SOME NON-ABELIAN COVERS OF KNOT COMPLEMENTS

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ABSTRACT

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Let K be a tame knot embedded in $\mathbf{S}^3$. We address the problem of finding the minimal degree non-cyclic cover $p : X \to \mathbf{S}^3 \smallsetminus K$. When K has non-trivial Alexander polynomial we construct finite non-abelian representations $\rho : \pi_1(\mathbf{S}^3 \smallsetminus K) \to G$, and provide bounds for the order of G in terms of the crossing number of K, which is an improvement on a result of Broaddus in this case. Using classical covering space theory along with the theory of Alexander stratifications we establish an upper and lower bound for the first betti number of the cover X_ρ associated to the $\ker(\rho)$ of $\mathbf{S}^3 \smallsetminus K$, consequently showing that it can be arbitrarily large, which provides an effective proof of a result involving peripheral subgroup separation. We also demonstrate that X_ρ contains non-peripheral homology for certain computable examples, which mirrors a famous result of Cooper, Long, and Reid when K is a knot with non-trivial Alexander polynomial.

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CHAPTER 1

INTRODUCTION

In 1987 Hempel [20] showed that the fundamental groups of Haken 3-manifolds are residually finite, i.e., $\bigcap H = \{1\}$ where H ranges over the finite index normal subgroups of the fundamental group of the 3-manifold. It follows that all topological 3-manifolds with single a torus boundary component are residually finite. A consequence of residual finiteness is that the fundamental group admits a rich family of finite quotients, and therefore a knot manifold M has an abundance of finite sheeted covers with varying topological properties.

For the remainder of this dissertation M_K always denotes the manifold $\mathbf{S}^3 \smallsetminus K$, and $\Gamma_K = \pi_1(M_K)$. There is a very well known construction which describes an infinite family of finite covers of a knot complement, namely those which arise from the kernels of finite cyclic quotients, known as cyclic covers. Such quotients come from the following construction. Denote $\Gamma_K^{\mathrm{ab}} = \Gamma_K / [\Gamma_K, \Gamma_K]$. Since $\Gamma_K^{\mathrm{ab}} \cong \mathbb{Z}$, for a knot K, there exists a homomorphism $\Gamma_K \to \mathbb{Z}/n\mathbb{Z}$. The kernel of this homomorphism corresponds to a regular cover, X_n, typically called the n-fold cyclic cover of M.[1]

When K is a non-trivial knot, residual finiteness ensures the existence of non-abelian quotients of Γ_K. Thus, there exist covers of M_K which do not arise from the cyclic quotients of $H_1(M_K)$ described above. In this dissertation we

[1]A similar, but different notion, is the cyclic covers of $\mathbf{S}^3$ branched over the knot K. We do not discuss these covers.

address the following question.

Question 1.1. *What is the minimal degree non-cyclic cover of M_K?*

The first systematic treatment of this problem was due to Broaddus [8]. In his thesis, he constructs explicit finite non-cyclic covers of the knot complements and provides an upper bound on the degree. Kuperberg [24] later decribed the growth rate of the degree of non-abelian covers as being NP modulo the Generalized Reimann Hypothesis. We improve on these results when K has non-trivial Alexander polynomial.

Other than Broaddus's and Kuperberg's work there is little in the literature that directly addresses the problem of minimal degree non-cyclic covers of knot complements. Moreover, Broaddus and Kuperberg both relate the degree of the non-cyclic covers to combinatorial invariants of the knot. Let D denote any diagram of K, recall that the crossing number of a knot is defined to be

$$c_K = \min|\{\text{Crossings} \quad \text{of} \quad D\}|,$$

where the minimum is taken over all diagrams, D, of the knot. Broaddus proved the following:

Theorem 1.1. *(Broaddus, [8]) For all non-trivial knots K, there exists an explicit function $\mathfrak{b} : \mathbb{N}_{3\geq} \to \mathbb{N}$,* [2] *and there exists a finite non-cyclic cover Z of M_K, with $[M_K : Z] \leq \mathfrak{b}(c_K)$.*

Similarly, Kuperberg proves the following result about the existence and order of finite non-abelian quotients of the group Γ_K. In the following "pol" and "exp" represent the existence of a polynomial and exponential functions in the variable c_K.

Theorem 1.2. *(Kuperberg, [24]) If K is a non-trivial knot, then there exists a finite quotient G of Γ_K with*

$$|G| = \exp(\exp(\text{pol}(c_K))).$$

[2] $\mathbb{N}_{3\geq}$ denotes Natural numbers greater than 3.

Assuming the Generalized Reimann Hypothesis, one has

$$|G| = \exp(\mathrm{pol}(c_K)).$$

Furthermore due to Fox's [14] work on dihedral coverings of knot complements, we know that when the determinant of a knot $|\Delta(-1)| \neq 1$ there exists an irregular cover $Y \to M_K$. This cover has degree bounded by $|\Delta(t)|$.

Now let $\delta \approx 1.83929$ be the inverse of

$$\delta^{-1} = -\frac{1}{3} - \frac{2}{3\sqrt[3]{17+3\sqrt{33}}} + \frac{\sqrt[3]{17+3\sqrt{33}}}{3}.$$

Stoimenow [36] then proved,

Theorem 1.3. *(Stoimenow, [36])*

For all knots K

$$|\Delta(-1)| \leq \delta^{c_K - 1}.$$

So it follows that what $|\Delta(t)| \neq 1$ there exists a non-cyclic cover with degree at most 2^{c_K-1}. We establish a larger bound, for all knots with non-trivial Alexander polynomial. Furthermore we also improve this result in terms of the degree of the cover in the results of Broaddus and Kuperberg, and drop the reliance on the Generalized Reimann Hypothesis, for a knot K with non-trivial Alexander polynomial. Explicitly, we establish an upper bound similar to the result of Broaddus, however our construction yields a computationally simpler bound, in the sense that the lower bound established by Broaddus exceeds computational capability of current computer software on a standard desktop computer even for $c_K = 3$. Furthermore the bound we establish is of the class $\exp(\mathrm{pol}(c_K))$, however both exp and pol are explicitly given.

1.1 Theorem 1

In Chapter 4 we prove the following Theorem.

Theorem 1. *If K is a knot with crossing number c_K and non-trivial Alexander polynomial, then there exists a regular non-abelian cover X_{ρ_α} of M_K with*

$$[M_K : X_{\rho_\alpha}] \leq 2^{4c_K^2},$$

and there exists an irregular non-cyclic cover Y_{ρ_α} with

$$[M_K : Y_{\rho_\alpha}] \leq 2^{2c_K^2}.$$

Notice that Theorem 1 addresses the minimality of *regular* non-cyclic covers, providing explicit constructions and bounds. This has not been previously studied in the literature. We we strengthen the conclusion of Theorem 1 for certain important families of knots.

Theorem 1.4.

1. *If K is a twist knot with $2n$ half twists, then*

$$[M_K : Y_{\rho_\alpha}] \leq 16n^2.$$

2. *If K is a fibered knot, with non-trivial Alexander polynomial we have*

$$[M_K : Y_{\rho_\alpha}] \leq 2^{c_K}.$$

3. *For knots with Alexander polynomial of degree n (it follows that $n \leq c_K - 1$,) hence we have*

$$[M_K : Y_{\rho_\alpha}] \leq 2^{2n^2}.$$

As we have mentioned, the Alexander polynomial is a well known invariant of the knot group, denoted $\Delta(t)$, defined in 1923 by J.W. Alexander [2]. Since then many authors have formulated equivalent definitions of the Alexander Polynomial ([2], [12], [10], [34], [31]). In order to prove Theorem 1 we generalize the construction due to de Rham [10], which which simultaneously defines the Alexander polynomial and constructs representations to affine-linear groups over $\mathbb{C}$. Using this point of view we are able to construct explicit, finite, metabelian representations of the knot group by generalizing de Rham's construction to an arbitrary finite field.

For $\alpha \in \widehat{\mathbb{F}_p}$ (the notation $\widehat{\kappa}$ denotes the algebraic closure of the field κ,) we can construct the following subgroup of $\mathrm{Aff}\left(\widehat{\mathbb{F}_p}\right)$. Let

$$G_\alpha = \langle \alpha z, z+1 \rangle < \mathrm{Aff}\left(\widehat{\mathbb{F}_p}\right),$$

since every $\alpha \in \widehat{\mathbb{F}_p}$ is contained in a finite extension of $\mathbb{F}_p$, this group is finite.

1.2 Theorem 2

Our generalization of de Rham's theorem is:

Theorem 2. *For all knots K, there exists a surjective homomorphism $\rho_\alpha : \Gamma_K \to G_\alpha$ if and only if α is a non-zero root of $\Delta(t) \pmod p$. The homomorphism ρ_α satisfies:*

- *$\rho_\alpha(\Gamma_K)$ is metabelian, in particular non-abelian.*
- *$|\rho_\alpha(\Gamma_K)| = np^d$, where $d = \deg_{\mathbb{F}_p}(\alpha)$ and $n = \mathrm{ord}_{\mathbb{F}^*_{p^d}}(\alpha)$ (the order of α in the group of units of $\mathbb{F}_{p^d}$.)*

A metabelian group G is group such that $[G,G]$ is abelian. Furthermore the quotients in Theorem 2 being metabelian should come as no suprise. The group ${}^{\Gamma_K}/_{\Gamma''_K} \cong \mathbb{Z} \ltimes {}^{\Gamma'_K}/_{\Gamma''_K}$, is a metabelian group, and such finite metabelian quotients of the knot group have been extensively studied. Fox, Artin, Hartley, and Neuwirth are the pioneers in the study of metabelian covers of knots. Fox [14], [11] describes the fundamental group of the branched cover corresponding to metacyclic representations for doubled knots. M. Artin [3] computed the first homology groups for the same covers described in [11] in his senior thesis at Princeton. R. Hartley [18] provided a necessary and sufficient criterion for a knot to admit a finite quotient to a specific class of metabelian groups; this criterion is given in terms of the abelianization of the fundamental group of the finite cyclic covers. Lastly L. P. Neuwirth [31] provided a criterion in terms of the Alexander polynomial similar to what we will describe to ensure that a knot group surjects onto a metacylic group. More recently, a general study of metabelian representations to $\mathrm{SL}(n, \mathbb{C})$ has been a fruitful area; for example see [15], [6], [22], [29], and [30].

1.3 Theorem 3

We then turn our attention to the topological properties of the regular covers X_{ρ_α}. From a computational point of view, the construction of X_{ρ_α} provides us with a large family new manifolds to examine and draw new intuition from. There are many questions to address with regards to these regular covers. For this dissertation we focus on the groups $H_1\left(X_{\rho_\alpha}\right)$, in particular the computation of $\beta_1\left(X_{\rho_\alpha}\right)$.

In an homage to Thurston's work on the virtual properties of 3-manifolds, Ian Agol's 2014 ICM address [1] highlighted the current state of the art for determining those properties of 3-manifolds. His address was focused on establishing a connection between results of Haglund and Wise and current geometric methods to answer 4 of Thurston's list of 24 problems involving virtual properties of 3-manifolds. One question of Thurston's involved the virtual first betti number. The virtual first betti number is defined to be

$$v\beta_1(M) = \sup\{\beta_1(X) \mid X \to M \text{ is a finite cover}\},$$

Thurston asks the question: Can a closed aspherical M have $v\beta_1(M) = \infty$? Agol goes on to answer this question in the positive, a consequence of the Virtual Special theorem for closed manifolds. However for manifolds, M with non-empty incompressible boundary it is a consequence of the The Seifert Fiber Theorem, The Torus Theorem, and facts about peripheral subgroup separation of Long and Niblo [27] that $v\beta_1(M) = \infty$. Furthermore the seminal paper of D. Cooper, D. Long, and A. Reid [9] from 1997 showed that for bounded 3 manifolds "non-peripheral" homology becomes unbounded in finite covers.

There is an extensive understanding of the topological and algebraic properties of finite cyclic covers of knot complements. In particular, complete information of the first homology groups of the cyclic covers X_n can be determined directly from the Alexander polynomial of the knot K. Ralph Fox [13] using his free differential calculus showed that both the free rank (the first betti number) and the order of the torsion subgroup of $H_1(X_n)$ can be directly computed from the Alexander polynomial. In particular, the first betti number of the X_n is 1 except when $\Delta(t)$ has an nth root of unity as a root [12]. An immediate consequence of this is that $\beta_1(X_n) \leq \deg(\Delta(t)) + 1$, for any n. The results of [9] provide the existence of covers with arbitrarily large betti number, and by such covers cannot be the cyclic covers of a knot complement.

We then turn our attention to the computation of $\beta_1(X_\alpha)$, as a first step in understanding such covers. In §5.1 we compute a lower bound for $\beta_1(X_\alpha)$ providing us with an alternate proof of Long and Niblo's result [27] in the case of a knot complement with non-trivial Alexander polynomial.

1.4 Theorem 3

Theorem 3. *Let p be a prime such that $\Delta(t) \pmod p$ is non-trivial, $\alpha \in \widehat{\mathbb{F}_p}$ a root of $\Delta(t) \pmod p$ with $d = \deg_{\mathbb{F}_p}(\alpha)$, and $\mathrm{ord}_{\mathbb{F}^*_{p^d}}(\alpha) = n$. Then the covers X_{ρ_α} satisfy*

$$p^d - 1 + \beta_1(X_n) \leq \beta_1(X_{\rho_\alpha}) \leq (n(c_K - 1))(p^d - 1) + \beta_1 (X_n) .$$

The lower bound is a direct computation of the number of boundary components of the cover X_{ρ_α}, along with basic facts about finite covering spaces. The upper bound here is a consequence of E. Hironaka's theory of Alexander stratifications and jumping loci [21].

An immediate corollary of Theorem 3 is:

Theorem 1.5. *(Long, Niblo [27]) When K is a knot with non-trivial Alexander polynomial,*

$$v\beta_1(M_K) = \infty.$$

We will show in §5 there are knots for which X_α has non-peripheral elements in first homology suggesting that our methods often lead to concrete constructions of covers whose non-peripheral first homology becomes arbitrarily large. It is possible from these computations that the covers X_{ρ_α} might provide a concrete construction to the famous result of Cooper, Long, and Reid in this case.

Finally we also study torsion in the first homology groups of these covers. For any compact manifold X the first homology group $H_1(X;\mathbb{Z})$ is a finitely generated abelian group, thus is isomorphic to the group $\mathbb{Z}^{\beta_1(X)} \oplus T(H_1(X;\mathbb{Z}))$, here $T(H_1(X;\mathbb{Z}))$ is the torsion subgroup. The study of $T(H_1(N_j;\mathbb{Z}))$ for finite sheeted covers N_j of a 3-manifold N is recently of significant interest. Fox's results [13] include an explicit formula for the order of the torsion subgroup of $T(H_1(X_n,\mathbb{Z}))$. This has lead to many results describing the growth of torsion in finite cyclic covers. In particular Gordon [17] showed linear growth in the torsion subgroup of $H_1(X_n)$ as $n \to \infty$ for infinite classes of knots. Independently Riley [33], Gonzalez-Acuña and Short [16] , and Weber [37] were able to build on Gordon's work to show exponential growth of the order of torsion through the cyclic covers of a non-trivial knot complement.

The torsion subgroup of $H_1(N_j)$ is of particularly importance when covers N_j arrange into a tower of covers

$$\cdots \to N_j \to \cdots \to N_1 \to N$$

so that $N_i \to N_{i-1}$ is finite sheeted for all i. Recent work of H. Baik, D. Bauer, I. Gekhtman, U. Hamenstädt, S. Hensel, T. Kastenholz, B. Petri, and D. Valenzuela [4] showed that exponential torsion growth is a generic property of random 3-manifolds. Furthermore, when the 3-manifold is endowed with a hyperbolic metric and $\bigcap_{i=1}^{\infty} \pi_1(N_i) = \{1\}$ the asymptotics of torsion growth is conjectured to have close relationship with the hyperbolic volume of M. N. Bergeron and A. Venkatesh conjectured [5] (Conjecture 1.3) describing this asymptotic growth phenomenon. The only cases for which there are complete results in this direction are in the case of the cyclic covers of hyperbolic knot complements this is due to T. Lê [25] and independently J. Raimbault [32], where the towers of cyclic covers are not exhaustive, but a similar behavior is exhibited. We will conclude the dissertation providing tables of computations and highlighting certain relationships between the torsion subgroups of $H_1(X_\alpha; \mathbb{Z})$ and $H_1(X_n; \mathbb{Z})$.

CHAPTER 2

THE ALEXANDER POLYNOMIAL

2.1 Background

2.1.1 The Alexander Polynomial

Let $\Delta_K(t)$ denote the Alexander polynomial of K, when the knot itself is undersood we will drop the K and write $\Delta(t)$. See [31] or [34] for some of the many definitions. We use the notation $\Gamma' = [\Gamma, \Gamma]$, $\Gamma'' = [\Gamma', \Gamma']$, and $\Gamma^{(i)} = [\Gamma^{(i-1)}, \Gamma^{(i-1)}]$ for the i^{th} iterated commutator subgroup. The classical definition of the Alexander Polynomial due to Alexander considers the split short exact sequence:

$$1 \to {}^{\Gamma'_K}/_{\Gamma''_K} \to {}^{\Gamma_K}/_{\Gamma''_K} \to \mathbb{Z} \to 1.$$

The group ${}^{\Gamma_K}/_{\Gamma''_K}$ is a semi-direct product $\mathbb{Z} \ltimes {}^{\Gamma'_K}/_{\Gamma''_K}$, hence ${}^{\Gamma'_K}/_{\Gamma''_K}$ is a finitely generated $\mathbb{Z}[t, t^{-1}]$ module [2]. Alexander proved that the annihilator of the module is a principal ideal in $\mathbb{Z}[t, t^{-1}]$, so it is generated by a single Laurent polynomial, $\Delta(t)$. Furthermore since ${}^{\Gamma'_K}/_{\Gamma''_K}$ is a finitely generated $\mathbb{Z}[t, t^{-1}]$ module, there exists a presentation matrix $A(t)$ of rank k over $\mathbb{Z}[t, t^{-1}]$, called the Alexander matrix. The ith Alexander ideal is then the principal ideal generated by the $(k-i)$ minors of $A(t)$, and therefore $\Delta(t)$ is the generator (up to multiplication by a unit of $\mathbb{Z}[t,t^{-1}]$) of the zeroth Alexander ideal. We denote by $\Delta_i(t)$ the generator of the ith Alexander ideal, thus $\Delta_i(t)$ is the ith invariant factor of $A(t)$.

The Alexander Polynomial with coefficients in an general field κ.

Denote by ι, the canonical ring homomorphism $\iota : \mathbb{Z} \to \kappa$ for any field κ, determined by

$$\mathbb{1}_{\mathbb{Z}} \mapsto \mathbb{1}_{\kappa}.$$

We are using the convention that $\mathbb{1}_R$ is the unit in the unital ring R. Considering the images of the coefficients of the entries of $A(t)$ under ι, we denote the resulting matrix by $A_\kappa(t)$. Thus $A_\kappa(t)$ presents ${}^{\Gamma'_K}/_{\Gamma''_K}$ as a $\left({}^{\mathbb{Z}}/_{\ker(\iota)}\right)[t,t^{-1}]$ module. Note that $\ker(\iota) = p\mathbb{Z}$ for $p=0$ or a prime (p is the characteristic of κ). Assuming that p is prime, and consequently non-zero, up to field isomorphism it follows that $A_\kappa(t) = A_{\mathbb{F}_p}(t)$, and presents ${}^{\Gamma'_K}/_{\Gamma''_K}$ as an $\mathbb{F}_p[t,t^{-1}]$ module. Furthermore since $\mathbb{F}_p[t,t^{-1}]$ is a principal ideal domain we define $\Delta_{(p,i)}(t)$ to be the ith invariant factor of $A_\kappa(t)$, hence $\Delta_p(t) = \Delta_{p_0}(t)$. We call $\Delta_p(t)$ the Alexander Polynomial with coefficients in $\mathbb{F}_{p^d}$. For $p=0$, the image is isomorphic to $\mathbb{Z}$ so this construction yields the classical Alexander polynomial $\Delta(t)$. In §3 we verify that $\Delta_p(t)$ and $\Delta(t) \pmod p$ are equivalent.

The Topological Interpretation of the Alexander polynomial

For any knot K, the abelianization Γ_K^{ab} is isomorphic to $\mathbb{Z}$. Furthermore let X_∞ denote the cover of M_K corresponding to Γ'_K. This is often called the *infinite cyclic* cover of M_K. By a construction of Seifert there is a spanning surface $\Sigma \subset M_K$ and $\partial\Sigma = K$, this subsurface is orientable, and has genus $g \geq 1$. This surface is dual to the generator of $H_1(M_K)$. Let Σ be a Seifert surface for the knot K. The infinite cyclic cover is constructed by first cutting M_K open along Σ the resulting manifold, which we denote by Y_0, and taking one copy, $\{Y_i\}_{i=-\infty}^{\infty}$, for each integer. The "top" of Y_i, which we denote Σ_{i+1}, is glued to the "bottom" of Y_{i+1} also denoted Σ_{i+1} via the $i+1$ power of the glueing map represented by the arrows in 2.1. For a full visual representation of the infinite cyclic cover see figure 2.1 below.

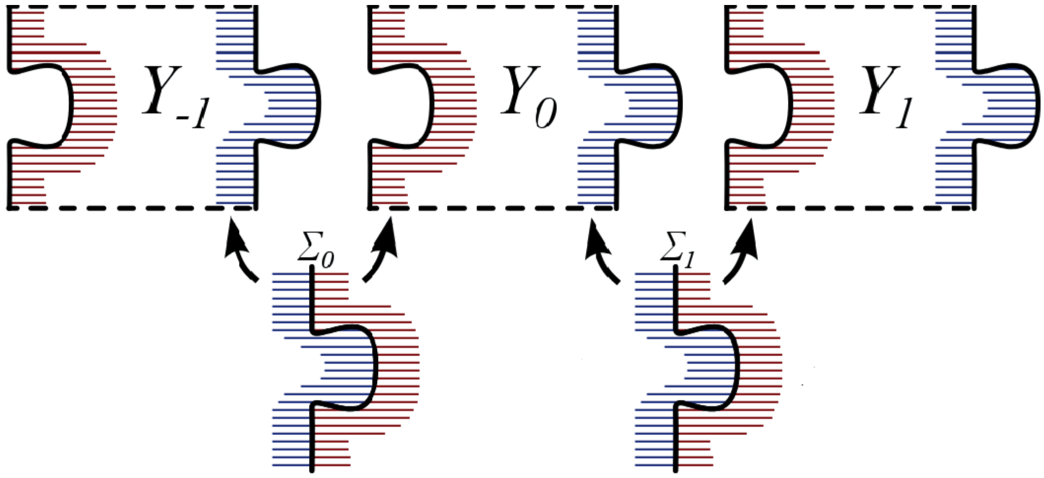

Figure 2.1: The infinite cyclic cover X_∞

The the deck group of the cover X_∞ is $\mathbb{Z} \cong \Gamma_K^{\mathrm{ab}}$. We assume that it is generated by and element t. Furthermore t acts on X_∞ by taking Σ_i to Σ_{i+1}, and Y_i to Y_{i+1}. Since t acts on $\pi_1(X_\infty) \cong \Gamma_K'$ it descend to and action of $H_1(X_\infty)$, since $\pi_1(X_\infty)'$ is characteristic. Therefore $H_1(X_\infty)$ is a finitely generated $\mathbb{Z}[t, t^{-1}]$ module and by Alexander's theorem [2] the annihilator is principal, generated by a single Laurent polynomial $\Delta(t)$. This discussion is just the topological analogue of the discussion in 2.1.1.

2.1.2 Facts About the Alexander polynomial

Lemma 2.1. *(Rolfsen, [34]) Let K be a knot, then its Alexander polynomial, $\Delta(t)$ satisfies:*

1) $\Delta(t) = t^{\pm i}\Delta(t^{-1})$ *for some* $i \geq 0$.

2) $\Delta(1) = \pm 1$.

We are able to summarize the above lemma in language that will be useful to the purposes of this dissertation.

Proposition 2.1 ([34]). *Let* $\Delta(t) = \Sigma_{i=-k}^{l} a_i t^i$ *be the Alexander polynomial of a knot* K *and define* $\deg(\Delta(t)) = l + k$*, then*

1) $\Delta(1) = \pm 1$*,*

2) $a_{l-j} = a_{j-k}$ *for* $j = 0, \dots, \frac{l+k}{2} - 1$*,*

3) $\deg_{\mathbb{Q}}(\Delta(t)) \leq c_K - 1$*, where* c_K *is the crossing number of* K*.*

Proof. The properties 1) and 2) are immediate consequences of Lemma 2.1. The third property is an observation of the fact that in the Wirtinger presentation of the knot K there are exactly $c_K - 1$ generators, so an Alexander matrix can be written as a $c_K \times c_K - 1$ matrix, thus $\deg_{\mathbb{Q}}(\Delta(t)) \leq c_K - 1$. □

We say $\Delta(t)$ is trivial if $\Delta(t) = \pm t^n$.

Lemma 2.2. *Suppose* $\Delta(t)$ *is non-trivial. Then it has at least* 3 *non-zero coefficients.*

Proof. Suppose $\Delta(t)$ has one non-zero coefficient a_0. Then by 2.1(1), $\Delta(1) = a_0 = \pm 1$, hence $\Delta(t)$ is trivial. Thus, $\Delta(t)$ has at least 2 non-zero coefficients a_0 and a_1 and by 2) they must be equal however, since $\Delta(1) = \pm 2a_0$ which cannot be 1, and the lemma follows. □

2.1.3 de Rham's Construction

We, however, bring attention to a definition of $\Delta(t)$ due to de Rham [10]. The definition of de Rham is of particular importance to us because it allows us to simultaneously define $\Delta_p(t)$ and construct non-abelian representations to finite groups.

For a field κ de Rham's construction begins by attempting to define a homomorphism,

$$\varphi : \Gamma_K \to \mathrm{Aff}(\kappa) = \{tz + x \mid t \in \kappa^*,\ x \in \kappa\}.$$

Using the Wirtinger presentation for Γ_K, we have that given a diagram of K with n crossings

$$\Gamma_K = \langle x_1, x_2, \dots x_n \mid x_i x_{j(i)} = x_{j(i)} x_{i+1} \text{ or } x_{i+1} x_{j(i)} = x_{j(i)} x_i \rangle. \tag{2.1}$$

It is important to note that this is a balanced presentation (the number of generators is equal to number of relators). See Figure 2.2 for the definition of $x_{j(i)}$ and the relations in Γ_K.

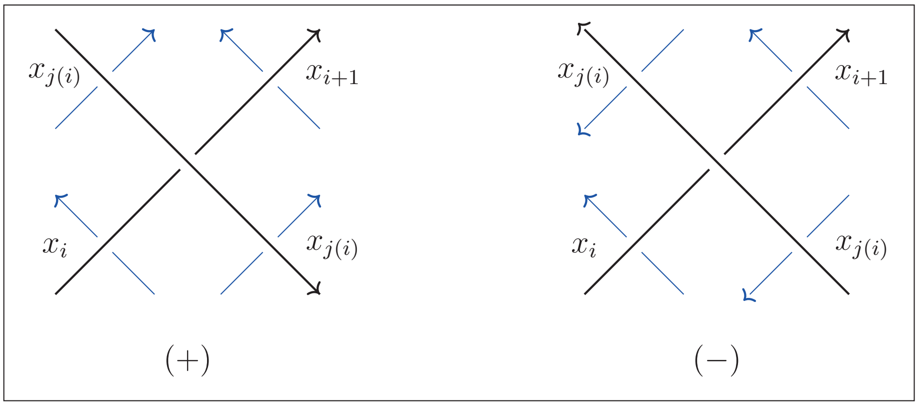

Figure 2.2: The two cases for relations in Γ_K.

Now, $\varphi : \Gamma_K \to \mathrm{Aff}(\kappa)$, thus each generator of Γ_K must satisfy $x_k \mapsto t_k z + y_k$ with $t_k \in \kappa^*$ and $y_k \in \kappa$ for $1 \leq k \leq n$. There are two equations that could hold, one coming from each case of the relations:

$$t_i t_{j(i)} z + t_i y_{j(i)} + y_i = t_{j(i)} t_{i+1} z + t_{j(i)} y_{i+1} + y_{j(i)} \tag{+}$$

$$t_{i+1} t_{j(i)} z + t_{i+1} y_{j(i)} + y_{i+1} = t_{j(i)} t_i z + t_{j(i)} y_i + y_{j(i)} \tag{−}$$

Analyzing the the coefficient of z we have $t_i t_{j(i)} - t_{j(i)} t_{i+1} = 0$, hence $t_i = t_{i+1}$ for all i, renaming $t_k := t$ for all k. The equations simplify to

$$(t-1) y_{j(i)} + y_i - t y_{i+1} = 0 \tag{+}$$

$$(t-1) y_{j(i)} - t y_i + y_{i+1} = 0. \tag{−}$$

Let $A_\kappa(t) \in \mathrm{Mat}_{n \times (n-1)}(\kappa[t, t^{-1}])$ be the presentation matrix for the above equations. The matrix $A(t)$ will denote the above presentation matrix when

κ has characteristic 0, and in the case of de Rham $\kappa = \mathbb{C}$. The polynomial $\Delta_\kappa(t) \in \kappa[t, t^{-1}]$ is defined to be the largest invariant factor of $A_\kappa(t)$. We will show in section 2.2, that $\Delta_\kappa(t) \equiv \Delta(t) \pmod p$ when κ is the finite field $\mathbb{F}_p$. In fact $\Delta_\kappa(t)$ can be taken to be a polynomial in $\kappa[t]$; we will often use this fact without explicitly stating it. Thus we are able to conclude that there exists a homomorphism $\varphi : \Gamma_K \to \mathrm{Aff}(\kappa(\alpha))$ if and only if α is a root of $\Delta_\kappa(t)$ in some finite extension of κ as described above for some non-zero $(y_1, y_2, \dots, y_n) \in \kappa(\alpha)^n$. In other words $(y_1, y_2, \dots, y_n)$ is a non-zero vector contained in the kernel of $A_\kappa(\alpha)$.

Suppose that α is a non-zero root of $\Delta_p(t)$ and $\alpha \in \mathbb{F}_{p^d}$, here $d = \deg_{\mathbb{F}_p}(\alpha)$. Recall the definition of the group G_α

$$G_\alpha = \langle \alpha z, z+1 \rangle < \mathrm{Aff}\left(\mathbb{F}_{p^d}\right).$$

It follows that if $\mathrm{ord}_{\mathbb{F}^*_{p^d}}(\alpha) = n$ then

$$G_\alpha = \{\alpha^i z + y \mid 0 \le i \le n-1, \text{ and } y \in \mathbb{F}_{p^d}\} \cong \langle \alpha \rangle \ltimes \mathbb{F}_{p^d}$$

We will simplify notation and denote the homomorphism φ by ρ_α to indicate that this homomorphism only depends on the root α of $\Delta_p(t)$.

Proposition 2.2. *Suppose that $\alpha \in \mathbb{F}_{p^d}$ is a non-zero root of $\Delta_p(t)$, then*

$$\rho_\alpha(\Gamma_K) = G_\alpha.$$

Proof. We first consider an alternate presentation of Γ_K, using the presentation 2.1, denote by R_j the relations for Γ_K. The new generating set is defined to be $\{s_i\}_{i=1}^n$, with $s_i := x_i x_1^{-1}$ for $i \neq 1$ and $s_1 = x_1$. New relations, R'_j, are formed from the relations R_j by setting $R'_j(s_1, \dots, s_n) := R_j(s_1, s_2 x_1, \dots, s_n x_1)$. With this presentation we have that each s_i for $i \ge 2$ is an element of Γ'_K, since the image $[s_i]$ of s_i in $H_1(M_K)$ is $[x_i] - [x_1] = 0$.

We have $\rho_\alpha(s_1) = \alpha z + y_1$ and we may assume that up to conjugation in $\mathrm{Aff}(\mathbb{F}_{p^d})$, we have $y_1 = 0$. However for $i \neq 1$, since $s_i \in \Gamma'_K$ we have

$$\rho_\alpha(s_i) \in \rho_\alpha(\Gamma'_K) < \{z + y_i \in \mathrm{Aff}(\mathbb{F}_{p^d}) \mid y_i \in \mathbb{F}_{p^d}\}$$

therefore if $i \geq 2$, then $\rho_\alpha(s_i) = z + y_i$ with all $y_i \in \mathbb{F}_{p^d}$. By construction of ρ_α there is a non-zero vector $(y_1, \ldots, y_n)$ contained in the kernel of $A_{\mathbb{F}_{p^d}}(\alpha)$, so we may assume that y_j is non-zero. Furthermore, by definition

$$\{\rho_\alpha(s_1^k s_j s_1^{-k})\}_{k=0}^d = \{z + \alpha^k y_j\}_{k=0}^d.$$

Now s_j and $s_1^k s_j s_1^{-k}$ have infinite order in Γ_K and the images $z+y_j$ and $z+\alpha^k y_j$ have additive order p. We have that $\alpha^k y_j \neq 0$ for all $0 \leq k \leq d$. Also $\alpha^k y_j \neq \alpha^l y_j$ for all $k \neq l$ with $0 \leq k \leq d$ and $0 \leq l \leq d$. Otherwise, if $\alpha^k y_j = \alpha^l y_j$, then without loss of generality assume $k > l$, so that $\left(\alpha^{k-l} - 1\right) y_j = 0$. However this cannot be the case because $\deg_{\mathbb{F}_p}(\alpha) < \mathrm{ord}_{\mathbb{F}^*_{p^d}}(\alpha)$. So we conclude that $\rho_\alpha(\Gamma'_K) = \mathbb{F}_{p^d}$. All that is left to do is to determine the image of the powers of s_1, but these are precisely the affine maps $\alpha^k z$ for $0 \leq k \leq \mathrm{ord}_{\mathbb{F}^*_{p^d}}(\alpha)$, we conclude that $\rho_\alpha(\Gamma_K) = G_\alpha \cong \langle \alpha \rangle \ltimes \mathbb{F}_{p^d}$. $\square$

For the rest of this dissertation the image $\rho_\alpha(\Gamma_K)$ is denoted by G_α, it is clear that $G_\alpha \cong \langle \alpha \rangle \ltimes (\mathbb{Z}/p\mathbb{Z})^d$, and that $|G_\alpha| = \mathrm{ord}_{\mathbb{F}^*_{p^d}}(\alpha) p^d$.

2.1.4 Examples

These examples will be lengthy and while we could simplify, we do not. It will be important to see how each part of §2.1.3 fits into the computation of the homomorphism.

Example 2.1. *The trefoil.*

Figure 2.3: The trefoil 3_1

The Wirtinger presentation of the trefoil is

$$\Gamma_K = \langle x_0, x_1, x_2 \mid x_1x_0 = x_2x_1,\ x_2x_1 = x_0x_2,\ x_0x_2 = x_1x_0 \rangle.$$

Let p be a prime number and d some positive integer; it will become clear what d should be by the end of this computation. There is a homomorphism $\rho_\alpha : \Gamma_K \to \mathrm{Aff}(\mathbb{F}_{p^d})$ if and only if

$$\begin{aligned} x_0 &\mapsto \alpha_0 z + b_0, \\ x_1 &\mapsto \alpha_1 z + b_1, \\ x_2 &\mapsto \alpha_2 z + b_2. \end{aligned}$$

Furthermore we may assume up to conjugation of the image of ρ_α in $\mathrm{Aff}(\mathbb{F}_{p^d})$ that $b_0 = 0$. So we have

$$\begin{aligned} x_0 &\mapsto \alpha_0 z, \\ x_1 &\mapsto \alpha_1 z + b_1, \\ x_2 &\mapsto \alpha_2 z + b_2. \end{aligned}$$

Now since

$$\begin{aligned} x_1x_0 &= x_2x_1, \\ x_2x_1 &= x_0x_2, \\ x_0x_2 &= x_1x_0, \end{aligned}$$

it follows that

$$\begin{aligned} \alpha_1\alpha_0 z &= \alpha_2\alpha_1 z, \\ \alpha_2\alpha_1 z &= \alpha_0\alpha_1 z, \\ \alpha_0\alpha_2 z &= \alpha_1\alpha_0 z, \end{aligned}$$

and we conclude that $\alpha_0 = \alpha_1 = \alpha_2 = \alpha$.

Now we have that a homomorphism $\rho_\alpha : \Gamma_K \to \mathrm{Aff}(\mathbb{F}_{p^d})$ exists if and only if the following equations hold

$$\begin{aligned} \alpha^2 z + b_1 &= \alpha^2 z + \alpha b_1 + b_2, \\ \alpha^2 z + \alpha b_1 + b_2 &= \alpha^2 z + \alpha b_2, \\ \alpha^2 z + \alpha b_2 &= \alpha^2 z + b_1. \end{aligned}$$

Simplifying these equations we obtain:

$$\begin{aligned} (\alpha - 1)b_1 + b_2 &= 0, \\ -\alpha b_1 + (\alpha - 1)b_2 &= 0, \\ b_1 - \alpha b_2 &= 0. \end{aligned}$$

This is described by the presentation matrix:

$$\begin{pmatrix} \alpha - 1 & 1 \\ -\alpha & \alpha - 1 \\ 1 & -\alpha \end{pmatrix}.$$

Recall that this is a matrix over $\mathbb{F}_p[\alpha, \alpha^{-1}]$, and it follows that a representation $\rho_\alpha : \Gamma_K \to \mathrm{Aff}(\mathbb{F}_{p^d})$ exists for some non-zero b_1, b_2 if and only if the largest invariant factor of the above matrix is generates the zero ideal of $\mathbb{F}_p[\alpha, \alpha^{-1}]$. The largest invariant factor of this matrix is the ideal generated by $\alpha^2 - \alpha + 1$, which is precisely the Alexander polynomial modulo the prime p. This ideal must be the zero ideal, thus α must be a root of $\Delta(t) \pmod p$. Furthermore d is the degree of the root α over $\mathbb{F}_p$ and p in this case can be any prime.

Example 2.2. *The figure 8 knot.*

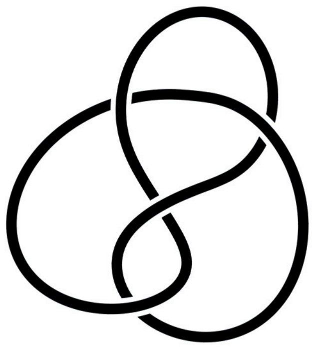

Figure 2.4: The Figure 8

We begin with the fiber bundle presentation for the figure 8 knot complement. We use this specific presentation of the figure 8 because defining a homomorphism such as in de Rham's construction does not depend on the presentation of the fundmental group.

$$\Gamma_K = \langle t, x, y \mid txt^{-1} = xyx, tyt^{-1} = yx \rangle$$

Let p be a prime number and d some positive integer; it will become clear what d should be by the end of this computation. There is a homomorphism $\rho_\alpha : \Gamma_K \to \mathrm{Aff}(\mathbb{F}_{p^d})$ if and only if

$$\begin{aligned} t &\mapsto \alpha_0 z + b_0, \\ x &\mapsto \alpha_1 z + b_1, \\ y &\mapsto \alpha_2 z + b_2 \end{aligned}$$

We may assume up to conjugation of the image of ρ_α in $\mathrm{Aff}(\mathbb{F}_{p^d})$ that $b_0 = 0$. Furthermore since $\Gamma_K' = \langle x, y \rangle$ and $\mathrm{Aff}(\mathbb{F}_{p^d})' = \{z + b \mid b \in \mathbb{F}_{p^d}\}$ it follows that $\alpha_1 = \alpha_2 = 1 \in \mathbb{F}_{p^d}^*$, so we rename $\alpha_0 = \alpha$. We have

$$\begin{aligned} t &\mapsto \alpha z, \\ x &\mapsto z + b_1, \\ y &\mapsto z + b_2 \end{aligned}$$

From the first relation, $txt^{-1} = xyx$, we have

$$\begin{aligned} \alpha z + \alpha b_1 &= \alpha z + 2b_1 + b_2, \\ 0 &= (2-\alpha)b_1 + b_2. \end{aligned}$$

From the second relation, $tyt^{-1} = yx$, we have

$$\begin{aligned} \alpha z + \alpha b_2 &= \alpha z + b_1 + b_2, \\ 0 &= b_1 + (1-\alpha)b_2. \end{aligned}$$

It is important to note that the coefficients of the b_1, b_2 are elements of $\mathbb{F}_{p^d}$.

Now we have the presentation matrix for the above relations, viewed as a matrix over $\mathbb{F}_{p^d}[\alpha, \alpha^{-1}]$:

$$\begin{pmatrix} 2-\alpha & 1 \\ 1 & 1-\alpha \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Therefore the above equation holds for

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

if and only if the largest invariant factors of the matrix

$$\begin{pmatrix} 2-\alpha & 1 \\ 1 & 1-\alpha \end{pmatrix}$$

generate the zero ideal. The largest invariant factor is $(\Delta_p(\alpha)) = (\alpha^2 - 3\alpha + 1)$ this is the zero ideal if and only if α is a non-zero root of $\Delta_p(t)$. The positive integer d is then seen to be the degree of the extension ${}^{\mathbb{F}_p(\alpha)}/_{\mathbb{F}_p}$. So in this case $d = 1$ or 2.

Suppose that $p = 11$. Then we have that $\Delta_p(t) = (t-5)(t-9)$. If $\alpha = 5 \in \mathbb{F}_{11}$, hence $d = 1$, and it follows that the homomorphism ρ_α is completely described in the following way.

$$\begin{aligned} t &\mapsto 5z, \\ x &\mapsto z+1, \\ y &\mapsto z+3 \end{aligned}$$

as elements of Aff $(\mathbb{F}_{11})$.

In figure 2.5 on page 21 the homeomorphism ϕ represents the monodromy of the figure 8 knot as presented as a fiber bundle over the circle, with fiber a once punctured torus. A quick check in MAGMA shows that the kernel of the homomorphism ρ_α is given by,

$$\langle t^5, x^{11}, yx^{-3}, \{x^i yx^{-3}x^{-i}\}_{i=1}^{10}\rangle \lhd \Gamma_K.$$

This allows us to compute $H_1\left(X_{\rho_5}\right)$, which is isomorphic to

$$\mathbb{Z}^{11} \oplus \mathbb{Z}/11\mathbb{Z}.$$

2.2 Fox's Free Differential Calculus

We investigate a defintion of $\Delta(t)$ due to Fox [12]. This description is given in [21], and we recall it here to expand on the details and adapt the definition to allow the derivative to take coefficients in an arbitray field κ. In this section we will explicitly show that the definition of $\Delta(t)$ due to de Rham agrees with the classical definition of $\Delta(t)$.

Let $\Lambda_r(\mathbb{Z}) = \mathbb{Z}[t_1, t_1^{-1}, \ldots, t_r, t_r^{-1}]$, the ring of integral Laurent polynomials in r variables. The Fox derivative can be defined in the following way. Suppose F_r is the free group on r generators i.e.

$$F_r = \langle x_1, \ldots, x_r\rangle.$$

Furthermore, denote ab : $\Gamma_K \to \Gamma_K^{\mathrm{ab}}$ as the canonical abelianizing homomorphism. Define the mapping $D_i : F_r \to \mathbb{Z}[F_r]$, to be

$$\begin{aligned} D_i(x_j) &= \delta_{ij} \\ D_i(uv) &= D_i(u) + uD_i(v). \end{aligned}$$

The map ab : $F_r \to F_r^{\mathrm{ab}}$ induces a mapping $(D_1, \ldots, D_r) : F_r \to \Lambda_r(\mathbb{Z})^r$ which we call the *Fox Derivative*, and the D_i are the ith partials. If we define

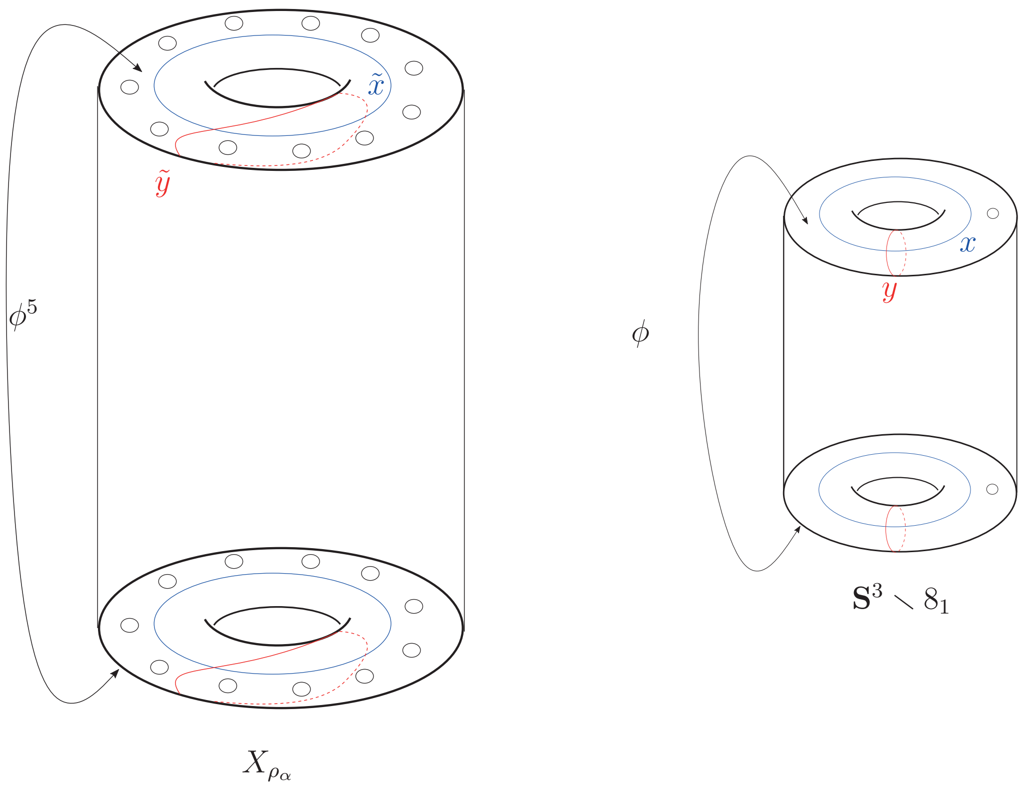

Figure 2.5: The cover X_{ρ_α} of the figure 8 for $p = 11$ and $\alpha = 5$.

the knot group Γ_K in the following way,

$$\Gamma_K = \langle F_r \mid \mathcal{R}_j \text{ for } j = 1, \ldots, s\rangle$$

we obtain the following

$$F_s \to F_r \xrightarrow{q} \Gamma_K.$$

Let q^* be the induced mapping $q^* : \mathbb{Z}[F_r^{\text{ab}}] \to \mathbb{Z}[\Gamma_K^{\text{ab}}]$. We are able to form the Alexander matrix of $r \times s$ partials also known as the Jacobian of Γ_K

$$M(F_r, \mathcal{R}_j) = [q^* D_i(\mathcal{R}_j)].$$

Using the presentation (1) for Γ_K from §2.1.3, with $\Gamma_K^{\text{ab}} \cong \langle t\rangle$ we have

$$\begin{aligned} q^* D_i(\mathcal{R}_i) &= 1 \text{ or } -t, \\ q^* D_{i+1}(\mathcal{R}_i) &= -t \text{ or } 1, \\ q^* D_{j(i)}(\mathcal{R}_i) &= t - 1, \\ q^* D_k(\mathcal{R}_i) &= 0 \text{ otherwise} \end{aligned}$$

Theorem 2.1 (Fox [12])**.** *The ith Alexander ideal* $(\Delta_i(t))$ *is the ideal generated by the* $(r-i) \times (r-i)$ *minors of* $M(F_r, \mathcal{R}_j)$*, thus* $\Delta(t)$ *is the largest invariant factor of* $M(F_r, \mathcal{R}_j)$*.*

CHAPTER 3

HOMOMORPHISMS FROM ROOTS OF $\Delta(t) \pmod p$

Recall the presentation matrix $A(t)$, as seen in §2.1.3 is defined over $\mathbb{Z}[t, t^{-1}]$. The following corollary summarizes the equivalence of Fox's Jacobian and de Rham's matrix $A(t)$. We may evaluate the entries of $M(F_r, \mathcal{R}_j)$ by the canonical homomorphism $\iota : \mathbb{Z} \to \kappa$, and the resulting matrix $M(F_r, \mathcal{R}_j)_{\mathbb{F}_p}$ has entries which lie in $\kappa[\Gamma_K^{\mathrm{ab}}]$.

Corollary 3.1 (de Rham [10])**.** *For the knot group Γ_K with the Wirtinger presentation* (2.1)*, $M(F_r, \mathcal{R}_j) = A(t)$.*

Furthermore the generalization of Fox's Jacobian and de Rham's presentation matrix $A_{\mathbb{F}_p}(t)$ to the finite field $\mathbb{F}_p$ yields a similar result.

Corollary 3.2. *For the knot group Γ_K with the Wirtinger presentation* (2.1)*, $M(F_r, \mathcal{R}_j)_{\mathbb{F}_p} = A_{\mathbb{F}_p}(t)$.*

Theorem 3.1. $\Delta_{(p,i)}(t) \equiv \Delta_i(t) \pmod p$

Proof. Let the knot group Γ_K have presentation $\langle F_r | \{\mathcal{R}_j\}\rangle$. Since $\Delta_i(t)$ is a principal generator for the ideal generated by the $(r-i) \times (r-i)$ minors of $M(F_r, \mathcal{R}_j)$ let $(f_1, \ldots, f_k) = (\Delta_i(t))$ for $f_j \in \mathbb{Z}[t, t^{-1}]$ i.e, the f_j are the $(r-i) \times (r-i)$ minors of $M(F_r, \mathcal{R}_j)$. Let $(q_1, \ldots, q_k) = (\Delta_{(p,i)}(t))$, where the

$q_j \in \mathbb{F}_p[t, t^{-1}]$ are the $(r-i) \times (r-i)$ minors of $M(F_r, \mathcal{R}_j)_{\mathbb{F}_p}$, and $\iota : Z \to \mathbb{F}_p$ be the canonical ring homomorphism. Corollaries 3.1 and 3.2 give us that evaluating ι at the coefficients of the entries of $M(F_r, \mathcal{R}_j)$ is the matrix $A_{\mathbb{F}_p}(t)$. Furthermore denote $\iota(f)$ for $f \in \mathbb{Z}[t, t^{-1}]$, the image of f after evaluating ι on the coefficients of f. Similarly if S is a matrix over $\mathbb{Z}[t, t^{-1}]$ then $\iota(S)$ denotes the matrix with entries in $\mathbb{F}_p[t, t^{-1}]$, having evaluated f at the coefficients of the entries of S. Each f_j comes from a determinant of a $(r-i) \times (r-i)$ sub-matrix S_j, and since ι is a homomorphism we have that

$$\iota(f_j) = \iota(\mathrm{Det}(S_j)) = \mathrm{Det}(\iota(S_j)) = q_j.$$

Therefore the image of $(\Delta_i(t)) = (f_1, \ldots, f_k)$ under ι is $(q_1, \ldots, q_k) = (\Delta_{(p,i)}(t))$, the image of $\Delta_i(t)$ under ι is $\Delta_i(t) \pmod p$. Hence we conclude that $\Delta_{(p,i)}(t) \equiv \Delta_i(t) \pmod p$. $\square$

Fix a root of $\Delta_p(t)$ and let ρ_α be the associated homomorphism. We recall for the reader that the image of ρ_α is the group G_α constructed in §2.1.3, that $G_\alpha \cong \langle\alpha\rangle \ltimes \mathbb{F}_{p^d}$, and that this semidirect product is defined via multiplication by $\alpha \in \mathbb{F}^*_{p^d}$.

Theorem 2. *There exists a homomorphism $\rho_\alpha : \Gamma_K \to \mathrm{GL}_2(\mathbb{F}_p(\alpha))$ for p a prime if and only if α is a non-zero root of $\Delta(t) \pmod p$ in some finite extension of $\mathbb{F}_p$. This representation satisfies:*

- *$\rho_\alpha(\Gamma_K)$ is metabelian, in particular non-abelian.*
- *$|\rho_\alpha(\Gamma_K)| = np^d$, where $n = \mathrm{ord}_{\mathbb{F}^*_p(\alpha)}(\alpha)$ and $d = [\mathbb{F}_p(\alpha) : \mathbb{F}_p]$.*

Proof. It follows from Theorem 3.1 that if α is a root of $\Delta(t) \pmod p$ in the extension $\mathbb{F}_p(\alpha) \cong \mathbb{F}_{p^d}$, for $d = [\mathbb{F}_p(\alpha) : \mathbb{F}_p]$, then α is also a root of $\Delta_p(t)$. Hence by Proposition 2.2, such a representation $\rho_\alpha : \Gamma_K \to G_\alpha$ exists if and only if α is a non-zero root of $\Delta(t) \pmod p$. $\square$

CHAPTER 4

BOUNDING THE COVERS IN TERMS OF CROSSING NUMBER

4.1 Bounds on the coefficients of $\Delta(t)$

In this section we provide conditions depending only on the crossing number c_K on the size of the smallest prime p so that $\Delta_p(t)$ is a non-trivial polynomial when $\Delta(t)$ is non-trivial. In particular this allows us to provide an upper bound on the index for which cover corresponding to $\ker(\rho_\alpha)$ exists. In other words we will provide a bound on the prime p so that a representation ρ_α of Γ_K onto G_α exists.

For such a representation ρ_α to exist, $\Delta_p(t)$ must be a non-constant Alexander polynomial, so that there are non-zero roots in some extension of $\mathbb{F}_p$. In this section we find a bound on the smallest prime in terms of the crossing number c_K, for which this holds. Consider the matrix $A(t) \in \mathrm{Mat}_n\left(\mathbb{Z}[t, t^{-1}]\right)$ in §2.1.3, and notice that it satisfies the following criteria;

1) The entries are in the set $\{0, 1, t, t-1\}$.

2) In each row the entries $1, t, t-1$ occur at most once, if at all.

3) No row is the zero vector.

Lemma 4.1. *Suppose $\mathcal{M}(t)$ is an matrix in* $\mathrm{Mat}_n\mathbb{Z}[t, t^{-1}]$ *is an $n \times n$, for an $n \geq 1$, is a matrix satisfying criteria 1), 2), and 3), then for any coefficient, a, of the determinant of $\mathcal{M}(t)$ we have $|a| \leq 4^{n-1}$.*

Proof. We proceed by induction on the size of the matrix $\mathcal{M}(t)$. In the base case $n = 1$, the largest coefficient is $4^0 = 1$. As an induction hypothesis, suppose for all k with $n \geq k \geq 1$ that for any matrix $\mathcal{M}(t)$ satisfying criteria 1), 2), and 3,) a coefficient a of the determinant of $\mathcal{M}(t)$ must satisfy $|a| \leq 4^{k-1}$. Now consider the case $k = n+1$. Denote by $B_{k-1}(t)$, $C_{k-1}(t)$, and $D_{k-1}(t)$ the $(k-1) \times (k-1)$ cofactor corresponding to $1, -t$ and $(t-1)$ along the first row of $\mathcal{M}(t)$, respectively. Then we have

$$\det(\mathcal{M}(t)) = \pm\det(B_{k-1}(t)) \pm t\det(C_{k-1}(t)) \pm (t-1)\det(D_{k-1}(t)).$$

Since $B_{k-1}(t)$, $C_{k-1}(t)$, and $D_{k-1}(t)$ all satisfy criteria 1), 2), and 3), it follows by the induction hypothesis that if b is any coefficient of $\det(B_{k-1}(t))$, c is any coefficient of $\det(C_{k-1}(t))$, and d is any coefficient of $\det(D_{k-1}(t))$, that $|b| \leq 4^{n-1}$, $|c| \leq 4^{n-1}$, and $|d| \leq 4^{n-1}$. Let a be any coefficient of $\det(\mathcal{M}(t))$, it follows from the above equation that

$$|a| \leq 4^{n-1} + 4^{n-1} + 4^{n-1} + 4^n = 4^n.$$

The lemma follows. $\square$

Lemma 4.2. *If $p \geq 4^{c_K - 2}$ and $\Delta(t)$ is non-trivial, then $\Delta(t)$ is non-trivial in $\mathbb{F}_p[t]$.*

Proof. The first observation is that the Wirtinger presentation for Γ_K has c_K generators since, there is exactly one generator for each crossing. Furthermore, (2.1) can be simplified to have $c_K - 1$ generators. Since $\Delta(t)$ is the largest invariant factor of the matrix $A(t)$, by Lemma 2.2 there are at least three non-zero coefficients of $\Delta(t)$. Furthermore any non-zero minor computed from

$A(t)$ comes from a sub-matrix $S(t)$ that satisfies criteria 1), 2), and 3). We have

$$\det(S(t)) = \sum_{i=0}^{k} s_i t^i, \;\; s_i \in \mathbb{Z}$$

and

$$\Delta(t) = \sum_{i=0}^{d} a_i t^i.$$

It follows that s_0 and s_k are non-zero, because, the leading and ending coefficients of $\Delta(t)$ are non-zero and, since $\Delta(t)$ is the largest invariant factor of $A(t)$, it must divide any maximal rank non-zero minor coming from $A(t)$. Therefore the absolute values of the coefficients satisfy $a_0 \mid s_0$ and $a_d \mid s_k$. Further, by Lemma 4.1, $|s_0| \leq 4^{c_K-2}$ and $|s_k| \leq 4^{c_K-2}$, so we have $|a_0| \leq 4^{c_K-2}$ and $|a_d| \leq 4^{c_K-2}$. If $p \geq 4^{c_K-2}$, then $a_d \neq 0 \pmod p$ and $a_0 \neq 0 \pmod p$. It follows that $\Delta_p(t)$ is non-constant, and by Theorem 3.1 $\Delta_p(t) \equiv \Delta(t) \pmod p$ is non-trivial. □

We are now ready to prove Theorem 1:

Theorem 4.1. *If K is a knot with non-trivial Alexander polynomial, then there exists a regular non-abelian cover X_{ρ_α} of M_K with*

$$[M_K : X_{\rho_\alpha}] \leq 4^{2c_K^2 - c_K},$$

and there exists an irregular non-cyclic cover Y_{ρ_α} with

$$[M_K : Y_{\rho_\alpha}] \leq 4^{c_K^2 - 2c_K}.$$

Proof. Let $\Delta(t)$ be the non-trivial Alexander polynomial for a knot K. Let p be a prime such that $4^{c_K-1} \leq p \leq 2{\cdot}4^{c_K-1} - 2$ which exists by Bertrand's postulate. By Theorem 4.2, it follows that $\Delta_p(t)$ is non-trivial, hence there exists a non-zero root α in some finite extension $\mathbb{F}_{p^d}$. Furthermore by §2.1.3 there exists a surjective homomorphism $\rho_\alpha : \Gamma_K \to G_\alpha$. Let X_{ρ_α} be the connected covering space of M_K corresponding to $\ker(\rho_\alpha)$. The index $[M : X_{\rho_\alpha}]$ is $\mathrm{ord}(\alpha)p^d$. We have the following;

$$\begin{aligned}[M : X_{\rho_\alpha}] &= \operatorname{ord}(\alpha)p^d, \\ &\leq (p^d - 1)(p^d), \\ &\leq (p^{c_K - 1} - 1)(p^{c_K - 1}), \\ &\leq ((2 \cdot 4^{c_K - 2} - 2)^{c_K - 1} - 1)((2 \cdot 4^{c_K - 2} - 2)^{c_K - 1}).\end{aligned}$$

The above bound is optimal for this argument, and the theorem follows from a simplification of the above.

If we take the subgroup of G_α generated by α and construct the cover corresponding to $\rho_\alpha^{-1}(\langle\alpha\rangle)$, which has index

$$\left[\Gamma_K : \rho_\alpha^{-1}(\langle x \rangle)\right] = [G_\alpha : \langle\alpha\rangle] = p^d.$$

We obtain a new non-cyclic cover of M_K which we denote Y_{ρ_α}, which is irregular because G_α is non-abelian. A similar computation follows;

$$\begin{aligned}[M : Y_{\rho_\alpha}] &= p^d, \\ &\leq (p^d), \\ &\leq (p^{c_K - 1}), \\ &\leq (2 \cdot 4^{c_K - 2} - 2)^{c_K - 1}).\end{aligned}$$

Again the above is optimal and the theorem follows from a simplification. $\square$

4.2 Special Cases

There are many infinite families of knots for which the Alexander polynomial takes on a specific form and the proof of Theorem 1 can be sharpened. Similarly there are certain properties of Alexander polynomials of knots which allow us rephrase Theorem 1 and simplify the bounds.

Fibered Knots

A knot K is *fibered* if the complement M_K is a fiber bundle over the circle. In this case the fundamental group is of the form $\Gamma_K = \langle t \rangle \ltimes \pi_1(\Sigma_g)$, where

Σ_g is the Seifert surface of K arising from Seifert's algorithm. By [34], the Alexander polynomial must be monic, and its degree is bounded above by $2g$.

Corollary 4.1. *If K is a fibered knot of genus g, with non-trivial Alexander polynomial, then for all primes p the representation $\rho_\alpha : \Gamma_K \to G_\alpha$ exists and*

$$[M_K : Y_{\rho_\alpha}] \leq 2^{2g}.$$

Proof. Since $\Delta_K(t)$ is monic, it is non-trivial modulo 2. □

A Family of Two Bridge Knots $J(k, l)$

The double twist knots which we denote $J(k, l)$ are a family of two bridge knots which have exactly two half-twist regions as seen in figure 4.1 below. Each region has k and l half-twists in their respective regions, we make the assignment that a twist is positive if it a right hand twist and negative if it is a left hand twist.

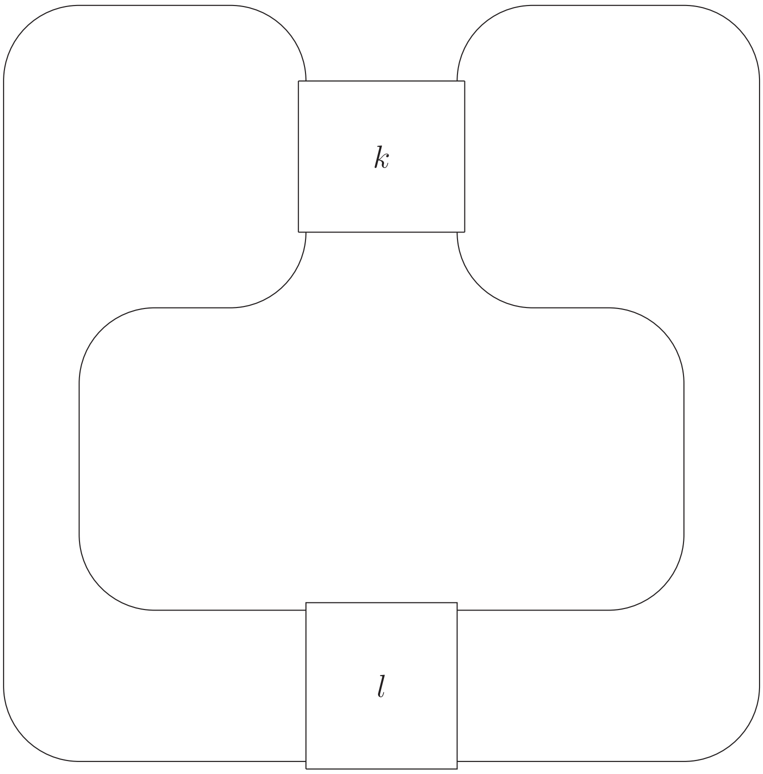

Figure 4.1: The Knot $J(k,l)$

The $J(k,l)$ are knots if kl is even otherwise they are two component links. Furthermore every knot $J(k,l)$ is isotopic to a knot $J(k,l)$ with l even.

Lemma 4.3 (Lemma 7.3 [28]). *For all non-zero integers k and even integers $l = 2n$, the knot $J(k,l)$ has Alexander polynomial:*

$$\Delta_{J(k,l)}(t) = \begin{cases} nmt^2 + (1-2mn)t + nm, & \textit{if } k = 2m \\ mt^{2n} + (1+2m)(-t^{2n-1} + \cdots - t) + m, & \textit{if } k = 2m+1 \textit{ and } l > 0 \\ (m+1)t^{-2n} + (1+2m)(-t^{-2n-1} + \cdots - t) + m + 1, & \textit{if } k = 2m+1 \textit{ and } l < 0 \end{cases}$$

Corollary 4.2. *If $J(k,l)$ is a twist knot with $l = 2n$, then for primes*

$$p \geq \begin{cases} mn, & \text{if } k = 2m \\ m+1, & \text{if } k = 2m+1 \text{ and } l = 2 \\ 2, & \text{if } k = 2m+1 \text{ and } l > 2 \\ 2, & \text{if } k = 2m+1 \text{ and } l < 0 \end{cases}$$

$\Delta_{J(k,l)}(t) \pmod p$ *is non-trivial and the surjective homomorphism* $\rho_\alpha : \Gamma_K \to G_\alpha$ *exists, and*

$$[M_K : Y_{\rho_\alpha}] \leq \begin{cases} (2mn)^2 - 4mn + 4, & \text{if } k = 2m \\ (2m)^2, & \text{if } k = 2m+1 \text{ and } l = 2 \\ 2^{2n-1}, & \text{if } k = 2m+1 \text{ and } l > 2 \\ 2^{2n}, & \text{if } k = 2m+1 \text{ and } l < 0 \end{cases}$$

Proof. One considers the coefficients for $\Delta(t)$ from Lemma 4.3, and computes the smallest degree in absolute value for which the $\Delta_p(t)$ would have at least 3 non-zero terms. $\square$

Pretzel Knots $K(p,q,r)$

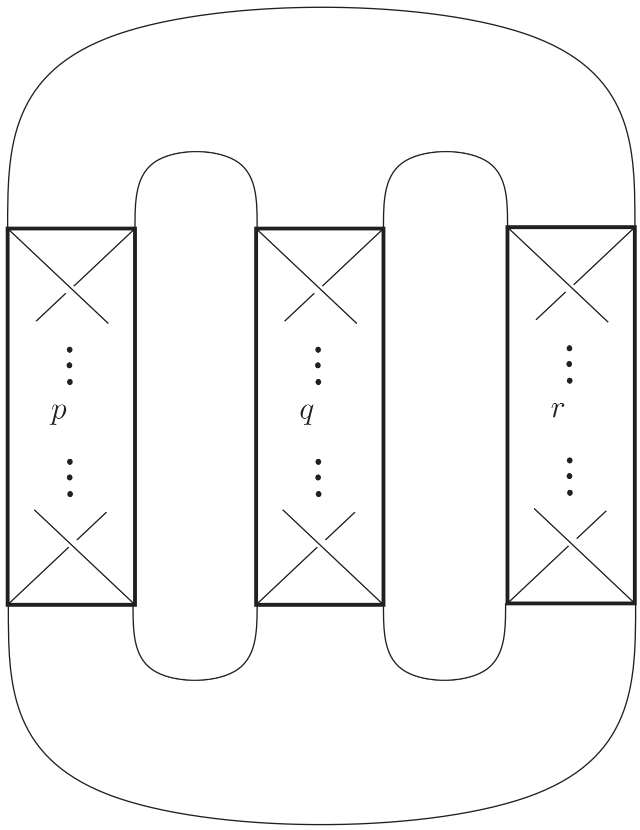

Figure 4.2: The pretzel knot $K(p,q,r)$.

The Alexander polynomial of a pretzel knot $K(p,q,r)$ (figure 4.2) , with p,q,r odd numbers is known and satisfies [26];

$$\Delta_{K(p,q,r)}(t) = \frac{1}{4}\left((pq+qr+rp)(t^2-2t-1)+t^2+2t+1\right).$$

Corollary 4.3. *Suppose $K(p,q,r)$ is a pretzel knot, normalized so that p is largest, and $pq+qr+pr \neq 1$, so $\Delta_{K(p,q,r)}(t)$ is non trivial then there exists a non-cyclic cover Y_{ρ_α} and:*

$$[M_K : Y_{\rho_\alpha}] \leq 4p^2$$

Dihedral Covers of Alternating Knots

Proposition 4.1 ([14]). *For primes $p||\Delta(-1)|$, there exists a homomorphism onto the dihedral group D_{2p} if and only if $m|\Delta(-1)| \neq 1$.*

This proposition is not stated in this way in Fox's article, in [31] on may find this exact statement. It is also possible to generalize the approach we describe in this dissertation to arrive at this conclusion, to be more precise. When we reduce the Alexander polynomial modulo $|\Delta(-1)| = n$, we have that $n-1$ is a root with multiplicative order 2. We are then able to construct a representation of the knot group to the finite dihedral group $\langle (n-1)z, z+1\rangle \leqslant \mathrm{Aff}\left(\mathbb{Z}/n\mathbb{Z}\right)$.

This proposition is a direct generalization of the the dihedral covers for two bridge knots described above. Furthermore the resulting non-cyclic cover Y_{ρ_α} has degree $|\Delta(-1)|$. The following result of Stoimenow [36] allows us to bound the degree of the non-cyclic cover Y_{ρ_α} in terms of the crossing number.

Lemma 4.4 ([36]). *For K a knot which admits an alternating diagram,*

$$|\Delta_K(-1)| =\leq 2^{c_K-1}.$$

The following theorem is a consequence of the fact that the determinant of a knot, $|\Delta(-1)|$, is equal to the number of spanning trees of the dual graph of the checkerboard coloring of an alternating diagram due to Kauffman [23].

Proposition 4.2 ([23]). *If K is an alternating knot and $|\Delta(-1)| = 1$, then K is the unknot.*

Theorem 4.2. *Let K be a knot such that $|\Delta_K(-1)| \neq 1$ with crossing number c_K, then there exists a regular non-abelian cover X_{ρ_α} of M_K with,*

$$[M_K : X_{\rho_\alpha}] \leq 2^{c_K},$$

and there exists an irregular non-cyclic cover Y_{ρ_α} of M_K with,

$$[M_K : Y_{\rho_\alpha}] \leq 2^{c_K - 1}.$$

This is a large improvement on the bound provided by Theorem 1 when K is alternating. Furthermore this non-cyclic cover is not minimal (in general), for instance the knots 4_1 and 5_2. There are examples for this non-cyclic cover (coming from a dihedral representation) which are minimal i.e. 3_1 and 6_1.

Knots With Non-trivial Alexander Polynomial

To finish the section on special families we note that using crossing number does not provide us with a good estimate on the degree of a root of the Alexander polynomial. The degree of the Alexander polynomial is as a polynomial in $\mathbb{Z}[t]$ is a more accurate bound in particular if the degree of $\Delta(t)$ is n, then $c_K - 1 \geq n$ [34].

Corollary 4.4. *Let K be a knot with non-trivial Alexander polynomial of degree d, then for all primes $p \geq 4^d$ the representation $\rho_\alpha : \Gamma_K \to G_\alpha$ exists and*

$$[M_K : Y_{\rho_\alpha}] \leq 2^{2d^2}.$$

CHAPTER 5

BOUNDS ON THE FIRST BETTI NUMBER

5.1 Lower Bound for $\beta_1(X_{\rho_\alpha})$

First recall that we define the multiplicative order of α to be n and $d = \deg_{\mathbb{F}_p}(\alpha)$, for α a non-zero root of $\Delta_p(t)$. Since

$$\Gamma_K \to G_\alpha \to {}^{\mathbb{Z}}/_{n\mathbb{Z}} \to 1,$$

we have $\ker(\rho_\alpha) < \ker\left(\Gamma_K \to {}^{\mathbb{Z}}/_{n\mathbb{Z}}\right)$, so $X_{\rho_\alpha} \to X_n$ is a regular covering space with deck group $\ker\left(G_\alpha \to {}^{\mathbb{Z}}/_{n\mathbb{Z}}\right) \cong \left({}^{\mathbb{Z}}/_{p\mathbb{Z}}\right)^d$, see figure 5.1 below. We may arrange the covers in the following commutative diagram of covers. The dashed arrows denote irregular covers and the solid arrows are regular, the corresponding deck group and index denoted above the arrows.

We first prove a lemma which relates classical results of cyclic covers of knot complements with the the covers X_{ρ_α}. First we recall the famous results of Fox and Burau [13].

Theorem 5.1 ([13])**.** *If X_n is the n-fold cyclic cover of a knot complement M_K, then we have the following*

1) $\beta_1(X_n) = 1 + |\{\xi \in \mathbb{C} \mid \xi^n = 1,\ \Delta(\xi) = 0\}|$,

$$\begin{tikzcd}
 & X_{\rho_\alpha} \arrow[dl, "n"'] \arrow[dd, "np^d"] \arrow[dr, "p^d"] & \\
Y_{\rho_\alpha} \arrow[dr, dashed, "p^d"'] & & X_n \arrow[dl, "n"] \\
 & M_K &
\end{tikzcd}$$

Figure 5.1: Commutative Diagrams of X_{ρ_α} and Y_{ρ_α}.

2) $|\mathrm{Torsion}(H_1(X_n;\mathbb{Z}))| = \prod_{\{\xi\in\mathbb{C}\ \mid\ \xi^n=1,\ \Delta(\xi)\neq 0\}} \Delta(\xi)$

First observe that Γ_K may be presented as $\langle t\rangle \rtimes \Gamma'_K$ which is a direct consequence of the split exact sequence

$$1 \to \Gamma'_K \to \Gamma_K \xrightarrow{\mathrm{ab}} \mathbb{Z} \to 1$$

hence the letter t can be represented by a meridian of the knot K. Now, since $\pi_1(X_n) \cong \ker(\Gamma_K \to \mathbb{Z}/n\mathbb{Z})$ we have that $\pi_1(X_n) \cong \mathrm{ab}^{-1}(n\mathbb{Z})$, and as a consequence we have

$$1 \to L \to \mathrm{ab}^{-1}(n\mathbb{Z}) \xrightarrow{\mathrm{ab}|_{\mathrm{ab}^{-1}(n\mathbb{Z})}} n\mathbb{Z} \to 1.$$

In the above $L = \ker(\mathrm{ab}^{-1}(n\mathbb{Z}) \to n\mathbb{Z})$, thus

$$\pi_1(X_n) \cong \langle t^n\rangle \ltimes L.$$

When analyzing the first statement in Theorem 5.1, the "$+1$" in $\beta_1(X_n)$ is exactly the contribution from the letter t^n, which we will call the *meridian* of X_n. Now suppose that $c \in H_1(X_n;\mathbb{Z})$ is a class which generates a free factor coming from and element of $\{\xi \in \mathbb{C} \mid \xi^n = 1,\ \Delta(\xi) = 0\}$. Let $\{c_1, \ldots, c_j\}$ be the collection of all such classes, i.e., the j roots of unity which are also roots of $\Delta(t)$. It follows that

$$H_1(X_n;\mathbb{Q}) = \mathrm{span}_{\mathbb{Q}}\{[t^n], c_1, \ldots, c_j\} \cong \mathbb{Q}^{j+1}.$$

The subspace $\mathrm{span}_{\mathbb{Q}}\{c_1, \ldots, c_j\}$ is the subgroup of *non-peripheral* free homology classes of X_n. The classes c_i will be called non-peripheral *generators* of $H_1(X_n; \mathbb{Q})$.

Lemma 5.1. *If $q : X_{\rho_\alpha} \to X_n$ is the cover constructed in §5.1, let $\{c_1, \ldots, c_j\}$ be the collection of non-peripheral generators of $H_1(X_n; \mathbb{Q})$ and for each $1 \leq i \leq j$ let $\gamma_i \in \pi_1(X_n)$ be a representative of c_i. Furthermore assume that*

$$\gamma_i \in \ker\left(\pi_1(X_n) \to (\mathbb{Z}/p\mathbb{Z})^d\right) \cong \pi_1\left(X_{\rho_\alpha}\right).$$

Then any lift $\tilde{\gamma}_i$ of γ_i to X_{ρ_α} has non trivial image in $H_1(X_{\rho_\alpha}; \mathbb{Q})$.

Proof. Let $[\tilde{\gamma}_i]$ be the image of $\tilde{\gamma}_i \in H_1(X_{\rho_\alpha}; \mathbb{Q})$. Since $q(\tilde{\gamma}_i) = \gamma_i$, if $q^* : H_1(X_{\rho_\alpha}; \mathbb{Q}) \to H_1(X_n; \mathbb{Q})$ is the induced mapping then $q^*([\tilde{\gamma}_i]) = c_i$. It follows that $[\tilde{\gamma}_i]$ is a non-trivial element of $H_1(X_{\rho_\alpha}; \mathbb{Q})$. □

Lemma 5.2. *If $q : X_{\rho_\alpha} \to X_n$ is the cover constructed above, let $\{c_1, \ldots, c_j\}$ be the collection of non-peripheral generators of X_n and for each $1 \leq i \leq j$ let $\gamma_i \in \pi_1(X_n)$ be a representative of c_i. Furthermore assume that*

$$\gamma_i \notin \ker\left(\pi_1(X_n) \to \mathbb{Z}/p\mathbb{Z}^d\right) \cong \pi_1\left(X_{\rho_\alpha}\right).$$

Then $\left[\tilde{\gamma_i^p}\right]$ is non trivial in $H_1(X_{\rho_\alpha}; \mathbb{Q})$.

Proof. We have by definition that $q_*(\tilde{\gamma_i^p}) = \gamma_i^p$, furthermore $[\gamma_i^p] = pc_i$. It then follows that $q^*\left(\left[\tilde{\gamma_i^p}\right]\right) = pc_i$, thus $\left[\tilde{\gamma_i^p}\right]$ is non-trivial in $H_1(X_{\rho_\alpha}; \mathbb{Q})$. □

Lemma 5.3. *Suppose that γ_i and γ_k are $\pi_1(X_n)$ representatives of distinct classes in $H_1(X_n; \mathbb{Q})$ satisfying the following conditions:*

1) *$[\gamma_i] \neq r[\gamma_k]$ for all $r \in \mathbb{Q}$,*

2) *$\tilde{\gamma}_i$ and $\tilde{\gamma}_k$ are any lifts of γ_i and γ_k respectively in X_{ρ_α}.*

Then $[\tilde{\gamma}_i] \neq r[\tilde{\gamma}_k]$ in $H_1(X_{\rho_\alpha}; \mathbb{Q})$ for all $r \in \mathbb{Q}$.

Proof. Suppose that $[\tilde{\gamma}_i] = r[\tilde{\gamma}_k]$ for some r in $\mathbb{Q}$, then $q^*([\tilde{\gamma}_i]) = q^*(r[\tilde{\gamma}_k])$ and thus $[\gamma_i] = r[\gamma_k]$ which is a contradiction. □

The following proposition is a direct consequence of this series of lemmas. We have that every non-peripheral generator c_i of X_n will lift and generate a free summand of $H_1(X_{\rho_\alpha};\mathbb{Q})$, and along with a computation of the number of boundary components, we will establish a lower bound on $\beta_1(X_{\rho_\alpha})$.

Proposition 5.1. *If K is a knot with non-trivial Alexander polynomial, X_{ρ_α} the associated non-abelian cover, and X_n the cyclic cover subordinate to X_{ρ_α}, we have*

$$\beta_1(X_n) \leq \beta_1(X_{\rho_\alpha}).$$

Proposition 5.2. *The number of torus boundary components of X_{ρ_α} is p^d.*

Proof. Denote by Γ_n by $\pi_1(X_n)$, and note that by the above there exists $p : \Gamma_n \to (\mathbb{Z}/p\mathbb{Z})^d$. Suppose that $\langle t, \lambda\rangle$ generate the peripheral subgroup of Γ_K, it follows that $\langle t^n, \lambda\rangle$ generate the image of $\pi_1(\partial X_n) \hookrightarrow \Gamma_n$, where ∂X_n denotes the single boundary component of X_n. The number of boundary components of X_α is equal to $p^d/|p(\langle t^n, \lambda\rangle)|$. Since $\alpha(t^n) = 1$ it follows that $\varphi(t^n) = 1$. Since λ bounds a Seifert surface F in M, and hence $\pi_1(F) \hookrightarrow \Gamma_K$ is contained in Γ'_K and it follows that λ also bounds the lift of F to X_n. Therefore $\lambda \in \Gamma'_n$, $p(\lambda) = 1$, and we have $|p(\langle t^n, \lambda\rangle)| = 1$. The number of boundary components of X_{ρ_α} is thus p^d. $\square$

By the half lives-half dies Lemma [19] the collection of lifts of meridinal boundary curve t^n, denoted $\{\tilde{t^n}_1, \ldots, \tilde{t^n}_{p^d}\}$ contribute to $\beta_1(X_{\rho_\alpha})$. Using Propositions 5.1 and 5.2 along with Lemmas 5.1, 5.2, and 5.3 we obtain the following lower bound.

Theorem 5.2. *If K is a knot with non-trivial Alexander polynomial, and $\rho_\alpha : \Gamma_K \to G_\alpha$ is the representation constructed in* §2.1.3, *then*

$$p^d + \beta(X_n) - 1 \leq \beta_1(X_{\rho_\alpha}).$$

5.2 Alexander Stratifications and The Upper Bound for $\beta_1(X_{\rho_\alpha})$

Recall the notation for the finitely presented group

$$\Gamma_K = \langle F_r \mid \mathcal{R}_j \text{ for } j = 1, \ldots, s\rangle.$$

The character group of Γ_K is defined to be $\widehat{\Gamma}_K = \mathrm{Hom}(\Gamma_K, \mathbb{C}^*)$. For any $f \in \widehat{\Gamma}_K$, the $r \times s$ matrix $M(F_r, \mathcal{R}_j)(f)$ defined in §2.2 is given by evaluation by f. The Alexander stratification of $\widehat{\Gamma}_K$ is

$$V_i(\Gamma_K) = \{f \in \widehat{\Gamma}_K \mid \mathrm{rank}(M(F_r, \mathcal{R}_j)(f)) < r - i\}.$$

The V_i are the subsets of $\widehat{\Gamma}_K$ defined by the ideals of the $(r-i)\times(r-i)$ minors of $M(F_r, \mathcal{R}_j)$. The nested sequence of algebraic subset $\widehat{\Gamma}_K \supset V_1 \supset \cdots \supset V_r$ is called the *Alexander stratification of* $\widehat{\Gamma}_K$.

The reason for introducing the Alexander stratification is to apply the following theorem.

Theorem 5.3 (Hironaka [21]). *Suppose that $p : Y \to X$ is a covering space of connected manifolds and*

$$\pi_1(X) \big/ p_*\pi_1(Y) = A$$

is a finite abelian group. Let $q : \pi_1(X) \to A$ be the quotient map and $\widehat{q} : \widehat{A} \hookrightarrow \widehat{\Gamma}_K$ the induced inclusion map. Then

$$\beta_1(Y) = \sum_{i=1}^{r-1} |V_i(\pi_1(X)) \cap \widehat{q}(\widehat{A} \setminus \widehat{1})| + \beta_1(X).$$

Since A is a finite abelian group every element of $\widehat{A}$ is determined by a root of unity, therefore $|\widehat{A}| = |A|$. An immediate consequence of this observation and Theorem 3.2 is the following corollary.

Corollary 5.1. *Let X_{ρ_α} be as above, let X_n denote the n-fold cyclic cover of the knot complement, and $r = \mathrm{rank}\,(\pi_1(X_n))$. Then $p : X_{\rho_\alpha} \to X_n$ is a* *covering map with deck group A, an elementary abelian p group, for p the smallest prime so that $\Delta_p(t)$ is non-trivial, then*

$$\beta_1(X_{\rho_\alpha}) = \sum_{x=1}^{r-1} |V_i(\pi_1(X_n)) \cap \widehat{q}(\widehat{A} \smallsetminus \widehat{1})| + \beta_1(X_n) \leq (r-1)(p^d - 1) + \beta_1(X_n).$$

This completes the proof of Theorem 3, we remark that the following Theorem is not Theorem 3 as stated. Specifically since the rank $(\pi_1(X_n)) \leq n(c_K - 1)$ it is an immediate consequence of the following.

Theorem 5.4. *Let K be a knot with non trivial Alexander polynomial and $\rho_\alpha : \Gamma_K \to G_\alpha$ be the representation constructed in* §2.1.3 *for some root $\alpha \in \mathbb{F}_{p^d}$ with $\mathrm{ord}_{\mathbb{F}^*_{p^d}}(\alpha) = n$, and r be the number of generators in a presentation of Γ_n. Then*

$$p^d + \beta_1(X_n) - 1 \leq \beta_1(X_{\rho_\alpha}) \leq (r-1)(p^d - 1) + \beta_1(X_n).$$

We have established an upper bound for the betti number of X_{ρ_α}. When K is fibered we are able to improve this bound, since in this case Γ'_K is a free group on $2g$ letters, where g is the genus of K. Therefore

$$\Gamma_K = \langle t, x_1, \ldots, x_{2g} \mid t x_i t^{-1} = w_i \text{ for } i = 1, \ldots, 2g \rangle,$$

the w_i are words in the x_i. This gives us a presentation

$$\Gamma_n = \langle t^n, x_1, \ldots, x_{2g} \mid t^n x_i t^n = u_i \rangle,$$

where u_i is a word in the x_j coming from the rule that $t x_i t^{-1} = w_i$. Thus there are $2g$ relations and $2g+1$ variables. We have that have that the Fox partial $D_0(t^n x_i t^n u_i^{-1})$ is $1 - x_i$, where the zeroth index is regarded as the index of the generator t^n.

Lemma 5.4. *Let X_n be the n-fold cyclic cover of a fibered knot complement, with presentation described above. If $p : Y \to X_n$ is a regular covering space with finite abelian deck group A and $q : \pi_1(X_n) \to A$ with $t^n \in \ker(q)$, then*

$$V_{(2g+1)-1}(\pi_1(X_n)) \cap \widehat{q}(\widehat{A} \smallsetminus \widehat{1}) = \emptyset$$

Proof. Assume by way of contradiction that $f \in V_{2g}(\pi_1(X_n)) \cap \widehat{q}(\widehat{A} \smallsetminus \widehat{1})$, hence $f(x) = a(q(x))$ for all $x \in K_n$ and some $a \in \widehat{A} \smallsetminus \widehat{1}$. Furthermore since $f \in V_{2g}(\pi_1(X_n))$, we have $D_0(t^n x_i t^n u_i^{-1})(f) = 0$ for all $i = 1, \ldots, 2g$. Thus $f(x_i) = 1$ and $x_i \in \ker(q)$ for all $i = 1, \ldots, 2g$. However A is a non-trivial quotient of K_n, hence with $t^n \in \ker(q)$ at least 1 generator x_j is not contained in the kernel of q. We have reached a contradiction, thus

$$V_{(2g+1)-1}(\pi_1(X_n)) \cap \widehat{q}(\widehat{A} \smallsetminus \widehat{1}) = \emptyset.$$

□

Corollary 5.2. *Let be α a root of $\Delta_p(t)$ of order n and degree d over $\mathbb{F}_p$, for a fibered knot of genus g. If $X_{\rho_\alpha} \to X_n$ the associated regular cover and $q : K_n \to \mathbb{F}_{p^d}$ then*

$$\beta_1(X_\alpha) = \sum_{x=1}^{2g-1} |V_i(\pi_1(X_n)) \cap \widehat{q}(\widehat{A} \smallsetminus \widehat{1})| + \beta_1(X_n) \leq (2g-1)(p^d - 1) + \beta_1(X_n).$$

Proof. All that we need to show is that $t^n \in \ker(q)$, however this follows directly from the fact that $t^n \in \ker(\alpha)$, and $\ker(q) = \ker(\alpha)$. □

This corollary is particularly interesting when we consider the figure 8 or trefoil knot complements, denoted 4_1 and 3_1 in the Rolfsen–Thistleswaithe table [34]. These knots are fibered and have genus 1, and furthermore $\beta_1(X_n) = 1$ for all $n > 1$. In this case Theorem 3.1 and Corollary 5.2 say that for any prime p and α a root of $\Delta_p(t)$ of order n and degree d

$$p^d + \beta_1(X_n) - 1 \leq \beta_1(X_{\rho_\alpha}) \leq p^d + \beta_1(X_n) - 1. \tag{5.1}$$

Therefore, for the figure 8, $\beta_1(X_{\rho_\alpha}) = p^d$, and since $\Delta(t) = t^2 - 3t + 1$, we have that $\beta_1(X_{\rho_\alpha}) = p$ if $\Delta(t)$ factors modulo p and $\beta_1(X_{\rho_\alpha}) = p^2$ if $\Delta(t)$ does not factor modulo p.

For the trefoil we have $\Delta(t) = t^2 - t + 1$, hence

$$\beta_1\left(X_{\rho_\alpha}\right) = \begin{cases} p+2 & \text{if } 6|n, \text{ and } \Delta_p(t) \text{ factors} \\ p^2+2 & \text{if } 6|n, \text{ and } \Delta_p(t) \text{ is irreducible} \\ p & \text{if } 6 \nmid n, \text{ and } \Delta_p(t) \text{ factors} \\ p^2 & \text{if } 6 \nmid n, \text{ and } \Delta_p(t) \text{ is irreducible} \end{cases}$$

In particular the bound in Corollary 4.5 is sharp, and more surprising is that this is not the only case for which it is sharp. It is sharp for 3_1, 5_1, as in Chapter 6, in the tables.

CHAPTER 6

COMPUTATIONS OF HOMOLOGY AND QUESTIONS

6.1 Some Questions

The computations in this section were done using both Sagemath [35] and Magma [7]. The following table below can be interpreted in the following way. For each knot there will be 2 rows, the upper row is $H_1(X_{\rho_\alpha})$ and the lower row is $H_1(X_n)$ of the cyclic cover X_n subordinate to X_{ρ_α}. The notation used in the table may be understood in the following way:

$$[0^{r_0}, n_1^{r_1}, \ldots, n_k^{r_k}] \leftrightarrow \mathbb{Z}^{r_0} \oplus (\mathbb{Z}/n_1\mathbb{Z})^{r_1} \oplus \cdots \oplus (\mathbb{Z}/n_k\mathbb{Z})^{r_k}\,.$$

Blank spaces in the table below indicate that MAGMA timed out in the computation of the abelianization of the kernels of ρ_α, and $\emptyset$ indicates that $\Delta(t)$ is trivial modulo p.

Question 6.1. *What is* $|\mathrm{Torsion}(H_1(X_{\rho_\alpha};\mathbb{Z}))|$*?*

It would be nice to compute the order of the torsion in terms of an invariant of the knot or the covers X_n or X_{ρ_α}, to mirror the classical computation of torsion for the cyclic covers X_n. There are many examples below for which

$$|\text{Torsion}\left(H_1\left(X_{\rho_\alpha};\mathbb{Z}\right)\right)| = \frac{|\text{Torsion}\left(H_1\left(X_n;\mathbb{Z}\right)\right)|}{p^d}. \tag{6.1}$$

Recall that p^d is the degree of the cover $X_{\rho_\alpha} \to X_n$.

For example equation 6.1 holds for 3_1 and the primes 2 and 3, for the other primes in the table first homology of the cyclic covers of 3_1 is torsion free. The same behavior is seen in the knot 5_1, for the prime 5 this is the case and for the other primes first homology of the cyclic covers is torsion free. For the knot 4_1 equation 6.1 holds for each prime presented in the table below. The knot 5_2 exhibits the same behavior, except for the prime 3 here the order of the torsion is larger than that of the cyclic cover. In the knot 6_1 equation 6.1 holds for all primes in the table, much like 4_1. The knot 6_2 sees equation 6.1 hold for one root of $\Delta(t) \pmod{11}$ but not the other root of the same polynomial, 6_3 also exhibits this behavior for the primes 13 and 7.

Question 6.2. *What feature of a knot makes* 6.1 *hold? For which primes does it hold?*

The for the knots 3_1, the $(3,2)$ torus knot, and 5_1, the $(5,2)$ torus knot, the upper bound for $\beta_1\left(X_{\rho_\alpha}\right)$ described in equation 6.1 is realized for certain values of p. Specifically the primes $3, 7$, and 11 for 5_1 and $5, 7, 11$, and 13 for 3_1. For 3_1 there is no torsion in $H_1\left(X_{\rho_\alpha};\mathbb{Z}\right)$ for primes such that $\mathrm{ord}_{\mathbb{F}^*_{p^d}}(\alpha)$ is $6 = 2\cdot 3$. Similarly for 5_1 and primes such that $\mathrm{ord}_{\mathbb{F}^*_{p^d}}(\alpha)$ is $10 = 5\cdot 2$. This phenomomenon also holds for one computable case of the $(7,2)$ torus knot, not appearing in the table below. What is even more interesting is that all the roots of the Alexander polynomial of the torus knot $T(p,q)$ are pq roots of unity [26].

Question 6.3. *If $T(p,q)$ is a torus knot, is the upper bound in Corollary* 5.2 *for $\beta_1\left(X_{\rho_\alpha}\right)$ realized for certain primes p, and is $H_1\left(X_{\rho_\alpha};\mathbb{Z}\right)$ torsion free for these primes?*

In many cases below the lower bound for $\beta_1(X_{\rho_\alpha})$ is realized however for the knot 6_3 and the prime 2 it is not, and neither is the upper bound. This phenomenon is also demonstrated in the the knot 6_2 for the primes 2 and 3.

Question 6.4. *For what knots is the upper bound in Theorem* 3 *or Corollary* 5.2 *realized? Similarly what knots is the lower bound of Theorem* 3 *realized?*

Another important feature of the table below is the light blue colored cells are indicating which metabelian covers X_{ρ_α} produce the minimal degree non-cylic cover as the quotient Y_{ρ_α}. It can be a seen that a for a few examples Y_{ρ_α} is the minimal degree non-cyclic cover for the knot in Question.

Question 6.5. *For the knots* 5_2*,* 6_2 *and* 6_3 *what is the minimal degree non-cyclic cover? Is the cover somehow related to* Y_{ρ_α}*?*

Question 6.6. *What feature must a knot have to make* Y_{ρ_α} *the minimal degree non-cyclic cover?*

6.2 Minimal Degree Non-cyclic Covers vs. Y_{ρ_α}

In what follows is a list of knots up to 7 crossings, and the number next to each knot is the minimal degree of a non-cyclic cover. The yes or no indicates if Y_{ρ_α} is the minimal degree non-cyclic cover, and $\Delta(t) \pmod p$ is written next to this which is used to determine whether Y_{ρ_α} is minimal.

- $3_1, 3$: Yes, $((t+1)^2, 3)$
- $4_1, 4$: Yes, $(t^2+t+1, 2)$
- $5_1, 5$: Yes, $((t+1)^4, 5)$
- $5_2, 5$: No, $((2)*(t^2+t+1), 5)$
- $6_1, 3$: Yes, $((-1)*(t+1)^2, 3)$
- $6_2, 5$: No, $((t^2+t+1)^2, 5)$

- $6_3, 5$: No, $(t^4 + 2 * t^3 + 2 * t + 1, 5)$
- $7_1, 7$: Yes, $((t + 1)^6, 7)$
- $7_2, 4$: Yes, $(t^2 + t + 1, 2)$
- $7_3, 4$: Yes, $(t * (t^2 + t + 1), 2)$
- $7_4, 3$: Yes, $((t + 1)^2, 3)$
- $7_6, 6$: No, $(t^2, 2), ((-1) * (t^4 + t^3 + t^2 + t + 1), 3)$
- $7_7, 3$: Yes, $((t + 1)^4, 3)$

6.3 Table of Homology

Now we give our table of computations for $H_1(X_{\rho_\alpha})$ and $H_1(X_n)$. Recall that the table can be interpreted in the following way. For each knot there will be 2 rows, the upper row is $H_1(X_{\rho_\alpha})$ and the lower row is $H_1(X_n)$ of the cyclic cover X_n subordinate to X_{ρ_α}. The notation used in the table may be understood in the following way:

$$[0^{r_0}, n_1^{r_1}, \ldots, n_k^{r_k}] \leftrightarrow \mathbb{Z}^{r_0} \oplus \left(\mathbb{Z}/n_1\mathbb{Z}\right)^{r_1} \oplus \cdots \oplus \left(\mathbb{Z}/n_k\mathbb{Z}\right)^{r_k}.$$

Blank spaces in the table below indicate that MAGMA timed out in the computation of the abelianization of the kernels of ρ_α, and $\emptyset$ indicates that $\Delta(t)$ is trivial modulo p. A blue shaded cell indicates that this is a minimal degree non-cyclic cover of a knot.

	2	3	5	7	11	13
3_1	$[0^4]$	$[0^3]$	$[0^{27}]$	$[0^9]$	$[0^{123}]$	$[0^{15}]$
	$[0, 2^2]$	$[0, 3]$	$[0^3]$	$[0^3]$	$[0^3]$	$[0^3]$
4_1	$[0^4, 2^2]$	$[0^9, 5]$	$[0^5]$	$[0^{49}, 3^2, 5]$	$[0^{11}, 11]$	$[0^{169}, 5, 29^2]$
	$[0, 4^2]$	$[0, 3^2, 5]$	$[0, 5]$	$[0, 3^2, 5, 7^2]$	$[0, 11^2]$	$[0, 5, 13^2, 29^2]$
5_1	$[0^{26}]$	$[0^{245}]$	$[0^5]$	$[0^{7205}]$	$[0^{35}]$	
	$[0, 2^4]$	$[0^5]$	$[0, 5]$	$[0^5]$	$[0^5]$	
5_2	$\emptyset$	$[0^9, 2^4, 7]$	$[0^{25}]$	$[0^7]$	$[0^{11}, 11]$	$[0^{169}]$
	$\emptyset$	$[0, 3^3]$	$[0, 5^2]$	$[0, 7]$	$[0, 11, 11]$	$[0, 13^2]$
6_1	$\emptyset$	$[0^3, 3]$	$[0^5, 9, 5]$	$[0^7, 7]$	$[0^{11}, 9, 11, 31^2]$	$[0^{13}, 3, 27, 5^2, 7^2, 13]$
	$\emptyset$	$[0, 9]$	$[0, 9, 5^2]$	$[0, 7^2]$	$[0, 9, 11^2, 31^2]$	$[0, 3, 27, 5^2, 7^2, 13^2]$
6_2	$[0^{21}, 4]$	$[0^{17}, 2^8, 3^2, 11]$	$[0^{25}, 5^3]$		$[0^{11}]$; $[0^{121}, 5^2, 11^{18}, 121^2, 23^{24}, 43^{12}]$	
	$[0, 2^4]$	$[0, 3^4, 11]$	$[0, 5^2]$		$[0, 11]$; $[0, 5^2, 11^3]$	
6_3	$[0^{31}, 2^5, 4]$	$[0^9, 3^4, 13]$		$[0^7, 7]$; $[0^{49}, 3^2, 13^{17}]$		$[0^{13}]$; $[0^{169}, 5^{28}, 13^{38}, 169^4, 43^2, 181^{28}]$
	$[0, 4^4]$	$[0, 3^2, 13]$		$[0, 7^2]$; $[0, 3^2, 7^2, 13]$		$[0, 13]$; $[0, 13^3, 43^2]$

Table 6.1: Table of $H_1\left(X_{\rho_\alpha}\right)$

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