HOMOGENIZATION OF DYNAMIC MATERIALS

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ABSTRACT

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In this work we study the homogenization problem associated with propagation of long wave disturbances in active materials—materials whose properties exhibit not only spacial but also temporal inhomogeneities and whose study was initiated by Lurie in his pioneering works of 1997. We study the possibility of extending the homogenization procedure developed for ordinary composites to the case of dynamic materials. We uncover dramatic differences between the hyperbolic and the elliptic cases. We also compute all exact relations for 3D composite conductors exhibiting the Hall effect.

ACKNOWLEDGEMENTS

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Finally, I gratefully acknowledge the research fellowship award funded through Dr. Grabovsky's NSF grant NSF-0094089.

DEDICATION

To my parents, Chaeyong and Bugum To

With all my love and gratitude.

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NOTATION

$\mathbb{Q}$	The set of rational numbers.
$\mathbb{Z}$	The set of integers.
$\mathbb{R}^d$	d-dimensional real Euclidean space, $\mathbb{R} = \mathbb{R}^1$.
Ω	Open, bounded subset in $\mathbb{R}^{d+1}$.
$\bar{\Omega}$	Closure of $\Omega = \partial\Omega \cup \Omega$, where $\partial\Omega$ is a boundary of Ω.
$C_0^\infty(\Omega)$	The space of infinitely differentiable functions with compact support in Ω.
$L^\infty(\Omega)$	The space of integrable functions $\phi : \Omega \to \mathbb{R}$, such that $\|\phi\|_{L^\infty(\Omega)} = esssup_\Omega|\phi|$.
$H^k(\Omega) = W^{k,2}(\Omega)$	Hilbert space.
$\boldsymbol{e}_i = (0, ..., 0, 1, ..., 0)$	i^{th} standard coordinate vector. e.g. $\boldsymbol{e}_1 = (1, 0, 0)$ in $\mathbb{R}^3$.
$\boldsymbol{X} = (\boldsymbol{x}, t)$	A typical point in $\mathbb{R}^d \times (0, +\infty)$. $\boldsymbol{x} = (x_1, ..., x_d)$ represents a row or column.
$\phi(\boldsymbol{X}) = \phi(x_1, ..., x_d, t)$	If $\phi : \Omega \to \mathbb{R}$. We say that ϕ smooth provided ϕ infinitely differentiable.
ϕ Lipschitz continuous	If $\phi : \Omega \to \mathbb{R}$ such that $\|\phi(\boldsymbol{x}) - \phi(\boldsymbol{y})\| \le C\|\boldsymbol{x} - \boldsymbol{y}\|$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^{d+1}(\Omega)$ and for some constant$C$.
$\nabla = \nabla_{\boldsymbol{X}}$	Gradient vector $\nabla = (\dfrac{\partial}{\partial x_1}, ..., \dfrac{\partial}{\partial x_d}, \dfrac{\partial}{\partial t})$.
$\mathrm{End}(\mathbb{R}^d)$	Space of real $d \times d$ matrices.
$\mathrm{Sym}(\mathbb{R}^d)$	Space of real symmetric $d \times d$ matrices.
$\mathrm{Tr}\,\boldsymbol{M}$	Trace of the matrix $\boldsymbol{M}$.
$\det \boldsymbol{M}$	Determinant of the matrix $\boldsymbol{M}$.
$\boldsymbol{M}^T$	Transpose of the matrix $\boldsymbol{M}$.
$\mathrm{adj}\boldsymbol{M}$	Adjoint of the matrix $\boldsymbol{M}$.
$A(f)$	Arithmetic mean of f, $A(f) = \theta f_1 + (1 - \theta) f_2$.
$H(f)$	Harmonic mean of f, $1/H(f) = \theta/f_1 + (1 - \theta)/f_2$.

CHAPTER 1

INTRODUCTION

In the rapidly advancing technology of today, new materials with desired characteristics are in demand. For example, one might need materials that are light but strong, flexible under some loads but rigid under others. In order to achieve these goals we can use composite materials.

Composites are mixtures of two or more materials on small length scales. Composites occur both in nature and in technology. Clouds, fog and rain are natural composites of air and water. Soil and rocks are other examples. Common metals are composites. Composites are materials that are homogeneous on a macroscopic scale but inhomogeneous on a microscopic scale. Our goal is to understand how microscopic properties influence the macroscopic behavior of a composite.

We simplify the problem by introducing an averaged description of a composite, replacing the original problem by a simpler averaged problem. This process is called the homogenization. In homogenization theory, we replace an equation with oscillating coefficients by a homogenized equation. The properties of a composite (called effective properties) depend on its microstructure. It is often difficult to describe the micro-structure of a composite. Therefore it is important to compute the set of all possible effective properties of composite materials made from the original materials; this is the so called G-closure set.

We consider a homogenization problem for active materials whose study was pioneered by Lurie [2, 10, 11, 12, 13, 14, 15, 16]. We consider the problem of periodic homogenization for two typical examples: the wave equation and the Maxwell system, where coefficients oscillate rapidly not only in space but also in time. Our goal is to examine under what conditions the homogenization procedure developed for elliptic systems also works for hyperbolic systems. Our idea is to place the homogenization problem in the abstract Hilbert space framework developed by Milton where there are a lot of formal similarities between the hyperbolic and the elliptic equations. Exploiting these similarities enables us to establish results for the hyperbolic case that are similar to the elliptic case. However, there are essential differences. In the hyperbolic case our results are based on a number of assumptions that are known facts for the elliptic case but need not hold in general in the hyperbolic case. For example, the classical theory of existence and uniqueness for the hyperbolic PDEs with variable coefficients relies in an essential way on the assumption of Lipschitz continuity of coefficients in the time variable [9]. The nature of our problems (space-time composites) forces us to deal with hyperbolic equations whose coefficients are discontinuous in both space and time. These discontinuities serve as boundaries separating regions in space-time with different material properties. Needless to say that one has to be careful in this territory. There are numerous studies of the homogenization of Maxwell's equations with coefficients oscillating in space but not in time. (See [8] and references therein.) It is the oscillations in time that lead to the new problems.

In this work we distinguish between two types of space-time composites: activated and kinetic materials [2]. The former are produced by external mechanisms that alter properties of material points in a pre-determined time-dependent manner. The latter involve the actual mechanical motion of various parts of the composite system. This distinction is important when we consider the electromagnetic phenomena described by Maxwell's equations. In the case of the actual mechanical motion we may not neglect the relativistic corrections. Even when the velocities involved are much smaller than the speed of light the relativistic corrections are still significant and readily measurable. An example is the magnetic field produced by a moving charge. This magnetic field can be understood as a relativistic correction to the electric field. We note that in the case of activated space-time composites the "interfaces" may propagate with any velocity (including infinite velocity). This does not contradict the relativistic principle because no information is transmitted along with the moving interface. The motion of the interface is the apparent motion, not the actual motion.

The structure of this work is as follows. Chapter 2 deals with general homogenization theorems for the wave and Maxwell's equations.

In Chapter 3, we consider a cell problem for periodic composites. We also compute the effective tensors of rank-one laminates for the one-dimensional wave equation and the full Maxwell's system explicitly.

Most G-closure sets have a non-empty interior, however researchers have found that sometimes G-closure sets have an empty interior, in fact, they lie on a surface. The equations describing the surface are called exact relations for effective properties of composites. An exact relation is a relation between the effective tensor of a composite material and the physical properties of its constituents, independent of their geometric arrangement. In Chapter 4, we compute all exact relations for 3-D composites with the Hall effect.

CHAPTER 2

HOMOGENIZATION FOR ACTIVE MATERIALS

2.1 Active materials

In this chapter we consider two types of PDE: the wave equation and Maxwell's system. We assume that the coefficients in these equations vary rapidly in both space and time. Our goal is to achieve an effective description of interaction of such active materials with long waves. This is a homogenization problem.

We consider the longitudinal wave propagation along elastic bars, described by the one-dimensional wave equation.

$$(\rho u_t)_t - (k u_x)_x = 0, \tag{2.1}$$

where the material parameters ρ and k are both space and time dependent. The model homogenization problem is to study the limit as $\epsilon \to 0$ of the solution u^ϵ of

$$(\rho(\frac{x}{\epsilon}, \frac{t}{\epsilon})u_t^\epsilon)_t - (k(\frac{x}{\epsilon}, \frac{t}{\epsilon})u_x^\epsilon)_x = 0, \tag{2.2}$$

where $\rho(y, \tau)$, and $k(y, \tau)$ are piecewise constant and periodic in y and τ. Similarly, an electromagnetic wave propagating along the z-axis is described by Maxwell's equations

$$\nabla \times \mathbf{E} = -\mathbf{B}_t, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{H} = \mathbf{D}_t, \quad \nabla \cdot \mathbf{D} = 0.$$

The magnetic and dielectric permeabilities of the material oscillate on a small length scale in both space and time:

$$\varepsilon_\epsilon = \varepsilon(\boldsymbol{x}/\epsilon, t/\epsilon), \quad \mu_\epsilon = \mu(\boldsymbol{x}/\epsilon, t/\epsilon).$$

We can compare the orders of magnitude of space and time variables by means of the characteristic velocity of propagation of disturbances *in the material*.

In section 2.2 we begin with a simple explicit example of the one dimensional wave equation in a two-phase active medium. We show that under some circumstances either existence or uniqueness may fail. The conditions for existence and uniqueness are exactly the conditions of the regular transport of the wave across the interface [11].

In section 2.3 we consider the homogenization problem for one-dimensional wave equation.

In section 2.4 we consider a similar homogenization problem for Maxwell's equations.

We follow the general outline of Tartar's proof of the homogenization theorem for elliptic equations [22] to establish the homogenization theorem for hyperbolic equations.

2.2 Non-existence and non-uniqueness in hyperbolic problems

The aim of this section is to study the existence and uniqueness of solutions to the one-dimensional wave equation. The conditions for existence and uniqueness that we obtain are exactly the conditions of the regular transport of the wave across the interface [11]. But we also examine what fails if these conditions are not satisfied. We consider the following initial boundary value problem :

$$\begin{gathered}(\rho(x,t)u_t)_t - (k(x,t)u_x)_x = 0,\ (x,t) \in \mathbb{R} \times (0,+\infty) \\ u(x,0) = u_0(x); \quad u_t(x,0) = v_0(x),\ x \in \mathbb{R}\end{gathered} \tag{2.3}$$

where

$$\rho(x,t) = \begin{cases} \rho_1,\ x < vt \\ \rho_2,\ x > vt, \end{cases}$$

$$k(x,t) = \begin{cases} k_1,\ x < vt \\ k_2,\ x > vt, \end{cases}$$

and where $v > 0$ is the velocity of the interface separating materials (ρ_1, k_1) and (ρ_2, k_2). Figure 2.1 shows the distributions of the two materials in space and time.

Let $c_i = \sqrt{\rho_i/k_i}$, $i = 1, 2$ be the two local phase velocities. We have

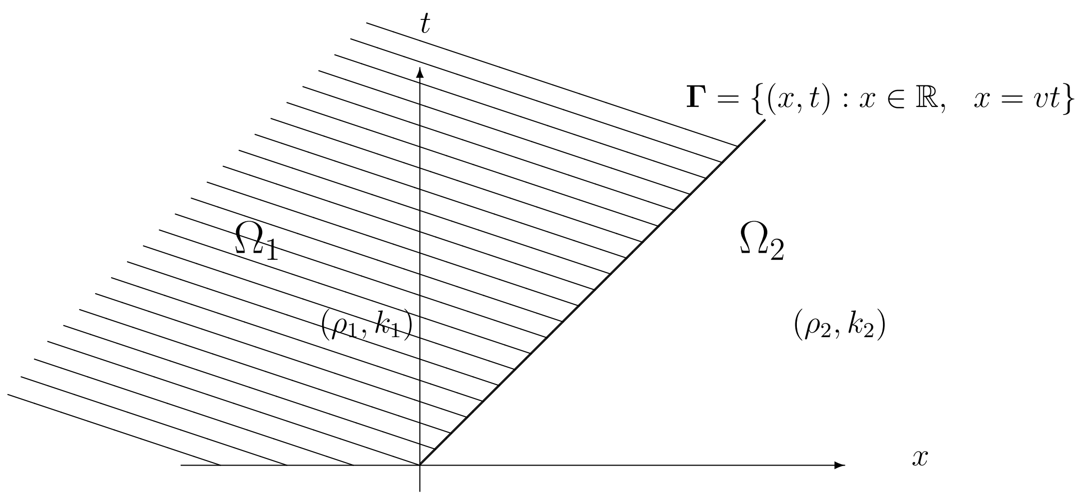


Figure 2.1: A two phase active medium

$$\begin{cases} u^1_{tt} - c_1^2 u^1_{xx} = 0 \ \text{ in } \Omega_1 \\ u^1(x,0) = u_0(x)\ ; \qquad u^1_t(x,0) = v_0(x) \ \text{ for } x < 0. \end{cases} \tag{2.4}$$

and

$$\begin{cases} u^2_{tt} - c_2^2 u^2_{xx} = 0 \ \text{ in } \Omega_2 \\ u^2(x,0) = u_0(x) \ ; \ u^2_t(x,0) = v_0(x) \ \text{ for } x > 0. \end{cases} \tag{2.5}$$

On the interface Γ we also have the continuity of the wave amplitude and the transmission conditions

$$\begin{cases} u^2(vt,t) = u^1(vt,t) & t > 0 \\ vu^2_t(vt,t) + c_2^2 u^2_x(vt,t) = \rho_1/\rho_2(vu^1_t(vt,t) + c_1^2 u^1_x(vt,t)) & t > 0. \end{cases} \tag{2.6}$$

By the d'Alembert's formula, our solution u^i has the form

$$u^i(x,t) = f_i(x + c_i t) + g_i(x - c_i t), \ (x,t) \in \Omega_i \tag{2.7}$$

with appropriate functions $f_i, \ g_i$ for each $i = 1, 2$. The initial conditions determine $f_1(\xi), \ g_1(\xi)$ for $\xi < 0$ and $f_2(\xi), \ g_2(\xi)$ for $\xi > 0$.

$$2f_i(x) = u_0(x) + \frac{1}{c_i}\int_0^x v_0(\xi)d\xi, \ x < 0 \text{ for } i = 1, \ x < 0 \text{ for } i = 2,$$

$$2g_i(x) = u_0(x) - \frac{1}{c_i}\int_0^x v_0(\xi)d\xi, \ x < 0 \text{ for } i = 1, \ x < 0 \text{ for } i = 2.$$

Substituting (2.7) into the first equation (2.6) we obtain

$$f_1((v + c_1)t) + g_1((v - c_1)t) = f_2((v + c_2)t) + g_2((v - c_2)t), \ t > 0. \tag{2.8}$$

Substituting (2.7) into the second equation (2.6) we obtain

$$\begin{aligned} &(v + c_1)f_1'((v + c_1)t) - (v - c_1)g_1'((v - c_1)t) \\ &= \alpha\big\{(v + c_2)f_2'((v + c_2)t) - (v - c_2)g_2'((v - c_2)t)\big\}, \end{aligned} \tag{2.9}$$

where $\alpha = \rho_2 c_2/\rho_1 c_1$. Integrating (2.9) over t we obtain :

$$f_1((v + c_1)t) - g_1((v - c_1)t) = \alpha\big\{f_2((v + c_2)t) - g_2((v - c_2)t)\big\}. \tag{2.10}$$

The constant of integration in (2.10) is non-essential because f_i, g_i are determined up to an additive constant for each $i = 1, 2$. Finally, solving (2.8) and (2.10) for f_1 and g_1 we obtain

$$\begin{cases} 2f_1((v + c_1)t) = (1 + \alpha)f_2((v + c_2)t) + (1 - \alpha)g_2((v - c_2)t), \\ 2g_1((v - c_1)t) = (1 - \alpha)f_2((v + c_2)t) + (1 + \alpha)g_2((v - c_2)t) \end{cases} \tag{2.11}$$

for all $t > 0$. We are now ready to study the existence and uniqueness questions for (2.4), (2.5), and (2.11).

In order determine u_1 in $\Omega_1 = \{(x,t) : x < vt,\ t > 0\}$ we need to know $f_1(\cdot)$ on $\mathbb{R}$ and $g_1(\cdot)$ on $\mathbb{R}$, if $v > c_1$ or $g_1(\cdot)$ on $(-\infty, 0]$, if $v \leq c_1$. In order determine u_2 in $\Omega_2 = \{(x,t) : x > vt,\ t > 0\}$ we need to know $f_2(\cdot)$ on $(0, +\infty)$ and $g_2(\cdot)$ on $\mathbb{R}$, if $v < c_2$ or $g_2(\cdot)$ on $[0, +\infty)$, if $v \geq c_2$. We see now that we have to consider the following four cases.

Case I: $v > \max\{c_1,\ c_2\}$. We see from (2.11) that f_1 and g_1 on $[0, +\infty)$ are uniquely determined by f_2 and g_2 on $[0, +\infty)$. Thus, we have a unique solution for the initial value problem (2.3).

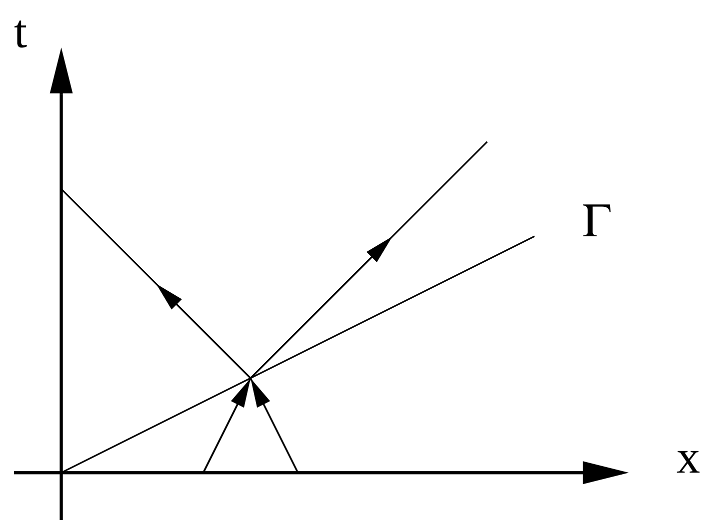


Figure 2.2: $v > c_1,\ v > c_2$.

Case II: $c_1 < v < c_2$. The equation (2.11) expresses f_1 and g_1 on $[0, +\infty)$ in terms of f_2 on $[0, +\infty)$ and g_2 on $(-\infty, 0]$. Thus, choosing g_2 on $(-\infty, 0]$ arbitrarily we obtain an infinite family of solutions.

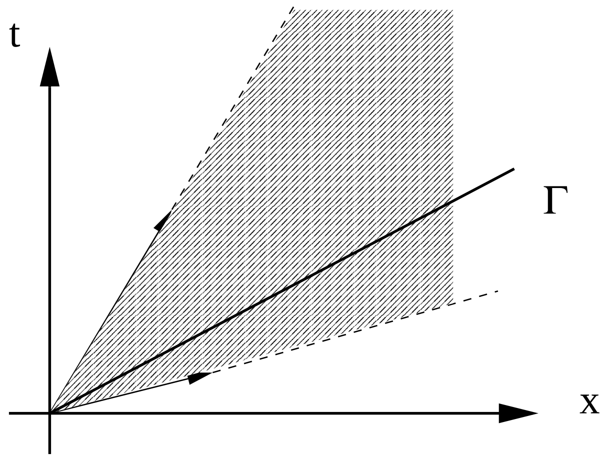


Figure 2.3: $v > c_1, \;\; v < c_2$.

Case III: $c_2 \leq v \leq c_1$. The second equation in (2.11) expresses g_1 on $(-\infty, 0]$ in terms of f_2 and g_2 on $[0, +\infty)$. All three functions are determined in the indicated regions by initial data. Thus, unless initial data is specially chosen, we will have a contradictory set of constraints. In this case we have non-existence.

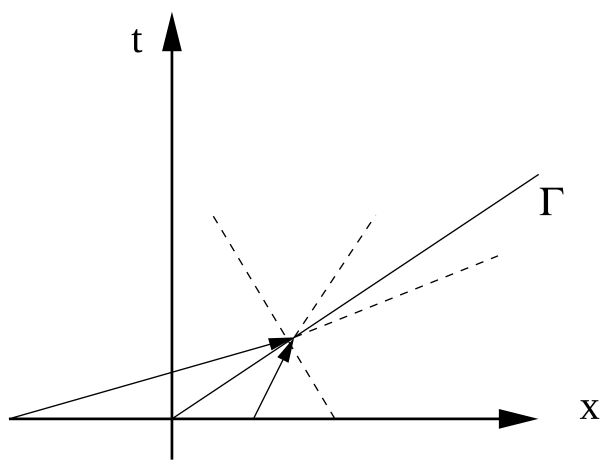


Figure 2.4: $v < c_1, \;\; v > c_2$.

Case IV: $v < \min\{c_1, c_2\}$. Solving (2.11) for f_1 and g_2 we obtain

$$\begin{cases} (1+\alpha)f_1((v+c_1)t) = 2\alpha f_2((v+c_2)t) + (1-\alpha)g_1((v-c_1)t) \\ (1+\alpha)g_2((v-c_2)t) = -(1-\alpha)f_2((v+c_2)t) + 2g_1((v-c_1)t). \end{cases} \tag{2.12}$$

We see that f_1 on $[0, +\infty)$ and g_2 on $(-\infty, 0]$ are defined in terms of g_1 on $(-\infty, 0]$ and f_2 on $[0, +\infty)$. It follows then that the functions u^1 and u^2 are uniquely defined on $t > 0$. Thus we have existence and uniqueness in this case.

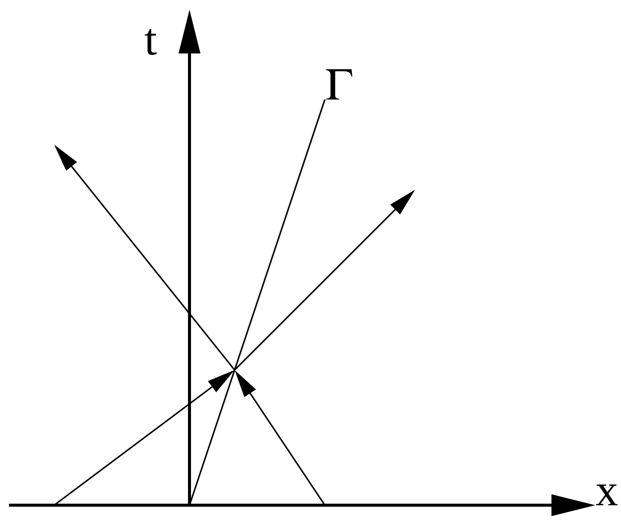


Figure 2.5: $v < c_1, \;\; v < c_2$.

Physical interpretation. The physical interpretation for the first and the last case, where we have existence and uniqueness was explained by Lurie [14, p. 289]. Here we give the physical interpretation of the remaining cases. The physical interpretation of the first and the last case is given here for the sake of completeness. In the case $v > \max\{c_1, c_2\}$ the moving interface overtakes the forward moving wave in region Ω_1 and gains on the forward moving wave in region Ω_2. In the coordinate system moving together with the interface Γ we will see that the forward moving wave in region Ω_2 is actually moving backwards with respect to us and regularly refracts/reflects in the stationary interface. In fact, every point in space-time ($t > 0$) is hit by exactly one forward moving wave, either from region Ω_1 or from region Ω_2. This situation is depicted in Figure 2.2. Thus, we have existence and uniqueness.

In the case $c_1 < v < c_2$ the forward moving wave in Ω_1 is still slower than the interface, while the forward moving wave in Ω_2 is faster than the interface. Thus the two forward moving characteristics emanating from the origin in Figure 2.3 bound the sector in space-time containing Γ that is not visited by any of the forward moving waves. The solution in that region is not uniquely determined by the initial data. That is why this case corresponds to non-uniqueness.

In the case $c_2 \leq v \leq c_1$ shown in Figure 2.4 the forward wave in Ω_1 moves faster than the interface, passes through it and collides with the forward moving wave in region Ω_2 which is moving slower than the interface. In this case there are points in space-time through which pass two forward waves leading to contradictory values of the amplitude. Thus we have non-existence in this case.

In the remaining case $v < \min\{c_1, c_2\}$ every point in space is again covered by exactly one forward moving characteristic emanating either from region Ω_1 or Ω_2 as shown in Figure 2.5. Thus in this case we have both existence and uniqueness. We have considered only the forward moving characteristics because we have assumed that the interface moves forward ($v > 0$). In this case every point in space is covered exactly once by the backward characteristics regardless of which case we are in.

2.3 Homogenization for the wave equation

We study the limit as $\epsilon \to 0$ of the solution u^ϵ of

$$\begin{gathered} \frac{\partial}{\partial t}(\rho(\frac{\boldsymbol{x}}{\epsilon}, \frac{t}{\epsilon})u_t^\epsilon) - \nabla_{\boldsymbol{x}} \cdot (\boldsymbol{k}(\frac{\boldsymbol{x}}{\epsilon}, \frac{t}{\epsilon})\nabla_{\boldsymbol{x}} u^\epsilon) = 0; \\ u(\boldsymbol{x}/\epsilon, 0) = u_0(\boldsymbol{x}),\ u_t(\boldsymbol{x}/\epsilon, 0) = v_0(\boldsymbol{x}), \end{gathered} \tag{2.13}$$

where $\rho(\boldsymbol{y}, \tau)$, and $\boldsymbol{k}(\boldsymbol{y}, \tau)$ are periodic in space-time with a parallelepiped of periods $Q \subset \mathbb{R}^{d+1}$ and where $u_0,\ v_0 \colon \mathbb{R}^d \longrightarrow \mathbb{R}$ are given Lipschitz functions with compact support.

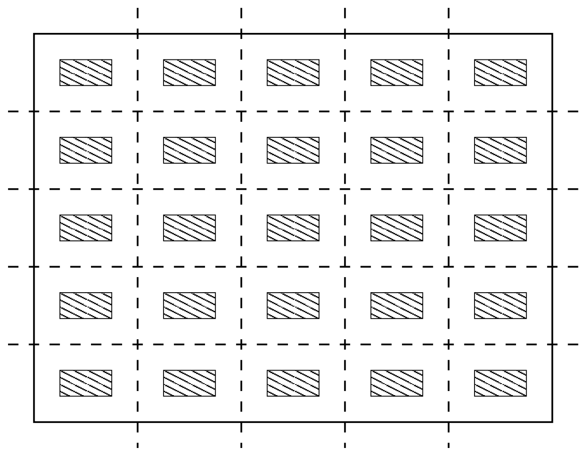


Figure 2.6: A periodic composite

Definition 1. $H^1_p(Q) = \{h \ : \ h \text{ is } Q-\text{periodic and } h \in H^1_{\text{loc}}(\mathbb{R}^{d+1})\}$

Notation 1.

$$\boldsymbol{\sigma}_\epsilon = \boldsymbol{\sigma}(\boldsymbol{x}/\epsilon, t/\epsilon) = \begin{bmatrix} \boldsymbol{k}(\boldsymbol{x}/\epsilon, t/\epsilon) & \mathbf{0} \\ \mathbf{0} & -\rho(\boldsymbol{x}/\epsilon, t/\epsilon) \end{bmatrix}. \tag{2.14}$$

Notation 2. *Set* $\boldsymbol{X} = (\boldsymbol{x}, t) \in \mathbb{R}^d \times (0, +\infty)$, *and denote* $\nabla = \nabla_{\boldsymbol{X}}$.

In our new notation the problem (2.13) becomes

$$\begin{gathered} \nabla \cdot (\boldsymbol{\sigma}_\epsilon \nabla u^\epsilon) = 0; \\ u^\epsilon(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), \quad \tfrac{\partial u^\epsilon}{\partial t}(\boldsymbol{x}, 0) = v_0(\boldsymbol{x}). \end{gathered} \tag{2.15}$$

We would like to investigate when does u^ϵ converge to the solution u^0 of the homogenized equation

$$\begin{gathered} -\nabla \cdot (\boldsymbol{\sigma}^* \nabla u^0) = 0 \ \text{ in } \mathbb{R}^d \times (0, +\infty); \\ u^0(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), \quad \tfrac{\partial u^0}{\partial t}(\boldsymbol{x}, 0) = v_0(\boldsymbol{x}). \end{gathered} \tag{2.16}$$

Here $\boldsymbol{\sigma}^*$ is defined by its action on an arbitrary vector $\boldsymbol{\gamma} \in \mathbb{R}^{d+1}$ by

$$\boldsymbol{\sigma}^* \boldsymbol{\gamma} = \fint_Q \boldsymbol{\sigma}(\mathbf{X})(\nabla w + \boldsymbol{\gamma}) d\mathbf{X}, \tag{2.17}$$

where w is the solution of the cell problem

$$\begin{cases} \nabla \cdot (\boldsymbol{\sigma}(\boldsymbol{X})(\nabla w + \boldsymbol{\gamma})) = 0, \\ w \in H^1_p(Q). \end{cases} \tag{2.18}$$

By contrast with the elliptic case we cannot prove that the cell problem has a unique solution. Therefore, we are led to make the following assumption.

Assumption 1. *We assume that the problem (2.18) has a solution for all* $\boldsymbol{\gamma} \in \mathbb{R}^{d+1}$. *In addition assume that if* $w^0 \in H^1_p(Q)$ *solves* $\nabla \cdot (\boldsymbol{\sigma}(\boldsymbol{X})(\nabla w^0)) = 0$ *then*

$$\int_Q \boldsymbol{\sigma}(\boldsymbol{X}) \nabla w^0(\boldsymbol{X}) d\boldsymbol{X} = 0. \tag{2.19}$$

Definition 2. *If Assumption 1 is satisfied then we say that* $\boldsymbol{\sigma}^*$ *is well-defined.*

Our main tool will be the div-curl lemma.

Lemma 2.1. Div-Curl Lemma

Let Ω *be an open subset in* $\mathbb{R}^d$. *Suppose* $\boldsymbol{p}^\epsilon$ *and* $\boldsymbol{v}^\epsilon$ *converges weakly to* $\boldsymbol{p}^0$ *and* $\boldsymbol{v}^0$ *in* $L^2(\Omega; \mathbb{R}^d)$, *respectively.*
If $\nabla \cdot \boldsymbol{p}^\epsilon \longrightarrow \nabla \cdot \boldsymbol{p}^0$ *in* $H^{-1}(\Omega)$ *strong and* $\nabla \times \boldsymbol{v}^\epsilon = 0$. *Then* $\boldsymbol{p}^\epsilon \cdot \boldsymbol{v}^\epsilon \overset{*}{\rightharpoonup} \boldsymbol{p}^0 \cdot \boldsymbol{v}^0$ *in the sense of measures.*

The proof can be found in [21]. We also recall the Riemann-Lebesgue lemma.

Lemma 2.2. Riemann-Lebesgue Lemma

If $f(\boldsymbol{y})$ *is* Q*-periodic,* $L^\infty(\mathbb{R}^d)$ *and* $g(\boldsymbol{x}) \in L^1(\mathbb{R}^d)$. *Then*

$$\int_{\mathbb{R}^d} f(n\boldsymbol{x}) g(\boldsymbol{x}) d\boldsymbol{x} \longrightarrow \fint_Q f(\boldsymbol{y}) d\boldsymbol{y} \int_{\mathbb{R}^d} g(\boldsymbol{x}) d\boldsymbol{x} \tag{2.20}$$

In other words,

$$f(n\boldsymbol{x}) \overset{*}{\rightharpoonup} \fint_Q f(\boldsymbol{y}) d\boldsymbol{y} \text{ weak-}* \text{ in } L^\infty(\mathbb{R}^d).$$

Theorem 2.1. Convergence of Arbitrary Solutions
Let $\Omega \subset \mathbb{R}^d \times (0,T)$ be an open and bounded set. Assume that the sequence $u^\epsilon \in H^1(\Omega)$ and $u^\epsilon \rightharpoonup u^0$ in $H^1(\Omega)$ weak, solves $\nabla \cdot (\boldsymbol{\sigma}_\epsilon \nabla u^\epsilon) = 0$, where $\boldsymbol{\sigma}_\epsilon$ is defined by (2.14). Assume $\boldsymbol{\sigma}^*$ is well-defined in the sense of Definition 2. Then u^0 is a solution of the homogenized equation $\nabla \cdot (\boldsymbol{\sigma}^* \nabla u^0) = 0$ and $\boldsymbol{\sigma}(\boldsymbol{x}/\epsilon, t/\epsilon)\nabla u^\epsilon \rightharpoonup \boldsymbol{\sigma}^* \nabla u^0$ weakly in $L^2(\Omega; \mathbb{R}^{d+1})$.

Our proof follows the same steps as the proof of the analogous theorem for conductivity by Murat and Tartar [21, 20].

Proof. Let $\boldsymbol{p}^\epsilon = \boldsymbol{\sigma}_\epsilon \nabla u^\epsilon$. Then $\boldsymbol{p}^\epsilon$ is bounded in $L^2(\Omega; \mathbb{R}^d)$. Extract a weakly convergent subsequence of $\boldsymbol{p}^\epsilon$ in $L^2(\Omega; \mathbb{R}^d)$ and define $\boldsymbol{p}^0$ as its weak limit. Let w be a solution of (2.18), which exists because $\boldsymbol{\sigma}^*$ *is well-defined*. Set $\boldsymbol{\psi}^\epsilon(\boldsymbol{X}) = \nabla w(\boldsymbol{X}/\epsilon) + \boldsymbol{\gamma}$. Then

$$\nabla \times \boldsymbol{\psi}^\epsilon = 0, \quad \nabla \cdot (\boldsymbol{\sigma}_\epsilon \boldsymbol{\psi}^\epsilon) = 0.$$

By the Riemann-Lebesgue lemma we have : $\boldsymbol{\psi}^\epsilon \rightharpoonup \boldsymbol{\gamma}$, $\boldsymbol{\sigma}_\epsilon \boldsymbol{\psi}^\epsilon \rightharpoonup \boldsymbol{\sigma}^* \boldsymbol{\gamma}$ in $L^2(\Omega; \mathbb{R}^{d+1})$ weak. It is easy to see that

$$\boldsymbol{\psi}^\epsilon \cdot \boldsymbol{p}^\epsilon = \boldsymbol{\psi}^\epsilon \cdot (\boldsymbol{\sigma}_\epsilon \nabla u^\epsilon) = \boldsymbol{\sigma}_\epsilon \boldsymbol{\psi}^\epsilon \cdot \nabla u^\epsilon, \quad \forall \boldsymbol{X} \in \Omega. \tag{2.21}$$

By div-curl lemma, we can pass to the limit in (2.21) and get

$$\boldsymbol{\gamma} \cdot \boldsymbol{p}^0 = \boldsymbol{\sigma}^* \boldsymbol{\gamma} \cdot \nabla u^0. \tag{2.22}$$

Thus,

$$\boldsymbol{p}^0 = \boldsymbol{\sigma}^* \nabla u^0. \tag{2.23}$$

Moreover, $\nabla \cdot \boldsymbol{p}^\epsilon = 0$ and consequently, $\nabla \cdot \boldsymbol{p}^0 = 0$. Thus u^0 is a solution of the homogenized equation $\nabla \cdot (\boldsymbol{\sigma}^* \nabla u^0) = 0$. □

Theorem 2.2. Periodic Homogenization
Suppose u^ϵ solves (2.15). Assume the effective tensor $\boldsymbol{\sigma}^*$ is well-defined in the

sense of Definition 2. Assume there is $C(\Omega)$ such that $\|u^\epsilon\|_{H^1(\Omega)} \leq C(\Omega)$ independent of ϵ. Then $u^\epsilon \rightharpoonup u^0$ in $H^1(\Omega)$ weak, where u^0 is the unique solution of (2.16).

Proof. By our assumption of boundedness we can extract a weakly convergent subsequence $u^{\epsilon'}$ in $H^1(\Omega)$ such that $u^{\epsilon'} \rightharpoonup u^0$ in $H^1(\Omega)$ weak. Apply Theorem 2.1 and conclude that u^0 solves (2.16). Since $\boldsymbol{\sigma}^*$ is constant, (2.16) has a unique solution thus u^ϵ has a weak limit u^0. □

Next we show that the assumption of boundedness that was necessary to establish Theorem 2.2 is satisfied in the case of an activated composite where the properties appear to be moving with the uniform velocity $\boldsymbol{v}$. Let $\rho_\epsilon(\boldsymbol{x}, t) = \rho((\boldsymbol{x} - \boldsymbol{v}t)/\epsilon))$; $\boldsymbol{k}_\epsilon(\boldsymbol{x}, t) = \boldsymbol{k}((\boldsymbol{x} - \boldsymbol{v}t)/\epsilon))$, ρ, $\boldsymbol{k}$ are $[0,1]^3$-periodic in space. Let us make a linear change of variables in (2.15). Let $\boldsymbol{X}' = \boldsymbol{C}\boldsymbol{X}$ where $\boldsymbol{C} = \begin{bmatrix} \mathbf{I} & -\boldsymbol{v} \\ \mathbf{0} & 1 \end{bmatrix}$, and where $\mathbf{I}$ is a 3×3 identity matrix. Let $u'_\epsilon(\boldsymbol{X}') = u_\epsilon(\boldsymbol{C}^{-1}\boldsymbol{X}')$, $\boldsymbol{\sigma}'_\epsilon(\boldsymbol{X}') = \boldsymbol{C}\boldsymbol{\sigma}_\epsilon(\boldsymbol{C}^{-1}\boldsymbol{X}')\boldsymbol{C}^T$. Then $u'_\epsilon(\boldsymbol{X}')$ satisfies

$$\nabla' \cdot (\boldsymbol{\sigma}'_\epsilon(\boldsymbol{X}')\nabla' u'_\epsilon(\boldsymbol{X}')) = 0, \tag{2.24}$$

where $\nabla' = \nabla_{\boldsymbol{X}'}$. It follows that

$$\boldsymbol{\sigma}'_\epsilon(\boldsymbol{X}') = \begin{bmatrix} \boldsymbol{m}(\boldsymbol{x}'/\epsilon) & \boldsymbol{v}\rho(\boldsymbol{x}'/\epsilon) \\ \boldsymbol{v}\rho(\boldsymbol{x}'/\epsilon) & -\rho(\boldsymbol{x}'/\epsilon) \end{bmatrix},$$

where $\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{v}t$ and $\boldsymbol{m}(\boldsymbol{x}') = \boldsymbol{k}(\boldsymbol{x}') - (\boldsymbol{v} \otimes \boldsymbol{v})\rho(\boldsymbol{x}')$. The equation (2.24) becomes

$$\rho(\boldsymbol{x}'/\epsilon)\frac{\partial^2 u'_\epsilon}{\partial t^2} = \nabla_{\boldsymbol{x}'}\cdot\big(\boldsymbol{m}(\boldsymbol{x}'/\epsilon)\nabla_{\boldsymbol{x}'}u'_\epsilon + \rho(\boldsymbol{x}'/\epsilon)\frac{\partial u'_\epsilon}{\partial t}\boldsymbol{v}\big) + \rho(\boldsymbol{x}'/\epsilon)\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}'}\frac{\partial u'_\epsilon}{\partial t}. \tag{2.25}$$

Let

$$E_\epsilon(t) = \int_{\mathbb{R}^3}\left[\frac{1}{2}\boldsymbol{m}(\boldsymbol{x}'/\epsilon)\nabla_{\boldsymbol{x}'}u^\epsilon \cdot \nabla_{\boldsymbol{x}'}u^\epsilon + \frac{1}{2}\rho(\boldsymbol{x}'/\epsilon)(\frac{\partial u'_\epsilon}{\partial t})^2\right]d\boldsymbol{x}' \tag{2.26}$$

then

$$E'_\epsilon(t) = \int_{\mathbb{R}^3} \left[\boldsymbol{m}(\boldsymbol{x}'/\epsilon) \nabla_{\boldsymbol{x}'} u^\epsilon \cdot \nabla_{\boldsymbol{x}'} \frac{\partial u'_\epsilon}{\partial t} + \rho(\boldsymbol{x}'/\epsilon) \frac{\partial u'_\epsilon}{\partial t} \frac{\partial^2 u'_\epsilon}{\partial t^2} \right] d\boldsymbol{x}'. \tag{2.27}$$

Using (2.25) and the integration by parts we get

$$\begin{aligned} E'_\epsilon(t) = & \int_{\mathbb{R}^3} \big[\boldsymbol{m}(\boldsymbol{x}'/\epsilon) \nabla_{\boldsymbol{x}'} u'_\epsilon \cdot \nabla_{\boldsymbol{x}'} \frac{\partial u'_\epsilon}{\partial t} \big] d\boldsymbol{x}' + \\ & \int_{\mathbb{R}^3} \big[\big(\nabla_{\boldsymbol{x}'} \cdot \big(\boldsymbol{m}(\boldsymbol{x}'/\epsilon) \nabla_{\boldsymbol{x}'} u'_\epsilon + \rho(\boldsymbol{x}'/\epsilon) \frac{\partial u'_\epsilon}{\partial t} \boldsymbol{v} \big) + \rho(\boldsymbol{x}'/\epsilon) \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}'} \frac{\partial u'_\epsilon}{\partial t} \big) \frac{\partial u'_\epsilon}{\partial t} \big] d\boldsymbol{x}' \\ = & \int_{\mathbb{R}^3} \big[\boldsymbol{m}(\boldsymbol{x}'/\epsilon) \nabla_{\boldsymbol{x}'} u'_\epsilon \cdot \nabla_{\boldsymbol{x}'} \frac{\partial u'_\epsilon}{\partial t} \big] d\boldsymbol{x}' + \\ & \int_{\mathbb{R}^3} \big[\rho(\boldsymbol{x}'/\epsilon) \frac{\partial u'_\epsilon}{\partial t} \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}'} \frac{\partial u'_\epsilon}{\partial t} - (\boldsymbol{m}(\boldsymbol{x}'/\epsilon) \nabla_{\boldsymbol{x}'} u'_\epsilon + \rho(\boldsymbol{x}'/\epsilon) \frac{\partial u'_\epsilon}{\partial t} \boldsymbol{v}) \cdot \nabla_{\boldsymbol{x}'} \frac{\partial u'_\epsilon}{\partial t} \big] d\boldsymbol{x}' \\ = & \; 0. \end{aligned} \tag{2.28}$$

This implies $E_\epsilon(t)$ is independent of t.

Lemma 2.3. *Let $\mathbf{A} = \boldsymbol{k}(\boldsymbol{x})/\rho(\boldsymbol{x})$ be the acoustic tensor. Assume that the "properties wave" is slower than all of the characteristic speeds, i.e. $|\boldsymbol{v}|^2 < \|\mathbf{A}(\mathbf{x})^{-1}\|^{-1}$ for all $\boldsymbol{x} \in \mathbb{R}^3$. Then $\boldsymbol{m}(\boldsymbol{x})$ is a positive definite matrix.*

Proof. A symmetric matrix is positive definite if and only if all of its eigenvalues are positive. Since $\boldsymbol{k}(\boldsymbol{x})$ is a positive definite matrix, we can write

$$\boldsymbol{k}(\boldsymbol{x})^{-1/2} \boldsymbol{m}(\boldsymbol{x}) \boldsymbol{k}(\boldsymbol{x})^{-1/2} = \mathbf{I} - (\boldsymbol{k}(\boldsymbol{x})^{-1/2} \boldsymbol{v} \otimes \boldsymbol{k}(\boldsymbol{x})^{-1/2} \boldsymbol{v}) \rho(\boldsymbol{x}) = \mathbf{I} - \boldsymbol{a} \otimes \boldsymbol{a}, \tag{2.29}$$

where $\boldsymbol{a} = \sqrt{\rho} \boldsymbol{k}^{-1/2} \boldsymbol{v}$. Thus, the eigenvalues of $\boldsymbol{k}(\boldsymbol{x})^{-1/2} \boldsymbol{m}(\boldsymbol{x}) \boldsymbol{k}(\boldsymbol{x})^{-1/2}$ are $1 - |\boldsymbol{a}|^2$ and 1. We have

$$|\boldsymbol{a}|^2 = |\sqrt{\rho} \boldsymbol{k}^{-1/2} \boldsymbol{v}|^2 = \rho(\boldsymbol{k}^{-1/2} \boldsymbol{v}, \boldsymbol{k}^{-1/2} \boldsymbol{v}) = (\mathbf{A}^{-1} \boldsymbol{v}, \boldsymbol{v}) \le \|\mathbf{A}^{-1}\| |\boldsymbol{v}|^2 < 1. \tag{2.30}$$

So, we have $|\boldsymbol{a}|^2 < 1$. Therefore, $\boldsymbol{m}(\boldsymbol{x})$ is positive definite. $\square$

Lemma 2.3 implies that there exists $\alpha > 0$ such that $\boldsymbol{m}(\boldsymbol{x}') \geq \alpha \mathbf{I}$, for all $\mathbf{x} \in \mathbb{R}^3$. Thus we have

$$\begin{aligned} E_\epsilon(t) &= \int_{\mathbb{R}^3} \left[\frac{1}{2} \boldsymbol{m}(\boldsymbol{x}'/\epsilon) \nabla_{\boldsymbol{x}'} u^\epsilon \cdot \nabla_{\boldsymbol{x}'} u^\epsilon + \frac{1}{2} \rho(\boldsymbol{x}'/\epsilon) (\frac{\partial u'_\epsilon}{\partial t})^2 \right] d\boldsymbol{x}' \\ &\geq \frac{1}{2} \int_{\mathbb{R}^3} \left[\alpha |\nabla_{\boldsymbol{x}'} u^\epsilon|^2 + \beta (\frac{\partial u'_\epsilon}{\partial t})^2 \right] d\boldsymbol{x}' \end{aligned} \tag{2.31}$$

for some $\alpha > 0,\ \beta > 0$. Therefore

$$\int_{\mathbb{R}^3} |\nabla' u'_\epsilon(\boldsymbol{x}', t)|^2 d\boldsymbol{x}' \ \leq C E_\epsilon(t) = C E_\epsilon(0) \ \leq C' \int_{\mathbb{R}^3} \left[|\nabla_{\boldsymbol{x}} u_0|^2 + |v_0|^2 \right] d\boldsymbol{x}'. \tag{2.32}$$

We thus conclude that

$$\forall\ T > 0, \quad \int_0^T \int_{\mathbb{R}^3} |\nabla' u'_\epsilon(\boldsymbol{X}')|^2 d\boldsymbol{X}' \ \leq \ C' T \int_{\mathbb{R}^3} \left[|\nabla_{\boldsymbol{x}} u_0|^2 + |v_0|^2 \right] d\boldsymbol{x}' \tag{2.33}$$

independent of ϵ.

Theorem 2.3. *Suppose $\boldsymbol{\sigma}(\boldsymbol{y}, \tau) = \boldsymbol{\sigma}(\boldsymbol{y} - \boldsymbol{v}\tau)$ then $\boldsymbol{\sigma}^*$ is well-defined in the sense of Definition 2.*

Proof. Let $\boldsymbol{\gamma} = (\boldsymbol{\gamma}_{\boldsymbol{x}}, \gamma_t) \in \mathbb{R}^4$. When we change variables $\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{v}t$ in (2.18) We obtain

$$\begin{cases} \nabla \cdot \begin{bmatrix} \boldsymbol{m}(\boldsymbol{x}') & \boldsymbol{v}\rho(\boldsymbol{x}') \\ \boldsymbol{v}\rho(\boldsymbol{x}') & -\rho(\boldsymbol{x}') \end{bmatrix} \left(\begin{bmatrix} \nabla_{\boldsymbol{x}'} w \\ w_t \end{bmatrix} + \begin{bmatrix} \boldsymbol{\gamma}_{\boldsymbol{x}} \\ \gamma'_t \end{bmatrix} \right) = 0 \\ w(\boldsymbol{x}', t) \text{ is } [0,1]^4\text{-periodic,} \end{cases} \tag{2.34}$$

where $\gamma'_t = \boldsymbol{\gamma}_{\boldsymbol{x}} \cdot \boldsymbol{v} + \gamma_t$ and $w'(\boldsymbol{x}', t) = w(\boldsymbol{x}' + \boldsymbol{v}t, t)$.

We want to show that for all $\boldsymbol{\gamma} \in \mathbb{R}^4$ the equation (2.18) has a solution. Equivalently, we want to show that for any $(\boldsymbol{\gamma}_{\boldsymbol{x}}, \gamma'_t) \in \mathbb{R}^4$ the equation (2.34) has a solution. In fact, we show that there is a unique $[0,1]^3$-periodic function $w'(\boldsymbol{x}')$ such that $w'(\boldsymbol{x}', t) = w'(\boldsymbol{x}')$ is a solution of (2.34).

Substituting $w'(\boldsymbol{x}')$ for $w'(\boldsymbol{x}',t)$ in (2.34) we get

$$\begin{cases} \nabla_{\boldsymbol{x}'} \cdot (\boldsymbol{m}(\boldsymbol{x}')\nabla_{\boldsymbol{x}'} w'(\boldsymbol{x}')) + \nabla_{\boldsymbol{x}'} \cdot (\boldsymbol{m}(\boldsymbol{x}')\boldsymbol{\gamma}_{\boldsymbol{x}} + \rho \boldsymbol{v} \gamma'_t) = 0 \\ w \text{ is } [0,1]^3\text{-periodic.} \end{cases} \tag{2.35}$$

This elliptic problem has a unique solution $w' \in H^1_p([0,1]^3)$ because $\mathbf{m}(\mathbf{x}')$ is positive definite by Lemma 2.3.

Finally we show that if $w^0(\mathbf{X})$ is a solution of the cell problem (2.18) with $\boldsymbol{\gamma} = \mathbf{0}$ then (2.19) holds. We note that if $w^0(\mathbf{X})$ solves (2.18) then $w'(\mathbf{X}')$ solves (2.34) with $\boldsymbol{\gamma}' = \mathbf{0}$. Let

$$< w' >^t (\boldsymbol{x}') = \int_0^1 w'(\boldsymbol{x}',t)dt$$

be the time average of w'. Expanding (2.34) we get

$$\nabla_{\boldsymbol{x}'} \cdot \big(\boldsymbol{m}(\boldsymbol{x}')\nabla_{\boldsymbol{x}'} w' + \rho(\boldsymbol{x}')\boldsymbol{v}\frac{\partial w'}{\partial t}\big) + \frac{\partial}{\partial t}\big(\rho(\boldsymbol{x}')\boldsymbol{v} \cdot \nabla_{\boldsymbol{x}'} w' - \rho(\boldsymbol{x}')\frac{\partial w'}{\partial t}\big) = 0. \tag{2.36}$$

Averaging (2.36) in time we obtain:

$$\nabla_{\boldsymbol{x}'} \cdot \Big(\boldsymbol{m}(\boldsymbol{x}')\nabla_{\boldsymbol{x}'} < w' >^t\Big) = 0 \tag{2.37}$$

Lemma 2.3 says that $\boldsymbol{m}(\boldsymbol{x}')$ is positive definite, since we assume that the wave of properties is slower than any of the characteristic speeds. Consequently,

$$\nabla_{\boldsymbol{x}'} < w' >^t = 0. \tag{2.38}$$

So, $< w' >^t$ is independent of $\boldsymbol{x}'$.

Now we make the same change of variables: $\boldsymbol{X}' = \boldsymbol{C}\boldsymbol{X}$ in the integral

$$\int_Q \boldsymbol{\sigma}(\boldsymbol{X})\nabla_{\boldsymbol{x}} w^0(\boldsymbol{X})d\boldsymbol{X}.$$

We obtain

$$\int_Q \boldsymbol{\sigma}(\boldsymbol{X})\nabla_{\boldsymbol{x}} w^0(\boldsymbol{X})d\boldsymbol{X} = \frac{\boldsymbol{C}^{-1}}{\det \boldsymbol{C}} \int_{Q'} \boldsymbol{\sigma}'(\boldsymbol{X}')\nabla_{\boldsymbol{x}'} w'(\boldsymbol{X}')d\boldsymbol{X}'.$$

But

$$\boldsymbol{\sigma}'(\boldsymbol{X}')\nabla_{\boldsymbol{x}'}w'(\boldsymbol{X}') = \begin{bmatrix} \boldsymbol{m}(\boldsymbol{x}')\nabla_{\boldsymbol{x}'}w'(\boldsymbol{x}',t) + \ \rho(\boldsymbol{x}')\frac{\partial w'}{\partial t}(\boldsymbol{x}',t) \\ \rho(\boldsymbol{x}')\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}'}w'(\boldsymbol{x}',t) - \rho(\boldsymbol{x}')\frac{\partial w'}{\partial t}(\boldsymbol{x}',t) \end{bmatrix}. \tag{2.39}$$

Therefore,

$$\int_0^1 \boldsymbol{\sigma}'(\boldsymbol{X}')\nabla_{\boldsymbol{x}'}w'(\boldsymbol{X}')dt = \begin{bmatrix} \boldsymbol{m}(\boldsymbol{x}')\nabla_{\boldsymbol{x}'} < w' >^t \\ \rho'(\boldsymbol{x}')\boldsymbol{v}\nabla_{\boldsymbol{x}'} < w' >^t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{2.40}$$

Thus, the property (2.19) is proved. □

2.4 Homogenization for Maxwell's system

As before, we denote the points in $\mathbb{R}^4$ by $\boldsymbol{X} = (\boldsymbol{x}, t)$. Assume there are no free charges and no free currents in a given medium. Consider Maxwell's system:

$$\begin{cases} \nabla\times\mathbf{E} = -\partial\mathbf{B}/\partial t,\ \ \nabla\cdot\mathbf{B} = 0,\ \ \nabla\times\mathbf{H} = \partial\mathbf{D}/\partial t,\ \ \nabla\cdot\mathbf{D} = 0, \\ \mathbf{D} = \varepsilon\ \mathbf{E},\ \mathbf{B} = \mu\ \mathbf{H} \end{cases} \tag{2.41}$$

where $\mathbf{E}$ is the electric field, $\mathbf{D}$ is the electric displacement, $\mathbf{B}$ is the magnetic induction, and $\mathbf{H}$ is the magnetic field; $\varepsilon(\boldsymbol{x}, t)$ and $\mu(\boldsymbol{x}, t)$ is the dielectric permittivity and the magnetic permeability respectively.

In homogenization theory, we consider Maxwell's equations with oscillating coefficients.

$$\begin{cases} \nabla\times\mathbf{E}_\epsilon = -\partial\mathbf{B}_\epsilon/\partial t,\ \nabla\cdot\mathbf{B}_\epsilon = 0,\ \nabla\times\mathbf{H}_\epsilon = \partial\mathbf{D}_\epsilon/\partial t,\ \nabla\cdot\mathbf{D}_\epsilon = 0 \\ \mathbf{D}_\epsilon = \varepsilon_\epsilon\ \mathbf{E}_\epsilon,\ \mathbf{B}_\epsilon = \mu_\epsilon\ \mathbf{H}_\epsilon, \\ \mathbf{B}_\epsilon(\boldsymbol{x}, 0) = \mathbf{B}_0(\boldsymbol{x}),\ \mathbf{D}_\epsilon(\boldsymbol{x}, 0) = \mathbf{D}_0(\boldsymbol{x}) \text{ where } \nabla\cdot\mathbf{B}_0 = \nabla\cdot\mathbf{D}_0 = 0, \end{cases} \tag{2.42}$$

where $\mathbf{B}_0(\boldsymbol{x}),\ \mathbf{D}_0(\boldsymbol{x})$ have compact support and are in $L^2(\mathbb{R}^3)$. Here

$$\varepsilon_\epsilon = \varepsilon(\boldsymbol{x}/\epsilon, t/\epsilon);\ \ \mu_\epsilon = \mu(\boldsymbol{x}/\epsilon, t/\epsilon),$$

where functions $\varepsilon(\boldsymbol{y},\tau)$ and $\mu(\boldsymbol{y},\tau)$ are Q-periodic, and Q is a parallelepiped of periods in the 4-dimensional space-time.

For our purposes it will be convenient to denote

$$\mathbb{E}_\epsilon(\boldsymbol{x},t) = (\mathbf{E}_\epsilon(\boldsymbol{x},t), \mathbf{B}_\epsilon(\boldsymbol{x},t)) \tag{2.43}$$

and

$$\mathbb{J}_\epsilon(\boldsymbol{x},t) = (-\mathbf{D}_\epsilon(\boldsymbol{x},t), \mathbf{H}_\epsilon(\boldsymbol{x},t)). \tag{2.44}$$

Then the constitutive relations from (2.42) can be written as

$$\mathbb{J}_\epsilon(\boldsymbol{x},t) = \mathbb{L}(\boldsymbol{x}/\epsilon, t/\epsilon)\mathbb{E}_\epsilon(\boldsymbol{x},t), \tag{2.45}$$

where

$$\mathbb{L}(\boldsymbol{y},\tau) = \begin{bmatrix} -\varepsilon(\boldsymbol{y},\tau)\mathbf{I} & \mathbf{0} \\ \mathbf{0} & 1/\mu(\boldsymbol{y},\tau)\mathbf{I} \end{bmatrix}. \tag{2.46}$$

The problem is to study the limit as $\epsilon \to 0$ of the solution $\mathbb{E}_\epsilon$, $\mathbb{J}_\epsilon$ of (2.42). We will show that under the appropriate assumptions $\mathbb{E}_\epsilon \rightharpoonup \mathbb{E}^*$, $\mathbb{J}_\epsilon \rightharpoonup \mathbb{J}^*$ weakly in L^2 where $(\mathbb{E}^*, \mathbb{J}^*)$ is the solution of the homogenized equation

$$\begin{cases} \nabla\times\mathbf{E}^* = -\partial\mathbf{B}^*/\partial t,\ \nabla\cdot\mathbf{B}^* = 0,\ \nabla\times\mathbf{H}^* = \partial\mathbf{D}^*/\partial t,\ \nabla\cdot\mathbf{D}^* = 0 \\ \mathbb{J}^* = \mathbb{L}^*\mathbb{E}^*, \\ \mathbf{B}^*(\boldsymbol{x},0) = \mathbf{B}_0(\boldsymbol{x}),\ \mathbf{D}^*(\boldsymbol{x},0) = \mathbf{D}_0(\boldsymbol{x}). \end{cases} \tag{2.47}$$

where $\mathbb{J}^* = (-\mathbf{D}^*, \mathbf{H}^*)$ and $\mathbb{E}^* = (\mathbf{E}^*, \mathbf{B}^*)$. The effective tensor $\mathbb{L}^* \in \mathrm{Sym}(\mathbb{R}^6)$ is defined by its action on the arbitrary vector $\bar{\mathbb{E}}_0 \in \mathbb{R}^6$ by

$$\mathbb{L}^*\bar{\mathbb{E}}_0 = \fint_Q \mathbb{L}(\boldsymbol{X})\bar{\mathbb{E}}(\boldsymbol{X})d\boldsymbol{X}, \tag{2.48}$$

where $\bar{\mathbb{E}} = (\bar{\mathbf{E}}, \bar{\mathbf{B}})$ solves the periodic cell problem

$$\begin{cases} \nabla\times\bar{\mathbf{E}} = -\partial\bar{\mathbf{B}}/\partial t,\ \nabla\cdot\bar{\mathbf{B}} = 0,\ \nabla\times\bar{\mathbf{H}} = \partial\bar{\mathbf{D}}/\partial t,\ \nabla\cdot\bar{\mathbf{D}} = 0 \\ \bar{\mathbb{J}}(\boldsymbol{X}) = \mathbb{L}(\boldsymbol{X})\bar{\mathbb{E}}(\boldsymbol{X}), \\ \bar{\mathbb{E}} \text{ is Q-periodic and } \fint_Q \bar{\mathbb{E}}(\boldsymbol{X})d\boldsymbol{X} = \bar{\mathbb{E}}_0. \end{cases} \tag{2.49}$$

In the hyperbolic case the cell problem may have no solutions or a multiplicity of solutions. Therefore it is necessary to require that $\mathbb{L}^*$ be well-defined.

Definition 3. *We say that $\mathbb{L}^*$ is well-defined if the cell problem (2.49) has solutions for every $\bar{\mathbb{E}}_0$ in $\mathbb{R}^6$ and*

$$\int_Q \mathbb{L}(\boldsymbol{X})\bar{\mathbb{E}}(\boldsymbol{X})d\boldsymbol{X} = 0. \tag{2.50}$$

for every solution $\bar{\mathbb{E}}(\boldsymbol{X})$ of (2.49) with $\bar{\mathbb{E}}_0 = 0$.

Theorem 2.4. Convergence of arbitrary solutions for Maxwell's equations

Let $\Omega \subset \mathbb{R}^3 \times (0,T)$ be open and bounded. Assume that $\mathbb{E}_\epsilon$, $\mathbb{J}_\epsilon$ are the solutions of (2.42). Suppose $\mathbb{E}_\epsilon \rightharpoonup \mathbb{E}^*$ in $L^2(\Omega;\mathbb{R}^6)$ weakly. Assume $\mathbb{L}^*$ is well-defined in the sense of definition 3. Then $(\mathbb{E}^*, \mathbb{J}^*)$ satisfies the partial differential equations from (2.47).

The proof follows the same outline as for the wave equation [21, 20].

Proof. ¿From the constitutive relation $\mathbb{J}_\epsilon = \mathbb{L}(\boldsymbol{X}/\epsilon)\mathbb{E}_\epsilon$ we see that $\mathbb{J}_\epsilon$ is bounded in $L^2(\Omega)$. Let $\mathbb{J}_{\epsilon'}$ be a weakly convergent subsequence of $\mathbb{J}_\epsilon$. Let $\bar{\mathbb{J}}_0$ be its weak limit. Let $(\bar{\mathbb{E}}, \bar{\mathbb{J}})$ be a solution of (2.49) that exists since $\mathbb{L}^*$ is well-defined. By the Riemann-Lebesgue Lemma,

$$\bar{\mathbb{E}}_\epsilon = \bar{\mathbb{E}}(\boldsymbol{X}/\epsilon) \overset{*}{\rightharpoonup} \bar{\mathbb{E}}_0, \quad \bar{\mathbb{J}}_\epsilon = \bar{\mathbb{J}}(\boldsymbol{X}/\epsilon) \overset{*}{\rightharpoonup} \bar{\mathbb{J}}_0 = \mathbb{L}^*\bar{\mathbb{E}}_0 \text{ in } L^\infty(\Omega) \text{ weak-}* \tag{2.51}$$

we then have

$$\bar{\mathbb{J}}_\epsilon \cdot \mathbb{E}_\epsilon \quad \overset{*}{\rightharpoonup} \quad \bar{\mathbb{J}}_0 \cdot \mathbb{E}^*, \quad \text{and} \quad \bar{\mathbb{E}}_\epsilon \cdot \mathbb{J}_\epsilon \quad \overset{*}{\rightharpoonup} \quad \bar{\mathbb{E}}_0 \cdot \mathbb{J}^* \tag{2.52}$$

in the sense of measures [22] by a corollary of the compensated compactness Theorem [22]. Since $\mathbb{L}$ is symmetric, we get

$$\bar{\mathbb{J}}_\epsilon \cdot \mathbb{E}_\epsilon = (\mathbb{L}(\boldsymbol{X}/\epsilon)\bar{\mathbb{E}}_\epsilon) \cdot \mathbb{E}_\epsilon = \bar{\mathbb{E}}_\epsilon \cdot (\mathbb{L}(\boldsymbol{X}/\epsilon)\mathbb{E}_\epsilon) = \bar{\mathbb{E}}_\epsilon \cdot \mathbb{J}_\epsilon, \; \forall \boldsymbol{X} \in \Omega. \tag{2.53}$$

So, we conclude that

$$\bar{\mathbb{J}}_0 \cdot \mathbb{E}^* = \bar{\mathbb{E}}_0 \cdot \mathbb{J}^*. \tag{2.54}$$

Thus

$$\mathbb{J}^* = \mathbb{L}^*\mathbb{E}^*. \tag{2.55}$$

Moreover, the pair $(\mathbb{E}^*, \mathbb{J}^*)$ satisfies the partial differential equations in (2.47).

□

Theorem 2.5. Homogenization of Maxwell's System
Suppose $\mathbb{E}_\epsilon$, $\mathbb{J}_\epsilon$ satisfy (2.42). Assume that the sequence $\mathbb{E}_\epsilon$ is bounded in $L^2_{loc}(\mathbb{R}^4;\mathbb{R}^6)$. Assume that $\mathbb{L}^*$ is well-defined in the sense of Definition 3. Then $\mathbb{E}_\epsilon \rightharpoonup \mathbb{E}^*$, $\mathbb{J}_\epsilon \rightharpoonup \mathbb{J}^*$ weakly in L^2, where $(\mathbb{E}^*, \mathbb{J}^*)$ is the unique solution of (2.47) .

Proof. By our assumption we can extract a weakly convergent in L^2 subsequence $(\mathbb{E}_{\epsilon'}, \mathbb{J}_{\epsilon'})$ such that $\mathbb{E}_{\epsilon'} \rightharpoonup \mathbb{E}^*$, and $\mathbb{J}_{\epsilon'} \rightharpoonup \mathbb{J}^*$. Apply Theorem 2.4 and conclude that $(\mathbb{E}^*, \mathbb{J}^*)$ solves (2.47). Since $\mathbb{L}^*$ is constant, (2.47) has a unique solution. It follows that $(\mathbb{E}_\epsilon, \mathbb{J}_\epsilon)$ converges weakly to $(\mathbb{E}^*, \mathbb{J}^*)$. □

Now we show that in the case of an activated space-time composite whose "properties wave" moves with constant velocity $\boldsymbol{v}$, the boundedness assumption holds. The proof that $\mathbf{L}^*$ is well-defined for the Maxwell system is very similar to the proof of the analogous result for the wave equation in Section 2.3. We therefore omit the details.

We assume that $|\boldsymbol{v}|$ is smaller than the speed of light in any of the materials in our composite. Suppose that the micro-structure is determined by the $[0,1]^3$-periodic functions $\varepsilon_0(\boldsymbol{y})$ and $\mu_0(\boldsymbol{y})$ describing the local dielectric permittivity and magnetic permeability respectively. Let

$$\varepsilon_\epsilon(\boldsymbol{x}, t) = \varepsilon_0(\frac{\boldsymbol{x} - \boldsymbol{v}t}{\epsilon});\ \mu_\epsilon(\boldsymbol{x}, t) = \mu_0(\frac{\boldsymbol{x} - \boldsymbol{v}t}{\epsilon}), \tag{2.56}$$

for any $\boldsymbol{x} \in \mathbb{R}^3,\ \boldsymbol{v} \in \mathbb{R}^3,\ t > 0,\ \epsilon > 0$. The propagation of the electromagnetic waves through such a composite is governed by Maxwell's system (2.42). Let us show that there is a constant $\bar{C}$ independent of ϵ such that

$$\forall\ t > 0, \quad \int_{\mathbb{R}^3} \{|\mathbf{D}_\epsilon(\boldsymbol{x},t)|^2 + |\mathbf{B}_\epsilon(\boldsymbol{x},t)|^2\}\ dx\ \leq\ \bar{C}. \tag{2.57}$$

Let us the change of variables $\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{v}t,\ t' = t$ in (2.42). Let $\mathbf{E}'_\epsilon(\boldsymbol{x},t) = \mathbf{E}_\epsilon(\boldsymbol{x}' + \boldsymbol{v}t, t)$ and similarly for all fields $\mathbf{D}'_\epsilon$, $\mathbf{B}'_\epsilon$, and $\mathbf{H}'_\epsilon$. Then the Maxwell's system (2.42) becomes

$$\begin{cases} \nabla' \times \mathbf{E}'_\epsilon = \nabla'\mathbf{B}'_\epsilon \boldsymbol{v} - \partial \mathbf{B}'_\epsilon/\partial t; \quad \nabla' \cdot \mathbf{B}'_\epsilon = 0 \\ \nabla' \times \mathbf{H}'_\epsilon = \nabla'\mathbf{D}'_\epsilon \boldsymbol{v} + \partial \mathbf{D}'_\epsilon/\partial t; \quad \nabla' \cdot \mathbf{D}'_\epsilon = 0 \\ \mathbf{D}'_\epsilon = \varepsilon_0(\boldsymbol{x}'/\epsilon)\mathbf{E}'_\epsilon; \quad \mathbf{B}'_\epsilon = \mu_0(\boldsymbol{x}'/\epsilon)\mathbf{H}'_\epsilon. \end{cases} \tag{2.58}$$

Let us denote $\varepsilon_\epsilon = \varepsilon_0(\boldsymbol{x}'/\epsilon)$ and $\mu_\epsilon = \mu_0(\boldsymbol{x}'/\epsilon)$. Let

$$W_\epsilon(t) = \frac{1}{2}\int_{\mathbb{R}^3} \left\{ \varepsilon_\epsilon |\mathbf{E}'_\epsilon|^2 + \mu_\epsilon |\mathbf{H}'_\epsilon|^2 + 2(\boldsymbol{v}, \mathbf{B}'_\epsilon, \mathbf{D}'_\epsilon) \right\} d\boldsymbol{x}' \tag{2.59}$$

where $(\boldsymbol{v}, \mathbf{B}'_\epsilon, \mathbf{D}'_\epsilon)$ is the triple product : $(\boldsymbol{v}, \mathbf{B}'_\epsilon, \mathbf{D}'_\epsilon) = (\boldsymbol{v} \times \mathbf{B}'_\epsilon) \cdot \mathbf{D}'_\epsilon$.

In order to compute $\frac{dW_\epsilon}{dt}$ it will be convenient to rewrite W_ϵ in terms of $\mathbf{D}'_\epsilon$ and $\mathbf{B}'_\epsilon$ only. By the Cauchy-Schwarz inequality $|(\boldsymbol{v}, \mathbf{B}'_\epsilon, \mathbf{D}'_\epsilon)| \leq |\boldsymbol{v}||\mathbf{B}'_\epsilon||\mathbf{D}'_\epsilon|$ we get

$$\begin{aligned} W_\epsilon(t) &\geq \frac{1}{2}\int_{\mathbb{R}^3} \left\{ \frac{1}{\varepsilon_\epsilon}|\mathbf{D}'_\epsilon|^2 + \frac{1}{\mu_\epsilon}|\mathbf{B}'_\epsilon|^2 - 2|\boldsymbol{v}||\mathbf{B}'_\epsilon||\mathbf{D}'_\epsilon)| \right\} d\boldsymbol{x}' \\ &= \frac{1}{2}\int_{\mathbb{R}^3} \left\{ \begin{bmatrix} 1/\varepsilon_\epsilon & -|\boldsymbol{v}| \\ -|\boldsymbol{v}| & 1/\mu_\epsilon \end{bmatrix} \begin{bmatrix} |\mathbf{D}'_\epsilon| \\ |\mathbf{B}'_\epsilon| \end{bmatrix} \begin{bmatrix} |\mathbf{D}'_\epsilon| \\ |\mathbf{B}'_\epsilon| \end{bmatrix} \right\} d\boldsymbol{x}' \end{aligned} \tag{2.60}$$

By our assumptions $\varepsilon_\epsilon > 0$ and $1/\varepsilon_\epsilon\mu_\epsilon - |\boldsymbol{v}|^2 = c_\epsilon^2 - |\boldsymbol{v}|^2 > 0$. Thus the matrix $\begin{bmatrix} 1/\varepsilon & -|\boldsymbol{v}| \\ -|\boldsymbol{v}| & 1/\mu_\epsilon \end{bmatrix}$ is positive definite. It follows that there is some $\nu > 0$ such that

$$W_\epsilon(t)\ \geq\ \frac{1}{\nu}\int_{\mathbb{R}^3} \left\{ |\mathbf{D}'_\epsilon(\boldsymbol{x},t)|^2 + |\mathbf{B}'_\epsilon(\boldsymbol{x},t)|^2 \right\}\ d\boldsymbol{x}. \tag{2.61}$$

Thus we have

$$\int_{\mathbb{R}^3} \left\{ |\mathbf{D}'_\epsilon(\boldsymbol{x},t)|^2 + |\mathbf{B}'_\epsilon(\boldsymbol{x},t)|^2 \right\}\ d\boldsymbol{x}\ \leq\ \nu\ W_\epsilon(t). \tag{2.62}$$

Lemma 2.4. *$W_\epsilon(t)$ is a constant.*

Proof. We compute $dW_\epsilon(t)/dt$ and show that it is zero. Differentiating (2.59) we get

$$\frac{dW_\epsilon}{dt} = \int_{\mathbb{R}^3}\left\{\frac{1}{\varepsilon_\epsilon}\mathbf{D}'_\epsilon\cdot\frac{\partial\mathbf{D}'_\epsilon}{\partial t} + \frac{1}{\mu_\epsilon}\mathbf{B}'_\epsilon\cdot\frac{\partial\mathbf{B}'_\epsilon}{\partial t} + (\boldsymbol{v}, \frac{\partial\mathbf{B}'_\epsilon}{\partial t}, \mathbf{D}'_\epsilon) + (\boldsymbol{v}, \mathbf{B}'_\epsilon, \frac{\partial\mathbf{D}'_\epsilon}{\partial t})\right\}d\boldsymbol{x}$$

We replace $\partial\mathbf{D}'_\epsilon/\partial t$ and $\partial\mathbf{B}'_\epsilon/\partial t$ by their expressions from Maxwell's system (2.58) and we replace $(1/\varepsilon_\epsilon)\mathbf{D}'_\epsilon$, and $(1/\mu_\epsilon)\mathbf{B}'_\epsilon$ by $\mathbf{E}'_\epsilon$ and $\mathbf{H}'_\epsilon$ respectively. We obtain

$$\frac{dW_\epsilon}{dt} = \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3 + \mathrm{I}_4, \tag{2.63}$$

where

$$\begin{aligned}
\mathrm{I}_1 &= \int_{\mathbb{R}^3}\left\{\mathbf{E}'_\epsilon\cdot\nabla'\times\mathbf{H}'_\epsilon - \mathbf{H}'_\epsilon\cdot\nabla'\times\mathbf{E}'_\epsilon\right\}d\boldsymbol{x}'\\
\mathrm{I}_2 &= \int_{\mathbb{R}^3}\left\{\mathbf{E}'_\epsilon\cdot\nabla'\mathbf{D}'_\epsilon\boldsymbol{v} - (\boldsymbol{v}, \nabla'\times\mathbf{E}'_\epsilon, \mathbf{D}'_\epsilon)\right\}d\boldsymbol{x}'\\
\mathrm{I}_3 &= \int_{\mathbb{R}^3}\left\{\mathbf{H}'_\epsilon\cdot\nabla'\mathbf{B}'_\epsilon\boldsymbol{v} + (\boldsymbol{v}, \mathbf{B}'_\epsilon, \nabla'\times\mathbf{H}'_\epsilon)\right\}d\boldsymbol{x}'\\
\mathrm{I}_4 &= \int_{\mathbb{R}^3}\left\{(\boldsymbol{v}, \nabla'\mathbf{B}'_\epsilon\boldsymbol{v}, \mathbf{D}'_\epsilon) + (\boldsymbol{v}, \mathbf{B}'_\epsilon, \nabla'\mathbf{D}'_\epsilon\boldsymbol{v})\right\}d\boldsymbol{x}'
\end{aligned}$$

Now we show that $\mathrm{I}_j = 0$ for all $j = 1, \ldots, 4$.

1. Integration by parts gives

$$\int_{\mathbb{R}^3}\mathbf{E}'_\epsilon\cdot\nabla'\times\mathbf{H}'_\epsilon\ d\boldsymbol{x}' = \int_{\mathbb{R}^3}\nabla'\times\mathbf{E}'_\epsilon\cdot\mathbf{H}'_\epsilon\ d\boldsymbol{x}'. \tag{2.64}$$

So, $\mathrm{I}_1 = 0$.

2. Integration by parts yields

$$\begin{aligned}
-\int_{\mathbb{R}^3}(\boldsymbol{v}, \nabla'\times\mathbf{E}'_\epsilon, \mathbf{D}'_\epsilon)d\boldsymbol{x}' &= \int_{\mathbb{R}^3}\left\{(\nabla'\cdot\mathbf{D}'_\epsilon)\boldsymbol{v}\cdot\mathbf{E}'_\epsilon - (\nabla'\mathbf{D}'_\epsilon)\boldsymbol{v}\cdot\mathbf{E}'_\epsilon\right\}d\boldsymbol{x}'\\
&= -\int_{\mathbb{R}^3}(\nabla'\mathbf{D}'_\epsilon\boldsymbol{v}\cdot\mathbf{E}'_\epsilon)d\boldsymbol{x}',
\end{aligned} \tag{2.65}$$

because $\nabla'\cdot\mathbf{D}'_\epsilon = 0$. Thus $\mathrm{I}_2 = 0$.

3. If we replace $\mathbf{H}'_\epsilon$ by $\mathbf{E}'_\epsilon$ and $\mathbf{B}'_\epsilon$ by $\mathbf{D}'_\epsilon$ in I_2 we will obtain I_3. Moreover, $\nabla' \cdot \mathbf{B}'_\epsilon = 0$. Thus, the calculation we did for I_2 also applies to I_3.

4. $\nabla' \mathbf{D}'_\epsilon \boldsymbol{v}$ is a directional derivative along $\boldsymbol{v}$. We can then apply integration by parts to get

$$\int_{\mathbb{R}^3} (\boldsymbol{v}, \mathbf{B}'_\epsilon, \nabla' \mathbf{D}_\epsilon \boldsymbol{v}) \, d\boldsymbol{x}' = - \int_{\mathbb{R}^3} (\boldsymbol{v}, \nabla' \mathbf{B}'_\epsilon \boldsymbol{v}, \mathbf{D}'_\epsilon) \, d\boldsymbol{x}' \tag{2.66}$$

So we see that $\mathrm{I}_4 = 0$ as well.

Therefore, $dW_\epsilon(t)/dt = 0$. Consequently, $W_\epsilon(t)$ is a constant. Thus, we have

$$W(t) = W(0) \ \le \ \beta \int_{\mathbb{R}^3} \left\{ |\mathbf{D}'_\epsilon(\boldsymbol{x}, 0)|^2 + |\mathbf{B}'_\epsilon(\boldsymbol{x}, 0)|^2 \right\} d\boldsymbol{x} \ \le \ \bar{C}.$$

□

Remark 2.1. *The energy bound still holds if the composite moves with non-uniform velocity, i.e.,* $\boldsymbol{v} = \boldsymbol{v}(t)$*. In that case, let* $\boldsymbol{s}(t) = \int_0^t \boldsymbol{v}(\tau)\, d\tau$ *and let* $\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{s}(t)$*. Then the same formulae hold with* $\boldsymbol{v}$ *replaced by* $\boldsymbol{v}(t)$*. However, when we differentiate* $W(t)$ *there will be one extra term present :*

$$\frac{dW_\epsilon}{dt} = \int_{\mathbb{R}^3} (\boldsymbol{v}'(t), \mathbf{B}'_\epsilon, \mathbf{D}'_\epsilon) \, d\boldsymbol{x} \le C W_\epsilon.$$

Thus $dW_\epsilon/dt \le C W_\epsilon(t)$ *and consequently, by Gronwall's inequality*

$$W_\epsilon(t) \le e^{Ct} W_\epsilon(0).$$

Remark 2.2. *Strictly speaking, a composite moving with non-uniform velocity* $\boldsymbol{v}(t)$ *does not qualify to be called the space-time composite. If* $\boldsymbol{v}(\tau)$ *is periodic with period 1 and* ρ_0 *is periodic with period* $[0,1]^3$*, then* $\rho_0((\boldsymbol{x} - \boldsymbol{s}(t))/\epsilon)$ *has a period cell that does not shrink to a point as* $\epsilon \to 0$*. If instead we define* $\bar{\rho}(\boldsymbol{x}, t) = \rho_0(\boldsymbol{x} - \boldsymbol{s}(t))$ *and* $\rho_\epsilon(\boldsymbol{x}, t) = \bar{\rho}(\boldsymbol{x}/\epsilon, t/\epsilon) = \rho_0(\boldsymbol{x}/\epsilon - \boldsymbol{s}(t/\epsilon))$ *then* $\boldsymbol{v}_\epsilon(t) = \boldsymbol{v}(t/\epsilon)$ *and* $\boldsymbol{v}'_\epsilon(t) = \boldsymbol{v}'(t/\epsilon)/\epsilon$ *and the bound on energy cannot be proved by this method. It may very well be true that the energy is in fact unbounded as* $\epsilon \to 0$*.*

CHAPTER 3

CELL PROBLEMS AND LAMINATE FORMULAE

3.1 Cell problem for 1-dimensional wave equation

The cell problem (2.18) for the one-dimensional wave equation becomes

$$\begin{cases} \frac{\partial}{\partial t}(\rho(x,t)(\frac{\partial u}{\partial t}+\xi_1)) = \frac{\partial}{\partial x}(k(x,t)(\frac{\partial u}{\partial x}+\xi_2)) \\ u(x,t) \in H^1_p(Q) \end{cases} \tag{3.1}$$

Assume that

$$(\rho(x,t),k(x,t)) = \begin{cases} (\rho_1,k_1), & \text{if } (x,t)\in Q_1 \\ (\rho_2,k_2), & \text{if } (x,t)\in Q_2, \end{cases} \tag{3.2}$$

where Q_1 is a simply connected inclusion in a connected matrix Q_2 (see Figure 3.1). We also assume for simplicity that the period cell Q is a square $[0,1]^2$.

Let Q_1^p and Q_2^p denote the Q-periodic extension of Q_1 and Q_2 to all of $\mathbb{R}^2$ respectively. We denote $u^1(x,t)$ the restriction of $u(x,t)$ to Q_1 and by $u^2(x,t)$ the restriction of $u(x,t)$ to Q_2^p.

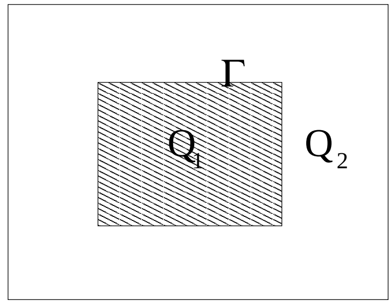


Figure 3.1: A periodic array of simply-connected space-time "inclusions"

Theorem 3.1. *Assume that $c = c_2 = \sqrt{k_2/\rho_2}$ is irrational. If Q_1 is a simply connected inclusion in a connected matrix Q_2 and if $\xi_1 \neq 0$ in (3.1), then the cell problem (3.1) has no solution.*

Proof. The key observation here is that since the set Q_2^p is connected, there exist a single pair of functions $f(\xi), g(\eta)$ defined on all of $\mathbb{R}$ such that for all $(x,t) \in Q_2^p$

$$u^2(x,t) = f(x+ct) + g(x-ct).$$

We are looking for a solution $u \in H_p^1(Q)$ to (3.1). This implies that

$$f'(x+ct) = \frac{1}{2}\left(\frac{\partial u^2}{\partial x} + \frac{1}{c}\frac{\partial u^2}{\partial t}\right) \in L^2_{\text{loc}}(Q_2^p)$$

and

$$g'(x-ct) = \frac{1}{2}\left(\frac{\partial u^2}{\partial x} - \frac{1}{c}\frac{\partial u^2}{\partial t}\right) \in L^2_{\text{loc}}(Q_2^p).$$

From which it follows that $\{f', g'\} \subset L^2_{\text{loc}}(\mathbb{R})$.
So, $\{f, g\} \subset H^1_{\text{loc}}(\mathbb{R})$. The Q-periodicity of $u(x,t)$ can be expressed as follows:

$$\begin{cases} f(x+ct+1) + g(x-ct+1) = f(x+ct) + g(x-ct) \\ f(x+ct+c) + g(x-ct-c) = f(x+ct) + g(x-ct) \end{cases}$$

for all $(x,t) \in Q_2^p$.

Now, we change variables:

$$x + ct = \xi, \;\; x - ct = \eta. \tag{3.3}$$

Let $\widetilde{Q}_2^p$ be the image of Q_2^p under this linear change of variables.

The set Q_2^p is open and connected and so is the set $\widetilde{Q}_2^p$. Also for any $(\xi, \eta) \in \widetilde{Q}_2^p$ we have

$$\begin{cases} f(\xi+1) + g(\eta+1) = f(\xi) + g(\eta), \\ f(\xi+c) + g(\eta-c) = f(\xi) + g(\eta). \end{cases} \tag{3.4}$$

This is equivalent to

$$\begin{cases} f(\xi+1) - f(\xi) = g(\eta) - g(\eta+1), \\ f(\xi+c) - f(\xi) = g(\eta) - g(\eta-c). \end{cases} \tag{3.5}$$

Now fix any $\xi_0 \in \mathbb{R}$. The line $x+ct = \xi_0$ cannot all lie in the set Q_1^p because this set is a disjoint union of bounded components. Thus there exists (x_0, t_0) such that $x_0+ct_0 = \xi_0$ and such that $(x_0, t_0) \in Q_2^p$. Since Q_2^p (and $\widetilde{Q}_2^p$) is an open set, it follows that there exists $\epsilon > 0$ such that $(\xi_0 - \epsilon, \xi_0 + \epsilon) \times (\eta_0 - \epsilon, \eta_0 + \epsilon) \in \widetilde{Q}_2^p$, where $\eta_0 = x_0 - ct_0$. But then for any $\xi \in (\xi_0 - \epsilon, \xi_0 + \epsilon)$,

$$f(\xi+1) - f(\xi) = g(\eta_0) - g(\eta_0+1) = \text{ constant.} \tag{3.6}$$

So, $f(\xi+1) - f(\xi)$ is locally constant on $\mathbb{R}$. Therefore, $f(\xi+1) - f(\xi)$ globally constant. This implies that $f'(\xi+1) - f'(\xi) = 0$, and so, $f'(\xi)$ is a 1-periodic function. A similar analysis applied to the equation

$$f(\xi+c) - f(\xi) = g(\eta) - g(\eta-c). \tag{3.7}$$

It follows that $f'(\xi)$ is a c-periodic function.

Then $f'(\xi) = \text{ constant } = f_0$ since we have assumed that $c \notin \mathbb{Q}$.

A similar conclusion holds for $g : g'(\eta) = g_0$. So,

$$f(\xi) = f_0\xi + \alpha; \quad g(\eta) = g_0\eta + \beta \tag{3.8}$$

and

$$f(\xi+1) - f(\xi) = f_0; \quad g(\eta) - g(\eta+1) = -g_0. \tag{3.9}$$

Therefore, $f_0 = -g_0$.

Substituting (3.8) into the second periodicity condition (3.5) we get $cf_0 = f(\xi + c) - f(\xi) = g(\eta) - g(\eta - c) = cg_0$. This implies that $f_0 = g_0$.

Thus we conclude that $f_0 = g_0 = 0$. So, $u^2(x,t) = u_0$ where u_0 is constant in Q_2^p. Since $u^2(x,t)$ is defined only up to a constant then without loss of generality $u^2(x,t) \equiv 0$ in Q_2^p. Let $U(x,t) = u(x,t) + \xi_1 x + \xi_2 t$. Then the cell problem can be written as follows

$$\begin{aligned} U_{tt}^1 &= c_1^2 U_{xx}^1, \ (x,t) \in Q_1, \\ U_{tt}^2 &= c_2^2 U_{xx}^2, \ (x,t) \in Q_2, \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} U^1(x,t) &= U^2(x,t), \ (x,t) \in \Gamma \\ \rho_1 U_t^1 n_t - k_1 U_x^1 n_x &= \rho_2 U_t^2 n_t - k_2 U_x^2 n_x, \ (x,t) \in \Gamma, \end{aligned} \tag{3.11}$$

where $\Gamma = \partial Q_1$ and $\boldsymbol{n} = (n_t, n_x)$ is the unit normal on Γ.

We use D'Alembert's representation of a solution :

$$\begin{aligned} U^1(x,t) &= f_1(x + c_1 t) + g_1(x - c_1 t), \ (x,t) \in Q_1, \\ U^2(x,t) &= f_2(x + c_2 t) + g_2(x - c_2 t), \ (x,t) \in Q_2. \end{aligned} \tag{3.12}$$

Lemma 3.1. *Condition (3.11) can be written as*

$$\begin{cases} 2f_1(x + c_1 t) = (1+\alpha) f_2(x + c_2 t) + (1-\alpha) g_2(x - c_2 t), & (x,t) \in \Gamma \\ 2g_1(x - c_1 t) = (1-\alpha) f_2(x + c_2 t) + (1+\alpha) g_2(x - c_2 t), & (x,t) \in \Gamma \end{cases} \tag{3.13}$$

where $\alpha = \rho_2 c_2 / \rho_1 c_1$.

Proof. Let $x = x(s), \ t = t(s)$ be a parameterization of Γ. Then $\boldsymbol{N} = (N_x, N_t) = (\dot{t}, -\dot{x})$ is the normal to Γ at $(x(s), t(s))$. Using (3.12) we obtain

$$\begin{aligned} &\rho_1 U_t^1 N_t - k_1 U_x^1 N_x = \\ &-\rho_1 c_1 (\dot{x} + c_1 \dot{t}) f_1'(x(s) + c_1 t(s)) + \rho_1 c_1 (\dot{x} - c_1 \dot{t}) g_1'(x(s) - c_1 t(s)). \end{aligned}$$

We observe that the right hand side is a full derivative, so

$$\rho_1 U_t^1 N_t - k_1 U_x^1 N_x = \rho_1 c_1 \frac{d}{ds}(g_1(x(s) - c_1 t(s)) - f_1(x(s) + c_1 t(s))).$$

Similarly,

$$\rho_2 U_t^2 N_t - k_1 U_x^2 N_x = \rho_2 c_2 \frac{d}{ds}(g_2(x(s) - c_2 t(s)) - f_2(x(s) + c_2 t(s))).$$

So, the equation (3.11) can be integrated:

$$f_1(x + c_1 t) - g_1(x - c_1 t) = \alpha(f_2(x + c_2 t) - g_2(x - c_2 t)), \ (x, t) \in \Gamma. \quad (3.14)$$

By the continuity of $u(x, t)$ on the smooth boundary we have

$$f_2(x + c_2 t) + g_2(x - c_2 t) = f_1(x + c_1 t) + g_1(x - c_1 t), \ (x, t) \in \Gamma. \quad (3.15)$$

Combining (3.14) with (3.15) we get (3.13). □

Recall that we have shown that $u^2(x, t) \equiv 0$ in Q_2. This implies that

$$f_2(x + c_2 t) + g_2(x - c_2 t) = \xi_1 x + \xi_2 t. \quad (3.16)$$

Therefore,

$$\begin{cases} f_2'(x + c_2 t) + g_2'(x - c_2 t) & = \xi_1 \\ c_2(f_2'(x + c_2 t) - g_2'(x - c_2 t)) & = \xi_2. \end{cases} \quad (3.17)$$

Consequently,

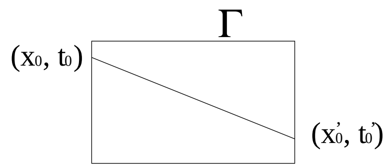
$$f_2(\lambda) = \frac{1}{2}(\xi_1 + \frac{\xi_2}{c_2})\lambda; \quad g_2(\lambda) = \frac{1}{2}(\xi_1 - \frac{\xi_2}{c_2})\lambda, \quad (3.18)$$

Thus, by (3.13)

$$\begin{aligned} 2f_1(x + c_1 t) &= \frac{1+\alpha}{2}(\xi_1 + \frac{1}{c_2}\xi_2)(x + c_2 t) + \frac{1-\alpha}{2}(\xi_1 - \frac{1}{c_2}\xi_2)(x - c_2 t) \\ &= c_2(\alpha\xi_1 + \frac{\xi_2}{c_2})t + (\xi_1 + \alpha\frac{\xi_2}{c_2})x, \ (x, t) \in \Gamma, \quad (3.19) \\ 2g_1(x - c_1 t) &= \frac{1-\alpha}{2}(\xi_1 + \frac{1}{c_2}\xi_2)(x + c_2 t) + \frac{1+\alpha}{2}(\xi_1 - \frac{1}{c_2}\xi_2)(x - c_2 t) \\ &= -c_2(\alpha\xi_1 + \frac{\xi_2}{c_2})t - (\xi_1 - \alpha\frac{\xi_2}{c_2})x, \ (x, t) \in \Gamma. \quad (3.20) \end{aligned}$$

Consider (3.19) first. It says that

$$\forall \ (x, t) \in \Gamma : \ f_1(x + c_1 t) = Ax + Bt$$



where

$$A = \frac{1}{2}(\xi_1 + \alpha\frac{\xi_2}{c_2});\;\; B = \frac{1}{2}c_2(\alpha\xi_1 + \frac{\xi_2}{c_2}). \tag{3.21}$$

Take $(x_0, t_0) \in \Gamma$ and consider the line $x + c_1 t = \xi_0$, where $\xi_0 = x_0 + c_1 t_0$. By our assumption this line will intersect Γ in at least two places (x_0, t_0) and (x_0', t_0'), where $x_0' + c_1 t_0' = \xi_0$. But then we have:

$$f_1(\xi_0) = f_1(x_0 + c_1 t_0) = A(\xi_0 - c_1 t_0) + B t_0 = A\xi_0 + (B - c_1 A)t_0. \tag{3.22}$$

On the other hand we may replace (x_0, t_0) by (x_0', t_0'). Then we have

$$f_1(\xi_0) = A\xi_0 + (B - c_1 A)t_0'. \tag{3.23}$$

Comparing (3.22) and (3.23) we get a contradiction, unless $B = c_1 A$. Similarly, considering the formula (3.20) we have

$$g_1(x - c_1 t) = A'x + B't, \tag{3.24}$$

where

$$A' = -\frac{1}{2}(\xi_1 - \alpha\frac{1}{c_2}\xi_2),\;\; B' = -\frac{1}{2}(\alpha\xi_1 + \frac{\xi_2}{c_2})c_2. \tag{3.25}$$

Consider now the line $x - c_1 t = \zeta_0 = x_0 - c_1 t_0$. This line will intersect Γ at another point (x_0'', t_0''). Then $x_0 - c_1 t_0 = \zeta_0 = x_0' - c_1 t_0'$ and

$$g_1(\zeta_0) = A'(\zeta_0 + c_1 t_0) + B' t_0 = A'\zeta_0 + (B' + c_1 A')t_0.$$

Also,

$$g_1(\zeta_0) = A'\zeta_0 + (B' + c_1 A')t_0'',$$

and we get a contradiction, unless $B' = -c_1 A'$. So, we get a contradiction, unless

$$\frac{c_2}{2}(\alpha\xi_1 + \frac{\xi_2}{c_2}) = \frac{c_1}{2}(\xi_1 + \alpha\frac{\xi_2}{c_2}) \;\;\text{and}\;\; \frac{c_2}{2}(\alpha\xi_1 + \frac{\xi_2}{c_2}) = \frac{c_1}{2}(\alpha\frac{\xi_2}{c_2} - \xi_1) \tag{3.26}$$

This implies $\xi_1 = 0$ and $\xi_2 = \alpha c_1 \xi_2 / c_2 = \rho_2 \xi_2 / \rho_1$. So,

$$\xi_1 = 0, \text{ and either } \xi_2 = 0 \text{ or } \rho_1 = \rho_2.$$

So, if $\xi_1 \neq 0$, and $c_2 \notin \mathbb{Q}$ the cell problem (3.1) has no solution. $\square$

We remark, that the regular transport conditions [11] (conditions for which the test problem from Section 2.2 has a unique solution) need not be violated by structures considered in this section. For example, a periodic array of rectangles in space-time uses only vertical or horizontal interfaces for which regular transport conditions always holds. Another example is a periodic array of rhombuses with slopes of sides corresponding to slow motions. These examples show that the regular transport conditions are insufficient to guarantee the existence of solutions to a periodic cell problem.

We conjecture that nucleation or disappearance of a new phase always generates shock waves. As the wave scatters over the periodic array of inclusions the strength of the shocks grow.

3.2 Lamination formula

Lamination formula for the 1D wave equation was derived and analyzed by Lurie in [11] and also in subsequent works [16, 14]. The lamination formula for the Maxwell system was derived by Dunaevskaya (unpublished, private communication by Lurie).

The aim of this section is to extend the Hilbert space formalism introduced by Milton in [19] to the setting of space-time composites. We illustrate its usefulness by rederiving the lamination formulas for the wave equation and the Maxwell system in Sections 3.2.1 and 3.2.2 respectively.

Milton observed that the cell problem in various contexts can be written in terms of two fields: $\mathbb{E}$, the intensity field and $\mathbb{J}$, the flux field. Let $Q = [0, 1]^d$ be the unit cube in $\mathbb{R}^d$. The fields $\mathbb{E}$ and $\mathbb{J}$ take their values in a finite dimensional tensor space $\mathcal{T}$ and are related by a linear map $\mathbb{L}$:

$$\mathbb{J} = \mathbb{L}\mathbb{E}, \tag{3.27}$$

where $\mathbb{L}(\boldsymbol{x}) \in L^\infty(Q) \otimes \mathrm{End}(\mathcal{T})$. In order to write the cell problem we introduce, following Milton [19], the Hilbert space $\mathcal{H} = L^2(Q) \otimes \mathcal{T}$.

For conducting composites, for example, $\mathcal{T} = \mathbb{R}^3$. For the 1-dimensional wave equation $\mathcal{T} = \mathbb{R}^2$, and for Maxwell's system $\mathcal{T} = \mathbb{R}^6$. The intensity field $\mathbb{E}$ and the flux field $\mathbb{J}$ are the electric field and the current density respectively in the context of conductivity. For Maxwell's equations $\mathbb{E}$ is a pair $(\mathbf{E}, \mathbf{B})$ and $\mathbb{J}$ is a pair $(-\mathbf{D}, \mathbf{H})$. For the 1-dimensional wave equation $\mathbb{E}$ is a pair (u_x, u_t) and $\mathbb{J}$ is a pair $(ku_x, -\rho u_t)$. Let $\mathcal{E}$ and $\mathcal{J}$ be the subspaces of $\mathcal{H}$ corresponding to the differential equations satisfied by $\mathbb{E}$ and $\mathbb{J}$. For example, for conductivity

$$\mathcal{E} = \{\nabla\phi \ : \ \phi \in H^1_p(Q)\} \tag{3.28}$$

$$\mathcal{J} = \{\boldsymbol{j} \in L^2(Q) \otimes \mathbb{R}^3 \ : \ \nabla \cdot \boldsymbol{j} = 0, \quad <\boldsymbol{j}>= 0\}. \tag{3.29}$$

For the 1-dimensional wave equation, the subspaces $\mathcal{E}$ and $\mathcal{J}$ are given by (3.28) and (3.29), except $\mathbb{R}^3$ is replaced by $\mathbb{R}^2$. For the Maxwell's system

$$\mathcal{E} = \{\mathbb{E} = (\mathbf{E}, \mathbf{B}) \in L^2(Q) \otimes \mathbb{R}^6 \mid \nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t, \ \nabla \cdot \mathbf{B} = 0, \quad <\mathbb{E}>= 0\}$$

$$\mathcal{J} = \{\mathbb{J} = (-\mathbf{D}, \mathbf{H}) \in L^2(Q) \otimes \mathbb{R}^6 \mid \nabla \times \mathbf{H} = -\partial\mathbf{D}/\partial t, \ \nabla \cdot \mathbf{D} = 0, \quad <\mathbb{J}>= 0\}.$$

Finally, let $\mathcal{U} = \mathbb{R} \otimes \mathcal{T} \subset \mathcal{H}$ be the space of uniform fields. Then the cell problem can be written as

$$\mathbb{E} \in \mathcal{E} \oplus \mathcal{U}, \quad \mathbb{J} \in \mathcal{J} \oplus \mathcal{U}, \quad \mathbb{J} = \mathbb{L}\mathbb{E}, \tag{3.30}$$

and, the effective tensor $\mathbb{L}^*$ is defined by

$$\mathbb{L}^* < \mathbb{E} >=< \mathbb{J} > . \tag{3.31}$$

Let $\Gamma : \mathcal{H} \to \mathcal{H}$ be an orthogonal projection onto $\mathcal{E}$. Then there are finite dimensional orthogonal projection matrices $\mathbf{\Gamma}(\boldsymbol{n}), \quad |\boldsymbol{n}| = 1$, such that for any $f \in \mathcal{H}$,

$$\widehat{\Gamma f}(\boldsymbol{k}) = \begin{cases} \mathbf{\Gamma}(\frac{\boldsymbol{k}}{|\boldsymbol{k}|})\hat{f}(\boldsymbol{k}), & \boldsymbol{k} \in \mathbb{Z}^d \backslash \{\mathbf{0}\} \\ \mathbf{0} & \boldsymbol{k} = \mathbf{0}. \end{cases} \tag{3.32}$$

In the case of conductivity and 1-dimensional wave equation $\mathbf{\Gamma}(\boldsymbol{n}) = \boldsymbol{n}\otimes\boldsymbol{n}$. In the case of Maxwell's system $\mathbf{\Gamma}(\boldsymbol{n})$ is more complicated and will be computed in section 3.2.2.

Now consider a simple laminate made of materials $\mathbb{L}_1$ and $\mathbb{L}_2$ taken in volume fractions θ and $1-\theta$. Let $\boldsymbol{n}$ be a unit normal to the layers. Then there is a nice formula for $\mathbb{L}^*$, the effective tensor of the laminate, due to Milton :

$$W_{\boldsymbol{n}}(\mathbb{L}^*) = \theta W_{\boldsymbol{n}}(\mathbb{L}_1) + (1-\theta)W_{\boldsymbol{n}}(\mathbb{L}_2), \tag{3.33}$$

where

$$W_{\boldsymbol{n}}(\mathbb{L}) = \left[(\mathbf{I}-\mathbb{L}^{-1})^{-1} - \mathbf{\Gamma}(\boldsymbol{n})\right]^{-1}. \tag{3.34}$$

This general formula reduces to the lamination formulas for conductivity [23, 24] and elasticity [1] in the corresponding contexts.

In the next two sections we obtain the explicit formulas for the effective parameters of an activated composite laminate in the contexts of 1-dimensional wave equation and Maxwell's system using (3.33).

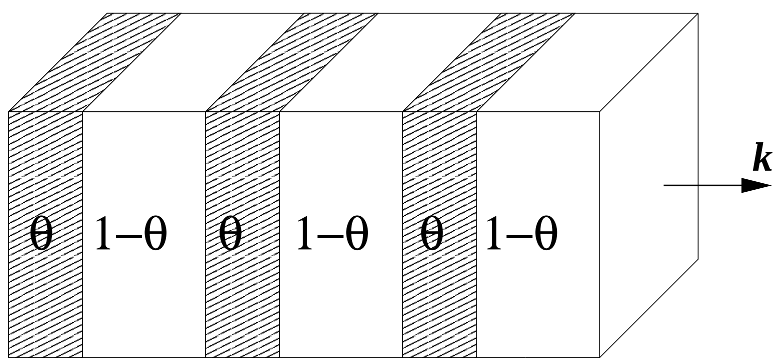


Figure 3.2: Rank-1 laminate

3.2.1 Lamination formula for the 1-dimensional wave equation

In this section we illustrate the usefulness of the machinery developed above by rederiving the lamination formula for the 1D wave equation [11].

Recall that for 1-dimensional wave equation we have :

$$\begin{aligned}\mathcal{T} &= \mathbb{R}^2 \\ \mathcal{E} &= \{\nabla w \ : \ w \in H^1_p(Q)\} \\ \mathcal{J} &= \{\boldsymbol{J} \in L^2(Q) \otimes \mathbb{R}^2 \ : \ \nabla \cdot \boldsymbol{J} = 0, \quad <\boldsymbol{J}>= 0\},\end{aligned} \tag{3.35}$$

with a linear map $\mathbb{L}(x,t) = \begin{bmatrix} k(x-vt) & 0 \\ 0 & -\rho(x-vt) \end{bmatrix}$.

Suppose $\boldsymbol{n} = \begin{bmatrix} \xi \\ \eta \end{bmatrix}$ where $\xi = 1/\sqrt{1+v^2}$, $\eta = -v/\sqrt{1+v^2}$, then

$$\boldsymbol{\Gamma}(\boldsymbol{n}) = \boldsymbol{n} \otimes \boldsymbol{n} = \begin{bmatrix} \xi^2 & \xi\eta \\ \xi\eta & \eta^2 \end{bmatrix}. \tag{3.36}$$

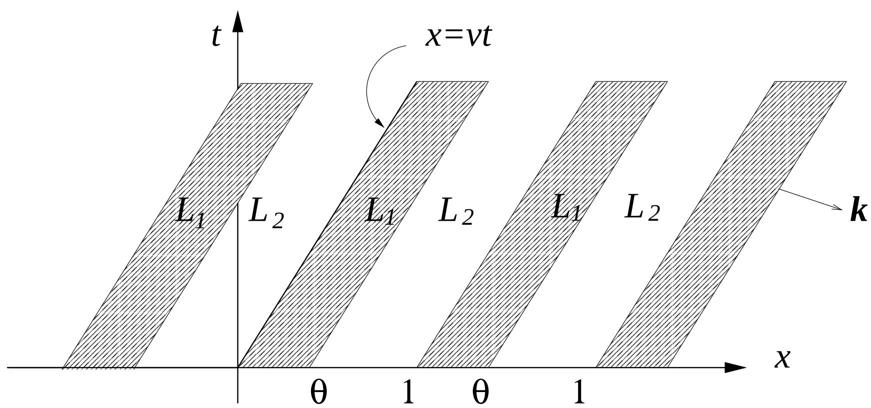


Figure 3.3: One dimensional dynamic laminate

If the two materials that are layered have densities ρ_1 and ρ_2 and stiffnesses k_1 and k_2 respectively, then

$$\mathbb{L}_1 = \begin{bmatrix} k_1 & 0 \\ 0 & -\rho_1 \end{bmatrix}; \quad \mathbb{L}_2 = \begin{bmatrix} k_2 & 0 \\ 0 & -\rho_2 \end{bmatrix}. \tag{3.37}$$

We compute

$$W_{(\xi,\eta)}(\mathbb{L}_j) = \frac{1}{k_j\xi^2 - \rho_j\eta^2} \begin{bmatrix} (1-k_j)(\xi^2 - \rho_j\eta^2) & (1-k_j)(1+\rho_j)\xi\eta \\ (1-k_j)(1+\rho_j)\xi\eta & (1+\rho_j)(\eta^2 + k_j\xi^2) \end{bmatrix} \tag{3.38}$$

and

$$\mathbb{L}^* = W^{-1}_{(\xi,\eta)}(\theta W_{(\xi,\eta)}(\mathbb{L}_1) + (1-\theta)W_{(\xi,\eta)}(\mathbb{L}_2)). \tag{3.39}$$

Suppose $\mathbb{L}^* = \begin{bmatrix} L^*_{11} & L^*_{12} \\ L^*_{21} & L^*_{22} \end{bmatrix}$. We interpret this $\mathbb{L}^*$ as $\begin{bmatrix} k^* & 0 \\ 0 & -\rho^* \end{bmatrix}$ in the coordinate system moving with velocity v_*.

Then

$$\mathbb{L}^* = \begin{bmatrix} k^* - \rho^*(v_*)^2 & -\rho^* v_* \\ -\rho^* v_* & -\rho^* \end{bmatrix} \tag{3.40}$$

Thus $\rho^* = -L^*_{22},\ v_* = L^*_{12}/L^*_{22}$ and $k^* = (\det \mathbb{L}^*)/L^*_{22}$. The computation is straightforward with Maple software. We simply substitute (3.38) into (3.39) and $\xi = 1/\sqrt{1+v^2},\ \eta = -v/\sqrt{1+v^2}$.

The final result is

$$\begin{aligned}
\rho^* &= H(\rho)\frac{A(\rho) - \gamma H(k)}{H(\rho) - \gamma H(k)}, \\
v^* &= v\frac{\theta(1-\theta)(\rho_1 - \rho_2)(k_2 - k_1)H(k)}{k_1 k_2(A(\rho) - \gamma H(k))}, \\
k^* &= H(k)\frac{A(\rho) - \gamma A(k)}{A(\rho) - \gamma H(k)},
\end{aligned} \tag{3.41}$$

where $\gamma = v^2/c_1^2 c_2^2$ and $A(f),\ H(f)$ denote an arithmetic mean of f and a harmonic mean of f, respectively.

3.2.2 Lamination formula for Maxwell's System

In this section we derive the lamination formula for the full Maxwell's system by employing the computational machinery of the Hilbert space formalism of Milton. Our work is independent from the unpublished results of Dunaevskaya.

Consider Maxwell's equations (2.41). Recall that $\mathbb{E} = (\mathbf{E}, \mathbf{B})$, $\mathbb{J} = (-\mathbf{D}, \mathbf{H})$ and

$$\begin{aligned}
\mathcal{T} &= \mathbb{R}^6 \\
\mathcal{E} &= \{\mathbb{E} \in L^2(Q) \otimes \mathbb{R}^6 \mid \nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t,\ \nabla \cdot \mathbf{B} = 0, \quad <\mathbb{E}>= 0\} \\
\mathcal{J} &= \{\mathbb{J} \in L^2(Q) \otimes \mathbb{R}^6 \mid \nabla \times \mathbf{H} = -\partial \mathbf{D}/\partial t,\ \nabla \cdot \mathbf{D} = 0, \quad <\mathbb{J}>= 0\}.
\end{aligned}$$

The constitutive relation is given by $\mathbb{L}(\boldsymbol{x},t) = \begin{bmatrix} -\varepsilon(\boldsymbol{x}-\boldsymbol{v}t)\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}/\mu(\boldsymbol{x}-\boldsymbol{v}t) \end{bmatrix}$. In terms of the Fourier coefficients we get

$$\begin{aligned} \mathcal{E} &= \{(\widehat{\mathbf{E}},\widehat{\mathbf{B}}) \in \mathcal{E}_{(\boldsymbol{\xi},\eta)} \ \forall\ (\boldsymbol{\xi},\eta) \in \mathbb{R}^4 \backslash \{\mathbf{0}\}\}, \\ \mathcal{J} &= \{(-\widehat{\mathbf{D}},\widehat{\mathbf{H}}) \in \mathcal{J}_{(\boldsymbol{\xi},\eta)} \ \forall\ (\boldsymbol{\xi},\eta) \in \mathbb{R}^4 \backslash \{\mathbf{0}\}\}, \end{aligned}$$

where

$$\mathcal{E}_{(\boldsymbol{\xi},\eta)} = \{(\widehat{\mathbf{E}},\widehat{\mathbf{B}}) \mid \boldsymbol{\xi} \times \widehat{\mathbf{E}} = -\eta\widehat{\mathbf{B}}\ ;\ \boldsymbol{\xi}\cdot\widehat{\mathbf{B}} = 0\},$$

and

$$\mathcal{J}_{(\boldsymbol{\xi},\eta)} = \begin{cases} \{(-\boldsymbol{\xi}\times\mathbf{H},\eta\mathbf{H}) \mid \mathbf{H}\in\mathbb{R}^3\} & \text{if } \eta \neq 0 \\ \{(\mathbf{D},\boldsymbol{t}\boldsymbol{\xi}) \mid \boldsymbol{t}\in\mathbb{R}^3,\ \mathbf{D}\in\mathbb{R}^3,\ \mathbf{D}\cdot\boldsymbol{\xi} = \mathbf{0}\} & \text{if } \eta = 0. \end{cases}$$

Obviously, for any $\mathbf{E} \in \mathcal{E}_{(\boldsymbol{\xi},\eta)},\ \mathbf{J} \in \mathcal{J}_{(\boldsymbol{\xi},\eta)}$, we have

$$(\mathbf{E},\mathbf{J}) = 0. \tag{3.42}$$

But also $\dim \mathcal{E}_{(\boldsymbol{\xi},\eta)} = \dim \mathcal{J}_{(\boldsymbol{\xi},\eta)} = 3$. Thus, $\mathbb{R}^6 = \mathcal{E}_{(\boldsymbol{\xi},\eta)} \oplus \mathcal{J}_{(\boldsymbol{\xi},\eta)}$. Now, let us compute the projection $\boldsymbol{\Gamma}(\boldsymbol{\xi},\eta)$ onto $\mathcal{E}_{(\boldsymbol{\xi},\eta)}$

$$\boldsymbol{\Gamma}(\boldsymbol{\xi},\eta)\begin{bmatrix}\boldsymbol{u}\\ \boldsymbol{v}\end{bmatrix} = (\boldsymbol{e}, -\frac{1}{\eta}\boldsymbol{\xi}\times\boldsymbol{e}) + (-\frac{1}{\eta}\boldsymbol{\xi}\times\boldsymbol{h},\boldsymbol{h}) \tag{3.43}$$

where $\boldsymbol{e}\in\mathbb{R}^3,\ \boldsymbol{h}\in\mathbb{R}^3$. Thus,

$$\boldsymbol{e} - \frac{1}{\eta}\boldsymbol{\xi}\times\boldsymbol{h} = \boldsymbol{u}\ ;\ \boldsymbol{h} - \frac{1}{\eta}\boldsymbol{\xi}\times\boldsymbol{e} = \boldsymbol{v}. \tag{3.44}$$

Solving for $\boldsymbol{e}$ and $\boldsymbol{h}$ we obtain

$$\boldsymbol{\Gamma}(\boldsymbol{\xi},\eta)\begin{bmatrix}\boldsymbol{u}\\ \boldsymbol{v}\end{bmatrix} = (\eta^2\boldsymbol{u} + \boldsymbol{\xi}(\boldsymbol{\xi}\cdot\boldsymbol{u}) + \eta\boldsymbol{\xi}\times\boldsymbol{v},\ -\eta\boldsymbol{\xi}\times\boldsymbol{u} - \boldsymbol{\xi}\times(\boldsymbol{\xi}\times\boldsymbol{v})), \tag{3.45}$$

provided $\eta^2 + |\boldsymbol{\xi}|^2 = 1$.

Notation 3. *Let $\pi(\boldsymbol{\xi})$ denote the skew-symmetric matrix such that $\pi(\boldsymbol{\xi})\boldsymbol{a} = \boldsymbol{\xi}\times\boldsymbol{a}$.*

Then in this notation

$$\mathbf{\Gamma}(\boldsymbol{\xi},\eta)=\begin{bmatrix}\eta^2\mathbf{I}+\boldsymbol{\xi}\otimes\boldsymbol{\xi} & \eta\pi(\boldsymbol{\xi})\\ -\eta\pi(\boldsymbol{\xi}) & -\pi(\boldsymbol{\xi})^2\end{bmatrix} \tag{3.46}$$

Let $(\varepsilon_1,\mu_1),(\varepsilon_2,\mu_2)$ be pairs of dielectric permeability and magnetic permittivity of two materials that we layer. Then

$$\mathbb{L}_1=\begin{bmatrix}-\varepsilon_1\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}/\mu_1\end{bmatrix},\ \mathbb{L}_2=\begin{bmatrix}-\varepsilon_2\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}/\mu_2\end{bmatrix}. \tag{3.47}$$

We want to compute

$$\mathbb{L}^*=W^{-1}_{(\boldsymbol{\xi},\eta)}(\theta W_{(\boldsymbol{\xi},\eta)}(\mathbb{L}_1)+(1-\theta)W_{(\boldsymbol{\xi},\eta)}(\mathbb{L}_2)). \tag{3.48}$$

The components of the space-time unit normal $(\boldsymbol{\xi},\eta)$ to the layers receive the following interpretation

$$\boldsymbol{\xi}=\frac{\boldsymbol{e_1}}{\sqrt{1+v^2}},\ \eta=-\frac{v}{\sqrt{1+v^2}}, \tag{3.49}$$

where v is the normal velocity of the layers and $\boldsymbol{e_1}=(1,0,0)$ is the spacial normal to the layers. The computation is a straightforward calculation with Maple software. We simply substitute (3.47) and (3.49) into (3.48). The W-transformation is given here by (3.34) and $\mathbf{\Gamma}(\boldsymbol{\xi},\eta)$ is given by (3.46). Performing the computation with Maple we obtain $\mathbb{L}^*=\begin{bmatrix}\boldsymbol{L}^*_{11} & \boldsymbol{L}^*_{12}\\ -\boldsymbol{L}^*_{12} & \boldsymbol{L}^*_{22}\end{bmatrix}$, where

$$\boldsymbol{L}^*_{11}=\begin{bmatrix}a_1&0&0\\0&b_1&0\\0&0&b_1\end{bmatrix},\quad \boldsymbol{L}^*_{12}=\begin{bmatrix}0&0&0\\0&0&d\\0&-d&0\end{bmatrix},\quad \boldsymbol{L}^*_{22}=\begin{bmatrix}a_2&0&0\\0&b_2&0\\0&0&b_2\end{bmatrix}, \tag{3.50}$$

and where

$$\begin{aligned}
a_1&=-H(\varepsilon),\quad b_1=H(\varepsilon)\frac{\varepsilon_1\varepsilon_2\mu_1\mu_2\eta^2-A(\varepsilon)A(\mu)}{-\eta^2\varepsilon_1\varepsilon_2\mu_1\mu_2+A(\mu)H(\varepsilon)},\\
a_2&=1/H(\mu),\quad b_2=\frac{1}{H(\mu)}\,\frac{-\varepsilon_1\varepsilon_2\mu_1\mu_2\eta^2+H(\varepsilon)H(\mu)}{-\eta^2\varepsilon_1\varepsilon_2\mu_1\mu_2+A(\mu)H(\varepsilon)},\\
d&=\frac{H(\varepsilon)}{H(\mu)}\,\frac{-\eta\,\theta\,\mu_2(\varepsilon_2-\varepsilon_1)(\mu_1-H(\mu))}{-\eta^2\varepsilon_1\varepsilon_2\mu_1\mu_2+A(\mu)H(\varepsilon)}.
\end{aligned} \tag{3.51}$$

$A(f)$, $H(f)$ denote an arithmetic mean of f and a harmonic mean of f, respectively.

We may interpret $\mathbb{L}^*$ as the tensor of electromagnetic properties of a moving dielectric via Minkowski material relations.

$$\mathbf{D} + \boldsymbol{v}^* \times \mathbf{H} = \boldsymbol{\varepsilon}^*(\mathbf{E} + \boldsymbol{v}^* \times \mathbf{B}), \tag{3.52}$$

$$\mathbf{B} - \boldsymbol{v}^* \times \mathbf{E} = \boldsymbol{\mu}^*(\mathbf{H} - \boldsymbol{v}^* \times \mathbf{D}). \tag{3.53}$$

Let $v_* = |\boldsymbol{v}^*|$. Then, solving (3.52), (3.53) for $\mathbf{D}$ and $\mathbf{H}$ we get

$$(1 - v_*^2)\boldsymbol{\mu}^*\mathbf{D} = (\boldsymbol{\mu}^*\boldsymbol{\varepsilon}^* - v_*^2\mathbf{I})\mathbf{E} + (\boldsymbol{\mu}^*\boldsymbol{\varepsilon}^* - \mathbf{I})[\boldsymbol{v}^* \times \mathbf{B} - \boldsymbol{v}^*(\boldsymbol{v}^* \cdot \mathbf{E})], \tag{3.54}$$

$$(1 - v_*^2)\boldsymbol{\mu}^*\mathbf{H} = (\mathbf{I} - v_*^2\boldsymbol{\mu}^*\boldsymbol{\varepsilon}^*)\mathbf{B} + (\boldsymbol{\mu}^*\boldsymbol{\varepsilon}^* - \mathbf{I})[\boldsymbol{v}^* \times \mathbf{E} + \boldsymbol{v}^*(\boldsymbol{v}^* \cdot \mathbf{B})]. \tag{3.55}$$

We want to identify the constitutive relation

$$\begin{bmatrix} -\mathbf{D} \\ \mathbf{H} \end{bmatrix} = \begin{bmatrix} \boldsymbol{L}_{11}^* & \boldsymbol{L}_{12}^* \\ -\boldsymbol{L}_{12}^* & \boldsymbol{L}_{22}^* \end{bmatrix} \begin{bmatrix} \mathbf{E} \\ \mathbf{B} \end{bmatrix}. \tag{3.56}$$

with the Minkowski material relations (3.54) and (3.55). Observe that

$$\begin{aligned} \boldsymbol{L}_{11}^* &= b_1\mathbf{I} + (a_1 - b_1)\boldsymbol{e_1} \otimes \boldsymbol{e_1}, \\ \boldsymbol{L}_{12}^* &= d\pi(\boldsymbol{e_1}), \\ \boldsymbol{L}_{22}^* &= b_2\mathbf{I} + (a_2 - b_2)\boldsymbol{e_1} \otimes \boldsymbol{e_1}. \end{aligned} \tag{3.57}$$

Therefore, we look for $\boldsymbol{\varepsilon}^*$, $\boldsymbol{\mu}^*$ in the form

$$\boldsymbol{\mu}^* = \mu_*'\mathbf{I} + \mu_*''\boldsymbol{e_1} \otimes \boldsymbol{e_1}, \quad \boldsymbol{\varepsilon}^* = \varepsilon_*'\mathbf{I} + \varepsilon_*''\boldsymbol{e_1} \otimes \boldsymbol{e_1}. \tag{3.58}$$

Substituting $\mathbf{D}$ and $\mathbf{H}$ from (3.56) and $\boldsymbol{\varepsilon}^*$ and $\boldsymbol{\mu}^*$ from (3.58) into (3.54) and (3.55) we get

$$\begin{aligned} (1 - v_*^2)\mu_*'\boldsymbol{L}_{11}^* &= (v_*^2 - \mu_*'\varepsilon_*')\mathbf{I} + (v_*^2(\mu_*'\varepsilon_*' - 1) - \mu_*'\varepsilon_*''(1 - v_*^2))\boldsymbol{e_1} \otimes \boldsymbol{e_1}, \\ (1 - v_*^2)\mu_*'\boldsymbol{L}_{12}^* &= v_*(1 - \mu_*'\varepsilon_*')\pi(\boldsymbol{e_1}), \\ (1 - v_*^2)\mu_*'\boldsymbol{L}_{22}^* &= (1 - v_*^2\mu_*'\varepsilon_*')\mathbf{I} + (v_*^2(\mu_*'\varepsilon_*' - 1) - \frac{\mu_*''(1 - v_*^2)}{\mu_*' + \mu_*''})\boldsymbol{e_1} \otimes \boldsymbol{e_1}. \end{aligned} \tag{3.59}$$

Substituting (3.57) into (3.59) we get the following equations for $\varepsilon'_*,\ \varepsilon''_*,\ \mu'_*,\ \mu''_*$ and v_*.

$$\begin{aligned} a_1 &= -(\varepsilon'_* + \varepsilon''_*), \quad b_1 = \frac{v_*^2 - \mu'_*\varepsilon'_*}{\mu'_*(1-v_*^2)}, \\ a_2 &= \frac{1}{\mu'_* + \mu''_*}, \quad b_2 = \frac{1-\mu'_*\varepsilon'_* v_*^2}{\mu'_*(1-v_*^2)}, \quad d = \frac{v_*(\mu'_*\varepsilon'_* - 1)}{\mu'_*(1-v_*^2)}. \end{aligned} \tag{3.60}$$

Solving (3.60) for $\varepsilon'_*,\ \varepsilon''_*,\ \mu'_*,\ \mu''_*$ and v_* we obtain

$$\begin{aligned} \varepsilon'_* &= \frac{b_2 v_*^2 - b_1}{1+v_*^2}, \\ \varepsilon''_* &= -\frac{b_2 v_*^2 - b_1}{1+v_*^2} - a_1, \\ \mu'_* &= \frac{1}{b_2 - v_* d}, \\ \mu''_* &= \frac{1}{a_2} + \frac{1}{v_* d - b_2}, \\ v_* &= \frac{b_1 + b_2 \pm \sqrt{(b_1+b_2)^2 - 4d^2}}{2d}, \end{aligned} \tag{3.61}$$

where a_1, a_2, b_1, b_2 and d are given by (3.51). In the formula for v_* in (3.61) we take "+" sign if $b_1 + b_2 < 0$ and "−" sign if $b_1 + b_2 > 0$.

CHAPTER 4

EXACT RELATIONS FOR 3D HALL EFFECT

Consider composites made with conductors exhibiting the Hall effect. The goal is to describe all exact relations in this physical context. An example of an exact relation is the set of conductivity tensors that *do not* exhibit Hall effect : any mixture made with such materials will not exhibit the Hall effect. From the geometric point of view, a six dimensional surface consisting of 3×3 symmetric matrices lying in a nine dimensional space of all 3×3 matrices is an exact relation.

We are going to use the general theory of exact relations developed in [4, 5, 6, 7]. According to the general theory, exact relations passing through a given reference medium $\mathbf{L}_0 \in \mathrm{End}^+(\mathbb{R}^3)$ are in one-to-one correspondence with all subspaces $\Pi \in \mathrm{End}(\mathbb{R}^3)$ satisfying the condition

$$\boldsymbol{KAK} \in \Pi \quad \text{for all } \boldsymbol{K} \in \Pi \quad \text{and for all } \boldsymbol{A} \in \mathcal{A}, \tag{4.1}$$

where $\mathcal{A} = \mathrm{Span}\{\Gamma'(\boldsymbol{n}) - \Gamma'(\boldsymbol{e}_1); \quad |\boldsymbol{n}| = 1\}$ and $\Gamma'(\boldsymbol{n}) = \frac{\boldsymbol{L}_0\boldsymbol{n}\otimes\boldsymbol{n}}{(\boldsymbol{L}_0\boldsymbol{n},\boldsymbol{n})}$. The exact relation $\mathbb{M}$ corresponding to Π is then given by

$$\mathbb{M} = \{\boldsymbol{L} \in \mathrm{End}^+(\mathbb{R}^3) : \quad \boldsymbol{L} = \boldsymbol{L}_0 - [\boldsymbol{I} + \boldsymbol{K}\Gamma'(\boldsymbol{e}_1)]^{-1}\boldsymbol{K}\boldsymbol{L}_0, \ \mathbf{K} \in \Pi\}. \tag{4.2}$$

In order to simplify the problem of solving (4.1) we make the following transformation :
$\Pi_0 = \boldsymbol{Y}^{-1}\Pi\boldsymbol{X}^{-1}$, $\quad \mathcal{A}_0 = \boldsymbol{X}\mathcal{A}\boldsymbol{Y}$, where $\mathbf{X}$ and $\mathbf{Y}$ can be any invertible 3×3 matrices. Then Π_0 satisfies

$$\boldsymbol{KAK} \in \Pi_0 \quad \text{for all } \boldsymbol{K} \in \Pi_0 \quad \text{and for all } \boldsymbol{A} \in \mathcal{A}_0. \tag{4.3}$$

Let $\boldsymbol{\sigma}_0 = (\boldsymbol{L}_0 + \boldsymbol{L}_0^T)/2 \in \mathrm{Sym}^+(\mathbb{R}^3)$. Let $\boldsymbol{X} = \boldsymbol{\sigma}_0^{1/2}\boldsymbol{L}_0^{-1}$ and $\boldsymbol{Y} = \boldsymbol{\sigma}_0^{1/2}$ then

$$\mathcal{A}_0 = \mathrm{Span}\left\{\frac{\boldsymbol{\sigma}_0^{1/2}\boldsymbol{n}\otimes\boldsymbol{\sigma}_0^{1/2}\boldsymbol{n}}{(\boldsymbol{L}_0\boldsymbol{n},\boldsymbol{n})} - \frac{\boldsymbol{\sigma}_0^{1/2}\boldsymbol{e}_1\otimes\boldsymbol{\sigma}_0^{1/2}\boldsymbol{e}_1}{(\boldsymbol{L}_0\boldsymbol{e}_1,\boldsymbol{e}_1)} : \quad |\boldsymbol{n}| = 1\right\}. \tag{4.4}$$

Obviously, $\mathcal{A}_0 \subset \{\boldsymbol{A} \in \mathrm{Sym}(\mathbb{R}^3) : \quad \mathrm{Tr}\,\boldsymbol{A} = 0\}$. Suppose $\boldsymbol{B} \in \mathrm{Sym}(\mathbb{R}^3)$ is orthogonal to $\mathcal{A}_0$. Then for all $|\boldsymbol{n}| = 1$

$$\frac{(\boldsymbol{B}\boldsymbol{\sigma}_0^{1/2}\boldsymbol{n},\boldsymbol{\sigma}_0^{1/2}\boldsymbol{n})}{(\boldsymbol{L}_0\boldsymbol{n},\boldsymbol{n})} = \frac{(\boldsymbol{B}\boldsymbol{\sigma}_0^{1/2}\boldsymbol{e}_1,\boldsymbol{\sigma}_0^{1/2}\boldsymbol{e}_1)}{(\boldsymbol{L}_0\boldsymbol{e}_1,\boldsymbol{e}_1)}.$$

In other words there is $\alpha \in \mathbb{R}$ such that for all $|\boldsymbol{n}| = 1$

$$(\boldsymbol{\sigma}_0^{1/2}\boldsymbol{B}\boldsymbol{\sigma}_0^{1/2}\boldsymbol{n},\boldsymbol{n}) = \alpha(\boldsymbol{\sigma}_0\boldsymbol{n},\boldsymbol{n}).$$

Since both $\boldsymbol{\sigma}_0$ and $\boldsymbol{\sigma}_0^{1/2}\boldsymbol{B}\boldsymbol{\sigma}_0^{1/2}$ are symmetric we conclude that $\boldsymbol{\sigma}_0^{1/2}\boldsymbol{B}\boldsymbol{\sigma}_0^{1/2} = \alpha\boldsymbol{\sigma}_0$. Thus, $\boldsymbol{B} = \alpha\boldsymbol{I}$. Therefore,

$$\mathcal{A}_0 = \{\boldsymbol{A} \in \mathrm{Sym}(\mathbb{R}^3) \ : \ \mathrm{Tr}\,(\boldsymbol{A}) = 0\}. \tag{4.5}$$

We solve (4.3) with the aid of the computer algebra package Maple.

We can represent a basis of a subspace $\mathcal{L} \subset \mathrm{End}(\mathbb{R}^3)$ by a matrix, whose rows are basis elements of $\mathcal{L}$ written in terms of the chosen basis for $\mathrm{End}(\mathbb{R}^3)$. If we row-reduce this matrix to the row-reduced echelon form (rref) then the rows of rref still form a basis of $\mathcal{L}$. Moreover, there is a unique basis of $\mathcal{L}$ that corresponds to rref. The structure of the rref is defined by the position of pivots. We go through all possible rref structures and determine all subspaces with that structure of rref that satisfy (4.1). This is accomplished via the computer algebra package, Maple that uses the grobner basis technique to solve the resulting system of cubic polynomial equations in as many as 20 variables (at the most). The program has returned 62 families of subspaces.

Before we list our results we would like to mention one particularly simple type of exact relations, the uniform field relations [3, 17, 18]. If the local tensor $\mathbf{L}(\mathbf{x})$ satisfies p vector identities

$$\mathbf{L}(\mathbf{x})\boldsymbol{e}_i = \boldsymbol{j}_i, \; i = 1, \ldots, p$$

for some fixed constant vectors $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_p$ and $\boldsymbol{j}_1, \ldots, \boldsymbol{j}_p$, then $\mathbf{L}^*\boldsymbol{e}_i = \boldsymbol{j}_i$, $i = 1, \ldots, p$ because the pairs of uniform fields $(\boldsymbol{e}_i, \boldsymbol{j}_i)$ solve the cell problem. In the language of subspaces Π these exact relations correspond to the annihilators

$$\mathrm{Ann}(\mathcal{L}) = \{\mathbf{K} \in \mathrm{End}(\mathbb{R}^3) : \mathbf{K}\boldsymbol{a} = \mathbf{0} \text{ for all } \boldsymbol{a} \in \mathcal{L}\},$$

where $\mathcal{L}$ is a subspace of $\mathbb{R}^3$ of dimension p (for a non-trivial exact relation p must be either 1 or 2 in this context.)

By our design the resulting families of subspaces are parameterized by $\mathbb{R}^k$, where k is the number of free parameters in Maple output. From the geometric point of view this is not always natural. For example, the family $\Pi_{\boldsymbol{a}} = \{\boldsymbol{a} \otimes \boldsymbol{v} \mid \boldsymbol{v} \in \mathbb{R}^3\}, \quad \boldsymbol{a} \neq \mathbf{0}$ is parameterized by $\mathbb{RP}^2$. Therefore, in Maple output this family of subspaces would be replaced by 3 outputs.

$$\begin{aligned}
\Pi_{\tilde{\boldsymbol{a}}} &= \{(1, \tilde{\boldsymbol{a}}) \otimes \boldsymbol{v} \mid \boldsymbol{v} \in \mathbb{R}^3\}, &\quad \tilde{\boldsymbol{a}} \in \mathbb{R}^2 \\
\Pi_{\alpha} &= \{(0, 1, \alpha) \otimes \boldsymbol{v} \mid \boldsymbol{v} \in \mathbb{R}^3\}, &\quad \alpha \in \mathbb{R} \\
\Pi_{\infty} &= \{(0, 0, 1) \otimes \boldsymbol{v} \mid \boldsymbol{v} \in \mathbb{R}^3\}.
\end{aligned}$$

This representation corresponds to the cell-decomposition of a CW-complex $\mathbb{RP}^2$ into its CW-cells.

Consequently, I had to go through the Maple output and collect together all the cells that belong to the same CW-complex in the space of parameters. At the end the set of subspaces Π satisfying (4.1) is split into the smaller number of larger families parameterized by various Grassmanians.

Here is the summary of our analysis of the 62 families of subspaces Π returned by Maple.

1. There are 9 families of one-dimensional subspaces in the Maple output. They can be aggregated into one larger family

$$\Pi = \mathbb{R}(\boldsymbol{a} \otimes \boldsymbol{b}), \quad \boldsymbol{a} \in \mathbb{R}^3 \setminus \{\boldsymbol{0}\},\ \boldsymbol{b} \in \mathbb{R}^3 \setminus \{\boldsymbol{0}\}, \tag{4.6}$$

We may write the family Π as $(\mathrm{Ann}((\mathbb{R}\boldsymbol{a})^\perp))^T \cap \mathrm{Ann}((\mathbb{R}\boldsymbol{b})^\perp)$.

2. There are 18 families of two-dimensional subspaces in the Maple output. They can be aggregated into two larger families

$$\Pi = \{\boldsymbol{v} \otimes \boldsymbol{a} \mid \boldsymbol{v} \in \mathcal{L} \subset \mathbb{R}^3, \quad \dim \mathcal{L} = 2\}, \tag{4.7}$$

and

$$\Pi = \{\boldsymbol{a} \otimes \boldsymbol{v} \mid \boldsymbol{v} \in \mathcal{L} \subset \mathbb{R}^3, \quad \dim \mathcal{L} = 2\}. \tag{4.8}$$

The family of subspaces Π given by (4.7) is equal to $\mathrm{Ann}((\mathbb{R}\boldsymbol{a})^\perp) \cap (\mathrm{Ann}(\mathcal{L}^\perp))^T$ and the family of subspaces Π given by (4.8) is equal to $\mathrm{Ann}((\mathbb{R}\boldsymbol{a})^\perp)^T \cap \mathrm{Ann}(\mathcal{L}^\perp)$.

3. There are 9 families of three-dimensional subspaces in the Maple output. They can be aggregated into three larger families

$$\Pi = \{\boldsymbol{v} \otimes \boldsymbol{a} \mid \boldsymbol{v} \in \mathbb{R}^3\}, \tag{4.9}$$

$$\Pi = \{\boldsymbol{a} \otimes \boldsymbol{v} \mid \boldsymbol{v} \in \mathbb{R}^3\}, \tag{4.10}$$

and

$$\mathrm{Ann}(\mathbb{R}\boldsymbol{a}) \cap \mathrm{Sym}(\mathbb{R}^3). \tag{4.11}$$

The family of subspaces Π given by (4.9) is equal to $\mathrm{Ann}((\mathbb{R}\boldsymbol{a})^\perp)$ and the family of subspaces Π given by (4.10) is equal to $(\mathrm{Ann}((\mathbb{R}\boldsymbol{a})^\perp))^T$.

4. There are 9 families of four-dimensional subspaces in the Maple output. They can be aggregated into one larger family

$$\Pi = \{\boldsymbol{v} \otimes \boldsymbol{a} + \boldsymbol{w} \otimes \boldsymbol{b} \mid \boldsymbol{v} \in \mathcal{L}, \boldsymbol{w} \in \mathcal{L},\ \mathcal{L} \subset \mathbb{R}^3,\ \dim \mathcal{L} = 2\}, \tag{4.12}$$

The family of subspaces Π given by (4.12) is equal to
$\mathrm{Ann}(\mathbb{R}\boldsymbol{c}) \cap (\mathrm{Ann}(\mathbb{R}\boldsymbol{n}))^T$, where $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$ and $\boldsymbol{n}$ is normal to $\mathcal{L}$.

5. There are families of 7 six-dimensional subspaces in the Maple output. They can be aggregated into three larger families

$$\Pi = \{\boldsymbol{v} \otimes \boldsymbol{a} + \boldsymbol{w} \otimes \boldsymbol{b} \mid \boldsymbol{v} \in \mathbb{R}^3, \boldsymbol{w} \in \mathbb{R}^3\}, \tag{4.13}$$

$$\Pi = \{\boldsymbol{a} \otimes \boldsymbol{v} + \boldsymbol{b} \otimes \boldsymbol{w} \mid \boldsymbol{v} \in \mathbb{R}^3, \boldsymbol{w} \in \mathbb{R}^3\}, \tag{4.14}$$

and

$$\Pi = \mathrm{Sym}(\mathbb{R}^3). \tag{4.15}$$

The family of subspaces Π given by (4.13) is equal to $\mathrm{Ann}(\mathbb{R}\boldsymbol{c})$, where $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$. The family of subspaces Π given by (4.14) is equal to $(\mathrm{Ann}(\mathbb{R}\boldsymbol{c}))^T$.

There are no 5, 7 or 8 dimensional subspaces in the Maple output. The remaining 10 families of subspaces are complex valued and therefore are discarded.

In conclusion we point out that all exact relations for 3D Hall effect are generated by the exact relation $\mathrm{Sym}(\mathbb{R}^3)$ and the uniform field relations by means of taking transposes and intersections. The most important result of this chapter is that aside from the exact relations described above there are no others.

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