest possible dimension of an inner-faithful H-module algebra with no nonzero nilpotent elements is n, and that if such a lower bound is met, then these H-module algebras would be exactly $A(H)$ as in Notation 1.3.2.

Remark 2.5.7. For each of the Hopf algebras we consider, the group of grouplike elements is a finite cyclic group. Thus, for convenience, we describe the general structure of $A(H)$ in case $G(H)$ is cyclic. Fix a generator $g \in G(H)$. Then, for a generator $u \in A(H)$ such that $A(H) \cong \Bbbk[u]/(u^n - 1)$, there is a primitive n^{th} root of unity $q \in \Bbbk$ with $g \cdot u = qu$. Alternatively, for a fixed q, we can choose $u \in A(H)$ such that $g \cdot u = qu$ and $A(H) = \Bbbk[u]/(u^n - 1)$. Here, we write the eigenspaces of the g-action

$$A_i = \{a \in A : g \cdot a = q^i a\},$$

noting that $A = \bigoplus_{i=0}^{n-1} A_i$ and $A_i = \Bbbk u^i$. We will use this notation throughout.

CHAPTER 3

THE TAFT ALGEBRAS

In this section, we will consider Question 1.3.3 in Chapter 1 for the Taft algebras $T_n(q)$ (see Example 2.1.11).

3.1 Work of Montgomery–Schneider ($n \geq 3$)

Recall that Montgomery and Schneider have answered Question 1.3.3(a,b) for actions of the Taft algebras $T_n(q)$ on the algebra $A(T_n(q))$ given in Notation 1.3.2; see Theorem 1.3.1. They further answered Question 1.3.3(c) on actions of the double $D(T_n(q))$ on $A(T_n(q))$ for the case $n > 2$ as recalled in the next two results.

Lemma 3.1.1. *[24, Lemma 4.4] The Hopf algebra $D(T_n(q))$ is generated by grouplike elements g and G, a $(g, 1)$-skew primitive element x, and a $(1, G)$-skew primitive element X, subject to the relations*

$$g^n = G^n = 1, \quad x^n = X^n = 0, \quad gx = qxg, \quad GX = qXG,$$
$$gG = Gg, \quad xG = qGx, \quad gX = q^{-1}Xg, \quad xX = Xx + G - g. \qquad \square$$

Note that X is $(1, G)$-skew primitive in $D(T_n(q))$, whereas it is $(G, 1)$-skew primitive in $T_n(q)^* \cong T_n(q)$, because $D(T_n(q))$ contains a copy of $T_n(q)^{*cop}$.

Theorem 3.1.2. *[24, Theorem 4.5] Take $n > 2$. Let $A = \Bbbk[u]/(u^n - \beta)$ for $\beta \in \Bbbk^\times$ be an n-dimensional inner-faithful $T_n(q)$-module algebra with no nonzero nilpotent* *elements, such that $g \cdot u = qu$ and $x \cdot u = \gamma 1_A$ for $0 \neq \gamma \in \mathbb{k}$. Then, by defining $G \cdot u = q^{-1}u$ and $X \cdot u = \gamma^{-1}(q^{-1} - 1)u^2$, we obtain that $A(T_n(q))$ is a $D(T_n(q))$-module algebra. Moreover, all $D(T_n(q))$-module algebra structures on $A(T_n(q))$ are of this form.* $\square$

The original theorem in [24] has the assumption $n > 1$, not $n > 2$. We now discuss this disparity.

3.2 The Sweedler algebra ($n = 2$)

We begin with the following remark pertaining to Theorem 3.1.2 in the case when $n = 2$, i.e. for the Sweedler algebra $T_2(-1)$.

Remark 3.2.1. The proof of Theorem 3.1.2 in [24] fails for $n = 2$ at the point when one considers the action of $H^{*cop} \subset D(H)$, and applies [24, Theorem 2.2]. To specify the action of H^{*cop}, one uses integers $0 \leq s, t \leq n-1$ with $t(1-s) \equiv 1 \mod n$. It is shown then that $t = n - 1$, from which it is concluded that $s = 2$. This is valid if $n > 2$. However, for $n = 2$, we get that $s = 0$, and [24, Theorem 2.2] actually gives us different information than when $n > 2$. We explore here the case when $n = 2$, that is, when H is the Sweedler Hopf algebra, $T_2(-1)$.

Note that by Theorem 1.3.1 and Remark 2.5.7, we know all the actions (as in Notation 1.3.2) of $T_2(-1)$ on $A(T_2(-1))$, namely that as an algebra, $A(T_2(-1)) \cong \mathbb{k}[u]/(u^2 - 1)$, with the action given by $g \cdot u = -u$ and $x \cdot u = \gamma 1_A$ for some nonzero $\gamma \in \mathbb{k}$. Considering the remark above, we now examine Question 1.3.3(c) for $H = T_2(-1)$.

Proposition 3.2.2. *Recall the notation of Lemma 3.1.1 for $n = 2$, and thus $q = -1$. Fix an action of $T_2(-1)$ on $A(T_2(-1)) = \mathbb{k}[u]/(u^2 - 1)$ as in Theorem 1.3.1,*

$$g \cdot u = -u, \quad x \cdot u = \gamma 1_A,$$

for some nonzero $\gamma \in \mathbb{k}$. Then, for any $\delta \in \mathbb{k}$, by defining

$$G \cdot u = -u, \quad X \cdot u = \delta 1_A,$$

we obtain that $A(T_2(-1))$ is a $D(T_2(-1))$-module algebra. Moreover, all extensions of the action of $T_2(-1)$ on $A(T_2(-1))$ to $D(T_2(-1))$ are of this form.

Proof. That $A(T_2(-1))$ is a $D(T_2(-1))$-module algebra with the given action of G and X is easily verified, so we show that all extensions of the action of $T_2(-1)$ on $A(T_2(-1))$ to an action $D(T_2(-1))$ are of this form. Fix an action of $T_2(-1)$ on $A(T_2(-1))$. That is, we have $A := A(T_2(-1)) = \Bbbk[u]/(u^2 - 1)$ with the action of $T_2(-1)$ on A given by $g \cdot u = -u$ and $x \cdot u = \gamma 1_A$. We can decompose A by the eigenspaces of the action of g as in Remark 2.5.7: $A = A_0 \oplus A_1$ with $A_0 = \Bbbk 1_A$ and $A_1 = \Bbbk u$. Now assume this action can be extended to an action of $D(T_2(-1))$. Since A is a $D(T_2(-1))$-module algebra, $g \cdot (G \cdot u) = G \cdot (g \cdot u) = -G \cdot u$. Hence, $G \cdot u \in A_1$, so $G \cdot u = \alpha u$ for some $\alpha \in \Bbbk$. Also, we have $x \cdot (G \cdot u) = -G \cdot (x \cdot u) = -G \cdot \gamma 1_A = -\gamma 1_A$, so $\alpha = -1$. Finally, $g \cdot (X \cdot u) = -X \cdot (g \cdot u) = X \cdot u$ implies that $X \cdot u \in A_0 = \Bbbk 1_A$, so $X \cdot u = \delta 1_A$ for some $\delta \in \Bbbk$. (Note that $(xX - Xx) \cdot u = (G - g) \cdot u = 0$, so no restrictions on δ need to be imposed.) $\square$

All the results about the Taft algebras, including the Sweedler algebra — Montgomery and Schneider's results (stated in Theorem 1.3.1 and Theorem 3.1.2) as well as Proposition 3.2.2 — can be realized as a corollary of results about a generalization of Taft algebras, which we consider next.

CHAPTER 4

$H_n(\zeta, m, t)$, A CORADICALLY GRADED GENERALIZATION OF TAFT ALGEBRAS

We wish to answer Question 1.3.3 for a family of coradically graded Hopf algebras that contains the Taft algebras $T_n(q)$ (Example 2.1.11). In the language of Nichols algebras (Section 2.4), $T_n(q)$ is of Cartan type A_1 and has rank 1. In fact, $T_n(q) \cong \mathfrak{B}(V)\#\Bbbk\Gamma$, where $(V, c) = \Bbbk x$ is a one-dimensional braided vector space with braiding $c(x \otimes x) = qx \otimes x$, $\Gamma = \langle g \rangle$ the cyclic group of order n, $g \cdot x = qx$, and $\rho(x) = g \otimes x$. That is, $T_n(q)$ is a bosonization of the quantum linear space $\mathfrak{B}(V)$ of rank 1. Thus, we consider more generally bosonizations of all rank 1 quantum linear spaces over finite cyclic groups.

4.1 The Hopf Algebras $H_n(\zeta, m, t)$

We start with Andruskiewitsch and Schneider's construction in [5] of rank θ quantum linear spaces over a finite abelian group Γ, and then restrict to the case that Γ is cyclic and $\theta = 1$. Let $V \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$ of dimension θ, say with basis $x_1, \ldots, x_\theta$. By Remark 2.2.3, $\rho(x_i) = g_i \otimes x_i$ for some $g_i \in \Gamma$, and for all $g \in \Gamma$, $g \cdot x_i = \chi_i(g)x_i$

for some $\chi_i \in \Gamma$. Thus, by (2.13), there is a braiding on V given by

$$c(x_i \otimes x_j) = g_i \cdot x_j \otimes x_i = \chi_j(g_i) x_j \otimes x_i.$$

As stated in Section 2.4, if V is of type $(A_1)^{\times\theta}$, that is, if $\chi_i(g_i) \neq 1$ for all i and $\chi_i(g_j)\chi_j(g_i) = 1$ for all i, j, then $\mathfrak{B}(V)$ is a quantum linear space. One can verify that $\mathfrak{B}(V)$ is generated by the x_i with relations

$$x_i^{N_i} = 0 \text{ with } N_i = \mathrm{ord}(\chi_i(g_i)), \quad \text{and} \quad x_i x_j = \chi_j(g_i) x_j x_i \; (i \neq j).$$

Definition 4.1.1. [5, Section 3] For V as above, the *quantum linear space* $\mathfrak{B}(V)$ in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ just defined is denoted by $\mathcal{R}(g_1, \ldots, g_\theta; \chi_1, \ldots, \chi_\theta)$.

To classify all rank 1 quantum linear spaces over a finite cyclic group, we will use the following result in group theory, which can be verified using some basic number theory and the Chinese Remainder Theorem.

Lemma 4.1.2. *If Γ is a cyclic group of order n, and an element $g \in \Gamma$ has order n/k for some $k|n$, then there exists a generator y of Γ such that $g = y^k$.* □

Now let Γ be a finite cyclic group of order n. A quantum linear space of rank 1 over Γ, denoted $\mathcal{R}(g; \chi)$, is entirely determined by a choice of $g \in \Gamma$ and $\chi \in \widehat{\Gamma}$ such that $\chi(g) \neq 1$. Fix a non-identity element $g \in \Gamma$. Then g has order n/m for some $m|n$, and by Lemma 4.1.2, we can choose a generator y of Γ so that $g = y^m$. Similarly, fix a non-identity element $\chi \in \widehat{\Gamma}$. Then $\chi(y)$ is an n^{th} root of unity, say of order n/t with $t|n$, and again by Lemma 4.1.2, we can choose a primitive n^{th} root of unity, ζ, such that $\chi(y) = \zeta^t$. We have $\chi(g) = \zeta^{mt}$, and $N = \mathrm{ord}(\chi(g)) = \frac{n}{\gcd(n,mt)}$. Our assumption that $\chi(g) \neq 1$ means precisely that $n \nmid mt$. In this case, $\mathcal{R}(g; \chi)$ has a single generator, x, and a single relation, $x^N = 0$. By definition, $\mathcal{R}(g; \chi)$ is a braided Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ with $\rho(x) = g \otimes x = y^m \otimes x$ and $y \cdot x = \chi(y)x = \zeta^t x$. The structure of the bosonization $\mathcal{R}(g; \chi)\#\Bbbk\Gamma$ is similar to that of a Taft algebra, and we denote it with similar notation.

Definition-Proposition 4.1.3. Let m, t be positive integer divisors of n such that $n \nmid mt$ and let ζ be a primitive n^{th} root of unity. Define $H_n(\zeta, m, t)$ as the $\Bbbk$-algebra generated by y and x, subject to the relations

$$y^n = 1, \qquad x^N = 0 \text{ (for } N = \mathrm{ord}(\zeta^{mt})), \qquad yx = \zeta^t xy.$$

The algebra $H_n(\zeta, m, t)$ has a unique Hopf algebra structure determined by

$$\Delta(y) = y \otimes y, \quad \Delta(x) = y^m \otimes x + x \otimes 1,$$
$$\epsilon(y) = 1, \quad \epsilon(x) = 0, \quad S(y) = y^{-1}, \quad S(x) = -y^{-m}x.$$

Here, $H_n(\zeta, m, t) \cong \mathcal{R}(g;\chi)\#\Bbbk\Gamma$, where $g = y^m$ and $\chi(y) = \zeta^t$. Such Hopf algebras have dimension Nn. $\square$

For a fixed n, a natural first question is whether each choice of ζ, m, and t determines a unique Hopf algebra. Unsurprisingly, the answer is negative; however, an isomorphism class does uniquely determine n, m, and t. To show this, we require the following lemma characterizing certain primitive elements.

Lemma 4.1.4. *[5, Corollary 5.3] Let* $0 \leq b < n$. *Then*

$$P_{y^b,1}(H_n(\zeta, m, t)) = \begin{cases} \Bbbk x + \Bbbk(y^b - 1), & \text{if } b \equiv m \mod n \\ \Bbbk(y^b - 1), & \text{otherwise}. \end{cases}$$

$\square$

Proposition 4.1.5. *Let $m, \widehat{m}, t, \widehat{t}$ be positive divisors of n such that n divides neither mt nor $\widehat{m}\widehat{t}$. Let $\zeta, \widehat{\zeta}$ be primitive n^{th} roots of unity in $\Bbbk$. Then $H_n(\zeta, m, t) \cong H_{\widehat{n}}(\widehat{\zeta}, \widehat{m}, \widehat{t})$ if and only if $n = \widehat{n}$, $m = \widehat{m}$, $t = \widehat{t}$, and there exists $f \in (\mathbb{Z}/n\mathbb{Z})^\times$ such that $(\widehat{\zeta})^{ft} = \zeta^t$ and $fm \equiv m \mod n$. As a consequence, for fixed $n \in \mathbb{N}$ and a fixed primitive n^{th} root of unity ζ, each choice of $m, t \in \mathbb{N}$ with both dividing n and $n \nmid mt$ yields a unique isomorphism class of Hopf algebras $H_n(\zeta, m, t)$.*

Proof. Let y, x denote the generators of $H_n(\zeta, m, t)$, and $\hat{y}, \hat{x}$ the generators of $H_{\widehat{n}}(\widehat{\zeta}, \widehat{m}, \widehat{t})$. Assume the conditions on $\widehat{n}$, $\widehat{m}$, $\widehat{t}$, and f. The isomorphism between the two is defined by sending y to $\widehat{y}^f$ and x to $\widehat{x}$. One can easily check that this defines a Hopf algebra isomorphism.

On the other hand, suppose $H_n(\zeta, m, t)$ and $H_{\widehat{n}}(\widehat{\zeta}, \widehat{m}, \widehat{t})$ are isomorphic and let ϕ denote an isomorphism between them. By counting grouplike elements, $n = \widehat{n}$.

Moreover, $\phi(y)$ must be a grouplike element of order n. Thus, there exists some $f \in (\mathbb{Z}/n\mathbb{Z})^\times$ such that $\phi(y) = \widehat{y}^f$. Since $(\phi \otimes \phi) \circ \Delta = \Delta \circ \phi$, we must have $\phi(x) \in P_{\widehat{y}^{fm},1}(H_{\widehat{n}}(\widehat{\zeta}, \widehat{m}, \widehat{t}))$. By Lemma 4.1.4,

$$P_{\widehat{y}^{fm},1}(H_{\widehat{n}}(\widehat{\zeta}, \widehat{m}, \widehat{t})) = \begin{cases} \Bbbk\widehat{x} + \Bbbk(\widehat{y}^{fm} - 1), & fm \equiv \widehat{m} \mod n \\ \Bbbk(\widehat{y}^{fm} - 1), & \text{otherwise.} \end{cases}$$

Since $\phi(x)$ and $\widehat{y}^f$ must generate $H_{\widehat{n}}(\widehat{\zeta}, \widehat{m}, \widehat{t})$, it must be that $fm \equiv \widehat{m} \mod n$ and $\phi(x) = \alpha\widehat{x} + \beta(\widehat{y}^{fm} - 1)$ for some $\alpha, \beta \in \Bbbk$ with $\alpha \neq 0$. Now, since m and $\widehat{m}$ both divide n, and f is a unit mod n, the equation $fm \equiv \widehat{m} \mod n$ implies $m = \widehat{m}$. We must have

$$\begin{aligned} 0 = \phi(yx - \zeta^t xy) &= \widehat{y}^f(\alpha\widehat{x} + \beta(\widehat{y}^{fm} - 1)) - \zeta^t(\alpha\widehat{x} + \beta(\widehat{y}^{fm} - 1))\widehat{y}^f \\ &= ((\widehat{\zeta})^{f\widehat{t}} - \zeta^t)\alpha\widehat{x}\widehat{y}^f + (1 - \zeta^t)\beta\widehat{y}^f(\widehat{y}^{fm} - 1). \end{aligned}$$

Thus, since $\zeta^t \neq 1$ and $\widehat{y}^{fm} \neq 1$ (as $n \nmid mt$), we must have $\beta = 0$. Also, since $\alpha \neq 0$, we have $(\widehat{\zeta})^{f\widehat{t}} = \zeta^t$. Since ζ and $\widehat{\zeta}$ are primitive n^{th} roots of unity, $\widehat{\zeta} = \zeta^e$ for some $e \in (\mathbb{Z}/n\mathbb{Z})^\times$. Therefore, $ef\widehat{t} \equiv t \mod n$, and just as for $m = \widehat{m}$, we see that $t = \widehat{t}$. □

Not only do the Hopf algebras just presented include Taft algebras; they also include the coradically graded generalized Taft algebras.

Definition 4.1.6. For natural numbers n, N satisfying $N \mid n$, a primitive N^{th} root of unity $q \in \Bbbk$, and $\alpha \in \Bbbk$ arbitrary, the *generalized Taft algebra* $T(n, N, \alpha)$ is the Hopf algebra generated by a grouplike element g and a $(g, 1)$-skew primitive element x, subject to the relations

$$g^n = 1, \quad x^N = \alpha(g^N - 1), \quad gx = qxg.$$

Note that, if ζ is a primitive n^{th} root of unity with $\zeta^{\frac{n}{N}} = q$, then $T(n, N, \alpha)$ is a lifting of $H_n(\zeta, 1, \frac{n}{N})$ in the sense that $\mathrm{gr}(T(n, N, \alpha)) \cong H_n(\zeta, 1, \frac{n}{N})$ (Section 2.1.3).

Corollary 4.1.7. *The Hopf algebra $H_n(\zeta, m, t)$ is isomorphic to a generalized Taft algebra of the form $T(n, N, 0)$ if and only if $m = 1$. In this case, $q = \zeta^t$, and $N = n/t$. Moreover, any generalized Taft algebra of the form $T(n, N, 0)$ can be realized as such.*

Proof. Assume $m = 1$. Then $N = n/t$ and $q = \zeta^t$ is a primitive N^{th} root of unity by definition. Define a homomorphism $\phi : H_n(\zeta, 1, t) \to T(n, N, 0)$ by $y \mapsto g$ and $x \mapsto x$. We see that ϕ is surjective, so by a dimension count, ϕ is an isomorphism. Moreover, we see that any generalized Taft algebra, $T(n, N, 0)$ is isomorphic $H_n(\zeta, 1, n/N)$, where ζ is chosen so that $\zeta^t = q$.

Now, assume that $H_n(\zeta, m, t)$ is isomorphic to a coradically graded generalized Taft algebra $T(n, N, 0) \cong H_{\widehat{n}}(\widehat{\zeta}, 1, n/N)$. By Proposition 4.1.5, $m = 1$. □

Now the following consequence is clear.

Corollary 4.1.8. *The Hopf algebra $H_n(\zeta, m, t)$ is isomorphic to a Taft algebra if and only if $m = t = 1$. In that case, $H_n(\zeta, m, t) \cong T_n(\zeta)$. Thus, if n is prime, then every Hopf algebra of the form $H_n(\zeta, m, t)$ is a Taft algebra.*

Proof. Assume $m = t = 1$. Then $N = \text{ord}(\zeta^{mt}) = n$, so by Corollary 4.1.7, $H_n(\zeta, m, t) \cong T(n, n, 0) = T_n(q)$. By the same proposition, $q = \zeta^t = \zeta$.

On the other hand, assume $H_n(\zeta, m, t) \cong T_n(q) = T(n, n, 0)$. By Corollary 4.1.7, $m = 1$ and $n = n/t$, so $t = 1$. □

A consequence of Lemma 4.1.4 and Corollary 2.5.3 is the following:

Corollary 4.1.9. *A left $H_n(\zeta, m, t)$-module M is inner-faithful if and only if $G(H_n(\zeta, m, t)) = \langle y \rangle$ acts faithfully on M and $x \cdot M \neq 0$.*

Proof. The forward direction is clear. Assume, then, that $\langle y \rangle$ acts faithfully and that $x \cdot M \neq 0$. Then every nonzero multiple of $y^b - 1$ does not act by zero for every b. Thus, we only need to check that each nonzero element of $\Bbbk x + \Bbbk(y^m - 1)$ acts by nonzero by Corollary 2.5.3 and Lemma 4.1.4. Since x and $y^m - 1$ do not act by zero, this is equivalent to showing that x does not act as any nonzero scalar multiple of $y^m - 1$. Let $M_i = \{a \in M : y \cdot a = \zeta^i a\}$ denote the eigenspaces of the action of y on

M. Note that if $u \in M_i$, then $x \cdot u \in M_{i+t}$, since $y \cdot (x \cdot u) = \zeta^t x \cdot (y \cdot u) = \zeta^{i+t} x \cdot u$. If $x \cdot M_i = 0$ for all i, then $x \cdot M = 0$, contradicting our hypothesis. Thus, choose i and $u \in M_i$ such that $x \cdot u \neq 0$. Then $(y^m - 1) \cdot u = (\zeta^{mi} - 1)u \in M_i$, but $x \cdot u \in M_{i+t}$. Since $n \nmid mt$, $M_i \neq M_{i+t}$. Thus, $x \cdot u$ is not equal to any nonzero scalar multiple of $(y^m - 1) \cdot u$. □

Our next goal is to answer Question 1.3.3 for $H_n(\zeta, m, t)$. That is, we are interested in the existence of structures $A(H_n(\zeta, m, t))$ as in Notation 1.3.2, and whether or not such structures can be extended to admit actions of $D(H_n(\zeta, m, t))$. Before considering this, we compute $D(H_n(\zeta, m, t))$ explicitly. This is made easier by first giving a nice presentation of the dual.

4.2 The dual and double of $H_n(\zeta, m, t)$

4.2.1 The dual $H_n(\zeta, m, t)^*$

In [9], Beattie computed the duals of quantum linear spaces. As an application of [9, Corollary 2.3], we get the following result:

Lemma 4.2.1. *[9] As Hopf algebras, $H_n(\zeta, m, t)^* \cong H_n(\zeta, t, m)$.* □

Proof. As a reminder, $H_n(\zeta, m, t) \cong \mathfrak{B}(V)\#\Bbbk\Gamma$, for Γ a cyclic group of order n and $V = \Bbbk x \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$ with $x \in V_g^{\chi}$. Recall from the discussion before Definition 4.1.3, for y a generator of Γ, g and χ are defined by $g = y^m$ and $\chi(y) = \zeta^t$. Thus, by [9, Corollary 2.3], $H_n(\zeta, m, t)^* \cong \mathfrak{B}(W)\#\Bbbk\widehat{\Gamma}$, with $W = \Bbbk\hat{x}$, and $\hat{x} \in V_{\chi}^{g}$. Since Γ is abelian, $\widehat{\Gamma} \cong \Gamma$. Recalling the construction of $H_n(\zeta, m, t)$ in Section 4.1, we see that switching χ and g amounts to switching m and t. Hence, we get the result that $H_n(\zeta, m, t)^* \cong H_n(\zeta, t, m)$. □

Since we have a presentation of the dual, for computing the double, we would like to know the dual pairing between $H_n(\zeta, m, t)$ and $H_n(\zeta, t, m)$. Thus, we exhibit a perfect duality between these two Hopf algebras.

Proposition 4.2.2. *With y, x denoting the generators of $H_n(\zeta, m, t)$, and Y, X the generators $H_n(\zeta, t, m)$, the bilinear form defined by*

$$\langle X^i Y^j, x^k y^\ell \rangle = \delta_{i,k}\ (i)_q!\ \zeta^{j\ell}, \tag{4.1}$$

is a perfect duality.

In particular, we get that the dual pairing is given on generators by

$$\langle Y, y \rangle = \zeta, \quad \langle Y, x \rangle = 0, \quad \langle X, y \rangle = 0, \quad \langle X, x \rangle = 1.$$

Note the following equalities, which will be useful for our calculations:

$$\Delta(X^i Y^j) = \sum_{s=0}^{i} \binom{i}{s}_q X^{i-s} Y^{ts+j} \otimes X^s Y^j \quad \text{and} \tag{4.2}$$

$$S(x^i y^j) = S(y^j) S(x^i) = (-1)^i y^{-j} q^{i-1} y^{-im} x^i \ = \ (-1)^i q^{i-1} \zeta^{-ti(im+j)} x^i y^{-im-j}. \tag{4.3}$$

Proof of Proposition 4.2.2. We show that (4.1) is a duality, i.e. that (2.12) holds. First, we check that

$$\langle X^a Y^b, x^i y^j x^k y^\ell \rangle = \langle (X^a Y^b)_{(1)}, x^i y^j \rangle \langle (X^a Y^b)_{(2)}, x^k y^\ell \rangle. \tag{4.4}$$

On the one hand,

$$\langle X^a Y^b, x^i y^j x^k y^\ell \rangle = \zeta^{tjk} \langle X^a Y^b, x^{i+k} y^{j+\ell} \rangle \overset{(4.1)}{=} \delta_{a,i+k}\ (a)_q!\ \zeta^{tjk+b(j+\ell)}.$$

On the other hand, we have

$$\begin{aligned}
&\langle (X^a Y^b)_{(1)}, x^i y^j \rangle \langle (X^a Y^b)_{(2)}, x^k y^\ell \rangle \\
&\qquad \overset{(4.2)}{=} \sum_{s=0}^{a} \binom{a}{s}_q \langle X^{a-s} Y^{ts+b}, x^i y^j \rangle \langle X^s Y^b, x^k y^\ell \rangle \\
&\qquad \overset{(4.1)}{=} \sum_{s=0}^{a} \binom{a}{s}_q \delta_{a-s,i}\ (a-s)_q!\ \zeta^{(ts+b)j}\ \delta_{s,k}\ (s)_q!\ \zeta^{b\ell} \\
&\qquad = \ \delta_{a,i+k} \binom{a}{k}_q\ (i)_q!\ (k)_q!\ \zeta^{tjk+bj+b\ell}.
\end{aligned}$$

Now (4.4) follows since when $a = i + k$, we have $\binom{a}{k}_q = \frac{(a)_q!}{(i)_q!(k)_q!}$.

The proof that $\langle X^aY^bX^cY^d, x^iy^j\rangle = \langle X^aY^b, (x^iy^j)_{(1)}\rangle\langle X^cY^d, (x^iy^j)_{(2)}\rangle$ follows similarly. We also have by (4.1) that $\langle X^aY^b, 1\rangle = \delta_{a,0} = \epsilon(X^aY^b)$ and $\langle 1, x^iy^j\rangle = \delta_{0,i} = \epsilon(x^iy^j)$.

Finally, we have that

$$\begin{aligned}
\langle X^aY^b, S(x^iy^j)\rangle &\overset{(4.3)}{=} (-1)^i q^{i-1}\zeta^{-ti(im+j)}\langle X^aY^b, x^iy^{-im-j}\rangle \\
&\overset{(4.1)}{=} \delta_{a,i}\ (a)_q!\ (-1)^i q^{i-1}\zeta^{-ti(im+j)}\ \zeta^{-b(im+j)} \\
&= \delta_{a,i}\ (a)_q!\ (-1)^a q^{a-1}\zeta^{-ma(at+b)}\ \zeta^{-j(at+b)} \\
&\overset{(4.1)}{=} (-1)^a q^{a-1}\zeta^{-ma(at+b)}\langle X^aY^{-at-b}, x^iy^j\rangle \\
&\overset{(4.3)}{=} \langle S(X^aY^b), x^iy^j\rangle.
\end{aligned}$$

Therefore, we have a duality. To show that this duality is perfect, we need to show that the maps $\phi : H_n(\zeta, t, m) \to H_n(\zeta, m, t)^*$ and $\psi : H_n(\zeta, m, t) \to H_n(\zeta, t, m)^*$ defined by $\phi(u)(x) = \langle u, x\rangle = \psi(x)(u)$ are injective. By a dimension count, verifying just one of these claims suffices. Let

$$f = \sum_{a=0}^{N-1}\sum_{b=0}^{n-1} \alpha_{a,b}X^aY^b \quad (\alpha_{a,b} \in \Bbbk)\,,$$

and suppose $\phi(f) = 0$. Then for any i, j,

$$\begin{aligned}
0 = \phi(f)(x^iy^j) = \langle f, x^iy^j\rangle &= \sum_{a=0}^{N-1}\sum_{b=0}^{n-1}\alpha_{a,b}\langle X^aY^b, x^iy^j\rangle \\
= \sum_{a=0}^{N-1}\sum_{b=0}^{n-1}\alpha_{a,b}\ \delta_{a,i}\ (a)_q!\ \zeta^{bj} &= \sum_{b=0}^{n-1}\alpha_{i,b}\ (i)_q!\ \zeta^{bj}.
\end{aligned}$$

Let $\beta_{i,j}$ denote $\sum_{b=0}^{n-1}\alpha_{i,b}\ \zeta^{bj}$. By the above, for every i, j, $\beta_{i,j} = 0$. Thus, for any fixed i and k,

$$0 = \sum_{j=0}^{n-1}\zeta^{-jk}\beta_{i,j} = \sum_{j=0}^{n-1}\zeta^{-jk}\sum_{b=0}^{n-1}\alpha_{i,b}\ \zeta^{bj} = \sum_{b=0}^{n-1}\left(\sum_{j=0}^{n-1}\zeta^{(b-k)j}\right)\alpha_{i,b} = n\ \alpha_{i,k}.$$

The last equality follows because for ξ a non-identity n^{th} root of unity, $\sum_{j=0}^{n-1}\xi^j = 0$, and $\zeta^{b-k} \neq 1$ for all $b \neq k$. Thus, since each $\alpha_{i,j} = 0$, we have $f = 0$, so ϕ is injective. Hence, we have proven that the duality is perfect. $\square$

Remark 4.2.3. In [20, Section 2], Krop and Radford give an algebra presentation of the dual of any lifting H of a quantum linear space. In their notation, for $H_n(\zeta, m, t)$, we have $x = v_1,\ y^m = a_1,\ X = \xi_1$, and $Y = \sum_{i=0}^{n-1} \zeta^i \epsilon_{y^i}$. They also compute an algebra presentation of $D(H)$ in the case that the datum $\mathcal{D}$ is "simply linked" [20, Sections 3.2, 4]. The only Hopf algebra we consider that satisfies this condition is $u_q(\mathfrak{sl}_2)$ (Chapter 6).

4.2.2 The Drinfel'd double $D(H_n(\zeta, m, t))$

Now we begin with the computation of $D(H_n(\zeta, m, t))$. By Lemma 2.3.2, as an algebra, $D(H_n(\zeta, m, t))$ is generated by the generators of $H_n(\zeta, m, t)$ and of its dual, and has the relations of both. We only need to find how these generators "commute" with each other, i.e. how to in general write an element as a linear combination of monomials with X and Y to the left of x and y. To find these relations, we use (2.17) .

Proposition 4.2.4. *The Drinfel'd double $D(H_n(\zeta, m, t))$ of $H_n(\zeta, m, t)$ is generated by grouplike elements y and Y, a $(y^m, 1)$-skew primitive element x, and a $(1, Y^t)$-skew primitive element X, subject to the relations*

$$y^n = Y^n = 1, \quad x^N = X^N = 0, \quad yx = \zeta^t xy, \quad YX = \zeta^m XY,$$
$$yY = Yy, \quad xY = \zeta^m Yx, \quad yX = \zeta^{-t} Xy, \quad xX - Xx = Y^t - y^m,$$

where $N = \mathrm{ord}(\zeta^{mt}) = \frac{n}{\gcd(n, mt)}$.

Proof. The generators and first row of relations follow from Lemma 2.3.2. The remaining relations come from moving generators of one across generators of the other, which is done as follows. First, note that

$$\Delta^2(x) = y^m \otimes y^m \otimes x + y^m \otimes x \otimes 1 + x \otimes 1 \otimes 1, \quad \Delta^2(y) = y \otimes y \otimes y,$$
$$\Delta^2(X) = Y^t \otimes Y^t \otimes X + Y^t \otimes X \otimes \epsilon + X \otimes \epsilon \otimes \epsilon, \quad \Delta^2(Y) = Y \otimes Y \otimes Y,$$

and that $S^{-1}(x) = -xy^{-m}$. Thus, using (2.17) and (4.1), we have the following computations

$$yY = \langle Y, y^{-1} \rangle \langle Y, y \rangle Yy = \zeta^{-1} \zeta Yy = Yy,$$

$$xY \;=\; \langle Y, -xy^{-m}\rangle\langle Y, y^m\rangle Yy^m + \langle Y, 1\rangle\langle Y, y^m\rangle Yx, + \langle Y, 1\rangle\langle Y, x\rangle Y1 \;=\; \zeta^m Yx,$$

$$yX \;=\; \langle Y^t, y^{-1}\rangle\langle X, y\rangle Y^t y + \langle Y^t, y^{-1}\rangle\langle \epsilon, y\rangle Xy + \langle X, y^{-1}\rangle\langle \epsilon, y\rangle \epsilon y \;=\; \zeta^{-t} Xy,$$

$$\begin{aligned} xX \;&= \langle Y^t, -xy^{-m}\rangle\langle X, y^m\rangle Y^t y^m \;+\; \langle Y^t, -xy^{-m}\rangle\langle \epsilon, y^m\rangle Xy^m \\ &\quad+\; \langle X, -xy^{-m}\rangle\langle \epsilon, y^m\rangle \epsilon y^m \\ &\quad+\; \langle Y^t, 1\rangle\langle X, y^m\rangle Y^t x \;+\; \langle Y^t, 1\rangle\langle \epsilon, y^m\rangle Xx \;+\; \langle X, 1\rangle\langle \epsilon, y^m\rangle \epsilon x \\ &\quad+\; \langle Y^t, 1\rangle\langle X, x\rangle Y^t 1 \;+\; \langle Y^t, 1\rangle\langle \epsilon, x\rangle X1 \;+\; \langle X, 1\rangle\langle \epsilon, x\rangle \epsilon 1 \\ &= -\,y^m + Xx + Y^t. \end{aligned}$$

□

4.3 The possible structures of $A(H_n(\zeta, m, t))$

We will see that $H_n(\zeta, m, t)$-module algebra structures on $A(H_n(\zeta, m, t))$ as in Notation 1.3.2 do not always exist, depending on the value of m and t. For considering actions of $H_n(\zeta, m, t)$ on $A(H_n(\zeta, m, t))$, we will use an infinite-dimensional Hopf algebra for which $H_n(\zeta, m, t)$ is a quotient. For an integer $n > 0$, a primitive n^{th} root of unity $\zeta \in \Bbbk$, and $m, t \in \mathbb{Z}$ both dividing n, we define

$$\widetilde{H}_n(\zeta, m, t) = \Bbbk\langle y, x \mid y^n = 1,\ yx = \zeta^t xy\rangle,$$

with y grouplike, and x a $(y^m, 1)$-skew primitive element. It is clear that $H_n(\zeta, m, t)$ is the quotient of $\widetilde{H}_n(\zeta, m, t)$ by the Hopf ideal generated by x^N. The following technical lemma will help us determine when structures as in Notation 1.3.2 do exist. We will see that the obstruction comes from the condition that x^N acts by zero. We again use the notation from Remark 2.5.7 for eigenspaces of the action of y: $A_i = \{a \in A \mid y \cdot a = \zeta^i a\}$, noting that by a dimension count $A_i = \Bbbk u^i$ for all i.

Lemma 4.3.1. *Let $A = \Bbbk[u]/(u^n - 1)$ and suppose A is an $\widetilde{H}_n(\zeta, m, t)$-module algebra with $y \cdot u = \zeta u$ and $x \cdot u \neq 0$. Then there exists nonzero $\gamma \in \Bbbk$ such that for any $p, q > 0$,*

$$x \cdot u^p = \gamma\ (p)_{\zeta^m}\ u^{p+t} \quad \textit{and} \quad x^q \cdot u^p = \gamma^q \left(\prod_{i=0}^{q-1} (p + it)_{\zeta^m}\right) u^{p+qt}.$$

In particular, $x^N \cdot u^p = 0$ if and only if n/m divides $p + it$ for some $0 \leq i < N$.

Proof. First, since $yx \cdot u = \zeta^t xy \cdot u = \zeta^{t+1} x \cdot u$, we see that $x \cdot u \in A_{t+1} = \Bbbk u^{t+1}$. Thus, there exists nonzero $\gamma \in \Bbbk$ such that $x \cdot u = \gamma u^{1+t}$. We have established the first equality for the case $p = 1$. Thus, we proceed by induction, assuming the result for $p - 1$. We compute:

$$\begin{aligned} x \cdot u^p &= (y^m \cdot u)(x \cdot u^{p-1}) + (x \cdot u)(1 \cdot u^{p-1}) \\ &= (\zeta^m u)(\gamma\ (p-1)_{\zeta^m}\ u^{p-1+t}) + (\gamma u^{t+1})(u^{p-1}) \\ &= \gamma\ [\zeta^m\ (p-1)_{\zeta^m}\ + 1]\ u^{p+t}\ =\ \gamma\ (p)_{\zeta^m}\ u^{p+t}. \end{aligned}$$

This establishes the first result for all p, as well as the second equality in the case $q = 1$. We now prove the second equality for all q and p, by induction on q. Assume the result for $q - 1$. Then we compute:

$$\begin{aligned} x^q \cdot u^p = x \cdot (x^{q-1} \cdot u^p) = x \cdot &\left(\gamma^{q-1}\ \left(\prod_{i=0}^{q-2} (p+it)_{\zeta^m}\right)\ u^{p+(q-1)t}\right) \\ &= \gamma^{q-1}\ \left(\prod_{i=0}^{q-2} (p+it)_{\zeta^m}\right) \gamma\ (p+(q-1)t)_{\zeta^m}\ u^{p+(q-1)t+t} \\ &= \gamma^q \left(\prod_{i=0}^{q-1} (p+it)_{\zeta^m}\right) u^{p+qt}. \end{aligned}$$

The final statement holds as $(n)_q = 0$ if and only if $\mathrm{ord}(q) \mid n$, and as $\mathrm{ord}(\zeta^m) = n/m$. □

Proposition 4.3.2. *There exist $H_n(\zeta, m, t)$-module algebra structures on $A(H_n(\zeta, m, t))$ as in Notation 1.3.2 if and only if one of the following equivalent conditions holds:*

(a) $\gcd(t, n/m) = 1$

(b) $\gcd(mt, n) = m$

(c) $n/m = N$ $(= \mathrm{ord}(\zeta^{mt}))$

In particular, if $t = 1$, then there are $H_n(\zeta, m, t)$module algebra structures on $A(H_n(\zeta, m, t))$ as in Notation 1.3.2. On the other hand, if these structures exist, we must have that $t|m$, and in this case, the module structure is given by $y \cdot u = \zeta u$ and $x \cdot u = \gamma u^{t+1}$ for some nonzero $\gamma \in \Bbbk$.

Proof. The equivalence of the three conditions follows from elementary group theory and number theory. First, assume these conditions hold. Then by definition, $A(H_n(\zeta, m, t)) = \Bbbk[u]/(u^n - 1)$. For any nonzero $\gamma \in \Bbbk$, by defining $y \cdot u = \zeta u$ and $x \cdot u = \gamma u^{1+t}$, it is easy to check that $A(H_n(\zeta, m, t))$ is a $\widetilde{H}_n(\zeta, m, t)$-module algebra. In order to get a $H_n(\zeta, m, t)$-module algebra structure, we need only check that x^N acts by zero. By Lemma 4.3.1, we must check that for each p, we get that n/m divides $p + it$ for some $0 \leq i < N$. By assumption, $n/m = N$ is relatively prime to t. Thus, for any value of p, $\{p + it\}_{i=0}^{N-1}$ consists of N distinct values mod N. Thus, for exactly one value of i, we have $p + it \equiv 0 \mod N$. Therefore, $x^N \cdot u^p = 0$ for all p, so we have an $H_n(\zeta, m, t)$-module algebra structure. By Corollary 4.1.9, this action is inner-faithful.

On the other hand, fix an $H_n(\zeta, m, t)$-module algebra structure on

$$A := A(H_n(\zeta, m, t)) \cong \Bbbk[u]/(u^n - 1).$$

Since the $H_n(\zeta, m, t)$-module structure on $A(H_n(\zeta, m, t))$ is inner-faithful, by Corollary 4.1.9, $x \cdot u \neq 0$. By pulling back along the projection

$$\widetilde{H}_n(\zeta, m, t) \to H_n(\zeta, m, t),$$

A is a $\widetilde{H}_n(\zeta, m, t)$-module algebra, with $x^N \cdot u = 0$. Thus, by Lemma 4.3.1, we have $x \cdot u = \gamma u^{t+1}$. Moreover, by the same lemma, $1 + it \equiv 0 \mod n/m$ for some $0 \leq i < N$. That is, we can write $1 = -it + bn/m$ for some $i, b \in \mathbb{Z}$. Therefore, $\gcd(t, n/m) = 1$. $\square$

Proposition 4.3.2 generalizes Montgomery and Schneider's result (stated in Theorem 1.3.1), which examines the Taft algebras (the case that $m = t = 1$). Note that in their work, x acts by lowering the degree of u rather than raising it. This is due to the fact that they use the relation $xy = \zeta yx$ rather than $yx = \zeta xy$. By Corollary 4.1.7 and Corollary 4.1.8, we obtain the following result for coradically graded generalized Taft algebras, in general.

Corollary 4.3.3. *Consider a coradically graded generalized Taft algebra* $T(n, N, 0) = H_n(\zeta, 1, n/N)$ *for some* N *dividing* n*. Then* $T(n, N, 0)$*-module alge-*

bra structures on $A(T(n,N,0))$ as in Notation 1.3.2 exist if and only if $n = N$, i.e. if and only if $T(n,N,0)$ is a Taft algebra. $\square$

Thus, we have answered Question 1.3.3(a,b) for coradically graded generalized Taft algebras. We will consider the non-coradically graded generalized Taft algebras in Chapter 5.

4.4 Extensions to $D(H_n(\zeta, m, t))$

Recall that the Hopf algebra $H_n(\zeta, m, t)$ is determined by a primitive n^{th} root of unity ζ in $\Bbbk$ and two positive integer divisors of n: m, which is used to define the coalgebra structure, and t which is used to define the algebra structure. It is also assumed that $n \nmid mt$. We now assume $H_n(\zeta, m, t)$-module algebra structures on $A(H_n(\zeta, m, t))$ as in Notation 1.3.2 exist (that is, that gcd$(t, n/m) = 1$, by Proposition 4.3.2) and explore when such structures extend to be $D(H_n(\zeta, m, t))$-module algebras. Recall from Section 4.2.2 that $D(H_n(\zeta, m, t))$ is generated by grouplike elements y and Y, a $(y^m, 1)$-skew primitive element x, and a $(1, Y^t)$-skew primitive element X, subject to the relations

$$y^n = Y^n = 1, \quad x^N = X^N = 0, \quad yx = \zeta^t xy, \quad YX = \zeta^m XY,$$
$$yY = Yy, \quad xY = \zeta^m Yx, \quad yX = \zeta^{-t} Xy, \quad xX - Xx = Y^t - y^m$$

where $N = \text{ord}(\zeta^{mt}) = n/m$.

Theorem 4.4.1. *Fix an $H_n(\zeta, m, t)$-module algebra structure on the algebra $A := A(H_n(\zeta, m, t)) = \Bbbk[u]/(u^n - 1)$ as in Notation 1.3.2. If the action of $H_n(\zeta, m, t)$ extends to make A a $D(H_n(\zeta, m, t))$-module algebra, then there exists a nonzero scalar γ and scalar $\delta \in \Bbbk$, and a natural number $0 < d < n$ with $m \equiv -dt \mod n$ such that:*

$$y \cdot u = \zeta u, \quad Y \cdot u = \zeta^d u, \quad x \cdot u = \gamma u^{1+t}, \quad \textit{and} \quad X \cdot u = \delta u^{1-t}.$$

If $m \neq n/2$ (that is, if $N \neq 2$), then γ and δ are related by the identity

$$\gamma\delta = \frac{\zeta^{-m} - 1}{(n-t)_{\zeta^m}}.$$

In this case, the action of X on A is determined by the $H_n(\zeta, m, t)$-module algebra structure, and if further, $t = 1$, then the action of Y is as well.

On the other hand, if $m = n/2$, there is no such equation relating γ and δ.

Conversely, the conditions imposed above on δ and d are sufficient to define a $D(H_n(\zeta, m, t))$-module algebra structure on A.

We will need the following lemma about q-symbols in the proof of Theorem 4.4.1. It follows from the definitions.

Lemma 4.4.2. *Let $q \neq 1 \in \Bbbk$. Then the following statements hold.*

(a) *Suppose that* $\mathrm{ord}(q) = n$ *and* $p \equiv r \mod n$ *for integers* $p, r > 0$. *Then* $(p)_q = (r)_q$;

(b) *If* $\mathrm{ord}(q)|m$ *and* $0 \leq p \leq m$, *then* $(p)_{q^{-1}} = -q(m-p)_q$. □

Proof of Theorem 4.4.1. All actions of $H_n(\zeta, m, t)$ on $A(H_n(\zeta, m, t))$ as in Notation 1.3.2 are given by Proposition 4.3.2. In particular, $y \cdot u = \zeta u$, and $x \cdot u = \gamma u^{1+t}$ for some nonzero $\gamma \in \Bbbk$. Since $yY \cdot u = Yy \cdot u = \zeta Y \cdot u$, we see that $Y \cdot u \in A_1 = \Bbbk u$. Thus, $Y \cdot u = \delta u$ for some $\delta \in \Bbbk$. However, because Y^n must act by the identity, δ must be an n^{th} root of unity. That is, $\delta = \zeta^d$ for some $0 \leq d < n$.

On one hand, $xY \cdot u = \zeta^d x \cdot u = \zeta^d \gamma u^{1+t}$. On the other hand, $\zeta^m Yx \cdot u = \zeta^m \gamma Y \cdot u^{1+t} = \zeta^{m+d(1+t)} \gamma u^{1+t}$. Therefore, since $\gamma \neq 0$, it must be the case that $d \equiv m + d(1+t) \mod n$. That is, $m \equiv -dt \mod n$. In particular, this implies $d \neq 0$.

We also have $yX \cdot u = \zeta^{-t} Xy \cdot u = \zeta^{1-t} X \cdot u$, showing that $X \cdot u \in A_{1-t} = \Bbbk u^{1-t}$. Thus, $X \cdot u = \delta u^{1-t}$ for some $\delta \in \Bbbk$. One sees by induction that $X \cdot u^p = \delta\, (p)_{\zeta^{dt}} u^{p-t}$. Thus, on one hand, by Lemma 4.3.1 and Lemma 4.4.2,

$$(xX - Xx) \cdot u = \delta x \cdot u^{1-t} - \gamma X \cdot u^{1+t} = \delta\gamma (n+1-t)_{\zeta^m} u^{n+1} - \gamma\delta (1+t)_{\zeta^{dt}} u^{n+1},$$

and on the other hand, $(Y^t - y^m) \cdot u = (\zeta^{dt} - \zeta^m) u$. Therefore, since $m \equiv -dt \mod n$, we have

$$\zeta^{-m} - \zeta^m = \gamma\delta \left((n+1-t)_{\zeta^m} - (1+t)_{\zeta^{-m}} \right).$$

Note that $\mathrm{ord}(\zeta^m)$ divides n and that $1 < 1+t \leq n$. Thus, using Lemma 4.4.2,

$$\begin{aligned}(n+1-t)_{\zeta^m} - (1+t)_{\zeta^{-m}} &= \zeta^m(n-t)_{\zeta^m} + 1 + \zeta^m(n-1-t)_{\zeta^m} \\ &= (\zeta^m+1)(n-t)_{\zeta^m}.\end{aligned}$$

Therefore, if $\zeta^m \neq -1$ (or equivalently, if $m \neq n/2$), then since $(\zeta^m+1)(\zeta^{-m}-1) = \zeta^{-m} - \zeta^m$, we have

$$\gamma\delta = \frac{\zeta^{-m}-1}{(n-t)_{\zeta^m}}.$$

If $\zeta^m = -1$, then $\zeta^{-m} - \zeta^m = 0$, so we gain no new restrictions on δ.

We also have $YX \cdot u = \delta Y \cdot u^{1-t} = \delta\zeta^{d(1-t)}u^{1-t}$, and $\zeta^m XY \cdot u = \zeta^{m+d}X \cdot u = \delta\zeta^{m+d}u^{1-t}$. Therefore, $\delta = 0$ or $m + d \equiv d(1-t) \mod n$. However, we already know $m \equiv -dt \mod n$, so we have no further restrictions on δ or d.

Finally, we must have $X^N \cdot u^p = 0$ for all p. A simple calculation shows that

$$X^N \cdot u^p = \delta^N \left(\prod_{i=0}^{N-1} (p + i(n-t))_{\zeta^{dt}} \right) u^{p-Nt}.$$

If $\delta = 0$, we are done. Otherwise, $X^N \cdot u^p = 0$ if and only if $\mathrm{ord}(\zeta^{dt})$ divides some element of $\{p + i(n-t)\}_{i=0}^{N-1}$. Since $\mathrm{ord}(\zeta^{dt}) = \mathrm{ord}(\zeta^m) = n/m = N$ and $\gcd(t, N) = 1$ by Proposition 4.3.2, the set consists of N distinct values mod N. Therefore, N divides exactly one of them. Thus, $X^N \cdot u^p = 0$ for all p.

The converse statement, that the conditions imposed on δ and d are sufficient for making $A(H_n(\zeta, m, t))$ a $D(H_n(\zeta, m, t))$-module algebra, is straightforward to check. □

Note that this result generalizes the work of Montgomery and Schneider (stated in Theorem 3.1.2) and shows that there are other Hopf algebras closely related to Taft algebras, for which there is a unique extension of the action of H on $A(H)$ to $D(H)$, namely $H_n(\zeta, m, 1)$ for any $m \mid n$ with $m \neq n/2$.

Corollary 4.4.3. *Suppose $H_n(\zeta, m, t)$-module algebra structures on the algebra $A := A(H_n(\zeta, m, t))$ as in Notation 1.3.2 exist. If $m \neq n/2$ (e.g., if n is odd), then there are precisely t ways to extend this action to make A a $D(H_n(\zeta, m, t))$-module* *algebra. In particular, if $t = 1$, then the desired $H_n(\zeta, m, t)$-module algebra structure on $A(H_n(\zeta, m, t))$ exists, and the way to extend the action to $D(H_n(\zeta, m, t))$ is unique.*

If $m = n/2$, then in order to extend the action of $H_n(\zeta, m, t)$ on A to an action of $D(H_n(\zeta, m, t))$, there are t ways to define the action of the generator Y and the choice for the action of X is parametrized by $\Bbbk$.

Proof. By Proposition 4.3.2, $t|m$. Thus, there are t distinct choices for d such that $0 < d < n$ and $m \equiv -dt \mod n$. If $m \neq n/2$, the action of X is fixed by Theorem 4.4.1. Otherwise, any choice of $\delta \in \Bbbk$ will suffice to define the action of X. $\square$

While the Hopf algebras $H_n(\zeta, m, t)$ generalize the Taft algebras as bosonizations of quantum linear spaces over finite cyclic groups, there are other coradically graded generalizations and directions to consider for further study. We discuss this in Chapter 7.

CHAPTER 5

NON CORADICALLY-GRADED GENERALIZED TAFT ALGEBRAS

Let $n \in \mathbb{N}$, $n \geq 2$, and let $N \mid n$. Let $q \in \Bbbk$ be a primitive N^{th} root of unity. Recall that the generalized Taft algebra $T(n, N, \alpha)$ [Definition 4.1.6] is generated by $g \in G(T(n, N, 1))$ and $x \in P_{g,1}(T(n, N, 1))$, subject to the relations

$$g^n = 1, \qquad x^N = \alpha(g^N - 1), \qquad gx = qxg,$$

for some $\alpha \in \Bbbk$. If $\alpha = 0$, then $T(n, N, 0) \cong H_n(\zeta, 1, \frac{n}{N})$ [Corollary 4.1.7]. If $\alpha \neq 0$, then by scaling x, we can assume without loss of generality that $\alpha = 1$. In this section, we consider the generalized Taft algebras which are not of the form $H_n(\zeta, m, t)$ and hence covered by the previous section, i.e. we consider the non coradically-graded generalized Taft algebras $T(n, N, 1)$. As stated in Section 4.1, $T(n, N, 1)$ is a lifting of $H_n(\zeta, 1, \frac{n}{N})$, i.e. $\mathrm{gr}(T(n, N, 1)) \cong H_n(\zeta, 1, \frac{n}{N})$.

The proof of the following is standard and follows similarly to the proof of Lemma 4.1.4.

Lemma 5.0.1. *For $b \in \{0, 1, \ldots, N-1\}$, we have that*

$$P_{g^b,1}(T(n,N,1)) = \begin{cases} \Bbbk x + \Bbbk(g^b - 1), & \text{if } b = 1, \\ \Bbbk(g^b - 1), & \text{otherwise.} \end{cases} \qquad \square$$

Corollary 5.0.2. *A left $T(n,N,1)$-module M is inner-faithful if and only if $G(T(n,N,1)) = \langle g \rangle$ acts faithfully and $x \cdot M \neq 0$.*

Proof. The proof is essentially the same as that of Corollary 4.1.9. $\square$

5.1 The possible structures of $A(T(n,N,1))$

Since $G(T(n,N,1))$ is cyclic of order n, the module algebra $A(T(n,N,1))$ in Notation 1.3.2 is isomorphic to $\Bbbk[u]/(u^n - 1)$ as an algebra. (See Remark 2.5.7.) We determine all such possible $T(n,N,1)$-module algebra structures. Let $\zeta \in \Bbbk$ be a primitive n^{th} root of unity so that $\zeta^{\frac{n}{N}} = q$.

Proposition 5.1.1. *Let $A = \Bbbk[u]/(u^n - 1)$. By defining $g \cdot u = \zeta u$ and $x \cdot u = \gamma u^{\frac{n}{N}+1}$ for $\gamma \in \Bbbk$ satisfying $\gamma^N = -(1-\zeta)^N$, we obtain that $A = A(T(n,N,1))$ is a $T(n,N,1)$-module algebra as in Notation 1.3.2. Moreover, this gives all possible $T(n,N,1)$-module algebra structures on $A(T(n,N,1))$.*

Proof. For the first statement, it is easy to check that A, as defined, will be a $T(n,N,1)$-module algebra. By Corollary 5.0.2, since $x \cdot u \neq 0$, the action on A is inner-faithful.

To see that these are the only possible $T(n,N,1)$-module algebra structures on A as in Notation 1.3.2, fix such a structure. By Remark 2.5.7, we have that

$$A = \bigoplus_{i=0}^{n-1} A_i \text{ where } A_i = \{a \in A \mid g \cdot a = \zeta^i a\} = \Bbbk u^i.$$

Now, since $g \cdot (x \cdot u) = qx \cdot (g \cdot u) = \zeta^{\frac{n}{N}+1} x \cdot u$, we have that $x \cdot u \in A_{\frac{n}{N}+1} = \Bbbk u^{\frac{n}{N}+1}$. Therefore, $x \cdot u = \gamma u^{\frac{n}{N}+1}$ for some $\gamma \in \Bbbk$. We must also have that $x^N \cdot u = (g^N - 1) \cdot u = (\zeta^N - 1) \cdot u$. Inductively, we have

$$x^N \cdot u = \gamma^N \left(\prod_{i=0}^{N-1} \left(\frac{in}{N} + 1 \right)_{\zeta} \right) u.$$

Thus, we must have

$$\gamma^N \left(\prod_{i=0}^{N-1} \left(\frac{in}{N} + 1 \right)_\zeta \right) = \zeta^N - 1. \tag{5.1}$$

Now, we have

$$\prod_{i=0}^{N-1} \left(\frac{in}{N} + 1 \right)_\zeta = \prod_{i=0}^{N-1} \frac{\zeta^{\frac{in}{N}+1} - 1}{\zeta - 1} = \frac{\prod_{i=0}^{N-1} (\zeta q^i - 1)}{(\zeta - 1)^N} = \frac{(-1)^{N-1} \left(\zeta^N - 1\right)}{(\zeta - 1)^N}.$$

The last equality follows from the fact that q, being a primitive N^{th} root of unity, satisfies $\zeta^N - 1 = \prod_{i=0}^{N-1}(\zeta - q^i)$. Therefore, from (5.1), we see that $\gamma^N = -(1 - \zeta)^N$. $\square$

5.2 The dual and double of $T(n, N, 1)$

We must now compute the Drinfel'd double of $T(n, N, 1)$ so that we can examine the extensions of actions of $T(n, N, 1)$ on A to actions of $D(T(n, N, 1))$. First, we compute a presentation of the dual. We proceed in a similar fashion to Section 4.2.

5.2.1 The dual $T(n, N, 1)^*$

Definition 5.2.1. Let $\zeta \in \Bbbk$ be a primitive n^{th} root of unity chosen so that $\zeta^{\frac{n}{N}} = q$, and let $K_\zeta(n, N)$ denote the algebra generated by G and X, subject to the relations

$$G^n = 1, \qquad X^N = 0, \qquad GX = \zeta XG.$$

By a Diamond Lemma argument, $K_\zeta(n, N)$ has basis $\{X^i G^j\}_{0 \le i < N,\ 0 \le j < n}$.

We will show that $K_\zeta(n, N)$ is a Hopf algebra which is isomorphic as a Hopf algebra to $T(n, N, 1)^*$. For making calculations easier, we introduce the following notation.

Notation 5.2.2. Let $N \in \mathbb{N}$ and $q \in \Bbbk$ a primitive N^{th} root of unity. For $0 \le i < j < N \in \mathbb{N}$, define

$$\widetilde{\binom{i}{j}}_q = \frac{(i)_q!}{(i + N - j)_q!(j)_q!}.$$

For $1 \leq i < j < N$, these satisfy the equation

$$\widetilde{\binom{i}{j}}_q = \widetilde{\binom{i-1}{j}}_q + q^{i-j}\widetilde{\binom{i-1}{j-1}}_q = q^j\widetilde{\binom{i-1}{j}}_q + \widetilde{\binom{i-1}{j-1}}_q \tag{5.2}$$

We will need the following technical lemma to prove that $K_\zeta(n, N)$ is a Hopf algebra.

Lemma 5.2.3. *With q a primitive N^{th} root of unity and $0 \leq a < d < N$,,*

$$\sum_{c=0}^{a} \binom{a}{c}_q \frac{q^{c(c-d)}}{(d-c)_q!(N-d+c)_q!} = \widetilde{\binom{a}{d}}_q.$$

Proof. First, we note that the left hand side of the equation is equivalent to

$$\widetilde{\binom{a}{d}}_q \sum_{c=0}^{a} q^{c(c-d)} \binom{a+N-d}{c+N-d}_q \binom{d}{c}_q. \tag{5.3}$$

Thus, we have only to show that the above summation is equal to 1 for $0 \leq a < d < N$. We show this for $0 \leq a < d \leq N$. To this end, if $d < N$, then the summation is equivalent to

$$\begin{aligned}
&\sum_{c=0}^{a-1} q^{c(c-d)} \left[\binom{a+N-d-1}{c+N-d-1}_q + q^{c-d}\binom{a+N-d-1}{c+N-d}_q\right]\binom{d}{c}_q + q^{a(a-d)}\binom{d}{a}_q \\
&\quad = \sum_{c=0}^{a} q^{c(c-d)} \binom{a+N-d-1}{c+N-d-1}_q \binom{d}{c}_q \\
&\qquad\qquad + \sum_{c=0}^{a} q^{c(c-d-1)} \binom{a+N-d-1}{c+N-d-1}_q \binom{d}{c-1}_q \\
&\quad = \sum_{c=0}^{a} q^{c(c-d-1)} \binom{a+N-d-1}{c+N-d-1}_q \binom{d+1}{c}_q.
\end{aligned}$$

The problem is thus reduced to proving the summation in (5.3) is equal to 1 for $d = N$. Since $q^N = 0$, we have $\binom{N}{c}_q = 0$ for each $0 < c \leq a$, so the problem is reduced to the obvious fact that $q^0\binom{a}{0}_q\binom{N}{0}_q = 1$. $\square$

Proposition 5.2.4. *By defining*

$$\Delta(G) = G \otimes G + (\zeta^N - 1)\sum_{a=1}^{N-1} \widetilde{\binom{0}{a}}_q X^{N-a}G^{1+\frac{an}{N}} \otimes X^a G,$$
$$\Delta(X) = G^{\frac{n}{N}} \otimes X + X \otimes 1,$$
$$\epsilon(G) = 1, \qquad \epsilon(X) = 0,$$
$$S(G) = G^{-1}, \qquad S(X) = -G^{\frac{-n}{N}}X,$$

the algebra $K_\zeta(n, N)$ from Definition 5.2.1 is a Hopf algebra.

Proof. First, we check that Δ and ϵ, as defined, are algebra maps. Using induction, we see that

$$\Delta(G^d) = G^d \otimes G^d + (\zeta^{dN} - 1)\sum_{a=1}^{N-1} \widetilde{\binom{0}{a}}_q X^{N-a}G^{d+\frac{an}{N}} \otimes X^a G^d. \tag{5.4}$$

In particular, $\Delta(G^n) = G^n \otimes G^n = 1 \otimes 1 = \Delta(1)$. Similarly, from (2.9), we get

$$\Delta(X^a) = \sum_{b=0}^{a} \binom{a}{b}_q X^{a-b}G^{\frac{nb}{N}} \otimes X^b. \tag{5.5}$$

Since $q^N = 1$, by (2.10), $\Delta(X^N) = X^N \otimes 1 + G^n \otimes X^N = 0$. Finally, we check the relation $GX = \zeta XG$. On the one hand, using (5.2),

$$\begin{aligned}
\Delta(GX) = \zeta G^{1+\frac{n}{N}} &\otimes XG + \zeta XG \otimes G \\
&+ \zeta(\zeta^N - 1)\sum_{a=1}^{N-2} \widetilde{\binom{0}{a}}_q X^{N-a}G^{1+\frac{(a+1)n}{N}} \otimes X^{a+1}G \\
&+ \zeta(\zeta^N - 1)\sum_{a=2}^{N-1} q^a \widetilde{\binom{0}{a}}_q X^{N-a+1}G^{1+\frac{an}{N}} \otimes X^a G \\
= \zeta G^{1+\frac{n}{N}} &\otimes XG + \zeta XG \otimes G \\
&+ \zeta(\zeta^N - 1)\sum_{a=1}^{N-2} \widetilde{\binom{1}{a+1}}_q X^{N-a}G^{1+\frac{(a+1)n}{N}} \otimes X^{a+1}G.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\Delta(XG) &= G^{1+\frac{n}{N}} \otimes XG + XG \otimes G \\ &\qquad + (\zeta^N - 1) \sum_{a=1}^{N-2} q^{-a} \widetilde{\binom{0}{a}}_q X^{N-a} G^{1+\frac{(a+1)n}{N}} \otimes X^{a+1} G \\ &\qquad + (\zeta^N - 1) \sum_{a=2}^{N-1} \widetilde{\binom{0}{a}}_q X^{N-a+1} G^{1+\frac{an}{N}} \otimes X^a G \\ &= G^{1+\frac{n}{N}} \otimes XG + XG \otimes G \\ &\qquad + (\zeta^N - 1) \sum_{a=1}^{N-2} \widetilde{\binom{1}{a+1}}_q X^{N-a} G^{1+\frac{(a+1)n}{N}} \otimes X^{a+1} G.\end{aligned}$$

Thus, it is clear that $\Delta(GX - \zeta XG) = 0$. That ϵ is an algebra map is clear. Note that to show Δ and ϵ define a coalgebra structure, we know only need to check the commutativity of (2.2) when applied to G and X. The second diagram is surely commutative, and for the first, we need the following fact.

Claim. For $0 \leq a < N$ and $0 \leq b < n$,

$$\begin{aligned}\Delta(X^a G^b) = \sum_{c=0}^{a} \binom{a}{c}_q X^{a-c} G^{b+\frac{cn}{N}} \otimes X^c G^b \\ + (\zeta^{bN} - 1) \sum_{c=a+1}^{N-1} \widetilde{\binom{a}{c}}_q X^{a+N-c} G^{b+\frac{cn}{N}} \otimes X^c G^b.\end{aligned}$$

Proof of Claim. First, recall (5.4) and (5.5) for formulas for $\Delta(G^b)$ and $\Delta(X^a)$ respectively. Combining these and recalling that $X^N = 0$, we get

$$\Delta(X^aG^b) = \sum_{c=0}^{a} \binom{a}{c}_q X^{a-c}G^{\frac{cn}{N}+b} \otimes X^cG^b$$
$$+ (\zeta^{bN}-1)\sum_{c=0}^{a}\sum_{d=c+1}^{N+c-a-1}\binom{a}{c}_q \frac{q^{cd}}{(d)_q!(N-d)_q!}$$
$$X^{a-c+d}G^{b+\frac{(c-d)n}{N}} \otimes X^{N+c-d}G^b$$
$$= \sum_{c=0}^{a} \binom{a}{c}_q X^{a-c}G^{\frac{cn}{N}+b} \otimes X^cG^b$$
$$+ (\zeta^{bN}-1)\sum_{d=1}^{N-a-1}\left(\sum_{c=0}^{a}\binom{a}{c}_q \frac{q^{c(c+d)}}{(c+d)_q!(N-c-d)_q!}\right)$$
$$X^{a+d}G^{b-\frac{dn}{N}} \otimes X^{N-d}G^b$$
$$= \sum_{c=0}^{a} \binom{a}{c}_q X^{a-c}G^{\frac{cn}{N}+b} \otimes X^cG^b$$
$$+ (\zeta^{bN}-1)\sum_{d=1}^{N-a-1}\frac{(a)_q!}{(a+d)_q!(N-d)_q!}X^{a+d}G^{b-\frac{dn}{N}} \otimes X^{N-d}G^b.$$

The last equality follows from Lemma 5.2.3. ■

Now, we compute both $X_{(1)(1)} \otimes X_{(1)(2)} \otimes X_{(2)}$ and $X_{(1)} \otimes X_{(2)(1)}X_{(2)(2)}$ as

$$G^{\frac{n}{N}} \otimes G^{\frac{n}{N}} \otimes X + G^{\frac{n}{N}} \otimes X \otimes 1 + X \otimes 1 \otimes 1.$$

Using the claim above, we compute $G_{(1)(1)} \otimes G_{(1)(2)} \otimes G_{(2)}$ as

$$\Delta(G) \otimes G + (\zeta^N-1)\sum_{a=1}^{N-1}\widetilde{\binom{0}{a}}_q \Delta(X^{N-a}G^{1+\frac{an}{N}}) \otimes X^aG$$
$$= G \otimes G \otimes G + (\zeta^N-1)\sum_{a=1}^{N-1}\widetilde{\binom{0}{a}}_q X^{N-a}G^{1+\frac{an}{N}} \otimes X^aG \otimes G$$
$$+ (\zeta^N-1)\sum_{a=1}^{N-1}\sum_{c=0}^{N-a}\widetilde{\binom{0}{a}}_q \binom{N-a}{c}_q X^{N-a-c}G^{1+\frac{(a+c)n}{N}} \otimes X^cG^{1+\frac{an}{N}} \otimes X^aG$$
$$+ (\zeta^N-1)^2\sum_{a=1}^{N-1}\sum_{c=N-a+1}^{N-1}\widetilde{\binom{0}{a}}_q \widetilde{\binom{N-a}{c}}_q$$
$$X^{2N-a-c}G^{1+\frac{(a+c)n}{N}} \otimes X^cG^{1+\frac{an}{N}} \otimes X^aG.$$

Similarly, we compute $G_{(1)} \otimes G_{(2)(1)} \otimes G_{(2)(2)}$ as

$$
\begin{aligned}
&G \otimes \Delta(G) + (\zeta^N - 1) \sum_{a=1}^{N-1} \widetilde{\binom{0}{a}}_q X^{N-a} G^{1+\frac{an}{N}} \otimes \Delta(X^a G) \\
&= G \otimes G \otimes G + (\zeta^N - 1) \sum_{a=1}^{N-1} \widetilde{\binom{0}{a}}_q G \otimes X^{N-a} G^{1+\frac{an}{N}} \otimes X^a G \\
&\quad + (\zeta^N - 1) \sum_{a=1}^{N-1} \sum_{c=0}^{a} \widetilde{\binom{0}{a}}_q \binom{a}{c}_q X^{N-a} G^{1+\frac{an}{N}} \otimes X^{a-c} G^{1+\frac{cn}{N}} \otimes X^c G \\
&\quad + (\zeta^N - 1)^2 \sum_{a=1}^{N-1} \sum_{c=a+1}^{N-1} \widetilde{\binom{0}{a}}_q \widetilde{\binom{a}{c}}_q X^{N-a} G^{1+\frac{an}{N}} \otimes X^{a+N-c} G^{1+\frac{cn}{N}} \otimes X^c G.
\end{aligned}
$$

Through manipulation of the sums and indices, one sees that the above are equivalent. Therefore, we have shown that $K_\zeta(n, N)$ is a bialgebra. It is an easy check that S is indeed an antipode. $\square$

The following is an easy consequence of (5.4).

Corollary 5.2.5. *The element $G^{\frac{n}{N}}$, and so each of $\{G^{\frac{an}{N}}\}_{a\in\mathbb{Z}}$ is grouplike. Thus, we also have $X \in P_{G^{\frac{n}{N}},1}(K_\zeta(n, N))$.* $\square$

Proposition 5.2.6. *With g, x denoting the generators of $T(n, N, 1)$, and G, X the generators $K_\zeta(n, N)$, the bilinear form defined by*

$$\langle X^a G^b, x^i g^j \rangle = \delta_{a,i}\ (a)_q!\ \zeta^{bj}, \tag{5.6}$$

is a perfect duality. Therefore, $T(n, N, 1)^* \cong K_\zeta(n, N)$.

In particular, we get that the dual pairing is given on generators by

$$\langle G, g \rangle = \zeta, \qquad \langle G, x \rangle = 0, \qquad \langle X, g \rangle = 0, \qquad \langle X, x \rangle = 1.$$

Proof of Proposition 5.2.6. First, on the one hand, for $0 \leq a, c, i < N$, we have

$$\langle X^a G^b X^c G^d, x^i g^j \rangle = \zeta^{bc} \langle X^{a+c} G^{b+d}, x^i g^j \rangle = \delta_{a+c,i}\ (i)_q!\ \zeta^{bc+bj+dj}.$$

On the other hand, using $\Delta(x^i g^j)$, which is computed using (2.9),

$$\begin{aligned}\langle X^a G^b X^c G^d, x^i g^j\rangle &= \sum_{k=0}^{i} \binom{i}{k}_q \langle X^a G^b, x^{i-k} g^{j+k}\rangle \langle X^c G^d, x^k g^j\rangle \\ &= \binom{i}{c}_q \delta_{a,i-c}\ (i-c)_q!\ (c)_q!\ \zeta^{bc+bj+dj}.\end{aligned}$$

By the definition of q-binomial symbols, these are equivalent.

Next, for $0 \leq a, i, k < N$, we have

$$\langle X^a G^b, x^i g^j x^k g^\ell\rangle = (a)_q!\ q^{jk}\ \zeta^{bj+b\ell} \left(\delta_{a,i+k} + \delta_{a,i+k-N}(\zeta^{bN}-1)\right).$$

On the other hand, using the claim in the proof of Proposition 5.2.4,

$$\begin{aligned}\langle X^a G^b, x^i g^j x^k g^\ell\rangle &= \sum_{c=0}^{a} \binom{a}{c}_q \langle X^{a-c} G^{b+\frac{cn}{N}}, x^i g^j\rangle \langle X^c G^b, x^k g^\ell\rangle \\ &\quad + (\zeta^{bN}-1) \sum_{c=a+1}^{N-1} \widetilde{\binom{a}{c}}_q \langle X^{a+N-c} G^{b+\frac{cn}{N}}, x^i g^j\rangle \langle X^c G^b, x^k g^\ell\rangle \\ &= \binom{a}{k}_q \delta_{a-k,i}\ (a-k)_q!\ (k)_q!\ q^{jk}\ \zeta^{bj+b\ell} \\ &\quad + (\zeta^{bN}-1) \widetilde{\binom{a}{k}}_q\ \delta_{a+N-k,i}\ (a+N-k)_q!\ (k)_q!\ q^{jk}\ \zeta^{bj+b\ell}.\end{aligned}$$

Thus, these are equivalent. We also have

$$\langle X^a G^b, 1\rangle = \delta_{a,0} = \epsilon(X^a G^b), \qquad \langle 1, x^i g^j\rangle = \delta_{0,i} = \epsilon(x^i g^j).$$

Thus, we have a duality between the underlying bialgebras. To see that it is a duality between Hopf algebras, we compute

$$\begin{aligned}\langle S(X^a G^b), x^i g^j\rangle &= (-1)^a\ q^{\frac{-a(a+1)}{2}}\ \zeta^{-ab}\ \langle X^a G^{\frac{-an}{N}-b}, x^i g^j\rangle \\ &= (-1)^a\ q^{\frac{-a(a+1)}{2}}\ \zeta^{-ab}\ \delta_{a,i}\ (a)_q!\ \zeta^{-bj}\ q^{-aj} \\ &= (-1)^i\ q^{\frac{-i(i+1)}{2}-ij}\ \delta_{a,i}\ (a)_q!\ \zeta^{b(-i-j)} \\ &= (-1)^i\ q^{\frac{-i(i+1)}{2}-ij}\ \langle X^a G^b, x^i g^{-i-j}\rangle \ = \ \langle X^a G^b, S(x^i g^j)\rangle.\end{aligned}$$

To see that the duality is perfect, we must show that the map

$$\phi : K_\zeta(n,N) \to T(n,N,1)^*$$

defined by $\phi(u)(x) = \langle u, x\rangle$ is injective. Let $f = \sum_{a=0}^{N-1}\sum_{b=0}^{n-1}\alpha_{a,b}X^aG^b$ with $\alpha_{a,b}\in\Bbbk$ and suppose $\phi(f)=0$. Then for any $0\le i<N$ and $0\le j<n$,

$$\begin{aligned} 0 &= \phi(f)(x^ig^j) = \langle f, x^ig^j\rangle = \sum_{a=0}^{N-1}\sum_{b=0}^{n-1}\alpha_{a,b}\langle X^aG^b, x^ig^j\rangle \\ &= \sum_{a=0}^{N-1}\sum_{b=0}^{n-1}\alpha_{a,b}\,\delta_{a,i}\,(a)_q!\,\zeta^{bj} = (i)_q!\sum_{b=0}^{n-1}\alpha_{i,b}\,\zeta^{bj}. \end{aligned}$$

Let $\beta_{i,j}$ denote $\sum_{b=0}^{n-1}\alpha_{i,b}\,\zeta^{bj}$, each of which is 0. Then, for any fixed $0\le i<N$ and $0\le k<n$,

$$0 = \sum_{j=0}^{n-1}\zeta^{-jk}\beta_{i,j} = \sum_{j=0}^{n-1}\zeta^{-jk}\sum_{b=0}^{n-1}\alpha_{i,b}\,\zeta^{bj} = \sum_{b=0}^{n-1}\left(\sum_{j=0}^{n-1}\zeta^{j(b-k)}\right)\alpha_{i,b} = n\alpha_{i,k}.$$

Therefore, since each $\alpha_{i,k}=0$, we have $f=0$, so ϕ is injective, and the duality is perfect. $\square$

5.2.2 The double $D(T(n,N,1))$

We can now prove the following result.

Proposition 5.2.7. *The Drinfel'd double $D(T(n,N,1))$ of $T(n,N,1)$ is generated by $g,\ x,\ G,$ and X, subject to the relations*

$$G^n = g^n = 1,\qquad x^N = g^N - 1,\qquad X^N = 0,\qquad gx = qxg,\qquad GX = \zeta XG$$
$$gG = Gg,\qquad gX = q^{-1}Xg,\qquad xX - Xx = G^{\frac{n}{N}} - g,$$
$$xG - \zeta Gx = (\zeta^N - 1)\widetilde{\binom{0}{1}}_q X^{N-1}G\left(G^{\frac{n}{N}} - qg\right)$$

The coalgebra structure is determined by

$$\Delta(x) = g\otimes x + x\otimes 1,\qquad \Delta(g) = g\otimes g,\qquad \Delta(X) = 1\otimes X + X\otimes G^{\frac{n}{N}}$$
$$\Delta(G) = G\otimes G + (\zeta^N - 1)\sum_{a=1}^{N-1}\widetilde{\binom{0}{a}}_q X^aG\otimes X^{N-a}G^{1+\frac{an}{N}},$$
$$\epsilon(g) = \epsilon(G) = 1,\qquad \epsilon(x) = \epsilon(X) = 0.$$

Proof. The generators and top row of relations follow from Lemma 2.3.2. For the rest, first recall that in $K_\zeta(n,N)$ and $T(n,N,1)$, we have

$$\Delta^2(X) = G^{\frac{n}{N}} \otimes G^{\frac{n}{N}} \otimes X + G^{\frac{n}{N}} \otimes X \otimes 1 + X \otimes 1 \otimes 1,$$
$$\Delta^2(g) = g \otimes g \otimes g, \qquad \Delta^2(x) = g \otimes g \otimes x + g \otimes x \otimes 1 + x \otimes 1 \otimes 1,$$
$$S^{-1}(g) = g^{-1}, \qquad S^{-1}(x) = -xg^{-1}.$$

Thus, using (2.17) and (5.6), we have the following computations:

$$gX = \langle G^{\frac{n}{N}}, g^{-1}\rangle\langle X, g\rangle G^{\frac{n}{N}} g + \langle G^{\frac{n}{N}} g^{-1}\rangle\langle 1, g\rangle Xg + \langle X, g^{-1}\rangle\langle 1, g\rangle g = q^{-1}Xg,$$

$$\begin{aligned} xX &= \langle G^{\frac{n}{N}}, g^{-1}\rangle\langle X, g\rangle G^{\frac{n}{N}} g \ + \ \langle G^{\frac{n}{N}}, -xg^{-1}\rangle\langle 1, g\rangle Xg \ + \ \langle X, -xg^{-1}\rangle\langle 1, g\rangle g \\ &\quad + \langle G^{\frac{n}{N}}, 1\rangle\langle X, g\rangle G^{\frac{n}{N}} x \ + \ \langle G^{\frac{n}{N}}, 1\rangle\langle 1, g\rangle Xx \ + \ \langle X, 1\rangle\langle 1, g\rangle x \\ &\quad + \langle G^{\frac{n}{N}}, 1\rangle\langle X, x\rangle G^{\frac{n}{N}} \ + \ \langle G^{\frac{n}{N}}, 1\rangle\langle 1, x\rangle X \ + \ \langle X, 1\rangle\langle 1, x\rangle 1 \\ &= -g + Xx + G^{\frac{n}{N}}. \end{aligned}$$

The remaining computations are in A.1.1 in Appendix A. □

5.3 Extensions to $D(T(n,N,1))$

Now that we have a presentation of $D(T(n,N,1))$, we come to the result that extensions of actions of $T(n,N,1)$ on $A(T(n,N,1))$ to actions of $D(T(n,N,1))$ only exist for particular choices of n and N.

Theorem 5.3.1. *An action of $T(n,N,1)$ on $A := A(T(n,N,1))$ as in Proposition 5.1.1 extends to an action of $D(T(n,N,1))$ on A if and only if $N = 2$ and $\frac{n}{2}$ is odd. In that case, the action is given by*

(a) $G \cdot u = \zeta^a u$ *for some odd* $a \in \mathbb{N}$*, and*

(b) $X \cdot u = \delta u$*, where* $\delta \in \Bbbk$ *satisfies* $\gamma\delta = \zeta - 1$.

Proof. It is straightforward to check that if $N = 2$ and $\frac{n}{2}$ is odd, the given equations define an action of $D(T(n,N,1))$ on $A(T(n,N,1))$.

Now suppose we have an action of $T(n,N,1)$ on $A(T(n,N,1))$ which extends to an action of $D(T(n,N,1))$. By Proposition 5.1.1, we have $A=\Bbbk[u]/(u^n-1)$, with $g\cdot u=\zeta u$, and $x\cdot u=\gamma u^{\frac{n}{N}+1}$ where $\gamma^N=-(1-\zeta)^N$. By the relation $gX=q^{-1}Xg$, we have that

$$g\cdot X\cdot u=q^{-1}X\cdot g\cdot u=\zeta^{1-\frac{n}{N}}X\cdot u.$$

Therefore, $X\cdot u\in A_{1-\frac{n}{N}}=\Bbbk u^{1-\frac{n}{N}}$, so $X\cdot u=\delta u^{1-\frac{n}{N}}$ for some $\delta\in\Bbbk$. Similarly, by the relation $gG=Gg$, we have $g\cdot G\cdot u=G\cdot g\cdot u=\zeta G\cdot u$, so $G\cdot u\in A_1=\Bbbk u$. Since $G^n=1$, we have $G\cdot u=\zeta^a u$ for some unique $a\in\{0,1,\ldots,N-1\}$.

Throughout this proof, we will interpret the q-symbol $(s)_q$ (for s a negative number) as $(t)_q$, where $t\equiv s \pmod{\mathrm{ord}(q)}$ and $t>0$. Now, inductively, we have $X^N\cdot u=\delta^N\left(\prod_{j=0}^{N-1}\left(1-\frac{jn}{N}\right)_{q^a}\right)u$. Since $X^N=0$, we must have $\delta=0$ or some $\left(1-\frac{jn}{N}\right)_{q^a}=0$. If $\delta=0$, then X acts by zero, in which case $(xX-Xx)\cdot u=0$, while $(G^{\frac{n}{N}}-g)\cdot u=(q^a-\zeta)u$, a contradiction since $\mathrm{ord}(q^a)|N$ and $\mathrm{ord}(\zeta)=n$. Therefore, there must be some j such that $\left(1-\frac{jn}{N}\right)_{q^a}=0$. That is, $q^a\neq 1$ and $\gcd(\mathrm{ord}(q^a),\frac{n}{N})=1$.

We show that, in fact, $\mathrm{ord}(q^a)=N$. Note that

$$X^{\mathrm{ord}(q^a)}\cdot u^s=\delta^{\mathrm{ord}(q^a)}\left(\prod_{j=0}^{\mathrm{ord}(q^a)-1}\left(s-\frac{jn}{N}\right)_{q^a}\right)u^{s-\frac{\mathrm{ord}(q^a)n}{N}}=0.$$

The last equality follows from the fact that each $s-\frac{jn}{N}$ is distinct mod $\mathrm{ord}(q^a)$, and thus one of them is congruent to 0 mod $\mathrm{ord}(q^a)$. Thus, $X^{\mathrm{ord}(q^a)}$ acts by zero. Now suppose for a contradiction that $\mathrm{ord}(q^a)\neq N$. Then, since $\mathrm{ord}(q^a)|\mathrm{ord}(q)=N$, we have $\mathrm{ord}(q^a)\leq\frac{N}{2}$, so for any $a\in\{1,2,\ldots,N-1\}$, we have $a\geq\mathrm{ord}(q^a)$ or $N-a\geq\mathrm{ord}(q^a)$. Thus,

$$G\cdot u^2=(G\cdot u)^2+(\zeta^N-1)\sum_{a=1}^{N-1}\widetilde{\binom{0}{a}_q}\left(X^aG\cdot u\right)\left(X^{N-a}G^{1+\frac{an}{N}}\cdot u\right)=\zeta^{2a}u^2.$$

Using induction, we see that $G\cdot u^s=\zeta^{sa}u^s$ for any s. Thus,

$$0=(GX-\zeta XG)\cdot u=\delta\zeta^a(q^{-a}-\zeta)u^{1-\frac{n}{N}}.$$

Since, $\delta \neq 0$, we must have $\zeta = q^{-a}$, another contradiction. Therefore, $\mathrm{ord}(q^a) = N$.

Hence, we must have $\gcd(N, \frac{n}{N}) = 1$ as well as $\gcd(N, a) = 1$, the latter following from the fact that q is a primitive N^{th} root of unity.

From the equation $(xX - Xx) \cdot u^s = (G^{\frac{n}{N}} - g) \cdot u^s$, we get that for all s,

$$\gamma\delta \left[\left(s - \frac{n}{N}\right)_\zeta (s)_{q^a} - \left(s + \frac{n}{N}\right)_{q^a} (s)_\zeta \right] = q^{as} - \zeta^s. \tag{5.7}$$

For $s = 1$ and $s = n - 1$, this yields respectively

$$\gamma\delta \left[\left(1 - \frac{n}{N}\right)_\zeta - \left(1 + \frac{n}{N}\right)_{q^a} \right] = q^a - \zeta; \tag{5.8}$$

$$\gamma\delta \left[-q^{-a} \left(-1 - \frac{n}{N}\right)_\zeta + \zeta^{-1} \left(-1 + \frac{n}{N}\right)_{q^a} \right] = q^{-a} - \zeta^{-1}.$$

From this, we get that

$$\frac{q^a - \zeta}{\zeta \left(-\frac{n}{N}\right)_\zeta - q^a \left(\frac{n}{N}\right)_{q^a}} = \frac{q^a - \zeta}{\left(-\frac{n}{N}\right)_\zeta - \left(\frac{n}{N}\right)_{q^a}},$$

giving $q^{-1} = q^{\frac{an}{N}}$. Applying (5.7) to $s = \frac{n}{N}$ yields

$$-\gamma\delta \left(\frac{2n}{N}\right)_{q^a} \left(\frac{n}{N}\right)_\zeta = q^{-1} - q.$$

Comparing this with (5.8), and using the fact that $q^{-1} = q^{\frac{an}{N}}$ yields that $q^2 = 1$. Therefore, $N = 2$, and since $\gcd(\frac{n}{2}, 2) = 1$, we also have $\frac{n}{2}$ is odd. Hence, the first statement is proven.

To see that δ must be as specified, apply (5.8) to get

$$\gamma\delta = \frac{-1 - \zeta}{\left(1 - \frac{n}{N}\right)_\zeta} = \frac{(-1 - \zeta)(\zeta - 1)}{-\zeta - 1} = \zeta - 1.$$

$\square$

CHAPTER 6

THE FROBENIUS-LUSZTIG KERNEL $u_q(\mathfrak{sl}_2)$

The next algebra we study is the Frobenius-Lusztig kernel, $u_q(\mathfrak{sl}_2)$. It is well-known that $u_q(\mathfrak{sl}_2)$ contains two isomorphic copies of Taft algebras, which generate the whole algebra. In a sense, the Taft algebra $T_n(q)$ is like a Borel subalgebra of $u_q(\mathfrak{sl}_2)$. More precisely, with the decomposition, $T_n(q) \cong \mathfrak{B}(V)\#\Bbbk\Gamma$ as at the beginning of Chapter 4, $\mathfrak{B}(V) \cong u_q^+(\mathfrak{sl}_2)$ ([6, Theorem 4.3]).

Definition 6.0.1. Let $n \geq 3$ be an odd integer and let $q \in \Bbbk$ be a primitive n^{th} root of unity. The quantum group $U_q(\mathfrak{sl}_2)$, often called the *quantized universal enveloping algebra* of $\mathfrak{sl}_2$, is the Hopf algebra generated by grouplike elements K and K^{-1}, a $(1, K)$-skew primitive element E, and a $(K^{-1}, 1)$-skew primitive element F, subject to the relations

$$KK^{-1} = K^{-1}K = 1, \qquad KE = q^2EK,$$
$$KF = q^{-2}FK, \qquad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

The Frobenius-Lusztig kernel $u_q(\mathfrak{sl}_2)$ is then the quotient of $U_q(\mathfrak{sl}_2)$ by the (Hopf) ideal generated by $K^n - 1$, E^n, and F^n. Note that $\{E^iF^jK^\ell\}_{0\leq i,j,\ell<n}$ is a basis of $u_q(\mathfrak{sl}_2)$.

6.1 The dual and double of $u_q(\mathfrak{sl}_2)$

Using the fact that $u_q(\mathfrak{sl}_2)$ is factorizable, [29, Theorem 2.9] gives that the double $D(u_q(\mathfrak{sl}_2)) \cong u_q(\mathfrak{sl}_2) \otimes u_q(\mathfrak{sl}_2)$ as an algebra. However, the coproduct becomes much more complicated. Similarly, as mentioned in Remark 4.2.3, Krop and Radford provide a presentation of $D(u_q(\mathfrak{sl}_2))$ in [20]. However, the generators of this presentation also have a complicated coproduct. For the method that we use to extend actions of a Hopf algebra, it is better to have an uncomplicated coproduct, so we provide here a different presentation for $D(u_q(\mathfrak{sl}_2))$.

This presentation is computed by first showing that $u_q(\mathfrak{sl}_2)$ is dual to a quotient of the *quantized coordinate ring* $\mathcal{O}_q(SL_2)$.

This result is well-known (see [11, III.7.10]), but we include here an explicit proof for completion, something which is seemingly absent in the literature.

Definition 6.1.1. The quantum group $\mathcal{O}_q(SL_2)$ is the Hopf algebra generated by a, b, c, d subject to the relations

$$ba = qab, \quad ca = qac, \quad db = qbd, \quad dc = qcd, \quad bc = cb,$$
$$ad = q^{-1}bc + 1, \quad da = qbc + 1,$$

with coalgebra structure and antipode given by

$$\Delta(a) = a \otimes a + b \otimes c, \quad \Delta(b) = a \otimes b + b \otimes d,$$
$$\Delta(c) = c \otimes a + d \otimes c, \quad \Delta(d) = c \otimes b + d \otimes d$$
$$\epsilon(a) = \epsilon(d) = 1, \quad \epsilon(b) = \epsilon(c) = 0,$$
$$S(a) = d, \quad S(b) = -qb, \quad S(c) = -q^{-1}c, \quad S(d) = a.$$

One can easily verify that the ideal J generated by $a^n - 1$, b^n, c^n, and $d^n - 1$ is a Hopf ideal, so we define $\overline{\mathcal{O}_q(SL_2)} := \mathcal{O}_q(SL_2)/J$.

In $\overline{\mathcal{O}_q(SL_2)}$, the generators a and d are invertible. Using this, the relation $da = qbc+1$ becomes vacuous. Also, we can use the relation $ad = q^{-1}bc+1$ to eliminate the generator a from the algebra presentation of $\overline{\mathcal{O}_q(SL_2)}$. If we do so, all other relations involving a become vacuous, so we have

$$\overline{\mathcal{O}_q(SL_2)} \cong \Bbbk\langle b, c, d \mid b^n,\ c^n,\ d^n - 1,\ bc - cb,\ db - qbd,\ dc - qcd\rangle$$

as algebras. Thus, the finite set $\{b^i c^j d^\ell\}_{0\le i,j,\ell\le n-1}$ is a basis for $\overline{\mathcal{O}_q(SL_2)}$, and $\dim_{\Bbbk}(\overline{\mathcal{O}_q(SL_2)}) = n^3$.

The first step toward showing that $\overline{\mathcal{O}_q(SL_2)} \cong u_q(\mathfrak{sl}_2)^*$ is exhibiting a duality between $\mathcal{O}_q(SL_2)$ and $U_q(\mathfrak{sl}_2)$. This is done in [18, VII.4] and we recall the duality here. Let $V_{1,1}$ denote the highest weight $U_q(\mathfrak{sl}_2)$-module with basis v_0, v_1 determined by

$$E \cdot v_1 = v_0, \quad F \cdot v_0 = v_1, \quad K \cdot v_0 = qv_0, \quad K \cdot v_1 = q^{-1}v_1,$$
$$E \cdot v_0 = F \cdot v_1 = 0.$$

In other words, if $\rho : U_q(\mathfrak{sl}_2) \to \mathrm{End}_{\Bbbk}(V_{1,1})$ denotes the representation, then, identifying $\mathrm{End}_{\Bbbk}(V_{1,1})$ with $\mathrm{M}_2(\Bbbk)$ on the ordered basis $\{v_0, v_1\}$, we have

$$\rho(E) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \rho(F) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad \rho(K) = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}.$$

Now, for any element $u \in U_q(\mathfrak{sl}_2)$, define

$$\rho(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

to get four elements A, B, C, and D of $U_q(\mathfrak{sl}_2)^*$.

Theorem 6.1.2 ([18, VII.4.4])**.** *Let $\phi : \mathcal{O}_q(SL_2) \to U_q(\mathfrak{sl}_2)^*$ be defined by $\phi(a) = A$, $\phi(b) = B$, $\phi(c) = C$, $\phi(d) = D$. Then ϕ is a Hopf algebra map, and the bilinear form $\langle u, x\rangle = \phi(u)(x)$ realizes a duality between the Hopf algebras $\mathcal{O}_q(SL_2)$ and $U_q(\mathfrak{sl}_2)$.* □

Lemma 6.1.3. *For the map $\phi : \mathcal{O}_q(SL_2) \to U_q(\mathfrak{sl}_2)^*$ given in Theorem 6.1.2, we have that* $\mathrm{Im}(\phi) \subseteq u_q(\mathfrak{sl}_2)^*$.

Proof. We only need to show that A, B, C, and D all vanish on the (Hopf) ideal I of $U_q(\mathfrak{sl}_2)$ generated by $K^n - 1$, E^n, and F^n, which amounts to showing that

$\rho(E^n) = \rho(F^n) = \rho(K^n - 1) = 0$. We have that $\rho(E^n) = \rho(F^n) = 0$ because $\rho(E)$ and $\rho(F)$ each have nilpotency order 2, while $n \geq 3$. That $\rho(K^n - 1) = 0$ follows because q is an n^{th} root of unity. $\square$

We now have a Hopf algebra map $\phi : \mathcal{O}_q(SL_2) \to u_q(\mathfrak{sl}_2)^*$. We wish to show that ϕ induces an isomorphism of Hopf algebras $\overline{\phi} : \overline{\mathcal{O}_q(SL_2)} \to u_q(\mathfrak{sl}_2)^*$. To do this, we will need the following calculations, which can be verified using the pairing from Theorem 6.1.2.

Lemma 6.1.4. *For i, j nonnegative integers,*

$$\langle a^n, E^i \rangle = \langle d^n, E^i \rangle = \delta_{i,0}, \quad \langle a^n, F^j \rangle = \langle d^n, F^j \rangle = \delta_{j,0},$$
$$\langle b^n, E^i \rangle = \langle b^n, F^j \rangle = 0.$$

$\square$

Proposition 6.1.5. *The map ϕ induces a Hopf algebra map $\overline{\phi} : \overline{\mathcal{O}_q(SL_2)} \to u_q(\mathfrak{sl}_2)^*$ determined by $\phi = \overline{\phi} \circ \pi$, where $\pi : \mathcal{O}_q(SL_2) \to \overline{\mathcal{O}_q(SL_2)}$ is the usual projection. Hence, the bilinear form $\langle u, x \rangle = \overline{\phi}(u)(x)$ realizes a duality between the Hopf algebras $\overline{\mathcal{O}_q(SL_2)}$ and $u_q(\mathfrak{sl}_2)$.*

Proof. We need to show that ϕ vanishes on $a^n - 1$, b^n, c^n, and $d^n - 1$. Note that $(b \otimes c)(a \otimes a) = q^2(a \otimes a)(b \otimes c)$, and that q^2 is a primitive n^{th} root of unity because n is odd. Thus, by [28, Corollary 7.2.2], $\Delta(a^n) = (a \otimes a + b \otimes c)^n = a^n \otimes a^n + b^n \otimes c^n$, and similarly for $\Delta(b^n)$, $\Delta(c^n)$, and $\Delta(d^n)$. Thus, using Lemma 6.1.4 and the duality of Theorem 6.1.2, we compute for i, j, and k nonnegative integers,

$$\begin{aligned}
\langle a^n, E^i F^j K^\ell \rangle &= \langle a^n, E^i F^j \rangle \langle a^n, K^\ell \rangle + \langle b^n, E^i F^j \rangle \langle c^n, K^\ell \rangle \;=\; \langle a^n, E^i F^j \rangle \\
&= \langle a^n, E^i \rangle \langle a^n, F^j \rangle + \langle b^n, E^i \rangle \langle c^n, F^j \rangle \;=\; \delta_{i,0}\delta_{j,0}, \\
\langle b^n, E^i F^j K^\ell \rangle &= \langle a^n, E^i F^j \rangle \langle b^n, K^\ell \rangle + \langle b^n, E^i F^j \rangle \langle d^n, K^\ell \rangle \;=\; \langle b^n, E^i F^j \rangle \;=\; 0, \\
\langle c^n, E^i F^j K^\ell \rangle &= \langle c^n, E^i F^j \rangle \langle a^n, K^\ell \rangle + \langle d^n, E^i F^j \rangle \langle c^n, K^\ell \rangle \;=\; \langle c^n, E^i F^j \rangle \;=\; 0, \\
\langle d^n, E^i F^j K^\ell \rangle &= \langle c^n, E^i F^j \rangle \langle b^n, K^\ell \rangle + \langle d^n, E^i F^j \rangle \langle d^n, K^\ell \rangle \;=\; \langle d^n, E^i F^j \rangle \\
&= \langle c^n, E^i \rangle \langle b^n, F^j \rangle + \langle d^n, E^i \rangle \langle d^n, F^j \rangle \;=\; \delta_{i,0}\delta_{j,0}.
\end{aligned}$$

We have thus shown that ϕ vanishes on b^n and c^n. Now, since ϵ is an algebra map, we have that $\langle 1, E^i F^j K^\ell \rangle = \epsilon(E^i F^j K^\ell) = \epsilon(E)^i \epsilon(F)^j \epsilon(K)^\ell = \delta_{i,0}\delta_{j,0}$. Thus, ϕ also vanishes on $a^n - 1$ and $d^n - 1$. $\square$

At this point, we want to establish that the duality just formed between $\overline{\mathcal{O}_q(SL_2)}$ and $u_q(\mathfrak{sl}_2)$ is a perfect duality. We do this by showing that $\overline{\phi}$ is surjective, for which we will need the following technical computation.

For the basis $\{E^i F^j K^\ell\}$ of $u_q(\mathfrak{sl}_2)$, we let $\{p_{i,j,\ell}\}$ denote the dual basis of $u_q(\mathfrak{sl}_2)^*$. Because $K^n = 1$ in $u_q(\mathfrak{sl}_2)$, we will take the last argument of these basis elements modulo n.

Now via elementary computations we have in terms of the dual basis $\{p_{i,j,\ell}\}$ of $u_q(\mathfrak{sl}_2)^*$, that

$$B^s C^t D^r = [s]_q!\,[t]_q! \sum_{\ell=0}^{n-1} q^{-\ell(r+s-t)-rs} p_{s,t,\ell}. \tag{6.1}$$

For details of this computation, see A.2.1 in Appendix A.

Proposition 6.1.6. *The map $\overline{\phi} : \overline{\mathcal{O}_q(SL_2)} \to u_q(\mathfrak{sl}_2)^*$ is surjective, and hence is an isomorphism. Thus, the bilinear form $\langle u, x\rangle = \overline{\phi}(u)(x)$ realizes a perfect duality between $\overline{\mathcal{O}_q(SL_2)}$ and $u_q(\mathfrak{sl}_2)$. Therefore, $u_q(\mathfrak{sl}_2)^* \cong \overline{\mathcal{O}_q(SL_2)}$.*

Proof. We show that each basis element $p_{i,j,k}$ of $u_q(\mathfrak{sl}_2)$ is in the image of $\overline{\phi}$. In particular, for fixed integers $0 \le s, t, k \le n-1$, we show that

$$n\,[s]_q!\,[t]_q!\,p_{s,t,k} = \sum_{r=0}^{n-1} q^{(k+s)r+(s-t)k} B^s C^t D^r.$$

We compute via (6.1)

$$\begin{aligned}\sum_{r=0}^{n-1} q^{(k+s)r+(s-t)k} B^s C^t D^r &= \sum_{r=0}^{n-1} q^{(k+s)r+(s-t)k}\,[s]_q!\,[t]_q! \sum_{\ell=0}^{n-1} q^{-\ell(r+s-t)-rs} p_{s,t,\ell} \\ &= [s]_q!\,[t]_q! \sum_{\ell=0}^{n-1} \left(\sum_{r=0}^{n-1} q^{(k-\ell)(r+s-t)} \right) p_{s,t,\ell}.\end{aligned}$$

If $k \neq \ell$, then since $q^{k-\ell}$ is an n^{th} root of unity not equal to 1,

$$\sum_{r=0}^{n-1} q^{(k-\ell)(r+s-t)} = 0.$$

On the other hand, if $k = \ell$, then $\sum_{r=0}^{n-1} q^{(k-\ell)(r+s-t)} = n$. $\square$

In the presentation provided by Krop and Radford in [20], we have $v_1 = E,\ v_2 = K^{-1}F,\ d = \sum_{i=0}^{n-1} q^{-i}\epsilon_{K^i}$, $b = q\xi_1 d$, and $c = q^2\xi_2 d^{-1}$. (See Remark 4.2.3.) Now that we have established that $u_q(\mathfrak{sl}_2)^* \cong \overline{\mathcal{O}_q(SL_2)}$, we can prove the following.

Theorem 6.1.7. *The Drinfel'd double $D(u_q(\mathfrak{sl}_2))$ of $u_q(\mathfrak{sl}_2)$ is generated as an algebra by $a,\ b,\ c,\ d,\ E,\ F,\ K$ subject to the relations*

$$a^n = d^n = K^n = 1, \quad b^n = c^n = E^n = F^n = 0,$$

$$ba = qab, \quad db = qbd, \quad ca = qac, \quad dc = qcd, \quad bc = cb, \quad ad = q^{-1}bc + 1,$$

$$KE = q^2EK, \quad KF = q^{-2}FK, \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}},$$

$$Ka = aK, \quad Kb = q^{-2}bK, \quad Kc = q^2cK, \quad Kd = dK,$$

$$Ea = q^{-1}aE - q^{-1}c, \quad Eb = q^{-1}bE + q^{-1}aK - q^{-1}d,$$

$$Ec = qcE, \quad Ed = qdE + qcK,$$

$$Fa = q^{-1}aF + b, \quad Fb = qbF,$$

$$Fc = q^{-1}cF = aK^{-1} + d, \quad Fd = qdF - q^2bK^{-1}$$

The comultiplication and counit are given by

$$\Delta(a) = a \otimes a + c \otimes b, \quad \Delta(b) = b \otimes a + d \otimes b,$$

$$\Delta(c) = a \otimes c + c \otimes d, \quad \Delta(d) = b \otimes c + d \otimes d,$$

$$\Delta(K) = K \otimes K, \quad \Delta(E) = K \otimes E + E \otimes 1, \quad \Delta(F) = 1 \otimes F + F \otimes K^{-1},$$

$$\epsilon(a) = \epsilon(d) = \epsilon(K) = 1, \quad \epsilon(b) = \epsilon(c) = \epsilon(E) = \epsilon(F) = 0.$$

The antipode is given by

$$S(a) = d, \quad S(b) = -q^{-1}b, \quad S(c) = -qc, \quad S(d) = a,$$

$$S(K) = K^{-1}, \quad S(E) = -EK^{-1}, \quad S(F) = -KF.$$

As pointed out above, the generator a (or d) could be eliminated from the presentation, using the relation $ad = q^{-1}bc + 1$ and the fact that a and d are invertible. While doing so would significantly lower the number of relations, it would complicate both the relations between generators of $u_q(\mathfrak{sl}_2)$ and $\overline{\mathcal{O}_q(SL_2)}$ and the comultiplication of the latter.

Proof of Theorem 6.1.7. The comultiplication and antipode and most of the relations of the generators follow from (2.15) and Lemma 2.3.2. For the relations involving elements of both $u_q(\mathfrak{sl}_2)$ and its dual, we use (2.17) and the perfect duality established in Proposition 6.1.6. Note that

$$\Delta^2(a) = a \otimes a \otimes a + a \otimes b \otimes c + b \otimes c \otimes a + b \otimes d \otimes c,$$
$$\Delta^2(E) = 1 \otimes 1 \otimes E + 1 \otimes E \otimes K + E \otimes K \otimes K,$$
$$\Delta^2(F) = K^{-1} \otimes K^{-1} \otimes F + K^{-1} \otimes F \otimes 1 + F \otimes 1 \otimes 1,$$
$$S^{-1}(E) = -K^{-1}E, \quad S^{-1}(F) = -FK.$$

For example, we have

$$\begin{aligned} Ea &= \langle a_{(1)}, S^{-1}(E_{(3)})\rangle\langle a_{(3)}, E_{(1)}\rangle a_{(2)}E_{(2)} \\ &= \quad \langle a, -K^{-1}E\rangle\langle a, 1\rangle a1 + \langle a, K^{-1}\rangle\langle a, 1\rangle aE + \langle a, K^{-1}\rangle\langle a, E\rangle aK \\ &\quad + \langle a, -K^{-1}E\rangle\langle c, 1\rangle b1 + \langle a, K^{-1}\rangle\langle c, 1\rangle bE + \langle a, K^{-1}\rangle\langle c, E\rangle bK \\ &\quad + \langle b, -K^{-1}E\rangle\langle a, 1\rangle c1 + \langle b, K^{-1}\rangle\langle a, 1\rangle cE + \langle b, K^{-1}\rangle\langle a, E\rangle cK \\ &\quad + \langle b, -K^{-1}E\rangle\langle c, 1\rangle d1 + \langle b, K^{-1}\rangle\langle c, 1\rangle dE + \langle b, K^{-1}\rangle\langle c, E\rangle dK \\ &= q^{-1}aE - q^{-1}c. \end{aligned}$$

$$\begin{aligned} Fa = &\quad \langle a, -FK\rangle\langle a, K^{-1}\rangle aK^{-1} + \langle a, 1\rangle\langle a, K^{-1}\rangle aF + \langle a, 1\rangle\langle a, F\rangle a1 \\ &\quad + \langle a, -FK\rangle\langle c, K^{-1}\rangle bK^{-1} + \langle a, 1\rangle\langle c, K^{-1}\rangle bF + \langle a, 1\rangle\langle c, F\rangle b1 \\ &\quad + \langle b, -FK\rangle\langle a, K^{-1}\rangle cK^{-1} + \langle b, 1\rangle\langle a, K^{-1}\rangle cF + \langle b, 1\rangle\langle a, F\rangle c1 \\ &\quad + \langle b, -FK\rangle\langle c, K^{-1}\rangle dK^{-1} + \langle b, 1\rangle\langle c, K^{-1}\rangle dF + \langle b, 1\rangle\langle c, F\rangle d1 \\ = &\ q^{-1}aF + b. \end{aligned}$$

The rest of the relations follow similarly and are in A.2.2 in Appendix A. □

6.2 The possible structures of $A(u_q(\mathfrak{sl}_2))$ and extensions to $D(u_q(\mathfrak{sl}_2))$

To help us determine when an action of $u_q(\mathfrak{sl}_2)$ is inner-faithful, we have the following standard fact.

Proposition 6.2.1. *Let $0 \leq b < n$. Then*

$$P_{K^b,1}(u_q(\mathfrak{sl}_2)) = \begin{cases} \Bbbk(K^{-1}-1) + \Bbbk F + \Bbbk EK^{-1}, & \textit{if } b = n-1 \\ \Bbbk(K^b - 1), & \textit{otherwise.} \end{cases} \qquad \square$$

The following is a direct result of Corollary 2.5.3 and Proposition 6.2.1.

Corollary 6.2.2. *A $u_q(\mathfrak{sl}_2)$-module algebra is inner-faithful if and only if $G(H)$ acts faithfully, and if no nonzero element of $\Bbbk(K^{-1}-1)+\Bbbk F+\Bbbk EK^{-1}$ acts by zero.* $\square$

We now consider $u_q(\mathfrak{sl}_2)$-module algebra structures on $A(u_q(\mathfrak{sl}_2))$ as in Notation 1.3.2. By definition, $A = \Bbbk[u]/(u^n - 1)$. To see the possible module structures of A, we use the following result of Montgomery and Schneider. The original statement was for q a primitive $2n^{th}$ root of unity. However, their proof is also valid for the case we are interested in, since it only relies on the fact that q^2 is a primitive n^{th} root of unity so that $H_1 = \Bbbk\langle K^{-1}, F\rangle \cong T_n(q^{-2})$ and $H_2 = \Bbbk\langle K^{-1}, EK^{-1}\rangle \cong T_n(q^2)$.

Proposition 6.2.3 ([24, Corollary 3.2])**.** *Let A be an n-dimensional $\Bbbk$-algebra with no non-zero nilpotent elements, and assume that A is a $u_q(\mathfrak{sl}_2)$-module algebra such that $F \cdot A \neq 0$ (or that $E \cdot A \neq 0$). Then there exists $u \in A$ and $\beta, \gamma, \delta \in \Bbbk$, all nonzero, such that*

(a) $A = \Bbbk(u),\ u^n = \beta,$ *and* $K \cdot u = q^2 u$*;*

(b) $F \cdot u = \gamma 1$ *and* $E \cdot u = \delta u^2$*;*

(c) $\gamma\delta = -q$*.*

Moreover u is unique up to a scalar multiple. $\square$

We point out here that by Corollary 6.2.2, the assumption that $F \cdot A \neq 0$ or $E \cdot A \neq 0$ is necessary for the action to be inner-faithful, and that the actions on A described are in fact inner-faithful, because no nonzero element of $\Bbbk(1 - K^{-1}) + \Bbbk F + \Bbbk EK^{-1}$ acts by zero. Therefore, by scaling u, Proposition 6.2.3 classifies the $u_q(\mathfrak{sl}_2)$-module algebra structures on $A(u_q(\mathfrak{sl}_2))$ as in Notation 1.3.2. It turns out that the action of $u_q(\mathfrak{sl}_2)$ on A extends to an action of $D(u_q(\mathfrak{sl}_2))$ in two distinct ways.

Theorem 6.2.4. *Fix a $u_q(\mathfrak{sl}_2)$-module algebra structure on the algebra $A(u_q(\mathfrak{sl}_2)) = \Bbbk[u]/(u^n-1)$ as in Notation 1.3.2 by*

$$K \cdot u = q^2 u, \quad F \cdot u = \gamma 1, \quad E \cdot u = \delta u^2,$$

with q a primitive n^{th} root of unity, and $\gamma\delta = -q$. Recall the presentation of $D(u_q(\mathfrak{sl}_2))$ as in Theorem 6.1.7. If the action of $u_q(\mathfrak{sl}_2)$ on A extends to an action of $D(u_q(\mathfrak{sl}_2))$ so that A is a $D(u_q(\mathfrak{sl}_2))$-module algebra, then the action is specified by one of the following two conditions:

(i) $a \cdot u = qu,\ b \cdot u = \gamma(q-q^{-1})1,\ c \cdot u = 0,\ d \cdot u = q^{-1}u,$ *or*

(ii) $a \cdot u = q^{-1}u,\ b \cdot u = 0,\ c \cdot u = \gamma^{-1}(q-q^{-1})u^2,\ d \cdot u = qu.$

Conversely, by defining the action of a, b, c, and d by either (i) or (ii), an action of $u_q(\mathfrak{sl}_2)$ on A extends to an action of $D(u_q(\mathfrak{sl}_2))$.

Proof. Since $K \cdot u = q^2 u$, we use notation similar to that in Remark 2.5.7 :

$$A_i = \{a \in A \mid K \cdot a = q^{2i} a\} = \Bbbk u^i.$$

First, since $Ka = aK$, we have $K \cdot a \cdot u = a \cdot K \cdot u = q^2 a \cdot u$, so $a \cdot u \in A_1 = \Bbbk u$. Similarly, since $Kb = q^{-2}bK$, $Kc = q^2 cK$, and $Kd = dK$, we get that $b \cdot u \in A_0$, $c \cdot u \in A_2$, and $d \cdot u \in A_1$. Therefore, there exists $\theta_a, \theta_b, \theta_c, \theta_d \in \Bbbk$ such that

$$a \cdot u = \theta_a u, \qquad b \cdot u = \theta_b 1, \qquad c \cdot u = \theta_c u^2, \qquad \text{and} \qquad d \cdot u = \theta_d u.$$

Now, note that $c \cdot 1 = \epsilon(c) = 0$. Thus, since $bc = cb$ and $ad = q^{-1}bc + 1$, we compute that

$$\theta_a \theta_d u = (ad) \cdot u = q^{-1} c \cdot (b \cdot u) + 1 \cdot u = q^{-1}\theta_b c \cdot 1 + u = u.$$

Therefore, $\theta_d = \theta_a^{-1}$. Using the fact that $a^n = 1$, for some integer i, we have $\theta_a = q^i$ and $\theta_d = q^{-i}$. Note that $b \cdot u^2 = (b \cdot u)(a \cdot u) + (d \cdot u)(b \cdot u) = \theta_b\theta_a u + \theta_d \theta_b u = \theta_b(\theta_a + \theta_d)u$. Thus,

$$\theta_c \theta_b (\theta_a + \theta_d) u = (bc) \cdot u = (cb) \cdot u = \theta_b c \cdot 1 = 0.$$

Since $\theta_a = q^i$ is an odd root of unity, $\theta_a \neq -\theta_d (= -\theta_a^{-1})$. Thus, we must have

$$\theta_b = 0 \quad \text{or} \quad \theta_c = 0. \tag{6.2}$$

We also compute, using $a \cdot 1 = \epsilon(a) = 1$ and $d \cdot 1 = \epsilon(d) = 1$, that

$$\theta_a \gamma 1 = (Fa) \cdot u = q^{-1}(aF) \cdot u + b \cdot u = (q^{-1}\gamma + \theta_b)1 \qquad \text{and}$$
$$\theta_d \gamma 1 = (Fd) \cdot u = q(dF) \cdot u - q^2(bK^{-1}) \cdot u = (q\gamma - \theta_b)1,$$

which shows that

$$\theta_a = q^{-1} + \theta_b \gamma^{-1} \quad \text{and} \quad \theta_d = q - \theta_b \gamma^{-1}. \tag{6.3}$$

Therefore,

$$1 = \theta_a \theta_d = (q^{-1} + \theta_b \gamma^{-1})(q - \theta_b \gamma^{-1}) = 1 + (q - q^{-1})\theta_b \gamma^{-1} - \theta_b^2 \gamma^{-2},$$

implying that $0 = \theta_b \gamma^{-1}(q - q^{-1} - \theta_b \gamma^{-1})$. Since $\gamma \neq 0$, we have

$$\theta_b = 0 \quad \text{or} \quad \theta_b = \gamma(q - q^{-1}).$$

The former will correspond to *(ii)* and the latter to *(i)*. In case *(i)*, by (6.2), $\theta_c = 0$, and by (6.3), $\theta_a = q$ and $\theta_d = q^{-1}$. On the other hand, in case *(ii)*, by (6.3), $\theta_a = q^{-1}$ and $\theta_d = q$. Also, using the fact that $\gamma\delta = -q$, $Ea = q^{-1}aE - q^{-1}c$, and $a \cdot u^2 = (a \cdot u)^2 + (c \cdot u)(b \cdot u) = q^{-2}u^2$, we have

$$-\gamma^{-1}u^2 = q^{-1}\delta u^2 \;=\; (Ea) \cdot u \;=\; q^{-1}(aE) \cdot u - q^{-1}c \cdot u$$
$$= q^{-1}\delta a \cdot u^2 - q^{-1}\theta_c u^2 \;=\; -(q^{-2}\gamma^{-1} + q^{-1}\theta_c)u^2.$$

Therefore, $\gamma^{-1} = q^{-2}\gamma^{-1} + q^{-1}\theta_c$, which implies $\theta_c = \gamma^{-1}(q - q^{-1})$. Therefore, we have shown that an action of $D(u_q(\mathfrak{sl}_2))$ is specified by either *(i)* or *(ii)*.

It is straightforward to check the converse: that A is a $D(u_q(\mathfrak{sl}_2))$-module algebra with either of these structures. $\square$

CHAPTER 7

FUTURE DIRECTIONS

We wish to briefly mention some future directions for further research. As we have seen, there are multiple ways to generalize Taft algebras, but we have only examined a couple of generalizations, namely quantum linear spaces of rank 1 over cyclic groups, $H_n(\zeta, m, t)$ [Chapter 4], and generalized Taft algebras, $T(n, N, 1)$ [Chapter 5]. We could also consider quantum linear spaces of higher rank and/or over abelian, non-cyclic groups. Or more generally, we could consider bosonizations of Nichols algebras (of Cartan type) in the Yetter-Drinfeld category ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ for some abelian group Γ. We saw that the answer to Question 1.3.3(c) for the Hopf algebras $H_n(\zeta, m, 1)$ is 1, i.e. that there is a unique way to extend an action of $H_n(\zeta, m, 1)$ on $A := A(H_n(\zeta, m, t))$ to an action of $D(H_n(\zeta, m, 1))$ on A [Corolarry 4.4.3]. This generalizes Theorem 3.1.2, that there is a unique way to extend the action of $T_n(q)$ on $A(T_n(q))$ to an action of $D(T_n(q))$ on $A(T_n(q))$. One way of characterizing the condition $t = 1$ is that χ generates $\widehat{\Gamma}$ in the quantum linear space $\mathcal{R}(g; \chi) \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$. (See Sections 2.4 and 4.1 on quantum linear spaces.) This leads to the following:

Question 7.0.1. What can be said about Question 1.3.3 for bosonizations of quantum linear spaces of higher rank and/or over abelian non-cyclic groups? In particular:

- Is there a unique extension of an action of $H := \Bbbk\Gamma\#\mathcal{R}(g_1, \dots, g_\theta; \chi_1, \dots \chi_\theta)$ on $A(H)$ to an action of $D(H)$ on $A(H)$ if and only if $\widehat{\Gamma}$ is generated by $\chi_1, \dots, \chi_\theta$?

- If so, is there a condition for bosonizations of braided vector spaces of different Cartan types (i.e. other than A_1^θ, which give quantum linear spaces) that generalizes the condition that $\chi_1, \dots, \chi_\theta$ generates $\widehat{\Gamma}$, and which guarantees that there is a unique extension of H on $A(H)$ to an action of $D(H)$ on $A(H)$?

Another future direction comes from the results about actions of $u_q(\mathfrak{sl}_2)$. Recall that $T_n(q)$ can be considered as a Borel subalgebra of $u_q(\mathfrak{sl}_2)$: with the decomposition, $T_n(q) \cong \mathfrak{B}(V)\#\Bbbk\Gamma$ as at the beginning of Chapter 4, we have $\mathfrak{B}(V) \cong u_q^+(\mathfrak{sl}_2)$ ([6, Theorem 4.3]). Perhaps unsurprisingly, while $T_n(q)$ had a unique extension of its action on $A(T_n(q))$ to its double, $u_q(\mathfrak{sl}_2)$ has exactly two extensions of its action on $A(u_q(\mathfrak{sl}_2))$ to its double. That is, the answer to Question 1.3.3(c) for $u_q(\mathfrak{sl}_2)$ is precisely twice the answer for $T_n(q)$. We are led to the following.

Question 7.0.2. For a semisimple finite-dimensional Lie algebra $\mathfrak{g}$, is the answer to Question 1.3.3(c) for $u_q(\mathfrak{g})$ twice what the answer would be for a Borel subalgebra?

Depending on the answers to Questions 7.0.1 and 7.0.2, it is possible that there are more general things to be said about Question 1.3.3 for pointed Hopf algebras in general. To gather examples to look for patterns, one could start by considering the actions of finite-dimensional pointed Hopf algebras presented in work of Etingof and Walton [15, 16].

Alternatively, one could consider semisimple Hopf algebras. The case of group algebras is answered in Remark 1.3.4. A good place to begin after that would be small-dimensional examples, such as the Kac-Paljutkin algebra H_8 of dimension 8.

Toward a final future direction, throughout this work, we computed presentations of duals and doubles of Hopf algebras which may be of independent interest. The method for computing these presentations makes use of the fact that a pointed Hopf algebra H is generated by grouplike and skew primitive elements if $G(H)$ is abelian [8, Theorem 2], which are precisely the Hopf algebras considered in Question 1.3.3. We could use this method to compute nice presentations of duals and Drinfel'd doubles of more pointed Hopf algebras with an abelian group of grouplike elements.

REFERENCES

[1] N. Andruskiewitsch. About finite dimensional Hopf algebras. In *Quantum symmetries in theoretical physics and mathematics (Bariloche, 2000)*, volume 294 of *Contemp. Math.*, pages 1–57. Amer. Math. Soc., Providence, RI, 2002.

[2] N. Andruskiewitsch. On finite-dimensional Hopf algebras. In *Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. II*, pages 117–141. Kyung Moon Sa, Seoul, 2014.

[3] N. Andruskiewitsch and I. Angiono. On finite dimensional Nichols algebras of diagonal type. *Bull. Math. Sci.*, 7(3):353–573, 2017.

[4] N. Andruskiewitsch and W. Ferrer Santos. The beginnings of the theory of Hopf algebras. *Acta Appl. Math.*, 108(1):3–17, 2009.

[5] N. Andruskiewitsch and H.-J. Schneider. Lifting of quantum linear spaces and pointed Hopf algebras of order p^3. *J. Algebra*, 209(2):658–691, 1998.

[6] N. Andruskiewitsch and H.-J. Schneider. Pointed Hopf algebras. In *New directions in Hopf algebras*, volume 43 of *Math. Sci. Res. Inst. Publ.*, pages 1–68. Cambridge Univ. Press, Cambridge, 2002.

[7] N. Andruskiewitsch and H.-J. Schneider. On the classification of finite-dimensional pointed Hopf algebras. *Ann. of Math. (2)*, 171(1):375–417, 2010.

[8] I. Angiono. On Nichols algebras of diagonal type. *J. Reine Angew. Math.*, 683:189–251, 2013.

[9] M. Beattie. Duals of pointed Hopf algebras. *J. Algebra*, 262(1):54–76, 2003.

[10] M. Beattie and G. A. García. Classifying Hopf algebras of a given dimension. In *Hopf algebras and tensor categories*, volume 585 of *Contemp. Math.*, pages 125–152. Amer. Math. Soc., Providence, RI, 2013.

[11] K. A. Brown and K. R. Goodearl. *Lectures on algebraic quantum groups*. Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser Verlag, Basel, 2002.

[12] H.-X. Chen and Y. Zhang. Four-dimensional Yetter-Drinfeld module algebras over H_4. *J. Algebra*, 296(2):582–634, 2006.

[13] M. Cohen, D. Fischman, and S. Montgomery. On Yetter-Drinfeld categories and H-commutativity. *Comm. Algebra*, 27(3):1321–1345, 1999.

[14] V. G. Drinfel'd. Quantum groups. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986)*, pages 798–820. Amer. Math. Soc., Providence, RI, 1987.

[15] P. Etingof and C. Walton. Pointed Hopf actions on fields, I. *Transform. Groups*, 20(4):985–1013, 2015.

[16] P. Etingof and C. Walton. Pointed Hopf actions on fields, II. *J. Algebra*, 460:253–283, 2016.

[17] M. Jimbo, editor. *Yang-Baxter equation in integrable systems*, volume 10 of *Advanced Series in Mathematical Physics*. World Scientific Publishing Co., Inc., Teaneck, NJ, 1989.

[18] C. Kassel. *Quantum Groups*, volume 155 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[19] R. Kinser and C. Walton. Actions of some pointed Hopf algebras on path algebras of quivers. *Algebra Number Theory*, 10(1):117–154, 2016.

[20] L. Krop and D. E. Radford. Representations of pointed Hopf algebras and their Drinfel'd quantum doubles. *J. Algebra*, 321(9):2567–2603, 2009.

[21] M. Lorenz. *A tour of representation theory*, volume 193 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2018.

[22] G. Lusztig. Finite-dimensional Hopf algebras arising from quantized universal enveloping algebra. *J. Amer. Math. Soc.*, 3(1):257–296, 1990.

[23] S. Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[24] S. Montgomery and H.-J. Schneider. Skew derivations of finite-dimensional algebras and actions of the double of the Taft Hopf algebra. *Tsukuba J. Math.*, 25(2):337–358, 2001.

[25] W. D. Nichols. Bialgebras of type one. *Comm. Algebra*, 6(15):1521–1552, 1978.

[26] W. D. Nichols and M. B. Zoeller. A Hopf algebra freeness theorem. *Amer. J. Math.*, 111(2):381–385, 1989.

[27] D. E. Radford. The structure of Hopf algebras with a projection. *J. Algebra*, 92(2):322–347, 1985.

[28] D. E. Radford. *Hopf Algebras*, volume 49 of *Series on Knots and Everything*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2012.

[29] N. Y. Reshetikhin and M. A. Semenov-Tian-Shansky. Quantum R-matrices and factorization problems. *J. Geom. Phys.*, 5(4):533–550 (1989), 1988.

[30] M. E. Sweedler. *Hopf algebras*. Mathematics Lecture Note Series. W. A. Benjamin, Inc., New York, 1969.

[31] M. Takeuchi. Some topics on $\mathrm{GL}_q(n)$. *J. Algebra*, 147(2):379–410, 1992.

[32] V. Turaev and A. Virelizier. *Monoidal categories and topological field theory*, volume 322 of *Progress in Mathematics*. Birkhäuser/Springer, Cham, 2017.

APPENDIX A

EXTRA CALCULATIONS

There are some calculations that we would like to include that would be too unwieldy in their proper place in the manuscript, so we include the calculations here and refer to them in the text.

A.1 Calculations from Chapter 5

Some of the relations of $D(T(n,N,1))$ are long to compute and were skipped in the text. They are included here for completeness.

A.1.1 Remainder of the proof of Proposition 5.2.7

We would like to show that the relations

$$gG = Gg \quad \text{and} \quad xG - \zeta Gx = (\zeta^N - 1)\widetilde{\binom{0}{1}}_q X^{N-1}G\left(G^{\frac{n}{N}} - qg\right)$$

hold in $D(T(n,N,1))$. Recall that in $K_\zeta(n,N)$ and $T(n,N,1)$, we have

$$\begin{aligned}\Delta^2(G) =& G\otimes G\otimes G + (\zeta^N-1)\sum_{i=1}^{N-1}\widetilde{\binom{0}{i}}_q G\otimes X^{N-i}G^{1+\frac{in}{N}}\otimes X^iG\\ &+(\zeta^N-1)\sum_{i=1}^{N-1}\sum_{j=0}^{i}\widetilde{\binom{0}{i}}_q\binom{i}{j}_q X^{N-i}G^{1+\frac{in}{N}}\otimes X^{i-j}G^{1+\frac{jn}{N}}\otimes X^jG\\ &+(\zeta^N-1)^2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N-1}\widetilde{\binom{0}{i}}_q\widetilde{\binom{i}{j}}_q X^{N-i}G^{1+\frac{in}{N}}\otimes X^{N+i-j}G^{1+\frac{jn}{N}}\otimes X^jG,\end{aligned}$$

$$\Delta^2(g) = g\otimes g\otimes g,\qquad \Delta^2(x) = g\otimes g\otimes x + g\otimes x\otimes 1 + x\otimes 1\otimes 1,$$
$$S^{-1}(g) = g^{-1},\qquad S^{-1}(x) = -xg^{-1}.$$

Thus, using (2.17) and (5.6), we have the following computations:

$$\begin{aligned}gG =& \langle G,g^{-1}\rangle\langle G,g\rangle Gg + (\zeta^N-1)\sum_{i=1}^{N-1}\widetilde{\binom{0}{i}}_q\langle G,g^{-1}\rangle\langle X^iG,g\rangle X^{N-i}G^{1+\frac{in}{N}}Gg\\ &+(\zeta^N-1)\sum_{i=1}^{N-1}\sum_{j=0}^{i}\widetilde{\binom{0}{i}}_q\binom{i}{j}_q\langle X^{N-i}G^{1+\frac{in}{N}},g^{-1}\rangle\langle X^j,g\rangle X^{i-j}G^{1+\frac{jn}{N}}Gg\\ &+(\zeta^N-1)^2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N-1}\widetilde{\binom{0}{i}}_q\widetilde{\binom{i}{j}}_q\langle X^{N-i}G^{1+\frac{in}{N}},g^{-1}\rangle\langle X^jG,g\rangle\\ &\qquad\cdot X^{N+i-j}G^{1+\frac{jn}{N}}g\\ =&\ \zeta^{-1}\zeta Gg\\ =&\ Gg.\end{aligned}$$

$$
\begin{aligned}
xG &= \langle G, -xg^{-1}\rangle\langle G, g\rangle Gg \\
&\quad + (\zeta^N - 1)\sum_{i=1}^{N-1}\widetilde{\binom{0}{i}}_q \langle G, -xg^{-1}\rangle\langle X^iG, g\rangle X^{N-i}G^{1+\frac{in}{N}}Gg \\
&\quad + (\zeta^N - 1)\sum_{i=1}^{N-1}\sum_{j=0}^{i}\widetilde{\binom{0}{i}}_q \binom{i}{j}_q \langle X^{N-i}G^{1+\frac{in}{N}}, -xg^{-1}\rangle\langle X^j, g\rangle X^{i-j}G^{1+\frac{jn}{N}}Gg \\
&\quad + (\zeta^N - 1)^2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N-1}\widetilde{\binom{0}{i}}_q \widetilde{\binom{i}{j}}_q \langle X^{N-i}G^{1+\frac{in}{N}}, -xg^{-1}\rangle\langle X^jG, g\rangle \\
&\qquad\qquad \cdot X^{N+i-j}G^{1+\frac{jn}{N}}g \\
&\quad + \langle G, 1\rangle\langle G, g\rangle Gx \\
&\quad + (\zeta^N - 1)\sum_{i=1}^{N-1}\widetilde{\binom{0}{i}}_q \langle G, 1\rangle\langle X^iG, g\rangle X^{N-i}G^{1+\frac{in}{N}}Gx \\
&\quad + (\zeta^N - 1)\sum_{i=1}^{N-1}\sum_{j=0}^{i}\widetilde{\binom{0}{i}}_q \binom{i}{j}_q \langle X^{N-i}G^{1+\frac{in}{N}}, 1\rangle\langle X^j, g\rangle X^{i-j}G^{1+\frac{jn}{N}}Gx \\
&\quad + (\zeta^N - 1)^2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N-1}\widetilde{\binom{0}{i}}_q \widetilde{\binom{i}{j}}_q \langle X^{N-i}G^{1+\frac{in}{N}}, 1\rangle\langle X^jG, g\rangle X^{N+i-j}G^{1+\frac{jn}{N}}x \\
&\quad + \langle G, 1\rangle\langle G, x\rangle G \\
&\quad + (\zeta^N - 1)\sum_{i=1}^{N-1}\widetilde{\binom{0}{i}}_q \langle G, 1\rangle\langle X^iG, x\rangle X^{N-i}G^{1+\frac{in}{N}}G \\
&\quad + (\zeta^N - 1)\sum_{i=1}^{N-1}\sum_{j=0}^{i}\widetilde{\binom{0}{i}}_q \binom{i}{j}_q \langle X^{N-i}G^{1+\frac{in}{N}}, 1\rangle\langle X^j, x\rangle X^{i-j}G^{1+\frac{jn}{N}}G \\
&\quad + (\zeta^N - 1)^2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N-1}\widetilde{\binom{0}{i}}_q \widetilde{\binom{i}{j}}_q \langle X^{N-i}G^{1+\frac{in}{N}}, 1\rangle\langle X^jG, x\rangle X^{N+i-j}G^{1+\frac{jn}{N}} \\
&= -(\zeta^N - 1)\widetilde{\binom{0}{N-1}}_q qX^{N-1}Gg \;+\; \zeta Gx \;+\; (\zeta^N - 1)\widetilde{\binom{0}{1}}_q X^{N-1}G^{1+\frac{n}{N}} \\
&= \zeta Gx + (\zeta^N - 1)\widetilde{\binom{0}{1}}_q X^{N-1}G(G^{\frac{n}{N}} - qg).
\end{aligned}
$$

A.2 Calculations from Chapter 6

Two long computations have been omitted from Chapter 6, the proof of (6.1), which is instrumental in proving that $u_q(\mathfrak{sl}_2)^* \cong \overline{\mathcal{O}_q(SL_2)}$, and the computation of some of the relations of $D(u_q(\mathfrak{sl}_2))$ from Theorem 6.1.7. These are not particularly complicated or long. There are just many of them.

A.2.1 Proof of (6.1)

Recall that the set $\{E^iF^jK^\ell\}_{0\leq i,j,\ell\leq n-1}$ is a basis for $u_q(\mathfrak{sl}_2)$ and that we denote the dual basis for $u_q(\mathfrak{sl}_2)^*$ by $\{p_{i,j,\ell}\}_{0\leq i,j,\ell\leq n-1}$. Because $K^n = 1$ in $u_q(\mathfrak{sl}_2)$, we will consider the last argument of these basis elements modulo n.

Proposition A.2.1. *In $u_q(\mathfrak{sl}_2)^*$, multiplication of the dual basis elements is given by the following:*

$$p_{a,b,c} \ * \ p_{A,B,C} = \begin{cases} \alpha \, p_{A+a,B+b,C-a} & \text{if } A+a<n, \ B+b<n, \\ & \text{and } C-a \equiv c+B \mod n \\ 0 & \text{otherwise}, \end{cases}$$

where $\alpha = q^{-2aB}\binom{A+a}{a}_{q^2}\binom{B+b}{B}_{q^2}$

Proof. We have the following preliminary calculations. First, by (2.9),

$$\begin{aligned}\Delta(E^i) = (1\otimes E + E\otimes K)^i &= \sum_{s=0}^{i}\binom{i}{s}_{q^2}(1\otimes E)^{i-s}(E\otimes K)^s \\ &= \sum_{s=0}^{i}\binom{i}{s}_{q^2}E^s\otimes E^{i-s}K^s;\end{aligned}$$

$$\begin{aligned}\Delta(F^j) = (F\otimes 1 + K^{-1}\otimes F)^j &= \sum_{t=0}^{j}\binom{j}{t}_{q^2}(F\otimes 1)^{j-t}(K^{-1}\otimes F)^t \\ &= \sum_{t=0}^{j}\binom{j}{t}_{q^2}F^{j-t}K^{-t}\otimes F^t.\end{aligned}$$

Finally, $\Delta(K^\ell) = K^\ell \otimes K^\ell$. Therefore,

$$\begin{aligned}
&\Delta(E^iF^jK^\ell)\\
&\quad= \left(\sum_{s=0}^{i}\binom{i}{s}_{q^2} E^s \otimes E^{i-s}K^s\right)\left(\sum_{t=0}^{j}\binom{j}{t}_{q^2} F^{j-t}K^{-t} \otimes F^t\right)\left(K^\ell \otimes K^\ell\right)\\
&\quad= \sum_{s=0}^{i}\sum_{t=0}^{j}\binom{i}{s}_{q^2}\binom{j}{t}_{q^2} E^sF^{j-t}K^{\ell-t} \otimes E^{i-s}K^sF^tK^\ell\\
&\quad= \sum_{s=0}^{i}\sum_{t=0}^{j} q^{-2st}\binom{i}{s}_{q^2}\binom{j}{t}_{q^2} E^sF^{j-t}K^{\ell-t} \otimes E^{i-s}F^tK^{\ell+s}.
\end{aligned}$$

Since $(p_{a,b,c} \otimes p_{A,B,C})(E^sF^{j-t}K^{\ell-t} \otimes E^{i-s}F^tK^{\ell+s})$ is nonzero only if $a = s,\ b = j-t,\ A = i-s,\ B = t,\ c \equiv \ell - t \mod n$ and $C \equiv \ell + s \mod n$, the only possible nonzero term of $(p_{a,b,c} * p_{A,B,C})(E^iF^jK^\ell)$, when expressed using the above sum, is the term where $s = a$ and $t = B$. The above equations show that this term is only actually nonzero if we also have that $B + b = j,\ A + a = i$, and $c + B \equiv C - a \pmod{n}$. $\square$

The products of certain basis elements will be needed later, so we list them here.

Corollary A.2.2. *For integers $0 \le x \le n-2,\ 0 \le s,t \le n-1$, and arbitrary integers ℓ and m,*

$$p_{x,0,\ell} * p_{1,0,m} = \begin{cases} \binom{x+1}{1}_{q^2} p_{x+1,0,\ell}, & \textit{if } m - x \equiv \ell \textit{ (mod } n\textit{)} \\ 0, & \textit{otherwise} \end{cases}, \tag{A.1}$$

$$p_{0,1,m} * p_{0,x,\ell} = \begin{cases} \binom{x+1}{1}_{q^2} p_{0,x+1,\ell}, & \textit{if } \ell \equiv m + x \textit{ (mod } n\textit{)} \\ 0, & \textit{otherwise} \end{cases}, \tag{A.2}$$

$$p_{0,0,\ell} * p_{0,0,m} = \delta_{\ell,m} p_{0,0,\ell}, \tag{A.3}$$

$$p_{s,0,m} * p_{0,t,\ell} = \begin{cases} q^{-2st} p_{s,t,m+t}, & \textit{if } \ell - s \equiv m + t \textit{ (mod } n\textit{)} \\ 0, & \textit{otherwise} \end{cases}, \tag{A.4}$$

$$p_{s,t,m} * p_{0,0,\ell} = \begin{cases} p_{s,t,m}, & \textit{if } \ell - s \equiv m \textit{ (mod } n\textit{)} \\ 0, & \textit{otherwise} \end{cases} \tag{A.5}$$

Proof. This follows directly from Proposition A.2.1. □

Lemma A.2.3. *In terms of the basis* $\{p_{i,j,\ell}\}_{0\leq i,j,\ell\leq n-1}$ *of* $u_q(\mathfrak{sl}_2)$*,*

$$B = \sum_{\ell=0}^{n-1} q^{-\ell} p_{1,0,\ell}, \quad C = \sum_{\ell=0}^{n-1} q^{\ell} p_{0,1,\ell}, \quad \textit{and} \quad D = \sum_{\ell=0}^{n-1} q^{-\ell} p_{0,0,\ell}.$$

Proof. Using Theorem 6.1.2, we compute

$$\begin{aligned} B(E^iF^jK^\ell) &= \langle b, E^iF^jK^\ell\rangle = \langle a, E^iF^j\rangle\langle b, K^\ell\rangle + \langle b, E^iF^j\rangle\langle d, K^\ell\rangle \\ &= q^{-\ell}\langle b, E^iF^j\rangle = q^{-\ell}\left[\langle a, E^i\rangle\langle b, F^j\rangle + \langle b, E^i\rangle\langle d, F^j\rangle\right] \\ &= q^{-\ell}\left[\delta_{i,1}\delta_{j,0}\right] \end{aligned}$$

$$\begin{aligned} C(E^iF^jK^\ell) &= \langle c, E^iF^jK^\ell\rangle = \langle c, E^iF^j\rangle\langle a, K^\ell\rangle + \langle d, E^iF^j\rangle\langle c, K^\ell\rangle \\ &= q^{\ell}\langle c, E^iF^j\rangle = q^{\ell}\left[\langle c, E^i\rangle\langle a, F^j\rangle + \langle d, E^i\rangle\langle c, F^j\rangle\right] \\ &= q^{\ell}\left[\delta_{i,0}\delta_{j,1}\right] \end{aligned}$$

$$\begin{aligned} D(E^iF^jK^\ell) &= \langle d, E^iF^jK^\ell\rangle = \langle c, E^iF^j\rangle\langle b, K^\ell\rangle + \langle d, E^iF^j\rangle\langle d, K^\ell\rangle \\ &= q^{-\ell}\langle d, E^iF^j\rangle = q^{-\ell}\left[\langle c, E^i\rangle\langle b, F^j\rangle + \langle d, E^i\rangle\langle d, F^j\rangle\right] \\ &= q^{-\ell}\left[\delta_{i,0}\delta_{j,0}\right] \end{aligned}$$

□

Lemma A.2.4. *In terms of the basis* $\{p_{i,j,\ell}\}_{0\leq i,j,\ell\leq n-1}$ *of* $u_q(\mathfrak{sl}_2)$*, for*
$1 \leq s, t, j \leq n-1$.

$$B^s = [s]_q! \sum_{\ell=0}^{n-1} q^{-s\ell} p_{s,0,\ell}, \quad C^t = [t]_q! \sum_{\ell=0}^{n-1} q^{t\ell} p_{0,t,\ell}, \quad \textit{and} \quad D^r = \sum_{\ell=0}^{n-1} q^{-r\ell} p_{0,0,\ell}.$$

Proof. We proceed by induction, appealing to Lemma A.2.3 for the base case. The cases when $s, t,$ or $j = 0$ are trivial.

$$\begin{aligned} B^s = B^{s-1} * B \ &= \ [s-1]_q! \left(\sum_{\ell=0}^{n-1} q^{-(s-1)\ell} p_{s-1,0,\ell}\right)\left(\sum_{m=0}^{n-1} q^{-m} p_{1,0,m}\right) \\ &\overset{(A.1)}{=} [s-1]_q! \binom{s}{1}_{q^2} \sum_{\ell=0}^{n-1} q^{-(s-1)\ell-(\ell+s-1)} p_{s,0,\ell} \\ &\overset{(2.11)}{=} [s-1]_q!\, q^{s-1} \begin{bmatrix} s \\ 1 \end{bmatrix}_q \sum_{\ell=0}^{n-1} q^{-s\ell-(s-1)} p_{s,0,\ell} \ = \ [s]_q! \sum_{\ell=0}^{n-1} q^{-s\ell} p_{s,0,\ell}. \end{aligned}$$

$$C^t = C * C^{t-1} \;=\; [t-1]_q! \left(\sum_{m=0}^{n-1} q^m p_{0,1,m}\right)\left(\sum_{\ell=0}^{n-1} q^{(t-1)\ell} p_{0,t-1,\ell}\right)$$
$$\overset{\text{(A.2)}}{=} [t-1]_q! \binom{t}{1}_{q^2} \sum_{\ell=0}^{n-1} q^{(t-1)\ell+\ell-t+1} p_{0,t,\ell}$$
$$\overset{\text{(2.11)}}{=} [t-1]_q!\, q^{t-1} \begin{bmatrix} t \\ 1 \end{bmatrix}_q \sum_{\ell=0}^{n-1} q^{t\ell-(t-1)} p_{0,t,\ell} \;=\; [t]_q! \sum_{\ell=0}^{n-1} q^{t\ell} p_{0,t,\ell}.$$

$$D^r = D^{r-1} * D \;=\; \left(\sum_{\ell=0}^{n-1} q^{-(r-1)\ell} p_{0,0,\ell}\right)\left(\sum_{m=0}^{n-1} q^{-m} p_{0,0,m}\right) \overset{\text{(A.3)}}{=} \sum_{\ell=0}^{n-1} q^{-r\ell} p_{0,0,\ell}.$$

□

Proposition A.2.5.

$$B^s C^t D^r = [s]_q!\, [t]_q! \sum_{i=0}^{n-1} q^{-i(r+s-t)-rs} p_{s,t,i}$$

Proof. We begin by computing $B^s C^t$. By Lemma A.2.4,

$$B^s C^t = [s]_q!\, [t]_q! \left(\sum_{i=0}^{n-1} q^{-si} p_{s,0,i}\right)\left(\sum_{\ell=0}^{n-1} q^{t\ell} p_{0,t,\ell}\right)$$
$$\overset{\text{(A.4)}}{=} [s]_q!\, [t]_q! \sum_{i=0}^{n-1} q^{-2st-si+t(i+s+t)} p_{s,t,i+t}$$
$$= [s]_q!\, [t]_q! \sum_{i=t}^{n-1+t} q^{-2st-s(i-t)+t(i+s)} p_{s,t,i}$$
$$= [s]_q!\, [t]_q! \sum_{i=0}^{n-1} q^{i(t-s)} p_{s,t,i}.$$

Thus, again using Lemma A.2.4,

$$B^s C^t D^r = [s]_q!\, [t]_q! \left(\sum_{i=0}^{n-1} q^{i(t-s)} p_{s,t,i}\right)\left(\sum_{\ell=0}^{n-1} q^{-r\ell} p_{0,0,\ell}\right)$$
$$= [s]_q!\, [t]_q! \sum_{i=0}^{n-1} q^{i(t-s)-r(i+s)} p_{s,t,i} \qquad \text{(by Equation A.5)}$$
$$= [s]_q!\, [t]_q! \sum_{i=0}^{n-1} q^{-i(r+s-t)-rs} p_{s,t,i}.$$

□

A.2.2 Remainder of the proof of Theorem 6.1.7

We would like to show the relations of $D(u_q(\mathfrak{sl}_2))$ involving elements of both $u_q(\mathfrak{sl}_2)$ and its dual. We use (2.17) and the perfect duality established in Proposition 6.1.6. Note that

$$\Delta^2(a) = a \otimes a \otimes a + a \otimes b \otimes c + b \otimes c \otimes a + b \otimes d \otimes c,$$
$$\Delta^2(b) = a \otimes a \otimes b + a \otimes b \otimes d + b \otimes c \otimes b + b \otimes d \otimes d,$$
$$\Delta^2(c) = c \otimes a \otimes a + c \otimes b \otimes c + d \otimes c \otimes a + d \otimes d \otimes c,$$
$$\Delta^2(d) = c \otimes a \otimes b + c \otimes b \otimes d + d \otimes c \otimes b + d \otimes d \otimes d,$$
$$\Delta^2(E) = 1 \otimes 1 \otimes E + 1 \otimes E \otimes K + E \otimes K \otimes K, \quad \Delta^2(K) = K \otimes K \otimes K,$$
$$\Delta^2(F) = K^{-1} \otimes K^{-1} \otimes F + K^{-1} \otimes F \otimes 1 + F \otimes 1 \otimes 1,$$
$$S^{-1}(E) = -K^{-1}E, \quad S^{-1}(F) = -FK.$$

Thus, we compute:

$$\begin{aligned} Eb = \quad & \langle a, -K^{-1}E\rangle\langle b, 1\rangle a1 + \langle a, K^{-1}\rangle\langle b, 1\rangle aE + \langle a, K^{-1}\rangle\langle b, E\rangle aK \\ & + \langle a, -K^{-1}E\rangle\langle d, 1\rangle b1 + \langle a, K^{-1}\rangle\langle d, 1\rangle bE + \langle a, K^{-1}\rangle\langle d, E\rangle bK \\ & + \langle b, -K^{-1}E\rangle\langle b, 1\rangle c1 + \langle b, K^{-1}\rangle\langle b, 1\rangle cE + \langle b, K^{-1}\rangle\langle b, E\rangle cK \\ & + \langle b, -K^{-1}E\rangle\langle d, 1\rangle d1 + \langle b, K^{-1}\rangle\langle d, 1\rangle dE + \langle b, K^{-1}\rangle\langle d, E\rangle dK \\ = {} & q^{-1}bE + q^{-1}aK - q^{-1}d. \end{aligned}$$

$$\begin{aligned} Ec = \quad & \langle c, -K^{-1}E\rangle\langle a, 1\rangle a1 + \langle c, K^{-1}\rangle\langle a, 1\rangle aE + \langle c, K^{-1}\rangle\langle a, E\rangle aK \\ & + \langle c, -K^{-1}E\rangle\langle c, 1\rangle b1 + \langle c, K^{-1}\rangle\langle c, 1\rangle bE + \langle c, K^{-1}\rangle\langle c, E\rangle bK \\ & + \langle d, -K^{-1}E\rangle\langle a, 1\rangle c1 + \langle d, K^{-1}\rangle\langle a, 1\rangle cE + \langle d, K^{-1}\rangle\langle a, E\rangle cK \\ & + \langle d, -K^{-1}E\rangle\langle c, 1\rangle d1 + \langle d, K^{-1}\rangle\langle c, 1\rangle dE + \langle d, K^{-1}\rangle\langle c, E\rangle dK \\ = {} & qcE. \end{aligned}$$

$$\begin{aligned}
Ed = \quad & \langle c, -K^{-1}E\rangle\langle b,1\rangle a1 + \langle c, K^{-1}\rangle\langle b,1\rangle aE + \langle c, K^{-1}\rangle\langle b,E\rangle aK \\
& + \langle c, -K^{-1}E\rangle\langle d,1\rangle b1 + \langle c, K^{-1}\rangle\langle d,1\rangle bE + \langle c, K^{-1}\rangle\langle d,E\rangle bK \\
& + \langle d, -K^{-1}E\rangle\langle b,1\rangle c1 + \langle d, K^{-1}\rangle\langle b,1\rangle cE + \langle d, K^{-1}\rangle\langle b,E\rangle cK \\
& + \langle d, -K^{-1}E\rangle\langle d,1\rangle d1 + \langle d, K^{-1}\rangle\langle d,1\rangle dE + \langle d, K^{-1}\rangle\langle d,E\rangle dK \\
= {} & qdE + qcK.
\end{aligned}$$

$$\begin{aligned}
Fb = \quad & \langle a, -FK\rangle\langle b,K^{-1}\rangle aK^{-1} + \langle a,1\rangle\langle b,K^{-1}\rangle aF + \langle a,1\rangle\langle b,F\rangle a1 \\
& + \langle a, -FK\rangle\langle d,K^{-1}\rangle bK^{-1} + \langle a,1\rangle\langle d,K^{-1}\rangle bF + \langle a,1\rangle\langle d,F\rangle b1 \\
& + \langle b, -FK\rangle\langle b,K^{-1}\rangle cK^{-1} + \langle b,1\rangle\langle b,K^{-1}\rangle cF + \langle b,1\rangle\langle b,F\rangle c1 \\
& + \langle b, -FK\rangle\langle d,K^{-1}\rangle dK^{-1} + \langle b,1\rangle\langle d,K^{-1}\rangle dF + \langle b,1\rangle\langle d,F\rangle d1 \\
= {} & qbF.
\end{aligned}$$

$$\begin{aligned}
Fc = \quad & \langle c, -FK\rangle\langle a,K^{-1}\rangle aK^{-1} + \langle c,1\rangle\langle a,K^{-1}\rangle aF + \langle c,1\rangle\langle a,F\rangle a1 \\
& + \langle c, -FK\rangle\langle c,K^{-1}\rangle bK^{-1} + \langle c,1\rangle\langle c,K^{-1}\rangle bF + \langle c,1\rangle\langle c,F\rangle b1 \\
& + \langle d, -FK\rangle\langle a,K^{-1}\rangle cK^{-1} + \langle d,1\rangle\langle a,K^{-1}\rangle cF + \langle d,1\rangle\langle a,F\rangle c1 \\
& + \langle d, -FK\rangle\langle c,K^{-1}\rangle dK^{-1} + \langle d,1\rangle\langle c,K^{-1}\rangle dF + \langle d,1\rangle\langle c,F\rangle d1 \\
= {} & q^{-1}cF - aK^{-1} + d.
\end{aligned}$$

$$\begin{aligned}
Fd = \quad & \langle c, -FK\rangle\langle b,K^{-1}\rangle aK^{-1} + \langle c,1\rangle\langle b,K^{-1}\rangle aF + \langle c,1\rangle\langle b,F\rangle a1 \\
& + \langle c, -FK\rangle\langle d,K^{-1}\rangle bK^{-1} + \langle c,1\rangle\langle d,K^{-1}\rangle bF + \langle c,1\rangle\langle d,F\rangle b1 \\
& + \langle d, -FK\rangle\langle b,K^{-1}\rangle cK^{-1} + \langle d,1\rangle\langle b,K^{-1}\rangle cF + \langle d,1\rangle\langle b,F\rangle c1 \\
& + \langle d, -FK\rangle\langle d,K^{-1}\rangle dK^{-1} + \langle d,1\rangle\langle d,K^{-1}\rangle dF + \langle d,1\rangle\langle d,F\rangle d1 \\
= {} & qdF - q^2bK^{-1}.
\end{aligned}$$

$$\begin{aligned}
Ka = {} & \langle a, K^{-1}\rangle\langle a,K\rangle aK + \langle a, K^{-1}\rangle\langle c,K\rangle bK \\
& + \langle b, K^{-1}\rangle\langle a,K\rangle cK + \langle b, K^{-1}\rangle\langle c,K\rangle dK = aK.
\end{aligned}$$

$$\begin{aligned}
Kb = {} & \langle a, K^{-1}\rangle\langle b,K\rangle aK + \langle a, K^{-1}\rangle\langle d,K\rangle bK \\
& + \langle b, K^{-1}\rangle\langle b,K\rangle cK + \langle b, K^{-1}\rangle\langle d,K\rangle dK = q^{-2}bK.
\end{aligned}$$

$$
\begin{aligned}
Kc = {} & \langle c, K^{-1}\rangle\langle a, K\rangle aK + \langle c, K^{-1}\rangle\langle c, K\rangle bK \\
& + \langle d, K^{-1}\rangle\langle a, K\rangle cK + \langle d, K^{-1}\rangle\langle c, K\rangle dK = q^2 cK.
\end{aligned}
$$

$$
\begin{aligned}
Kd = {} & \langle c, K^{-1}\rangle\langle b, K\rangle aK + \langle c, K^{-1}\rangle\langle d, K\rangle bK \\
& + \langle d, K^{-1}\rangle\langle b, K\rangle cK + \langle d, K^{-1}\rangle\langle d, K\rangle dK = dK.
\end{aligned}
$$

INDEX OF NOTATION AND TERMINOLOGY

$A \underline{\otimes} B$, 26
$B\#H$, 27
$[n]_q$, 21
$\widetilde{\binom{i}{j}}_q$, 56
$\binom{n}{m}_q$, 21
$(n)_q!$, 21
$(n)_q$, 21
$a \succ p$, 29
$p \bowtie a$, 29
$p \prec a$, 29
$\begin{bmatrix} n \\ k \end{bmatrix}_q$, 21
$[n]_q!$, 21

$\mathbb{1}$, 23

$A(H)$, 7
A°, 16
$A\#H$, 24
A_i, 34
algebra, 23
    action of H on an, 24
        inner-faithful, 32
    associative, 13
    generalized Taft, 41, 54
    group, 3
    H-module, 24
    Hopf, 18, 26
    smash product, 24
    Sweedler, 20, 36
    Taft, 6, 19, 35
    universal enveloping, 3
antipode, 18

$\mathfrak{B}(V)$, 31
B°, 18
B^{cop}, 18
bi-ideal, 18
bialgebra, 18, 26
    coradically graded, 20
        lifting of a, 20
    dual, 18
    filtered, 20
    graded, 20
    pointed, 20
biproduct, 27
B^{op}, 18
$B^{op\ cop}$, 18
bosonization, 27

braided vector space, 31
  Cartan type, 32
  diagonal type, 31

C_0, 15
$c_{A,B}$, 26
$C\natural H$, 24
$\mathrm{C}_n(\Bbbk)$, 16
coalgebra, 23
  coaction of H on a, 24
  coassociative, 13
  cocommutative, 16
  comatrix, 16
  coopposite, 16
  coradical of a, 15
  coradically graded, 16
  filtered, 15
  graded, 16
  H-comodule, 24
  pointed, 15
  simple, 15
  smash coproduct, 24
  tensor product, 16
coideal, 15
comodule, 17
comultiplication, 14
convolution, 18
coproduct, 14
counit, 14

Δ, 13
$D(H)$, 29
Drinfel'd double, 29
duality, 22
  perfect, 22

ϵ, 14

filtration
  bialgebra, 20
  coalgebra, 15
  coradical, 15
  Hopf algebra, 20

$G(C)$, 15
generalized Taft algebra, 41, 54
$\mathrm{gr}(B)$, 20
$\mathrm{gr}(C)$, 16
grading
  bialgebra, 20
  coalgebra, 16
  Hopf algebra, 20
grouplike, 4, 15

H-comodule coalgebra, 24
H-module algebra, 24
  inner-faithful, 32
H°, 19
H^{cop}, 19
$H_n(\zeta, m, t)$, 39
$\widetilde{H}_n(\zeta, m, t)$, 47
homomorphism
  bialgebra, 18
  coalgebra, 15
H^{op}, 19

$H^{op\ cop}$, 19
Hopf algebra, 18, 26
  braided, 26
  coradically graded, 20
    lifting of a, 20, 28
  filtered, 20
  generalized Taft, 41, 54
  graded, 20
  pointed, 20
    diagram of a, 28
    rank of a, 28
  quasitriangular, 28
  Sweedler, 20, 36
  Taft, 6, 19, 35

$\Bbbk$-algebra, 13
$\Bbbk$-bialgebra, 18
$\Bbbk$-coalgebra, 13
$\Bbbk$-**Vect**, 23
$\Bbbk G$, 3, 19
$K_\zeta(n, N)$, 56

${}_A\mathcal{M}$, 17
$\mathcal{M}_A$, 17
${}^C\mathcal{M}$, 17
$\mathcal{M}^C$, 17
${}_H\mathcal{M}$, 23
  algebra in, 24
${}^H\mathcal{M}$, 24
  coalgebra in, 24
$\mathrm{M}_n(\Bbbk)$, 16
module, 16
  H-Hopf, 25
  inner-faithful H, 32
  Yetter-Drinfel'd, 25
monoidal category, 23
  algebra in a, 23
  braided, 26
    bialgebra in a, 26
    Hopf algebra in a, 26
  coalgebra in a, 23
  comonoid in a, 23
  monoid in a, 23
  rigid, 23

Nichols algebra, 31
  Cartan type, 32
  rank of a, 31

$\mathcal{O}_q(SL_2)$, 68
$\overline{\mathcal{O}_q(SL_2)}$, 68

$P_{g,h}(C)$, 15
product
  convolution, 18

quantum binomial coefficients, 21
quantum linear space, 32
quantum Yang-Baxter equation, 5

ρ, 17
$\mathcal{R}((g_1, \ldots, g_\theta; \chi_1, \ldots, \chi_\theta)$, 39

S, 18
S°, 19
Set, 23

skew primitive, 15
smash coproduct, 24
smash product, 24
subcoalgebra, 15
Sweedler Hopf algebra, 20
Sweedler notation, 14

Taft algebra, 6, 19, 35
$T(n, N, \alpha)$, 41, 54
$T_n(q)$, 6, 19, 35

$U(\mathfrak{g})$, 3, 19
$U_q(\mathfrak{sl}_2)$, 67
$u_q(\mathfrak{sl}_2)$, 67

${}^{\Gamma}_{\Gamma}\mathcal{YD}$, 25
${}^{H}_{H}\mathcal{YD}$, 25
    Hopf algebra in, 26
${}_{H}\mathcal{YD}^{H}$, 28
Yetter-Drinfel'd module, 25