Frobenius Reciprocity

and

Grothendieck Groups of Hopf Galois Extensions

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ABSTRACT

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Let $A \subset B$ be an H-Galois extension where H is a finite dimensional Hopf algebra over a commutative field K. We study the Grothendieck groups $G_0(B)$ and $K_0(B)$ of finitely generated and finitely generated projective, respectively, modules over B. Via tensor products, both $G_0(B)$ and $K_0(B)$ are shown to become modules over the Grothendieck ring $G_0(H)$ of H. This allows us to prove: If H is involutory and not semisimple and A is commutative with no non-trivial idempotents and $1 \notin [B, B]$, then for every finitely generated projective B-module P, rank (P_A) is divisible by char K. A similar result is proved for $G_0(B)$ in the situation where $B = A\#H$ is a smash product. Namely: If $p =$ char K divides $[H] \in G_0(H)$, $1 \notin [B, B]$ and all finitely generated projective modules of A are stably free, then the image of $K_0(B)$ under the Cartan map in is contained in $p \cdot \mathbb{Z}[A_B] + ann_{G_0(B)}([H])$. As a consequence, we deduce that B cannot be Morita equivalent to a noetherian domain. The result for $G_0(B)$ depends on a version of Frobenius reciprocity for modules over smash products which we establish here.

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To Dan, for everything

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CHAPTER 1

INTRODUCTION

Hopf algebras and quantum groups. A Hopf algebra H is an enriched algebra over a commutative field K. Two important and "natural" examples of Hopf algebras are *group algebras* and *enveloping algebras of Lie algebras.* A feature of a Hopf algebra is self duality. Besides the usual algebra map of multiplication $m : H \otimes H \to H$ there is the map of *comultiplication* $\Delta : H \to H \otimes H$ and along with the algebra unit map $u : K \to H$ there is a *counit* map $\varepsilon : H \to K$. The current definition of Hopf algebra also includes an endomorphism $S : H \to H$ called the *antipode.* For cases where the Hopf algebra is a group algebra, the antipode of the group element g is its group inverse g^{-1}. For Hopf algebras in general, the antipode is analogous to the group inverse and is often thought of as a substitute for the inverse map in groups. The various structure maps are required to satisfy a system of compatibility axioms, for whose precise statements we refer to the standard texts on the subject, [Abe], [Mont] and [Sw]. As a consequence of these axioms, when H is finite dimensional, the duals of the structure maps of H make the linear dual $H^* = \mathrm{Hom}_K(H, K)$ of H into an Hopf algebra as well. It is the finite dimensional case which is of greatest interest here.

What is now called *Hopf algebras* was derived from the work of Hopf [H] published in 1941, on the *homology of manifolds.* A manifold M admitting a product $M \times M \rightarrow M$ induces a multiplication map on homology $H_*(M) \otimes H_*(M) \rightarrow H_*(M)$ while the diagonal map $M \rightarrow M \times M$ yields a comultiplication: $H_*(M) \rightarrow H_*(M) \otimes H_*(M)$. Hopf proved in [H] that the homology of classical groups is the same as the homology of a product of odd dimensional spheres. The structure theorem of Hopf concerning such algebras was generalized by Borel [B], and others. Sometime later, Hopf algebras arose naturally in the theory of *group schemes*: Affine Hopf algebras with commutative multiplication are precisely the algebras representing affine group schemes [CPS]. Hopf algebras were used by Hochschild and Mostow [HM] in *representation theory* as representation rings of Lie groups and in studies by Larson [L67]. By the end of the 1960's, Hopf algebras were being studied in their own right as abstract algebraic systems in the seminal works by [MM] and [Sw].

Interest in applications of Hopf algebras was renewed in the mid 1980's when Drinfeld [Dr] and Jimbo [J] independently introduced the notion of *quantum groups* . These are now usually defined as Hopf algebras for which neither the multiplication nor the comultiplication is commutative . The term "quantum" refers to the potential non-commutativity of the underlying structure maps. Quantum groups arose as symmetries of quantum statistical systems, in particular, the Quantum Yang-Baxter Equation (QYBE) in statistical mechanics.

Invariant theory and Galois extensions. The topic of invariant theory, in essence, is the study of the relationship between a ring S and its subring of invariants $R = S^G$ under the action of a group G by *automorphisms*. Alternatively, R is often taken to be the subring of constants $R = S^{\mathfrak{g}}$ for the action of a Lie algebra $\mathfrak{g}$ by *derivations* on S. This is a classical algebraic theme which permeates virtually all areas of pure mathematics. Invariant theory is also found in some areas of applied mathematics, notably coding theory (see [Sl] and the references given there), and certain parts of physics as well (e.g. [Ma]). Both types of actions, group actions by automorphisms and actions of Lie algebras by derivations, can be simultaneously treated under the common roof of *Hopf algebra actions* on rings. The monograph [Mont] gives a concise introduction to Hopf algebras and their invariant theory.

An early example of invariant theory is classical *Galois theory*. In this case, the ring S is assumed to be a commutative field and G is a finite group of automorphisms on S. The main theorem of Galois theory, in its present formulation, sets up a one-to-one correspondence between the subgroups of G and the fields lying between S and the fixed field $R = S^G$. This correspondence behaves well in various ways; in particular, it matches normal subgroups of G with intermediate fields that are normal over R.

In developing Hopf Galois theory, Chase and Sweedler [CS] replaced the finite group G of automorphisms by a finite dimensional Hopf algebra H acting not necessarily by automorphisms. They did this in hopes of shedding light on inseparable field extension and ramified extensions of rings. Chase and Sweedler began with what we now call a commutative right H-comodule algebra B. As we see in Chapter 2, B is then a left H^*-module. The set of H^*-invariants for B is defined by $B^{H^*} = \{a \in B | f \cdot a = \varepsilon(f)a, \forall f \in H^*\}$. For H finite dimensional the H^*-invariants coincide with the H-coinvariants. A right extension $B \supseteq A = B^{H^*}$ is said to be a Hopf or H-Galois extension if a certain Morita context that is associated with the H^*-action on B is "well behaved"; details will be presented in Chapter 2. With these definitions Chase and Sweedler constructed a theory which stated that the correspondence between subHopf algebras of H and the subfields of B containing A was injective rather than bijective (as in the classical case) [Ch]. They were also unable to state any normality relationships, though Ligon [Li] later partially filled in this gap.

Frobenius reciprocity and Grothendieck groups. The technique of induced representations is one of the most important in representation theory. Frobenius defined for any subgroup L of a finite group G and any class function $\lambda : L \to \mathbb{C}$, the induced class function $\lambda^G : G \to \mathbb{C}$ and established that the inner product formula

$$(\lambda^G, \mu) = (\lambda, \mu|_L)$$

for any class function $\mu : G \to \mathbb{C}$. This is the original form of Frobenius reciprocity. If W is an KL-module and V is a KG-module, then an abstract version, equivalent to the original formula in case $K = \mathbb{C}$, is the linear isomorphism

$$\mathrm{Hom}_{KG}(W \otimes_{KL} KG, V) \cong \mathrm{Hom}_{KL}(W, V|_{KL}).$$

For $K = \mathbb{C}$ both express the fact that dimensions of the Hom spaces are identical. Mackey in [M] proved the strongest version, sometimes also called the *Tensor Product Theorem*:

$$V \otimes_K (W \otimes_{KL} KG) \cong (V|_L \otimes_K W) \otimes_{KL} KG.$$

This is an isomorphism of KG-modules; the previous version of Frobenius reciprocity follows by taking G-fixed points. Frobenius reciprocity in the last form cannot readily be extended to algebras in general and has usually been used only in the context of group algebras. A version for skew group rings was shown by Lorenz [Lo86]. Frobenius Reciprocity is related to the so-called *Fundamental Theorem of Hopf Modules* which perhaps explains our success in extending Frobenius Reciprocity to some H-Galois extensions.

Grothendieck Groups were introduced by Grothendieck in 1955; they had far reaching affects in many branches of algebraic thought. In particular, Grothendieck groups provided an ideal framework for the development of group representation theory and are now equally useful in the representation theory of quantum groups. We find that the Grothendieck groups of A, B and H are intimately connected. In Sections 4.7 and 4.8 we find that, under certain conditions, knowing what the isomorphism classes of modules over A and H look like, we can make predictions concerning some isomorphism classes of modules over B.

Main Results. In this thesis, we consider general finite H-Galois extensions, that is, H will be assumed to be finite dimensional. In this setting, we prove in Section 3.3 that for any (right) B-module V, restricting V to A then inducing back up to B has the same effect as tensoring the original module V by H. This fact can be thought of as a rudimentary (but useful) form of Frobenius reciprocity for Hopf Galois extensions. Further, we show if W is a right H-module then $V \otimes W$ is a B-module with $V \otimes W$ projective when V is projective. Moreover, if W is finitely generated, then $V \otimes W$ is finitely generated (resp. pseudo-coherent, resp. coherent) when V is finitely generated (resp. pseudo-coherent, resp. coherent). This allows us to provide an arithmetic restriction on the possible ranks of projective A-modules resulting from restricting a projective B-module:

Theorem *Let $A \subset B$ be a right H-Galois extension and assume that*

(a) A is commutative without idempotents $\neq 0, 1$.

(b) H is involutory, not semisimple. (So, in particular, $p = \operatorname{char} K$ is positive, in fact, a divisor of $\dim_K H$.)

(c) $1 \notin [B, B]$.

Then $p = \operatorname{char} K$ divides $\operatorname{rank}(P_A)$ for every finitely generated projective B-module P.

The most specialized H-Galois extension is the smash product $B = A\#H$ that is associated with an action of H on A. Smash products include but are not limited to *skew group rings* (when H is a group algebra), *differential polynomial rings* (when H is the enveloping algebra of a Lie algebra), and the smash product $H\#H^*$ which is known as the *Heisenberg double* in the quantum group literature. For smash products we prove a generalized version of Mackey's Frobenius reciprocity theorem. We also use the above results to describe the image of the Cartan map from $K_0(B)$ to $G_0(B)$ under certain conditions:

Theorem *Assume that $B = A\#H$ a smash product and assume that*

(a) $p = char K$ *divides* $[H] \in G_0(H)$

(b) $K_0(A) = \mathbb{Z}[A]$

(c) $1 \notin [B, B]$.

Then $c(K_0(B)) \subseteq p \cdot \mathbb{Z}[A_B] + ann_{G_0(B)}([H])$.

The latter result easily yields examples of simple noetherian rings that are not Morita equivalent to a domain ("Faith's Conjecture").

We assume that the reader is not an expert in H-Galois extensions and so in Chapter 2, we lay out its foundations and give some examples of the various types of H-Galois extensions. Further, in Chapter 4, we present the basic underlying ideas from the theory of Grothendieck groups.

CHAPTER 2

PRELIMINARIES

2.1 Overview

We present here some of the basics of Hopf algebras, H-invariants and H^*-coinvariants. We familiarize the reader with H-Galois extensions by definition and examples. A brief description of the Morita context is given as this gives rise to many known properties of H-Galois extensions. It is also shown that if the set A of H-coinvariants in B is commutative then the categories of modules of A and of $B\#H^*$ are equivalent.

2.2 Notations and Conventions

For general background on Hopf algebras the standard texts are [Abe] and [Sw]. For more recent developments in the field and especially for the material on smash and crossed products and on H-Galois extensions we use [Mont].

Throughout this thesis we will keep the following notations:

K	denotes a commutative field,
$\otimes$	represents $\otimes_K$,
H	will be a Hopf algebra over K with counit ε, antipode S, and comultiplication $\triangle : H \to H \otimes H$, $\triangle(h) = \sum h_1 \otimes h_2$.

Recall that if H is finite dimensional (over K) or pointed, then S is bijective. The inverse of the antipode, if defined, will be denoted S^{-1}.

B	is a right H-comodule algebra with structure map $\rho = \rho_B : B \to B \otimes H$, $b \mapsto \sum b_0 \otimes b_1$ (cf. [Mont, 4.1.2]).
$A = B^{coH}$	is the set of H-coinvariants of B, i.e., $A = \{a \in B \mid \rho(a) = a \otimes 1\}$.

In this situation, one also says that $A \subseteq B$ is a *right H-extension.* When H is finite dimensional, right H-comodule algebras are identical with left H^*-module algebras, that is, K-algebras that are acted on by the dual Hopf algebra H^*. The action is given by $f \cdot b = \sum b_0 \langle f, b_1 \rangle$ for $f \in H^*, b \in B$; see [Mont, p. 41]. Here, and in the following,

$\langle f, h \rangle$	means $f(h)$, for $h \in H$, $f \in H^*$,

A left H^*-module becomes a right H-comodule in the following manner. Let $\{f_1, ..., f_n\}$ be a basis for H^* then its dual basis $\{x_1, ..., x_n\}$ is a basis for H. Define the right H-comodule structure on B as $\rho_B(b) = \Sigma_{i=1}^n \langle f_i, b \rangle \otimes x_i$. Under the identification of right H-comodule algebras B with left H^*-module algebras, the H-coinvariants $A = B^{coH}$ of B become the H^*-invariants B^{H^*} in B. Here, for any left H-module M, the H-invariants in M are defined by

$$M^H = \{m \in M \mid h \cdot m = \varepsilon(h)m \text{ for all } h \in H\}.$$

Similarly for right modules. For B^{H^*} in particular, one uses the counit ε_{H^*} of H^* that is given by $\varepsilon_{H^*}(f) = \langle f, 1\rangle$. Finally, for a ring R,

Mod_R	will denote the category of right R-modules,
mod_R	is the category of finitely generated right R-modules, and
proj_R	is the category of finitely generated projective right R-modules.

Similarly, ${}_R\mathsf{Mod}$ denote the category of left R-modules, etc.

Further notations will be introduced as we go along.

2.3 Hopf Galois Extensions

For completeness we give definitions and some well known facts about H-Galois extensions here.

Definition. With A and B as above, the extension $A \subset B$ is called *right H-Galois* if the map $\beta : B \otimes_A B \to B \otimes H$, $b' \otimes_A b \mapsto (b' \otimes 1)\rho(b) = \sum b' b_0 \otimes b_1$ is bijective.

As usual, we shall denote the inverse of β as β^{-1}. If H is finite dimensional or, more generally, whenever S is bijective, then the map $\beta' : B \otimes_A B \to B \otimes H$, $b \otimes b' \mapsto \rho(b)(b' \otimes 1) = \sum b_0 b' \otimes b_1$ is also bijective; cf. [Mont, p. 124].

A right H-Galois extension $A \subset B$ will be called *finite* if B is finitely generated as left and right module over A. In fact, it suffices to assume that ${}_AB$ is finitely generated:

Lemma 1. *The following are equivalent for a right H-Galois extension $A \subset B$:*

(i) ${}_A B$ *is finitely generated.*

(ii) H *is finite dimensional.*

(iii) ${}_A B$ *and* B_A *are both finitely generated and projective.*

Proof. The last assertion obviously implies the first. Now assume ${}_A B$ finitely generated. Then ${}_B(B \otimes_A B)$ is finitely generated as well, and hence via β so is ${}_B(B \otimes H) \cong {}_B B^{\dim_K H}$. The latter condition forces H to be finite dimensional.

The proof that finite dimensionality of H implies that ${}_A B$ and B_A are both finitely generated and projective will be given in (2.3.2) below; see the remark following Theorem 1. □

When S is bijective, one can use the isomorphism β' in place of β to show that finite generation of B_A also implies that H is finite dimensional.

2.3.1 Standard Examples

We describe some basic examples here. The first one is included so as to justify the terminology, while the second will play a fundamental role later on in this thesis. Further examples will be discussed in (2.4).

Classical Galois Field Extensions

Let G be a finite group of automorphisms of a field $E \supseteq K$, and let $F = E^G$ denote the fixed subfield of this action. Then E is a (left) module algebra for the group algebra KG, and hence a (right) $(KG)^*$-comodule algebra. The field extension $F \subseteq E$, Galois in the sense of field theory, is in fact also Galois in the sense of the above definition. For a detailed verification, we refer to [Mont, 8.1.2].

Crossed products and smash products

Assume that H *measures* the K-algebra A, that is, there is a K-linear map $H \otimes A \to A$, denoted $h \otimes a \mapsto h \cdot a$, satisfying $h \cdot 1 = \varepsilon(h)1$ and $h \cdot (aa') = \sum (h_1 \cdot a)(h_2 \cdot a')$ for all $h \in H$, $a, a' \in A$. Suppose further that there is a map $\sigma \in \mathrm{Hom}_K(H \otimes H, A)$ that is convolution invertible; cf. [Mont, 1.4]. Then the *crossed product* $A\#_\sigma H$ is the K-vector space $A \otimes H$ endowed with the following multiplication:

$$(a\#h)(a'\#h') = \sum a(h_1 \cdot a')\sigma(h_2, h'_1)\#h_3h'_2$$

for $a, a' \in A$ and $h, h' \in H$. Here, as is customary, we have written $a\#h$ for $a \otimes h$. This multiplication makes $A\#_\sigma H$ an associative K-algebra with identity element $1 = 1_A\#1_H$ precisely if A is a twisted H-module via the given action and σ is a cocycle. Explicitly, for $h, k, l \in H$ and $a \in A$,

$$h \cdot (k \cdot a) = \sum \sigma(h_1, k_1)(h_2k_2 \cdot a)\sigma^{-1}(h_3, k_3) \ , \tag{2.1}$$

and

$$\sum [h_1 \cdot \sigma(k_1, l_1)]\sigma(h_2, k_2l_2) = \sum \sigma(h_1, k_1)\sigma(h_2k_2, l) \ . \tag{2.2}$$

In this case, $A = A\#1$ is a subalgebra of $B = A\#_\sigma H$ (while H need not be a subalgebra).

In the special case where the cocycle σ is *trivial*, that is,

$$\sigma(h,k) = \varepsilon(h)\varepsilon(k)1$$

holds for all $h, k \in H$, equation (2.1) just says that A is a left H-module, and hence a *left H-module algebra*, and (2.2) becomes trivial. In this case, $B = A\#_\sigma H$ is simply denoted $B = A\#H$ and is called a *smash product*; see also Section 3.5

Crossed products $B = A\#_\sigma H$ are right H-comodule algebras via $\rho = \mathrm{Id}_A \#_\sigma \triangle : B = A\#_\sigma H \to A\#_\sigma H \otimes H = B \otimes H$, $a\#h \mapsto \sum a\#h_1 \otimes h_2$. Right H-extensions of the form $A \subseteq B = A\#_\sigma H$ are called *cleft*. See [Mont, 7.1, 7.2] for all this, in particular [Mont, Theorem 7.2.2]. All cleft extensions $A \subseteq B = A\#_\sigma H$ are Galois; in fact, cleft extensions are precisely those right H-Galois extensions that enjoy the so-called (right) normal basis property; cf. [Mont, Corollary 8.2.5].

2.3.2 The Finite Dimensional Case

Throughout this section, H is assumed finite dimensional.

Morita contexts

Morita contexts provide a means for the transfer of structure between the module categories of two rings, say R and S. Specifically, a *Morita context* for R and S is given by bimodules ${}_R V_S$ and ${}_S W_R$ and bimodule maps

$$[\ ,\ ] : V \otimes_S W \to R \quad \text{and} \quad (\ ,\ ) : W \otimes_R V \to S$$

satisfying the associativity conditions

$$[v,w]v' = v(w,v') \quad \text{and} \quad (w,v)w' = w[v,w']$$

for $v, v' \in V$, $w, w' \in W$. If both maps are bijective, the rings R and S are called *Morita equivalent*; tensoring with the bimodules V and W yields and equivalence of the module categories of R and S in this case. Good references for this material are [Ba68, Chapter II] and [McCR].

We note in particular the following lemma which is a reformulation of [Ba68, Theorem II.3.4].

Lemma 2. *In the above setting, assume that the map $[\ ,\ ] : V \otimes_S W \to R$ is surjective. Then:*

(i) $[\ ,\ ]$ *is an isomorphism.*

(ii) V *and* W *are generators as* R*-modules.*

(iii) V *and* W *are finitely generated and projective as* S*-modules.*

(iv) *The map* $(\ ,\ )$ *induces bimodule isomorphisms* $V \cong \mathrm{Hom}_S(W,S)$ *and* $W \cong \mathrm{Hom}_S(V,S)$.

(v) *The ring homomorphisms* $\mathrm{End}(V_S) \longleftarrow R \longrightarrow \mathrm{End}({}_SW)^{op}$, *induced by the bimodule structures, are isomorphisms.*

The Morita context associated with an H-action

Returning to Hopf algebras, let A be a left H-module algebra, with H-action $H \times A \to A$ written as $(h,a) \mapsto h{\cdot}a$. Then there is a Morita context between the smash product $B = A\#H$ and the algebra of H-invariants A^H. The requisite bimodules are both afforded by A which carries two bimodule structures, ${}_{A^H}A_B$ and ${}_BA_{A^H}$. Here, in both cases, A^H acts simply by multiplication in A. The left B-action on A is given by

$$(a\#h) \cdot a' = a(h \cdot a')$$

for $a, a' \in A$, $h \in H$, while the definition of the right B-action on A requires slightly more care; cf. Section 4.8. To define the bimodule maps, fix a nonzero left integral t of H (i.e. $ht = \varepsilon(h)t$) holds for all $h \in H$) and define

$$\begin{aligned} [\ ,\ ] &: A \otimes_{A^H} A \to B = A\#H, & [a, a'] &= ata', \\ (\ ,\ ) &: A \otimes_B A \to A^H, & (a, a') &= t \cdot (aa')\ . \end{aligned}$$

The map $A \to A^H$, $a \mapsto t \cdot a$, is called the (left) *trace map* for H on A. Since t is unique up to a scalar multiple, the choice of t is inessential.

Finite Galois Extensions

We shall give several equivalent reformulations of the notion of right H-Galois extension for finite dimensional H. Recall that right H-comodule algebras B are identical with left H^*-module algebras, and $A = B^{coH} = B^{H^*}$. In particular, we can form the smash product $B\#H^*$. The operative fact in the following well known theorem (see [Mont, Theorem 8.3.3]) is (iv) which states that Galois extensions can be characterized by the surjectivity of the map $[\ ,\ ]$ in the above Morita context between A and $B\#H^*$. Much of the result is therefore a consequence of Lemma 2.

Theorem 1. *Let H be a finite dimensional Hopf algebra and B a right H-comodule algebra. Then the following are equivalent:*

(i) $A = B^{coH} \subset B$ *is a right H-Galois extension.*

(ii) (a) *The map* $\pi : B\#H^* \to \mathrm{End}(B_A)$, $a\#h^* \mapsto (b \mapsto a(h^* \cdot b))$, *is an algebra isomorphism, and*

(b) *B is a finitely generated projective right A-module*

(iii) *B is a generator for the category* ${}_{B\#H^*}\mathsf{Mod}$ *of left $B\#H^*$-modules.*

(iv) *If $0 \neq t$ is a left integral for H then the map* $[\ ,\ ] : B \otimes_A B \to B\#H^*$ *given by* $[b, b'] = btb'$ *is surjective.*

(v) *For any* $M \in {}_{B\#H^*}\mathsf{Mod}$, *consider* $B \otimes_A M^{H^*}$ *as a left $B\#H^*$-module by letting $B\#H^*$ act on B via π. Then the map* $\phi : B \otimes_A M^{H^*} \to M$, *given by* $b \otimes m_0 \mapsto b \cdot m_0$, *is a left $B\#H^*$-module isomorphism.*

Finally, we note that the left-handed version of (ii)(b) above is also true. Namely, for right H-Galois extensions $A \subset B$, Lemma 2 implies that B is also finitely generated and projective as left A-module. This completes the proof of Lemma 1.

The commutative case

The following Corollary, in the special case of group actions, is due to Auslander and Goldman [AG2, Proposition A.3]. The following is presented by Kreimer and Takeuchi though in different form. [KT, Prop. 1.9 and Cor. 1.10] [DT].

Corollary 1. *Let $A \subset B$ be a right H-Galois extension, with H finite dimensional. Assume that A is commutative. Then the trace map $B \to A = B^{H^*}$ is surjective. In particular, A and $B\#H^*$ are Morita equivalent.*

Proof. Since the trace map is left and right A-linear, its image is an ideal of A which we shall denote by $\mathfrak{t}$. So $\mathfrak{t} = (B, B)$, in the notation of the Morita context. Since $[B, B] = B\#H^*$, by part (4) of the Theorem, the associativity condition for Morita contexts yields

$$B = [B, B]B = B(B, B) = B\mathfrak{t} \, .$$

Inasmuch as B is finitely generated as right A-module, by part (2b) of the Theorem, the "Cayley-Hamilton Theorem" [E, Corollary 4.7] implies that there is an element $a \in \mathfrak{t}$ such that $B(1 - a) = 0$. Thus, $a = 1 \in \mathfrak{t}$, and so $\mathfrak{t} = A$. This proves surjectivity of the trace map. Hence, both maps in the Morita context are surjective and therefore actually bijective, cf. [Ba68, Chapter II]. Therefore, the Morita context yields an equivalence. □

2.4 Further Examples

2.4.1 B_A Faithfully Flat, Not H-Cleft

Definition. The A-module B is said to be *faithfully flat* if $B \otimes_A X = 0$ implies $X = 0$, for any left A-module X.

[Mont, p.128] Let $B = M_3(K)$. Let $a, b, c, d, e, u, v, w, x \in K$. Let $A = B_1$

be the set of matrices with the configuration

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{pmatrix}$$

and let B_g be the set of matrices with the configuration

$$\begin{pmatrix} 0 & 0 & u \\ 0 & 0 & v \\ w & x & 0 \end{pmatrix}$$

Then B is a $\mathbf{Z_2}$-graded algebra and A is clearly a direct summand of B_A. Hence, B is a right H-comodule algebra for $H = K\mathbb{Z}_2$, and B_A is faithfully flat as a right A-module. However, note that B is **not** a crossed product since $\dim B = 9 \neq 5 \cdot 2 = \dim A_1 \cdot \dim H$.

2.4.2 Basic H-Galois Extension

The following example summerizes an example of Kreimer [K, Ex. 1.9]. Let B be the ring of 3×3 matrices over a field F of characteristic 2. Let $e_{i,j}$ denote the element of B with 1 in the (i, j)-position and $0's$ elsewhere, $(1 \leq i, j \leq 3)$. Let σ be the inner automorphism of B determined by $e_{1,2} + e_{2,1} + e_{3,3}$. Then for $b \in B, \sigma$ switches the first and second rows, then the first and second columns of b. We note that σ generates a subgroup G of order 2 in the group of all automorphisms of B. Let A be a subgroup of G-invariant elements of B, then $A \subset B$ is an H-Galois extension, but since the characteristic F is 2,

$(1_B(a_{i,j})+\sigma(a_{i,j}))$ has a 0 in the $a_{3,3}$ position. Therefore there is no element c of B such that $(1_B+\sigma)(c) = Id_{3\times 3}$ and so B is a finitely generated projective right A-module which is not faithfully flat as a right or left A-module; see [KT, Prop. 1.9].

2.4.3 Separable Field Extension, Not Classically Galois

Whenever an extension field E over K is Galois in the classic sense, we have $[E:K] = |G|$ where G is the Galois group. Then $K \subseteq E$ is also H-Galois where H is the group ring $(KG)^*$; cf Section 2.3.1. If E is a separable field extension then the degree $[E:K]$ in part determines whether or not E is an H-Galois extension, The following remarks are from [GPa], the example follows [Mont 8.1.5]. If E is a separable extension and $[E:K]=2$, then E is always classically Galois over K. If $[E:K]=3$ or 4 then $E|K$ is always H-Galois, but if $[E:K]=5$ then there are separable extensions which are not H-Galois. The extensions of degree 3 or 4 are called "almost classically Galois"; they have the property that the subHopf algebras of H are in bijective correspondence with the intermediate fields of E over K.

For any k, let H_k denote the Hopf algebras with algebra structure given by $H_k = k[c,s]/(c^2+s^2-1, cs)$ and with coalgebra structure given by $\Delta c = c\otimes c - s\otimes s$, $\Delta s = c\otimes s + s\otimes c$, $\varepsilon(c)=1$, $\varepsilon(s)=0$, $S(c)=c$, and $S(s)=-s$. H_k is called the *circle Hopf algebra.* Now let $F=\mathbf{Q}$ and $E=F(\omega)$ where ω is the real fourth root of 2; $F\subset E$ is not Galois for any group G. However, it is $(H_k)^*$-Galois for $k=\mathbf{Q}$. In this case H_k acts on E as follows:

$c \cdot 1 = 1$, $c \cdot \omega = 0$, $c \cdot \omega^2 = -\omega^2$, $c \cdot \omega^3 = 0$; $s \cdot 1 = 0$, $s \cdot \omega = -\omega$, $s \cdot \omega^2 = 0$, $s \cdot \omega^3 = \omega^3$. As shown in [GPa], $\mathbf{Q} \subset E$ is $(H_k)^*$-Galois. Further, when $k = \mathbf{Q}(i)$, $H_k \cong k\mathbf{Z_4}$ the group algebra. In fact $\mathbf{Q} \subset E$ is also H^*-Galois for a second Hopf algebra H; this second Hopf algebra is a $\mathbf{Q}(\sqrt[2]{-2})$-form of $\mathbf{Q}[\mathbf{Z_2} \times \mathbf{Z_2}]$. Thus an extension can be Hopf Galois with two different Hopf algebras.

CHAPTER 3

MODULES AND FROBENIUS RECIPROCITY

Throughout this chapter, we assume that H is finite dimensional. The notations B and $A = B^{coH}$ of (2.2) remain in effect.

3.1 Overview

In this chapter we look at the creation of "new" right B-modules from existing B- and H-modules. Specifically we show that the tensor product of a right B-module V and a right H-module W can be made into a right B-module. Certain properties continue to hold after tensoring: We prove in Section 3.4 that if the original B-module is projective (resp, pseudo-coherent, resp. coherent) then its tensor product with a finitely generated H-module is also projective (resp. pseudo-coherent, resp. coherent). For those not familiar with the terms we define *coherent* and *pseudo-coherent.*

If U is a A-module then the operation $\mathrm{Ind}_A^B U$ is of special interest. Doi and Takeuchi [DT] showed that when H has a bijective antipode and for $A \subset B$ is H-Galois, if there is an H-comodule map $\phi : H \to B$ such that $\phi(1_H) = 1_B$, then the category of right A-modules is equivalent to the category of right B-modules - right H-comodules.

As mentioned before, Frobenius reciprocity originally arose in the context of character theory over a field. Our motivation stems from Zalesskii and Neroslavskii's [ZN] construction of a simple noetherian ring which is not a domain but contains no non-trivial idempotents. This construction was in response to a conjecture by Faith [F], answering the conjecture in the negative. Lorenz used results from skew group rings [Lo85] and later Frobenius reciprocity for skew group rings [Lo86] to circumvent the need for the difficult computations of Zalesskii and Neroslavskii's work and to provide examples of simple noetherian rings with zero divisors but without non-trivial idempotents which were in addition not Morita equivalent to a domain. We extend Frobenius reciprocity to smash products of Hopf algebras which are a generalization of skew group rings. Our version mimics that of Lorenz.

3.2 Tensor Products of B- and H-Modules

Let V be a right B-module and let W be a right H-module. Then $V \otimes W$ becomes a right B-module by putting

$$(v \otimes w) \cdot b = \sum v b_0 \otimes w b_1$$

for $v \in V$, $w \in W$, and $b \in B$. The module axioms are straightforward to verify, as are the following basic properties:

Functoriality: If $f : V \to V'$ is a B-module map and $g : W \to W'$ is an H-module map, then $f \otimes g : V \otimes W \to V' \otimes W'$ is a B-module map.

Associativity: If W and W' are right H-modules then, viewing $W \otimes W'$ as right H-module via $\triangle$, we have $V \otimes (W \otimes W') \cong (V \otimes W) \otimes W'$.

3.3 Special Cases

We describe some important special cases of the above construction. In particular, we discuss B-modules V that are *induced* from A-modules. By definition, these are B-modules of the form $V = U \otimes_A B$, where U_A is a right A-module.

Lemma 3. *Let V be a right B-module and W a right H-module. View $V \otimes W$ as a right B-module as in (3.2). Then:*

The case $W = H$: *Assume that $A \subset B$ is right H-Galois. Then,*

$$V \otimes_A B \cong V \otimes H$$

as right B-modules.

The case of induced B-modules: *If U_A is a right A-module, then*

$$(U \otimes_A B) \otimes W \cong (U \otimes W) \otimes_A B$$

as right B-modules. Here, $U \otimes W$ is a right A-module via $(u \otimes w)a = ua \otimes w$; so $U \otimes W \cong U^{(\dim_K W)}$. In particular, $B \otimes W \cong B^{(\dim_K W)}$.

Proof. First note that, in the special case where $V = B$ and $W = H$, the bijection $\beta : B \otimes_A B \to B \otimes H$, $b' \otimes_A b \mapsto= \sum b' b_0 \otimes b_1$ of (2.3) is in fact a (B, B)-bimodule map. Here, the left actions of B, for both $B \otimes_A B$ and $B \otimes H$, are given by multiplication, while the right actions are: multiplication on $B \otimes_A B$ and the above right B-module action on $B \otimes H$.

For general V, the map $\phi : V \to V \otimes H$, $v \mapsto v \otimes 1$ is right A-linear, as $a \otimes 1 = \sum a_0 \otimes a_1$ holds for all $a \in A$. Thus we have the "induced" B-module map $\Phi = \phi \otimes_A \mathrm{Id}_B : V \otimes_A B \to V \otimes H$; explicitly, $\Phi(v \otimes_A b) = \phi(v)b = \sum v b_0 \otimes b_1$. Therefore, as right B-modules,

$$V \otimes_A B \cong V \otimes_B (B \otimes_A B) \cong V \otimes_B (B \otimes H) \cong V \otimes H ,$$

where the second isomorphism is given by $\mathrm{Id}_V \otimes_B \beta$. This establishes the case $W = H$.

We now turn to the case of induced modules $V = U \otimes_A B$. In particular, taking $U = A_A$, the asserted isomorphism implies the isomorphism $B \otimes W \cong B^{(\dim_K W)}$.

We construct a map $\lambda : (U \otimes W) \otimes_A B \to (U \otimes_A B) \otimes W$ as follows. The canonical map $\phi : U \to U \otimes_A B$, $u \mapsto u \otimes_A 1$, gives rise to the map of A-modules $\psi = \phi \otimes \mathrm{Id}_W : U \otimes W \to (U \otimes_A B) \otimes W$, $u \otimes w \mapsto (u \otimes_A 1) \otimes w$. Since $(U \otimes_A B) \otimes W$ is in fact a B-module, ψ in turn induces a map of B-modules

$$\begin{array}{rcl} \lambda = \psi \otimes_A \mathrm{Id}_B : (U \otimes W) \otimes_A B & \longrightarrow & (U \otimes_A B) \otimes W \\ (u \otimes w) \otimes_A b & \mapsto & \sum (u \otimes_A b_0) \otimes w b_1 \end{array}$$

For the inverse map, we define

$$\begin{aligned}\delta : (U \otimes_A B) \otimes W &\longrightarrow (U \otimes W) \otimes_A B \\ (u \otimes_A b) \otimes w &\mapsto \sum (u \otimes wS^{-1}b_1) \otimes_A b_0\end{aligned}$$

This map is indeed well-defined: The formula is obviously linear in u, w, and b. Moreover, the map is clearly K-balanced, that is, $\delta((u \otimes_A b)k \otimes w) = \delta((u \otimes_A b) \otimes kw)$ holds for all $k \in K$. To verify A-balancedness, we use the formula $a \otimes 1 = \sum a_0 \otimes a_1$. With this, we calculate for $a \in A$

$$\begin{aligned}\delta((u \otimes_A ab) \otimes w) &= \sum (u \otimes wS^{-1}(a_1 b_1)) \otimes_A a_0 b_0 \\ &= \sum (u \otimes wS^{-1}(b_1)S^{-1}(a_1)) \otimes_A a_0 b_0 \\ &= \sum (u \otimes wS^{-1}(b_1)) \otimes_A S^{-1}(a_1) a_0 b_0 \\ &= \sum (u \otimes wS^{-1}b_1) \otimes_A ab_0\end{aligned}$$

and

$$\begin{aligned}\delta((ua \otimes_A b) \otimes w) &= \sum (ua \otimes wS^{-1}b_1) \otimes_A b_0 \\ &= \sum (u \otimes wS^{-1}b_1)a \otimes_A b_0 \\ &= \sum (u \otimes wS^{-1}b_1) \otimes_A ab_0\ ,\end{aligned}$$

as required. It remains to check that λ and δ are indeed inverse to each other. We will carry out the verification of the identity $\lambda \circ \delta = \mathrm{Id}_{(U \otimes_A B) \otimes W}$; the check of $\delta \circ \lambda = \mathrm{Id}_{(U\otimes W)\otimes_A B}$ can be handled in an entirely analogous fashion.

$$\begin{aligned}(\lambda \circ \delta)((u \otimes_A b) \otimes w) &= \lambda\left(\sum (u \otimes wS^{-1}b_1) \otimes_A b_0\right) \\ &= \sum (u \otimes_A (b_0)_0) \otimes (wS^{-1}b_1)(b_0)_1 \\ &= \sum (u \otimes_A b_0) \otimes w(S^{-1}b_2)b_1 \\ &= \sum (u \otimes_A b_0) \otimes w\varepsilon(b_1) \\ &= (u \otimes_A b) \otimes w \ ,\end{aligned}$$

as required. This completes the proof of the lemma. □

3.4 Projective, Coherent, and Pseudo-coherent Modules

We recall some general definitions.

Definition. A module M over a ring R is called *projective* if M is a direct summand of some free R-module $R_R^{(I)}$. Following [SGA6, I.2.9], M is called *pseudo-coherent* (or of *type* FP_∞) if there is an infinite resolution

$$\ldots \to P_n \to \ldots \to P_1 \to P_0 \to M \to 0 \ ,$$

where all P_n are finitely generated projective R-modules. The module M is called *coherent* (or of *type* FP) if the resolution can be chosen so that all $P_n = 0$ for large enough n; in this case the resolution is called *finite*.

Obviously, pseudo-coherent modules are finitely generated (in fact, finitely presented). Conversely, if the ring R is right noetherian, then every finitely generated R-module is pseudo-coherent (but not necessarily coherent). For any ring R, finitely generated projective modules are clearly coherent, and coherent modules are obviously pseudo-coherent.

We now study these notions especially for the B-modules $V \otimes W$, where V continues to denote a right B-module and W a right H-module.

Proposition 1. *If both V and W are finitely generated then so is $V \otimes W$. Moreover, if V is projective then $V \otimes W$ is likewise projective, for any W. Finally, if V is pseudo-coherent (resp., coherent) and W is finitely generated, then $V \otimes W$ is pseudo-coherent (resp., coherent).*

Proof. For finite generation, note that if V and W are finitely generated then $V \otimes W$ is a homomorphic image of $B^n \otimes H^m$ for suitable finite n and m. But $B^n \otimes H^m \cong (B \otimes H)^{nm}$ and $B \otimes H \cong B^{(\dim_K H)}$, by Lemma 3. Since H is assumed finite dimensional, $B^{(\dim_K H)}$ is finitely generated as a B-module, and hence so is $V \otimes W$.

For the transfer of projectivity, it suffices to consider the special case $V = B$, since the functor $(\,.\,) \otimes W$ commutes with direct sums. But $B \otimes W$ is free as right B-module, by Lemma 3, and so the proof is complete.

Finally, if

$$\mathbf{P}: \quad \ldots \to P_n \to \ldots \to P_1 \to P_0 \to V \to 0\ ,$$

is a projective resolution of V, with all P_n are finitely generated, then

$$\mathbf{P} \otimes W: \quad \ldots \to P_n \otimes W \to \ldots \to P_1 \otimes W \to P_0 \otimes W \to V \otimes W \to 0\ ,$$

is a resolution of $V \otimes W$, and all $P_n \otimes W$ are finitely generated projective, by the foregoing. This proves our assertion about pseudo-coherent modules, and the proof for coherent modules is analogous, starting with a finite resolution $\mathbf{P}$. The proof of the proposition is thus complete. $\square$

3.5 Smash Products

3.5.1 Basic Definitions of Smash Products

Recall one defines the smash product of a left H-module algebra A and a Hopf algebras H to be the K-vector space $A \otimes H$. Writing $a\#h = a \otimes h$, the multiplication of $A\#H$ is then defined by the rule

$$(a\#h)(b\#l) = \Sigma a(h_1 \cdot b)\#h_2 l. \qquad a, b \in A,\ h, l \in H$$

This makes $A\#H$ an associative K-algebra with identity $1_A\#1_H$. Identifying A with $A\#1 \subseteq A\#H$ and H with $1\#H \subseteq A\#H$, we can view both A and H as subalgebras of $A\#H$. As an example, we mention that if K is viewed as the *trivial* H-module algebra, via $h \cdot c = \varepsilon(h)c$, then $K\#H \cong H$ as K-algebras. For $B \cong A\#H$, clearly $A \subset B$ is an H-Galois extension, because smash products are special cases of crossed products; see Section 2.3.1.

Recall, we say L is a *subHopf algebra* of a Hopf algebra H if L is a subalgebra of H and $\Delta(L) \subseteq L \otimes L$ and $S(L) \subseteq L$.

3.5.2 Tensor Products

Let A_1 and A_2 be H-module algebras. Then $A_1 \otimes A_2$ becomes a module algebra over the Hopf algebra $H \otimes H$ ([Sw],p. 49) by defining

$$(h \otimes l) \cdot (a_1 \otimes a_2) = h \cdot a_1 \otimes l \cdot a_2 \qquad (h, l \in H, a_1 \in A_1, a_2 \in A_2).$$

Indeed, this action certainly makes $A_1 \otimes A_2$ a left module over $H \otimes H$, and $(h \otimes l) \cdot 1_{A_1 \otimes A_2} = (h \otimes l) \cdot (1_{A_1} \otimes 1_{A_2}) = h \cdot 1_{A_1} \otimes l \cdot 1_{A_2} = \varepsilon(h) \otimes \varepsilon(l) = \varepsilon(h)\varepsilon(l)1_{A_1 \otimes A_2}$. Further, for $a_1, b_1 \in A_1$ and $a_2, b_2 \in A_2$ and $h, l \in H$, one computes

$$\begin{aligned}
(h \otimes l) \cdot [(a_1 \otimes a_2)(b_1 \otimes b_2)] &= (h \otimes l) \cdot (a_1 b_1 \otimes a_2 b_2) \\
&= h \cdot a_1 b_1 \otimes l \cdot a_2 b_2 \\
&= \sum (h_1 \cdot a_1)(h_2 \cdot b_1) \otimes (l_1 \cdot a_2)(l_2 \cdot b_2) \\
&= \sum ((h_1 \otimes l_1) \cdot (a_1 \otimes a_2))((h_2 \otimes l_2) \cdot (b_1 \otimes b_2)) \\
&= \sum ((h \otimes l)_1 \cdot (a_1 \otimes a_2))((h \otimes l)_2 \cdot (b_1 \otimes b_2)).
\end{aligned}$$

Thus the axioms of an H-module algebra are all satisfied. Consequently, we can now talk about the smash product $(A_1 \otimes A_2)\#(H \otimes H)$.

Lemma 4. (i) *Putting $B_i = A_i \# H$, we have an isomorphism of K-algebras*

$$T : B_1 \otimes B_2 \to (A_1 \otimes A_2)\#(H \otimes H), a_1\#h \otimes a_2\#l \mapsto (a_1 \otimes a_2)\#(h \otimes l).$$

(ii) *The maps $f_i : B_i \to (A_1 \otimes A_2)\#(H \otimes H)$ $(i = 1, 2)$ that are defined by*

$$f_1(a_1\#h) = (a_1 \otimes 1_{A_2})\#\Delta h \text{ and } f_2(a_2\#h) = (1_{A_1} \otimes a_2)\#\Delta h$$

are K-algebra embeddings.

Proof. (i) It is clear that T is a K-linear isomorphism which matches up the identities of both algebras, and so it suffices to check that T is multiplicative. Multiplication in $A_1 \otimes A_2 \# H \otimes H$ is given by the following formula with

$a_1, b_1 \in A_1$ and $a_2, b_2 \in A_2$, $h, h', l, l' \in H$, :

$$\begin{aligned}(a_1 \otimes a_2 \# h \otimes h')(b_1 \otimes b_2 \# l \otimes l') &= \sum (a_1 \otimes a_2)[(h \otimes h')_1 \cdot (b_1 \otimes b_2)] \\ &\quad \#(h \otimes h')_2 (l \otimes l') \\ &= \sum (a_1 \otimes a_2)[(h_1 \otimes h'_1) \cdot (b_1 \otimes b_2)] \\ &\quad \#(h_2 \otimes h'_2)(l \otimes l') \\ &= \sum (a_1 \otimes a_2)[(h_1 \cdot b_1) \otimes (h'_1 \cdot b_2)] \\ &\quad \#(h_2 \otimes h'_2)(l \otimes l') \\ &= \sum a_1(h_1 \cdot b_1) \otimes a_2(h'_1 \cdot b_2) \# h_2 l \otimes h'_2 l'\end{aligned}$$

and multiplication in $B_1 \otimes B_2$ is given by the following formula,

$$\begin{aligned}(a_1 \# h \otimes a_2 \# h')(b_1 \# l \otimes b_2 \# l') &= (a_1 \# h)(b_1 \# l) \otimes (a_2 \# h')(b_2 \# l') \\ &= \sum a_1(h_1 \cdot b_1) \# h_2 l \otimes a_2(h'_1 \cdot b_2) \# h'_2 l'\end{aligned}$$

Since the final two expressions in both calculations correspond to each other under T, the proof of (i) is complete.

(ii) We concentrate of f_1, the map f_2 being completely analogous. First, as K-linear map f_1 is the tensor product of the map $\mu_1 : A_1 \to A_1 \otimes A_2$, $a_1 \mapsto a_1 \otimes 1_{A_2}$, with Δ. Since both of these maps are injective, so is f_1. Further, f_1 respects the identities, since both μ_1 and Δ do. It remains to check multiplicativity. Recall that

$$\begin{aligned}\Delta_{H \otimes H}(\Delta h) &= \Delta(\sum h_1 \otimes h_2) \\ &= \sum (h_1 \otimes h_2)_1 \otimes (h_1 \otimes h_2)_2 \\ &= \sum (h_{1_1} \otimes h_{2_1}) \otimes (h_{1_2} \otimes h_{2_2}) \\ &= \sum h_1 \otimes h_3 \otimes h_2 \otimes h_4\end{aligned}$$

Thus

$$\begin{aligned}
f_1(a\#h)f_1(b\#l) &= (a\otimes 1_{A_2}\#\Delta h)(b\otimes 1_{A_2}\otimes \Delta l)\\
&= \sum (a\otimes 1_{A_2})[(\Delta h)_1\cdot b\otimes 1_{A_2}]\#(\Delta h)_2(l_1\otimes l_2)\\
&= \sum (a\otimes 1_{A_2})[(h_1\cdot b)\otimes (h_3\cdot 1_{A_2})]\#(h_2\otimes h_4)(l_1\otimes l_2)\\
&= \sum (a(h_1\cdot b_1)\otimes \varepsilon(h_3)1_{A_2})\#h_2l_1\otimes h_4l_2\\
&= \sum a(h_1\cdot b_1)\otimes 1_{A_2}\#h_2\varepsilon(h_3)l_1\otimes h_4l_2\\
&= \sum a(h_1\cdot b_1)\otimes 1_{A_2}\#h_2l_1\otimes h_3l_2 \qquad (*)
\end{aligned}$$

and, on the other hand,

$$\begin{aligned}
f_1((a\#h)(b\#l)) &= f_1(\sum a(h_1\cdot b)\#h_2l)\\
&= \sum a(h_1\cdot b)\otimes 1_{A_2}\#\Delta(h_2l)\\
&= \sum a(h_1\cdot b)\otimes 1_{A_2}\#(h_2l_1\otimes h_3l_2) \qquad (**)
\end{aligned}$$

Comparing (*) and (**) we have our desired result. □

3.6 Frobenius Reciprocity

In the setting of Section 3.5.2, assume V_i are right B_i-modules $(i=1,2)$. The the tensor product $V_1\otimes V_2$ is a module over $B_1\otimes B_2$. Hence $V_1\otimes V_2$ is a module over each $B_i=A_i\#H$ via the maps f_i and T^{-1} of Lemma 4. Specifically, for B_1,

$$\begin{aligned}
(v_1\otimes v_2)(a_1\#h) &= (v_1\otimes v_2)T^{-1}(a_1\otimes 1_{A_2}\#\Delta h)\\
&= (v_1\otimes v_2)(\sum a_1\#h_1\otimes 1_A\#h_2)\\
&= \sum v_1(a_1\#h_1)\otimes v_2(1_{A_2}\#h_2),
\end{aligned}$$

where $v_1 \in V_1$, $v_2 \in V_2$, $a_1 \in A_1$, and $h \in H$. Similarly for B_2.

We have the following extension of the classic Frobenius reciprocity isomorphism for group rings (e.g. [S], Theorem 2.2) which mimics the version for skew group rings in [Lo86].

Theorem 2. *Let L be a subHopf algebra of H and let $C_i = A_i \# L \subseteq B_i$, for $i = 1, 2$. If V is a B_1-module and W is a C_2-module then, as B_i-modules,*

$$(V|_{C_1} \otimes W) \otimes_{C_i} B_i \cong V \otimes (W \otimes_{C_2} B_2).$$

Proof. Let $\psi : W \to W \otimes_{C_2} B_2$ denote the K-linear map given by

$$\psi(w) = w \otimes_{C_2} 1_{B_2} = w \otimes_{C_2} (1_{A_2} \# 1_H),$$

and define

$$\phi = Id_V \otimes \psi : V|_{C_1} \otimes W \to V \otimes (W \otimes_{C_2} B_2)$$
$$\phi(v \otimes w) = v \otimes (w \otimes_{C_2} (1_{A_2} \# 1_H)).$$

This is clearly a $C_1 \otimes C_2$-module map, because Id_V is a C_1-module map and ψ is a C_2-module map. Therefore, ϕ can be viewed as a module map over each C_i, using f_i and T^{-1}. Since $V \otimes (W \otimes_{C_2} B_2)$ is a module over B_i (via f_i and T^{-1}), we obtain a B_i-module map

$$g : (V|_{C_1} \otimes W) \otimes_{C_i} B_i \to V \otimes (W \otimes_{C_2} B_2)$$
$$(v \otimes w) \otimes_{C_i} a_i \# h \mapsto \phi(v \otimes w) \cdot (a_i \# h).$$

We claim that g is in fact an isomorphism, that is g is bijective. Note that, as modules over $H = 1 \# H \subseteq B_i$ and $L = 1 \# L \subseteq C_i$, we have

$$(V \otimes W) \otimes_{C_i} B_i|_H \cong (V \otimes W) \otimes_L H$$

and

$$V \otimes (W \otimes_{C_2} B_2)|_H \cong V \otimes (W \otimes_L H)$$

Furthermore,

$$g((v \otimes w) \otimes_L h) = \Sigma v h_1 \otimes (w \otimes_L h_2).$$

Define $f : V \otimes (W \otimes_L H) \to (V \otimes W) \otimes_L H$ by

$$f(v \otimes (w \otimes_L h)) = \Sigma (vS(h_1) \otimes w) \otimes_L h_2$$

To see that this map is well-defined, note that f is clearly additive in v, w, and h and that f is K-balanced since $f(vc \otimes (w \otimes_L h)) = f(v \otimes (cw \otimes_L h))$ holds for all scalars $c \in K$. Furthermore, if $x \in L$ then

$$\begin{aligned}
f(v \otimes (w \otimes_L xh)) &= \Sigma (vS(x_1 h_1) \otimes w) \otimes_L x_2 h_2 \\
&= \sum (vS(h_1)S(x_1) \otimes w) \otimes_L x_2 h_2 \\
&= \sum (vS(h_1)S(x_1) \otimes w) x_2 \otimes_L h_2 \\
&= \sum (vS(h_1)S(x_1)x_2 \otimes w x_3) \otimes_L h_2 \\
&= \sum (vS(h_1)\varepsilon(x_1) \otimes w x_2) \otimes_L h_2 \\
&= \sum (vS(h_1) \otimes w\varepsilon(x_1)x_2 \otimes_L h_2 \\
&= \sum (vS(h_1) \otimes wx) \otimes_L h_2 \\
&= f(v \otimes (wx \otimes_L h)).
\end{aligned}$$

So f is well-defined. Finally, to check that f is indeed the required inverse for

g, we compute

$$\begin{aligned}
f \circ g((v \otimes w) \otimes_L h) &= f(\sum vh_1 \otimes (w \otimes_L h_2)) \\
&= \sum (vh_1 S(h_2) \otimes w) \otimes_L h_3 \\
&= \sum (v\varepsilon(h_1) \otimes w) \otimes_L h_2 \\
&= \sum (v \otimes w) \otimes_L \varepsilon(h_1) h_2 \\
&= (v \otimes w) \otimes_L h
\end{aligned}$$

and

$$\begin{aligned}
g \circ f(v \otimes (w \otimes_L h)) &= g(\sum vS(h_1) \otimes w) \otimes_L h_2 \\
&= \sum vS(h_1)h_2 \otimes (w \otimes_L h_3) \\
&= \sum v\varepsilon(h_1) \otimes (w \otimes_L h_2) \\
&= \sum v \otimes (w \otimes_L \varepsilon(h_1) h_2) \\
&= v \otimes (w \otimes_L h)
\end{aligned}$$

So $f \circ g$ and $g \circ f$ are identity maps which completes the proof. $\square$

Letting $\mathrm{Ind}_{C_i}^{B_i} = (.) \otimes_{C_i} B_i$ denote the induction map, as usual, and $\mathrm{Res}_{C_i}^{B_i}$ the restriction map, the above isomorphism can also be written as follows:

$$\mathrm{Ind}_{C_i}^{B_i}(\mathrm{Res}_{C_1}^{B_1} V \otimes W) \cong V \otimes \mathrm{Ind}_{C_2}^{B_2} W.$$

CHAPTER 4

THE GROTHENDIECK GROUPS G_0 AND K_0

4.1 Overview

In this chapter after reviewing the basics of Grothendieck groups, in Sections 4.3 and 4.4, how we may consider $G_0(H)$ a ring (non-commutative, in general) and how $G_0(B)$ and $K_0(B)$ are $G_0(H)$-modules. We describe the character map $G_0(H) \to H^*$ and the Hattori-Stallings trace map, i.e. a ring theoretic analogue of the character map. Putting these pieces together in Section 4.7, we then prove that, if K is a splitting field for H the following holds.

Theorem *Let $A \subset B$ be a right H-Galois extension and assume that*

(a) A is commutative without idempotents $\neq 0, 1$.

(b) H is involutory, not semisimple. (So, in particular, $p = \operatorname{char} K$ is positive, in fact, a divisor of $\dim_K H$.)

(c) $1 \notin [B, B]$.

Then $p = \operatorname{char} K$ *divides* $\operatorname{rank}(P_A)$ *for every finitely generated projective* B*-module* P.

As in the last chapter, here too, we show a special result for a smash product $B \cong A\#H$ a smash product. If fact, under certain technical assumptions we will be able to characterize the image of the Cartan map $c : K_0(B) \to G_0(B)$. This will then allow us to purpose examples of noetherian rings that are not Morita equivalent to a domain.

4.2 Main Definitions

We recall a few definitions from classical algebraic K-theory. Good references for this material are [Ba68], [Ro], and [W].

4.2.1 G_0

The Grothendieck group $G_0(R)$ of a ring R is an additive abelian group that is associated with the category $\mathsf{ps.coh}_R$ of all pseudo-coherent right R-modules; see (3.4). Traditionally, $G_0(R)$ is only considered for right noetherian rings R, in which case $\mathsf{ps.coh}_R = \mathsf{mod}_R$, the category of finitely generated right R-modules; the present definition for general rings R is taken from [W, Chapter II].

Specifically, let $\mathcal{F}$ be the free additive abelian group on the isomorphism classes of the pseudo-coherent right R-modules. We note that every pseudo-coherent (indeed, every finitely generated) R-module is a homomorphic image of some free R-module R^n. Hence, the collection of isomorphism classes of pseudo-coherent R-modules forms a set and $\mathcal{F}$ is well-defined. For each V in $\mathsf{ps.coh}_R$, we let $\langle V \rangle$ denote the isomorphism class of V. With this notation, we can now state the following

Definition. The Grothendieck group $G_0(R)$ is defined to be the factor group $\mathcal{F}/\mathcal{R}$, where $\mathcal{R}$ is the subgroup of $\mathcal{F}$ that is generated by all elements of the form $\langle B \rangle - \langle A \rangle - \langle C \rangle$, with A, B, and C modules in $\mathsf{ps.coh}_R$, so that there exists a short exact sequence $0 \to A \to B \to C \to 0$.

The image of $\langle V \rangle$ in $G_0(R)$ is written $[V]$; every element of $G_0(R)$ has the form $[V] - [W]$ for suitable V, W in $\mathsf{ps.coh}_R$.

4.2.2 K_0

The Grothendieck group $K_0(R)$ is defined in the same way as $G_0(R)$, except that the category proj_R of all finitely generated *projective* right R-modules replaces the category $\mathsf{ps.coh}_R$. Alternatively, one can use the category coh_R of all *coherent* right R-modules in place of $\mathsf{ps.coh}_R$; this leads to the same group $K_0(R)$. Since short exact sequences of projective modules are split, the definition, when formulated for proj_R, takes the following form

Definition. The Grothendieck group $K_0(R)$ is defined to be the factor group $\mathcal{P}/\mathcal{S}$, where $\mathcal{P}$ is the free abelian group on the isomorphism classes of modules in proj_R and $\mathcal{S}$ is the subgroup of $\mathcal{P}$ that is generated by all elements of the form $\langle B \rangle - \langle A \rangle - \langle C \rangle$, with A, B, C in proj_R, so that $B \cong A \oplus C$.

Using $[V]$ to denote elements of $K_0(R)$, in analogy with $G_0(R)$, every element of $K_0(R)$ again has the form $[V] - [W]$ for suitable V, W in proj_R.

Moreover, $[V] = [W]$ holds in $K_0(R)$ if and only if V and W are *stably isomorphic*, that is, $V \oplus R^n \cong W \oplus R^n$ holds for some finite n.

A pleasant property of $K_0(R)$ is its unproblematic behavior under change of rings. To wit:

Functoriality: Induction. Let $\phi : R \to S$ be a ring homomorphism. Then, for any finitely generated projective right R-module P, the module $\mathrm{Ind}_R^S(P) = P \otimes_R S$ is finitely generated projective over S. Since $(\,.\,) \otimes_R S$ respects direct sums, we obtain a well-defined group homomorphism, often called *induction* or *base change* from R to S,

$$\mathrm{Ind}_R^S : K_0(R) \to K_0(S) \ , \qquad [P] \mapsto [P \otimes_R S] \ .$$

Functoriality: Restriction. Let $\phi : R \to S$ again be a ring homomorphism, but now assume that S becomes a coherent right R-module via ϕ. Then the *restriction map* $P_S \mapsto P\big|_R$ yields an exact functor $\mathsf{proj}_S \to \mathsf{coh}_R$, and this functor in turn induces a homomorphism, called *restriction* or *transfer* from S to R,

$$\mathrm{Res}_R^S : K_0(S) \to K_0(R) \ , \qquad [P] \mapsto [P\big|_R] \ .$$

4.2.3 The Cartan Map

The inclusion $\mathsf{proj}_R \hookrightarrow \mathsf{ps.coh}_R$ induces a map on the level of Grothendieck groups, the so-called *Cartan map*. Explicitly,

$$c = c_R : K_0(R) \to G_0(R), \qquad [V] \mapsto [V] \ .$$

Despite this seemingly trivial formula, the Cartan map is neither injective nor surjective in general.

4.2.4 The Hattori-Stallings Trace Map

If P is a finitely generated projective module over the ring R then $P \cong eR^n$ for some n and some idempotent matrix $e = (e_{i,j}) \in M_n(R)$. Putting

$$\mathbf{r}(P) = \sum_{i=1}^{n} e_{i,i} + [R,R] \in R/[R,R]$$

one obtains a well-defined additive map $\mathbf{r} = \mathbf{r}_R : K_0(R) \to R/[R,R]$, the so-called *Hattori-Stallings trace map*; see [Ba76, W]. The group $T(R) = R/[R,R]$ is called the *trace group* of R.

A basic property of the Hattori-Stallings trace map is its

Functoriality: If $\phi : R \to S$ is a ring homomorphism then

$$\mathbf{r}(\mathrm{Ind}_R^S(P)) = T(\phi)(\mathbf{r}(P)) \in T(S) \ ,$$

where $T(\phi) : T(R) \to T(S)$ sends $r + [R,R]$ to $\phi(r) + [S,S]$. This is clear from the definitions.

4.2.5 Ranks

Assume that R is *commutative*. Then, for each prime ideal $\mathfrak{p}$ of R, the localization $R_\mathfrak{p}$ exists. Since all finitely generated projectives over the local ring $R_\mathfrak{p}$ are free, one has $K_0(R_\mathfrak{p}) = \langle [R_\mathfrak{p}] \rangle \cong \mathbb{Z}$, and hence the induction map $\mathrm{Ind}_R^{R_\mathfrak{p}} : K_0(R) \to K_0(R_\mathfrak{p})$ can be viewed as a homomorphism $K_0(R) \to \mathbb{Z}$. This map is called the *rank map at* $\mathfrak{p}$ and denoted $\mathrm{rank}_\mathfrak{p}$. Explicitly,

$$\mathrm{rank}_\mathfrak{p} : K_0(R) \twoheadrightarrow \mathbb{Z} \ , \qquad [P] \mapsto \dim_{Q(R/\mathfrak{p})} \big((P/P\mathfrak{p}) \otimes_{R/\mathfrak{p}} Q(R/\mathfrak{p}) \big) \ ,$$

where $Q(R/\mathfrak{p})$ denotes the field of fractions of $R/\mathfrak{p}$.

The following lemma is a reformulation of [W, Chapter II, Proposition 2.5].

Lemma 5. $\bigcap_{\mathfrak{p}} \mathrm{Ker}(\mathrm{rank}_{\mathfrak{p}}) \subset \mathrm{Ker}(\mathbf{r})$, *where* $\mathfrak{p}$ *runs over all prime ideals of* R.

Proof. By definition of $\mathrm{rank}_{\mathfrak{p}}$, the intersection $\bigcap_{\mathfrak{p}} \mathrm{Ker}(\mathrm{rank}_{\mathfrak{p}})$ is precisely the kernel of the map $K_0(R) \to \prod_{\mathfrak{p}} K_0(R_{\mathfrak{p}})$ that is given the induction maps $\mathrm{Ind}_R^{R_{\mathfrak{p}}}$. The lemma is thus a consequence of the commutative diagram

$$\begin{tikzcd} K_0(R) \arrow[r] \arrow[d, "\mathbf{r}_R"'] & \prod_{\mathfrak{p}} K_0(R_{\mathfrak{p}}) \arrow[d, "\prod_{\mathfrak{p}} \mathbf{r}_{R_{\mathfrak{p}}}"] \\ R \arrow[r, hook, "\text{can.}"] & \prod_{\mathfrak{p}} R_{\mathfrak{p}} \end{tikzcd}$$

Several comments are in order. First, since all rings under consideration are commutative, their trace groups are the actual rings; so $T(R) = R$ and similarly for $R_{\mathfrak{p}}$. Thus, commutativity of the diagram follows from functoriality of the Hattori-Stallings trace map. Finally, the "canonical" map $R \to \prod_{\mathfrak{p}} R_{\mathfrak{p}}$ is injective, by [Bou, Cor. 2 Ch II.3.3]. $\square$

Special case: Rings without idempotents

Assume now that R is commutative with no idempotent elements except 0 and 1. Then it is known that $\mathrm{rank}_{\mathfrak{p}} : K_0(R) \to \mathbb{Z}$ is the same map for all primes $\mathfrak{p}$; e.g., [W, Chapter I, Exercises 2.4, 2.5]. Hence, in this case, one has a well-defined map

$$\mathrm{rank} : K_0(R) \twoheadrightarrow \mathbb{Z} ,$$

given by $\mathrm{rank} = \mathrm{rank}_{\mathfrak{p}}$ for any $\mathfrak{p}$. By Lemma 5, $\mathrm{Ker}(\mathrm{rank}) \subset \mathrm{Ker}(\mathbf{r})$, and so we obtain the following factorization of the Hattori-Stallings trace map.

Corollary 2. *If R is commutative without idempotents $\neq 0, 1$, then the Hattori-Stalling trace map $\mathbf{r}_R$ factors through* rank, *that is,*

$$\begin{tikzcd} K_0(R) \arrow[r, "\mathrm{rank}"] \arrow[rr, bend right, "\mathbf{r}_R"'] & \mathbb{Z} \arrow[r, "\mathrm{can.}"] & R \end{tikzcd} .$$

4.3 The Case of H-comodule Algebras

For the remainder of this chapter, we assume that

> *B is a right H-comodule algebra, with coinvariants $A = B^{coH}$, as in (2.2). Moreover, we continue to assume that H a finite dimensional Hopf algebra.*

By Proposition 1, the tensor product $\otimes$ yields bifunctors

$$\otimes : \mathsf{ps.coh}_B \times \mathsf{mod}_H \to \mathsf{ps.coh}_B \ , \qquad (V, W) \mapsto V \otimes W$$

and

$$\otimes : \mathsf{proj}_B \times \mathsf{mod}_H \to \mathsf{proj}_B \ , \qquad (V, W) \mapsto V \otimes W \ .$$

As these functors are exact in both arguments, they induce group homomorphisms

$$G_0(B) \times G_0(H) \overset{\cdot}{\longrightarrow} G_0(B) \qquad \text{and} \qquad K_0(B) \times G_0(H) \overset{\cdot}{\longrightarrow} K_0(B) \ ,$$

both given by the formula $([V], [W]) \mapsto [V \otimes W]$. In particular, the diagram

$$\begin{tikzcd} K_0(B) \times G_0(H) \arrow[r, "\cdot"] \arrow[d, "c_B \times \mathrm{Id}_{G_0(H)}"'] & K_0(B) \arrow[d, "c_B"] \\ G_0(B) \times G_0(H) \arrow[r, "\cdot"] & G_0(B) \end{tikzcd}$$

is commutative.

4.4 The Grothendieck Ring $G_0(H)$

In the special case where $B = H$, the constructions of (4.3) make $G_0(H)$ into an associative ring (in general non-commutative), with identity element $[K]$, the class of the "trivial" H-module $K = K_\varepsilon$. Explicitly, multiplication in $G_0(H)$ is given by the formula

$$[W_1] \cdot [W_2] = [W_1 \otimes W_2] \ .$$

This ring is called the *Grothendieck ring* of H in the literature, e.g., [Lo97, NR]. The function

$$\dim : G_0(H) \to \mathbb{Z} \ , \quad [W] \mapsto \dim_K W$$

is a ring homomorphism, called the *dimension map*. Finally, $G_0(H)$ has a canonical ring antiautomorphism

$$(\,.\,)^* : G_0(H) \to G_0(H) \ , \quad [W] \mapsto [W^*] \ ,$$

where $W^* = \mathrm{Hom}_K(W, K)$ is viewed as right H-module via $(fh)(w) = f(wS(h))$ for $f \in W^*$, $w \in W$, and $h \in H$.

Returning to general H-comodule algebras B, we record for future reference the following proposition.

Proposition 2. *The tensor product maps* $\cdot : G_0(B) \times G_0(H) \to G_0(B)$ *and* $\cdot : K_0(B) \times G_0(H) \to K_0(B)$ *of (4.3) define* $G_0(H)$*-module structures on* $G_0(B)$ *and on* $K_0(B)$*. The Cartan map* $c_B : K_0(B) \to G_0(B)$ *is a* $G_0(H)$*-module map.*

Proof. This is all straightforward. E.g., the associativity axiom for modules is a consequence of the associativity isomorphism noted in (3.2), and the fact that c_B is a module map is immediate from the commutative diagram in (4.3). $\square$

4.5 The Character Map

The *character map* $\mathrm{ch} : G_0(H) \to H^*$ is defined by $[V] \mapsto \mathrm{ch}_V$, where

$$\mathrm{ch}_V(h) = \mathrm{trace}_{V/K}(v \mapsto vh) \ ,$$

the trace of the K-linear endomorphism $v \mapsto vh$ of V. The *character algebra* $R(H)$ is the K-subalgebra of H^* that is generated by the image of the character map.

In the following proposition, which is identical with [Lo97, Prop. 3.6], we list some fundamental properties of the character map.

Proposition 3. (i) *The character map* $\mathrm{ch} : G_0(H) \to H^*$ *is a ring homomorphism which satisfies* $\mathrm{ch}_{V^*} = S^*(\mathrm{ch}_V)$ *and* $\mathrm{ch}_V(1) = \dim_K V$.

(ii) *The character algebra* $R(H)$ *is contained in* $([H,H] + \mathrm{rad}\, H)^\perp$*, the space of all linear forms on* H *that vanish on the space of Lie commutators* $[H,H]$ *and on the Jacobson radical* $\mathrm{rad}\, H$ *of* H*. If* K *is a splitting field for* H *then* $R(H) = ([H,H] + \mathrm{rad}\, H)^\perp$*, and* ch *induces an isomorphism of* K*-algebras*

$$\mathrm{Id}_K \otimes \mathrm{ch} : K \otimes_{\mathbb{Z}} G_0(H) \xrightarrow{\cong} R(H) \ .$$

4.6 Some Standard Examples

4.6.1 Finite Group Algebras

Let $H = KG$ be the group algebra of the finite group G and assume that K is a splitting field for KG. Then

$$R(H) \cong K^{T(G)_{\mathrm{reg}}} \subset H^* = K^G \ .$$

Here, $T(G)_{\text{reg}}$ denotes the set of *p-regular conjugacy classes*, that is, the conjugacy classes of elements of G whose order is not divisible by $p = \operatorname{char} K$. Also, for any set I, K^I denotes the algebra of functions $f : I \to K$, with "pointwise" addition and multiplication of functions.

Furthermore, via Brauer characters,

$$G_0(H) \subseteq G_0(H) \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathbb{C}^{T(G)_{\text{reg}}} \ .$$

The standard reference for this extensively investigated case is [CR].

4.6.2 Duals of Finite Group Algebras

The case where $H = (KG)^*$, with G a finite group as above, is straightforward: $G_0(H) = \mathbb{Z}G$ and $R(H) = H^* = KG$.

4.7 Application to H-Galois Extensions: Ranks of Projectives

Recall that the Hopf algebra H is said to be *involutory* if the antipode S has order 2, that is, $S^2 = \text{Id}$. By results of Larson and Radford, one knows that, for H involutory,

> *H and H^* are both semisimple if and only if $p = \operatorname{char} K$ does not divide $\dim_K H$. Moreover, H is semisimple if and only if the "regular" character $\operatorname{ch}_H : H \to K$, that is, the character of the regular H-module H_H, is not the zero map.*

This is proved in [LR87, Theorem 1] and [LR87, Proposition 1(c)].

Now assume that $A \subset B$ be a right H-Galois extension. Then, by Theorem 1, we know that B_A is finitely generated projective. Consequently, for any finitely generated projective right B-module P, the restriction P_A is finitely generated projective over A. The following result gives an arithmetic restriction on the possible ranks of the projective A-modules arising in this fashion. We assume that K is a splitting field for H.

Theorem 3. *Let $A \subset B$ be a right H-Galois extension and assume that*

(a) *A is commutative without idempotents $\neq 0, 1$.*

(b) *H is involutory, not semisimple. (So, in particular, $p = \operatorname{char} K$ is positive, in fact, a divisor of $\dim_K H$.)*

(c) $1 \notin [B, B]$.

Then $p = \operatorname{char} K$ divides $\operatorname{rank}(P_A)$ for every finitely generated projective B-module P.

Proof. By Proposition 3, the character map $\mathrm{ch} : G_0(H) \to H^*$ has kernel $pG_0(H)$. As remarked above, condition (b) on H implies that the regular character ch_H vanishes. Consequently,

$$[H] = pX \qquad \text{for some } X \in G_0(H).$$

Now let P be a finitely generated projective right B-module. Our goal is to show that

$$p \mid \operatorname{rank}(P_A) \ .$$

To this end, consider the following commutative diagram which combines functoriality of the Hattori-Stallings trace map with Corollary 2 :

$$\begin{tikzcd}
K_0(B) \arrow[rr, "\mathbf{r}_B"] & & B/[B,B] \\
K_0(A) \arrow[u, "\mathrm{Ind}_A^B"] \arrow[r, "\mathrm{rank}"] \arrow[rr, bend right, "\mathbf{r}_A"'] & \mathbb{Z} \arrow[r, "\mathrm{can.}"] & A \arrow[u, "\mathrm{can.}"']
\end{tikzcd}$$

Since $1 \notin [B,B]$, by hypothesis (c), the map $\mathbb{Z} \to T(B) = B/[B,B]$ has kernel $p\mathbb{Z}$. Thus our assertion becomes

$$\mathbf{r}_B(P \otimes_A B) = 0 \ .$$

But, by Lemma 3 (case $W = H$), we know that $[P \otimes_A B] = [P][H]$, and this in turn implies that

$$\mathbf{r}_B(P \otimes_A B) = \mathbf{r}_B([P][H]) = p\mathbf{r}_B([P]X) = 0 \ ,$$

as desired. □

Special cases

The Theorem applies in particular to the case where $A = K \subset B = H$. Note that hypotheses (a) and (c) are trivially satisfied here. Thus, the theorem yields the following previously known facts; part (i) is [Lo97, Theorem 2.3(b)] and (ii) is [L71, Theorem 4.3].

Corollary 3. *Assume that H is involutory. Then:*

(i) *If H is not semisimple then $p \mid \dim_K P$ holds for all finitely generated projective H-modules P.*

(ii) *If* $\dim_K H$ *is not divisible by* p *then* H *is semisimple.*

Proof. In both assertions, we may extend scalers to a splitting field of H, if necessary; so the theorem applies. For (i), just note that rank $= \dim_K$ if $A = K$. Part (ii) follows from (i) by taking $P = H$. □

We remark that (ii) above is false in general if H is not involutory; a counterexample is provided by the so-called Sweedler algebra (cf. [Lo97, 4.1]). Thus, the involutory hypothesis in (b) is necessary for the Theorem to hold.

4.8 Application for the Smash Product: Image of the Cartan Map

For the remainder of this chapter, we assume that $B \cong A\#H$ *a smash product. We continue to assume that* H *is finite dimensional.*

4.8.1 A as a B-module

Recall from Section 2.3.2 if $B \cong A\#H$ is a smash product, we have A as both a left and a right B-module. This follows [Mont] p.53.

Let $a, b \in A, h \in H$ then A is a left B-module via

$$(b\#h) \rightarrow a = b(h \cdot a).$$

Further A is a right B-module as follows:

$$\begin{aligned} a \leftarrow b\#h &= \Sigma\alpha(h_2)(S^{-1}(h_1) \cdot ab) \\ &= \Sigma\alpha(h_3)(S^{-1}(h_2) \cdot a)(S^{-1}(h_1) \cdot b) \end{aligned}$$

where α is the distinguished grouplike element of H^*. With these module actions we can state the following corollary of Lemma 3.

Corollary 4. *If $B \cong A\#H$ is a smash product and A is a (right) B-module as above, then we have $A \otimes H \cong B$ as right B-modules.*

The proof is immediate from the Lemma 3.3 (case $W = H$) applied to $V = A_B$. Written in Grothendieck group terms, Lemma 3.3 can be stated as saying that the map

$$\mathrm{Ind}_A^B \circ \mathrm{Res}_A^B : G_0(B) \to G_0(A) \to G_0(B)$$

is multiplication by $[H] \in G_0(H)$.

4.8.2 Image of the Cartan Map

Recall from Section 4.2.2 that since B_A is finitely generated projective over A, the restriction map $Res_A^B : K_0(B) \to K_0(A)$ is well-defined.

The condition that $K_0(A) = \mathbb{Z}[A]$ we saw before in Section 4.2.5. It is another was of saying that all finitely generated projective modules are *stably free*. Here Q is said to be *stably free* when $Q \oplus A^m \cong A^n$ for suitable $m, n \in \mathbb{Z}$. It must be noted that not all stably free modules are free (e.g., [P90, p.165-67]). The following gives us the image of $K_0(B)$ under the Cartan map under the specified conditions.

Theorem 4. *Assume that $B = A\#H$ a smash product and assume that*

(a) $p = char K$ divides $[H] \in G_0(H)$

(b) $K_0(A) = \mathbb{Z}[A]$

(c) $1 \notin [B, B]$

then $c(K_0(B)) \subseteq p \cdot \mathbb{Z}[A_B] + ann_{G_0(B)}([H])$.

Proof. Let $[P] \in K_0(B)$. We know from the above discussion that $P|_A$ is finitely generated projective over A, so in $K_0(A)$, $[P_A] = n[A]$ for some $n \in \mathbb{Z}$, by hypothesis (b). Thus $\mathrm{Ind}_A^B \circ \mathrm{Res}_A^B[P] = n[B]$ holds in $K_0(B)$.

On the other hand, as above we have by hypothesis (a) $[H] = pX$ for some $X \in G_0(H)$ and by Lemma 3: $\mathrm{Ind}_A^B \circ \mathrm{Res}_A^B[P] = [P][H]$ so we deduce that

$$[P][H] = p[P]X = n[B]$$

Applying the Hattori-Stallings trace: $\mathbf{r} : K_0(B) \to B/[B, B]$, we have

$$0 = p \cdot \mathbf{r}([P]X) = n\mathbf{r}([B]) = n + [B, B]$$

In view of hypothesis (c) this implies that p divides n. By Corollary 4 $[B] = [A_B][H]$ holds in $G_0(B)$, that is, after applying the Cartan map the equality $[P] \cdot [H] = n[B]$ can be expressed as

$$([P] - n[A_B]) \cdot [H] = 0$$

or

$$[P] \in n[A_B] + ann_{G_0(B)}([H]).$$

Since p divides n, this proves our theorem. □

4.9 Application: Noetherian Rings

4.9.1 Goldie's Reduced Rank

For any right noetherian ring R, Goldie's reduced rank function is a homomorphism:

$$\rho = \rho_R : G_0(R) \to \mathbb{Z}$$

which can be defined as follows. Let N be the nilpotent radical of R, that is, the largest nilpotent ideal of R. Then R/N is a semiprime noetherian ring and Goldie's Theorem (cf. [P90, p. 258]) implies that R/N has a semisimple Artinian classical right ring of quotients Q. If V is a finitely generated R/N-module, then $V \otimes_{R/N} Q$ is a finitely generated Q-module and hence the composition length $len_Q(V \otimes_R Q)$ is well-defined. Goldie's reduced rank ρ_R is define as the composite map:

$$\rho : G_0(R) \xrightarrow{\cong} G_0(R/N) \xrightarrow{.\otimes_{R/N} Q} G_0(Q) \xrightarrow{len_Q(.)} \mathbb{Z}$$

here len_Q is the composition length of the module over Q.

4.9.2 Noetherian Rings Not Morita Equivalent to a Domain

Corollary 5. *Let $B = A\#H$ be given with $H \neq K$ local involutory. Assume (b) and (c) of Theorem 4 are satisfied. Then B is not Morita equivalent to a noetherian domain.*

Proof. First note that our hypothesis on H implies that the trivial module, K, is the only simple H-module. So $G_0(H) =< [K] >$ and $[H] = (\dim_K H)[K]$.

Furthermore, H is not semisimple, and hence $char K = p > 0$ and divides $\dim_K H$; see Section 4.7. Thus condition (a) in Theorem 4 is satisfied.

Now, suppose B is Morita equivalent to a noetherian domain, say D. The equivalence is given by the tensor product $. \otimes_B P_D$, where ${}_B P_D$ is a (B, D)-bimodule with suitable properties ([Ba68]). In particular, $. \otimes_B P_D$, yields an isomorphism $K_0(B) \cong K_0(D)$ and $G_0(B) \cong G_0(D)$ which makes the following diagram commute:

$$\begin{tikzcd} K_0(B) \arrow[r, "c_B"] \arrow[d, "\cong"'] & G_0(B) \arrow[d, "\cong"] \\ K_0(D) \arrow[r, "c_D"] & G_0(D) \end{tikzcd}$$

The Theorem therefore implies that $c_D(K_0(D)) \subseteq p \cdot G_0(D) + ann_{G_0(D)}(\dim_K H)$. Now since D is a noetherian domain, Goldie's reduced rank function $\rho : G_0(D) \to \mathbb{Z}$ satisfies $\rho([D]) = 1$. On the other hand, $[D]$ belongs to $c_D(K_0(D))$. Futhermore $ann_{G_0(D)}(\dim_K(H))$ is contained in the torsion subgroup of $G_0(D)$ and hence in $\mathrm{Ker}\, \rho$. So ρ sends $p \cdot G_0(D) + ann_{G_0(D)}(\dim_K(H))$ to $p\mathbb{Z}$, contradicting $\rho([D]) = 1$.

$\square$

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