Stochastic Differential Equations: Some Risk and Insurance Applications

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Sheng Xiong
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Examining Committee Members:

Wei-shih Yang, Advisory Chair, Department of Math
Shiferaw S. Berhanu, Department of Math
Michael R. Powers, Department of RIHM
Hua Chen, External Member, Department of RIHM

ABSTRACT

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Sheng Xiong
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Professor Wei-Shih Yang, Chair

In this dissertation, we have studied diffusion models and their applications in risk theory and insurance. Let X_t be a d-dimensional diffusion process satisfying a system of Stochastic Differential Equations defined on an open set $G \subseteq R^d$, and let U_t be a utility function of X_t with $U_0 = u_0$. Let T be the first time that U_t reaches a level u^*. We study the Laplace transform of the distribution of T, as well as the probability of ruin, $\psi(u_0) = Pr\{T < \infty\}$, and other important probabilities. A class of exponential martingales is constructed to analyze the asymptotic properties of all probabilities. In addition, we prove that the expected discounted penalty function, a generalization of the probability of ultimate ruin, satisfies an elliptic partial differential equation, subject to some initial boundary conditions. Two examples from areas of actuarial work to which martingales have been applied are given to illustrate our methods and results: 1. Insurer's insolvency. 2. Terrorism risk. In particular, we study insurer's insolvency for the Cramér-Lundberg model with investments whose price follows a geometric Brownian motion. We prove the conjecture proposed by Constantinescu and Thommann [1].

Keywords: Stochastic differential equation, Ruin theory, Martingale, Diffusion processes, Point processes, Terrorism risk.

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Dedicated to the memory of Brandon Rupp

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CHAPTER 1

INTRODUCTION

In actuarial risk management it is an important issue to estimate the performance of the portfolio of an insurer. Ruin theory, as a branch of actuarial science that examines an insurer's vulnerability to insolvency, is used to analyze the insurer's surplus and ruin probability which can be interpreted as the probability of insurer's surplus drops bellow a specified lower bond. Most of the techniques and methodologies adopted in ruin theory are based on the application of stochastic processes. In particular, diffusion processes have been of great interest in modeling an insurer's surplus. In this dissertation, we have studied diffusion models and their applications in risk theory and insurance.

Let X_t be a d-dimensional diffusion process satisfying a system of Stochastic Differential Equations defined on an open set $G \subseteq R^d$, and let U_t be a utility function of X_t with $U_0 = u_0$. Let T be the first time that U_t reaches a level u^*. We study the Laplace transform of the distribution of T, as well as the probability of ruin, $\psi(u_0) = Pr\{T < \infty\}$, and other important probabilities. A class of exponential martingales is constructed to analyze the asymptotic properties of all probabilities. In addition, we prove that the expected discounted penalty function, a generalization of the probability of ultimate ruin, satisfies an elliptic partial differential equation, subject to some initial boundary conditions. Two examples from areas of actuarial work to which martingales have been applied are given to illustrate our methods and results: 1. Insurer's insolvency. 2. Terrorism risk. In particular, we study insurer's insolvency for the Cramér-Lundberg model with investments whose price follows a geometric Brownian motion. We prove the conjecture proposed by Constantinescu and Thommann [1].

The thesis is organized as follow: in chapter 3 and 4, we study the insurer's surplus and terrorism risk based on continuous stochastic processes. We construct a class of exponential martingales to analyze the asymptotic properties of ruin probability and other important probabilities. Moreover, we show the Laplace transform of the distribution of T satisfies an elliptic partial differential equation subject to some boundary condition.

In chapter 5, we study a conjecture in the Cramér-Lundberg model with investments. By assuming there is a cap on the claim sizes, we prove that the probability of ruin has at least an algebraic decay rate if $2a/\sigma^2 > 1$. More importantly, we show that the probability of ruin is certain for all initial capital u, if $2a/\sigma^2 \leq 1$.

CHAPTER 2

PRELIMINARY

This chapter provides a minimal amount of basic theory of Stochastic Calculus and Risk Theory & Insurance necessary to describe and prove our results. Almost all of the results recorded here are either well known or are easily deduced from well known results.

2.1 Martingale theory

Definition 2.1.1. *Let $(\Omega; \mathcal{F}; \mathcal{P})$ be a probability space and let $\mathcal{G}$ be a sub-sigma field of $\mathcal{F}$. If X is an integrable random variable, then the conditional expectation of X given $\mathcal{G}$ is any random variable Z which satisfies the following two properties:*
(1) Z is $\mathcal{G}$-measurable;
(2) if $\Lambda \in \mathcal{G}$, then

$$\int_\Lambda Z \, d\mathcal{P} = \int_\Lambda X \, d\mathcal{P}.$$

We denote Z by $E[X \mid \mathcal{G}]$.

Remark 2.1.1. *It is implicit in (2) that Z must be integrable.*

Theorem 2.1.1. *Let X and Y be integrable random variables, a and b real numbers. Then*
(i) $E[E[X \mid \mathcal{G}]] = E[X]$.

(ii) If X is $\mathcal{G}$-measurable, $E[X \mid \mathcal{G}] = X$ a.e.

(iii) $E[aX + bY \mid \mathcal{G}] = aE[X \mid \mathcal{G}] + bE[Y \mid \mathcal{G}]$.

(v) If $X \geq 0$ a.e., $E[X \mid \mathcal{G}] \geq 0$ a.e.

(vi) If $X \leq Y$ a.e., $E[X \mid \mathcal{G}] \leq E[Y \mid \mathcal{G}]$ a.e.

(vii)Suppose Y is $\mathcal{G}$-measurable and XY is integrable. Then

$$E[X \mid \mathcal{G}] = YE[X \mid \mathcal{G}] \text{ a.e.}$$

(viii) If X_n and X are integrable, and if either $X_n \uparrow X$, or $X_n \downarrow X$, then

$$E[X_n \mid \mathcal{G}] \rightarrow E[X \mid \mathcal{G}] \text{ a.e.}$$

Jensen's inequality for expectations:

Theorem 2.1.2. *Let X be a r.v. and ϕ a convex function. If both X and $\phi(X)$ are integrable, then*

$$\phi(E[X]) \leq E[\phi(X)].$$

Jensen's inequality for conditional expectations:

Theorem 2.1.3. *Let X be a r.v. and ϕ a convex function on R. If both X and $\phi(X)$ are integrable, then*

$$\phi(E[X \mid \mathcal{G}]) \leq E[\phi(X)\ \mathcal{G}] \text{ a.e.}$$

Definition 2.1.2. *A filtration on the probability space $(\Omega; \mathcal{F}; \mathcal{P})$ is a sequence $\{\mathcal{F}_n; n = 0, 1, 2, \ldots\}$ of sub-sigma fields of $\mathcal{F}$ such that for all $n, \mathcal{F}_n \subset \mathcal{F}_{n+1}$.*

Definition 2.1.3. *Given a probability space $(\Omega; \mathcal{F}; \mathcal{P})$, a stochastic process is a collection of random variables $\{\mathcal{F}_t\}_{t \geq 0}$ with 'time' index.*

That is a fairly general definition—it is almost hard to think of something numerical which is not a stochastic process. However, we have something more specific in mind.

Definition 2.1.4. *A stochastic process* $X = \{X_n; n = 0, 1, 2, \ldots\}$ *, is adapted to the filtration* $(\mathcal{F}_n)$ *if for all* n, X_n *is* $\mathcal{F}_n$*-measurable.*

Definition 2.1.5. *A process* $X = \{X_n; \mathcal{F}_n, n = 0, 1, 2, \ldots\}$ *, is a martingale if for each* $n = 0, 1, 2, \ldots$ *,*
(i) $\mathcal{F}_n, n = 0, 1, 2, \ldots$ *is a filtration and* X *is adapted to* $\mathcal{F}_n$*;*
(ii) for each n, X_n *is integrable;*
(iii) for each $n, E[X_{n+1} \,|\mathcal{F}_n] = X_n$*.*

The process X is called a submartingale if (iii) is replaced by for each n,

$$E[X_{n+1} \,|\mathcal{F}_n] \geq X_n.$$

It is called a supermartingale if (iii) is replaced by for each n,

$$E[X_{n+1} \,|\mathcal{F}_n] \leq X_n.$$

Example 2.1.1. *Let* $Z_n; n = 0, 1, 2, \ldots$ *be a sequence of independent random variables with mean* 0*. Let* $X_n = Z_1 + Z_2 + \cdots + Zn$ *and* $X_0 = 0$*. Let* $\mathcal{F}_n = \sigma(X_0, X_1, \ldots, X_n)$*, Then*
(a) $X = \{X_n; \mathcal{F}_n, n = 0, 1, 2, \ldots\}$ *is a martingale.*
(b) If $E[Z_{n+1} \,|\mathcal{F}_n] \geq Z_n$*, then* X *is a submatingale.*
(c) If $E[Z_{n+1} \,|\mathcal{F}_n] \leq Z_n$*, then* X *is a supermatingale.*

Proof

$$E[X_{n+1} \,|\mathcal{F}_n] = E[X_n + Z_{n+1} \,|\mathcal{F}_n] = E[X_n \,|\mathcal{F}_n\} + E[Z_{n+1} \,|\mathcal{F}_n].$$

Since X_n *is* $\mathcal{F}_n$*-measurable,* $E[X_n \,|\mathcal{F}_n] = X_n$ *. Since* Z_{n+1} *and* $\mathcal{F}_n$ *are independent,* $E[Z_{n+1} \,|\mathcal{F}_n] = E[Z_{n+1}] = 0$*. Therefore* $E[X_{n+1} \,|\mathcal{F}_n] = X_n$*.*

Example 2.1.2. *Let* $X = \{X_n; \mathcal{F}_n, n = 0, 1, 2, \ldots\}$ *be a martingale. Let* $W_n \leq W_{n+1}$*be a sequence of* $\mathcal{F}_n$ *adapted random variable. Then* $\{X_n + W_n; \mathcal{F}_n, n = 0, 1, 2, \ldots\}$ *is a submartingale. In short, a martingale plus an increasing adapted sequence is a submartingale.*

Proof

$$E[|Y_n|] = E[|E[Y \, | \mathcal{F}_n\}|] \leq E[E[|Y| \, | \mathcal{F}_n\}] = E[|Y|] < \infty,$$

where the inequality follows from Jensen's inequality. Hence

$$E[Y_{n+1} \, | \mathcal{F}_n] = E[E[Y \, | \mathcal{F}_{n+1}] \, | \mathcal{F}_n] = E[Y \, | \mathcal{F}_n] = Y_n.$$

Definition 2.1.6. (X_n) *is called uniformly integrable (UI) if*

$$\lim_{A \to \infty} \sup_n \int_{|X_n| > A} |X_n| d\mathcal{P} = 0.$$

Note that

(1) Suppose $E|X| < \infty$. *Then* $\lim_{A \to \infty} \int_{|X| > A} |X| d\mathcal{P} = 0.$, *by the Dominated Convergence Theorem.*

(2) Suppose $E|X| < \infty$. *Then for all* $\epsilon > 0$, *there exists* $\delta > 0$ *such that* $\int_A |X| d\mathcal{P} < \epsilon$ *whenever* $P(A) < \delta$.

The martingale in the following example is uniformly integrable.

Example 2.1.3. *Let* $\mathcal{F}_n, n = 0, 1, 2, \ldots$ *be a filtration. Let* $E[|Y|] < \infty$. *Let* $Y_n = E[Y \, | \mathcal{F}_n]$. *Then* $Y = \{Y_n; \mathcal{F}_n, n = 0, 1, 2, \ldots\}$ *is a martingale.*

The above examples are very important because we will see all the submartingales must be of Example 2.1.2 (Doob's Decomposition Theorem) and all UI martingales must be of Example 2.1.3.

Theorem 2.1.4. *Suppose* $X = \{X_n; \mathcal{F}_n, n = 0, 1, 2, \ldots\}$ *is a martingale (supermartingale, submartingale). Then for all* $m \leq n$, *we have*

$$E[X_{n+1} \, | \mathcal{F}_n] = X_n, a.s.(martingale),$$

$$E[X_{n+1} \, | \mathcal{F}_n] \leq X_n, a.s.(supermartingale),$$

$$E[X_{n+1} \, | \mathcal{F}_n] \geq X_n, a.s.(submartingale).$$

Theorem 2.1.5. *Suppose* $X = \{X_n; \mathcal{F}_n, n = 0, 1, 2, \ldots\}$ *is a martingale. Let* ϕ *be a convex function such that* $E[\phi(X_n)] < \infty$. *Then for all* n, $\{\phi(X_n); \mathcal{F}_n, n = 0, 1, 2, \ldots\}$ *is a submartingale.*

Definition 2.1.7. *Let $\mathcal{F}_n, n = 0, 1, 2, \ldots$ is a filtration. A random variable $\tau : \Omega \to (0, 1, 2, \ldots, \infty)$ is called a stopping time (with respect to $\mathcal{F}_n, n = 0, 1, 2, \ldots$) if $\{\omega \in \Omega, \tau(\omega) \le i\} \in \mathcal{F}_n$, for all $i = 0, 1, \ldots$.*

Example 2.1.4. *Let $X_0, X_1, \ldots$ be a sequence of random variables. Let $\mathcal{F}_n = \sigma(X_0, X_1, \ldots, X_n)$. Let $\mathcal{B}$ be a Borel subset of R. The first hitting time of $\mathcal{B}$ by (X_n) is defined by $\tau_{\mathcal{B}} = \min(0 \le n, X_n \in \mathcal{B})$. Since*

$$\{\tau_{\mathcal{B}} = i\} = \{X_0 \in \mathcal{B}, X_1 \in \mathcal{B}, \ldots, X_{i-1} \notin \mathcal{B}, X_i \in \mathcal{B}\} \in \mathcal{F}_i.$$

Therefore, $\tau_{\mathcal{B}}$ is a stopping time with respected to $\{\mathcal{F}_n, n = 0, 1, 2, \ldots\}$.

It is clear that the event that the first hitting time of $\mathcal{B}$ by (X_n) occurs at i only depends on the outcomes of $X_0, X_1, \ldots, X_i$. This is the property that motivates the definition of general stopping times.

Theorem 2.1.6. *Let $X = \{X_n; \mathcal{F}_n, n = 0, 1, 2, \ldots\}$ be a martingale (submartingale, supermartingale). Let $0 \le \tau_1 \le \tau_2 \le \ldots \le \tau_m \le N$ be a sequence of stopping times. Then $\{X_{\tau_n}; \mathcal{F}_{\tau_n}, n = 0, 1, 2, \ldots\}$ is a martingale (submartingale, supermartingale).*

Consider stochastic processes indexed by closed half-line $R_+ = \{t; t \ge 0\}$. Let $(\Omega; \mathcal{F}; \mathcal{P})$ be a probability space and $(\mathcal{F}_t)_{t \in R_+}$ be a filtration of $\mathcal{F}$. Assume that the probability space is complete, and that each $\sigma-$field $\mathcal{F}_t$ contains all of the $\mathcal{P}$-null sets. Let $\mathcal{F}_{t+} = \cap_{s>t}\mathcal{F}_s$ and $\mathcal{F}_{t-} = \sigma(\cap_{s<t}\mathcal{F}_s)$.

Definition 2.1.8. *$(\mathcal{F}_t)$ is said to be right-continuous if $(\mathcal{F}_{t+}) = (\mathcal{F}_t)$, for all $t \in R_+$. A process (X_t) is right-continuous if $X_t(\omega)$ is right-continuous as a function of t, for $\mathcal{P}$-a.e. ω.*

Definition 2.1.9. *A filtration on the probability space $(\Omega; \mathcal{F}; \mathcal{P})$ is a collection $\{\mathcal{F}_t; 0 \le t < \infty\}$ of sub-sigma fields of $\mathcal{F}$ such that $s \le t$, implies $\mathcal{F}_s \subset \mathcal{F}_t$.*

Definition 2.1.10. *Let $\{\mathcal{F}_t; 0 \le t < \infty\}$ is a filtration. A random variable $\tau : \Omega \to R \bigcup \{\infty\}$ is called a stopping time (with respect to $\mathcal{F}_t$) if $\{\omega \in \Omega, \tau(\omega) \le t\} \in \mathcal{F}_t$, for all $t \ge 0$.*

Definition 2.1.11. *(Martingale in continuous time)*
Let $(\Omega; \mathcal{F}; \mathcal{P})$ *be a probability space and* $\{\mathcal{F}_t\}_{t\geq 0}$ *be a filtration of* $\mathcal{F}$*. An adapted family* $\{X_t\}_{t\geq 0}$ *of random variables on this space with* $E[|X_t|] < \infty$ *for all* $t \geq 0$ *is a martingale if, for any* $s \leq t$*,*

$$E[X_t|\ \mathcal{F}_s] = X_s.$$

Theorem 2.1.7. *(Doob's continuous Stopping Theorem)*
Let M_t *be a continuous martingale with respect to a filtration* $(\mathcal{F}_t)_{t\in R_+}$*. If* τ *is a stopping time for* $\mathcal{F}_t$*. Then the process defined by*

$$X_t = M_{t\wedge\tau}$$

is also a martingale relative to $\mathcal{F}_t$*.*

Definition 2.1.12. *The continuous-time stochastic process* $\{W_t :\ 0 \leq t < T\}$ *is called a Standard Brownian Motion (or Wiener Process) on* $[0, T)$ *if*
1. $W_0 = 0$;
2. W_t *is almost surely continuous*;
3. W_t *has independent increments with Gaussian distribution*

$$W_t - W_s \sim \mathcal{N}(0, t - s)\ \ for\ \ 0 \leq s \leq t < T.$$

Example 2.1.5. *If* $\{W_t\}_{t\geq 0}$ *is a Standard Brownian Motion generating the filtration* $\{\mathcal{F}_t\}_{t\geq 0}$*, then*
1. W_t *is an* $\mathcal{F}_t$*-martingale.*
2. W_t^2 *is an* $\mathcal{F}_t$*-martingale.*
3. $\exp\left(\sigma W_t - \frac{\sigma^2}{2}t\right)$ *is* $\mathcal{F}_t$*-martingale. (called an exponential martingale).*

Definition 2.1.13. *(Local Martingale)*
Let $(\Omega; \mathcal{F}; \mathcal{P})$ *be a probability space and* $\{\mathcal{F}_t\}_{t\geq 0}$ *be a filtration of* $\mathcal{F}$*. Let* $X : [0,\ \infty) \times \Omega \to S$ *be an* $\{\mathcal{F}_t\}_{t\geq 0}$*-adapted stochastic process. Then* X *is called an* $\{\mathcal{F}_t\}_{t\geq 0}$*-local Martingale if there exists a sequence of* $\{\mathcal{F}_t\}_{t\geq 0}$*-stopping times* $\tau_k : \Omega \to [0, \infty)$ *such that*

1. the τ_k are almost surely increasing: $P(\tau_k < \tau_{k+1}) = 1$;

2. the τ_k diverge almost surely: $P(\tau_k \to \infty \text{ as } k \to \infty) = 1$;

3. the stopped process

$$1_{\{\tau_k > 0\}} {X_t}^{\tau_k} := 1_{\{\tau_k > 0\}} X_{\min\{t,\tau_k\}}$$

is an $\{\mathcal{F}_t\}_{t \geq 0}$-martingale for every k.

Theorem 2.1.8. *Let M_t be a local martingale with respect to a filtration $(\mathcal{F}_t)_{t \in R_+}$. If τ is a stopping time for $\mathcal{F}_t$. Then the process defined by*

$$X_t = M_{t \wedge \tau}$$

is also a local martingale relative to $\mathcal{F}_t$.

Remark 2.1.2. *In mathematics, a local martingale is a type of stochastic process, satisfying the localized version of the martingale property. Every martingale is a local martingale; every bounded local martingale is a martingale; however, in general a local martingale is not a martingale, because its expectation can be distorted by large values of small probability. In particular, a diffusion process without drift is a local martingale, but not necessarily a martingale.*

Theorem 2.1.9. *(The Optional Stopping Theorem)[22]*
Let $(X_t)_{t \in R_+}$ be a right-continuous supermartingale relative to a right-continuous filtration $(\mathcal{F}_t)_{t \in R_+}$. Suppose there exits an integrable random variable Y such that $X_t \geq E[Y|\mathcal{F}_t]$, for all $t \in R_+$. Let S and T be stopping times such that $S \leq T$. Then (X_S, X_T) is a two-term supermartingale relative to $\mathcal{F}_S, \mathcal{F}_T$.

2.2 The Itô integral

The Itô calculus is about systems driven by *white noise*, which is the derivative of Brownian motion. To find the response of the system, we integrate the forcing, which leads to the *Itô integral*, of a function against the derivative of Brownian motion.

Definition 2.2.1. *Let $\mathcal{F}_t$ be the filtration generated by Brownian motion up to time t, and let $F(t) \in \mathcal{F}_t$ be an adapted stochastic process. we define the following approximations to the Itô integral*

$$Y_{\Delta t}(t) = \sum_{t_k < t} F(t_k)\Delta W_k, \tag{2.2.1}$$

with the usual notions $t_k = k\Delta t$, and $\Delta W_k = W(t_{k+1}) - W(t_k)$. If the limit exists, the Itô integral is

$$Y(t) = \lim_{\Delta t \to 0} Y_{\Delta t}(t). \tag{2.2.2}$$

Example 2.2.1. *The simplest interesting integral is*

$$Y(T) = \int_0^T W(t)dW(t).$$

The correct Itô answer is

$$\int_0^T W(t)dW(t) = \lim_{\Delta t \to 0} Y_{\Delta t}(t) = \tfrac{1}{2}\left(W(t)^2 - T\right). \tag{2.2.3}$$

Lemma 2.2.1. *Itô's Formula with Space and Time Variable*
For any function $f(w,t) \in C^{1,2}(R^+ \times R)$, we have the following representation

$$df(W(t),t) = \partial_w f(W(t),t)dW(t) + \tfrac{1}{2}\partial_w^2 f(W(t),t)dt + \partial_t f(W(t),t)dt. \tag{2.2.4}$$

or written as the Itô differential form

$$\begin{aligned} f(W(T),T) - f(W(0),0) = &\int_0^T \partial_w f(W(t),t)dW(t) \\ &+ \int_0^T \left(\partial_w^2 f(W(t),t) + \partial_t f(W(t),t)\right) dt \end{aligned}$$

Suppose $X(t)$ is an adapted stochastic process with

$$dX(t) = a(t)dW(t) + b(t)dt.$$

Then X is a martingale if and only if $b(t) = 0$. We call $a(t)dW(t)$ the *martingale part* and $b(t)dt$ *drift term.* For the martingale part, we have the following Itô isometry formula:

$$E\left[\left(\int_{T_1}^{T_2} a(t)dW(t)\right)^2\right] = \int_{T_1}^{T_2} E[a(t)^2]dt. \tag{2.2.5}$$

2.3 Stochastic differential equations

The theory of stochastic differential equations (SDE) is a framework for expressing dynamical models that include both random and non random forces. Solutions to Itô SDEs are Markov processes in that the future depends on the past only through the present.

Definition 2.3.1. *An Itô stochastic differential equation takes the form*

$$dX(t) = a(X(t), t)dt + \sigma(X(t), t)dW(t). \tag{2.3.1}$$

Remark 2.3.1. *A solution is an adapted process that satisfies (2.3.1) in the sense that*

$$X(T) - X(0) = \int_0^T a(X(t), t)dt + \int_0^T \sigma(X(t), t)dW(t), \tag{2.3.2}$$

where the first integral on the right is a Riemann integral and the second is an Itô integral.

As in the general Itô differential, $a(X(t), t)dt$ is the *drift term*, and $\sigma(X(t), t)dW(t)$ is the *martingale term*. We often call $\sigma(x, t)$ the *volatility*.

Definition 2.3.2. *a geometric Brownian motion is a stochastic process that satisfies the SDE*

$$dX(t) = \mu X(t)dt + \sigma X(t)dW(t), \tag{2.3.3}$$

with initial data $X(0) = 1$.

Since

$$X(t) = e^{\mu t - \sigma^2 t/2 + \sigma W(t)} \tag{2.3.4}$$

satisfies (2.3.3), which implies that a geometric Brownian motion has the above representation.

Remark 2.3.2. *Steele [15] pointed out a paradox of risk without possibility of rewards for the geometric Brownian motion: if $\frac{2\mu}{\sigma^2} < 1$, then $X(t) \to 0$ as $t \to \infty$ a.s., despite the fact that the expected value of $X(t)$ goes to positive infinity.*

Definition 2.3.3. *a diffusion process is a solution to a stochastic differential equation. It is a continuous-time Markov process with continuous sample paths.*

Definition 2.3.4. *The backward equation is*

$$\partial_t f(x,t) + a(x,t)\partial_x f(x,t) + \frac{\sigma^2(x,t)}{2}\partial_x^2 f(x,t) = 0. \tag{2.3.5}$$

Definition 2.3.5. *The Forward equation is*

$$\partial_t u(x,t) = -\partial_x \left(a(x,t)u(x,t)\right) + \tfrac{1}{2}\partial_x^2 \left(\sigma^2(x,t)u(x,t)\right) \, . \tag{2.3.6}$$

Definition 2.3.6. *The* generator *of an Itô process is the* operator *containing the spatial part of the backward equation*[1]

$$L(t) = a(x,t)\partial_x + \tfrac{1}{2}\sigma^2(x,t)\partial_x^2.$$

For a general continuous time Markov process, the generator is defined by the requirement that

$$\frac{d}{dt}E[g(X(t),t)] = E\left[(L(t)g)(X(t),t) + g_t(X(t),t)\right], \tag{2.3.7}$$

for a sufficiently rich (dense) family of functions g.

This applies not only to diffusion processes, but also to jump diffusions, continuous time birth/death processes, continuous time Markov chains, etc.

Definition 2.3.7. *Let $(X, \mathcal{B}_X)$ be a measurable space. By a point function p on X we mean a mapping $p : D_p \subset (0,\infty) \mapsto X$, where the domain D_p is a countable subset of $(0,\infty)$. p defines a counting measure $N_p(dtdx)$ on $(0,\infty) \times X$ by*

$$N_p((0,t] \times U) = \sharp\{s \in D_p;\ s \le t,\ p(s) \in U\},\ t > 0,\ U \in \mathcal{B}_X.$$

A point process is obtained by randomizing the notion of point function. Let Π_X be the totality of point functions on X and $\mathcal{B}(\Pi_X)$ be the smallest σ-field on Π_X with respect to which all $p \mapsto N_p((0,t] \times U),\ t > 0,\ U \in \mathcal{B}_X$, are measurable.

[1]Some people include the time derivative in the definition of the generator.

Definition 2.3.8. *A point process p on X is a $(\Pi_X,\ \mathcal{B}(\Pi_X))$-valued random variable, that is, a mapping $p : \Omega \mapsto \Pi_X$ defined on a probability space $(\Omega; \mathcal{F}; \mathcal{P})$ which is $\mathcal{F}|\mathcal{B}(\Pi_X)$-measurable.*

A point process is called Poisson if $N_p(dtdx)$ is a Poisson random measure on $(0, \infty) \times X$.

Definition 2.3.9. *Let $(\Omega; \mathcal{F}; \mathcal{P})$ be a probability space and $(\mathcal{F})_{t\geq 0}$ be a filtration. A point process $p = (p(t))$ on X defined on Ω is called $\mathcal{F}_t$-adapted if every $t > 0$ and $U \in \mathcal{B}(X)$, $N_p(t, U) = \sum_{s\in D_p,\, s\leq t} I_U(p(s))$ is $\mathcal{F}_t$-measurable. p is called σ-finite, if there exist $U_n \in \mathcal{B}(X), n = 1, 2, \ldots,$ such that $U_n \uparrow X$ and $E[N_p(t, U_n)] < \infty$, for all $t > 0$ and $n = 1, 2, \ldots$.*

For a given $\mathcal{F}_t$-adapted, σ-finite point process p, let

$$\Gamma_p = \{U \in \mathcal{B}(X),\ E[N_p(t, U)] < \infty,\ for\ all\ t > 0\ and\ n = 1, 2, \ldots\}.$$

We define

Definition 2.3.10. *An $\mathcal{F}_t$-adapted point process p on $(\Omega; \mathcal{F}; \mathcal{P})$ is said to be of the class (QL) (Quasi left-continuous) if it is σ-finite and there exists $\hat{N}_p = (\hat{N}_p(t, U))$ such that*
(i) for $U \in \Gamma_p, t \mapsto \hat{N}_p(t, U)$ is a continuous $(\mathcal{F})_t$-adapted increasing process,
(ii) for each t and a.e. $\omega \in \Omega, t \mapsto \hat{N}_p(t, U)$ is a σ-finite measure on $(X, \mathcal{B}_X)$,
(iii) for $U \in \Gamma_p, t \mapsto \hat{N}_p(t, U) = N_p(t, U) - \hat{N}_p(t, U)$ is a $\mathcal{F}_t$-martingale.

we introduce the following classes:

$$F_p = \{f(t, x, \omega);\ \text{f is}\, \mathcal{F}_t\text{−predictable and for each}\ t > 0, \int_0^{t^+}\int_x |f_1(s, x, \cdot)| N_p(dsdx) < \infty\}$$

$$F_p{}^2 = \{f(t, x, \omega);\ \text{f is}\, \mathcal{F}_t - \text{predictable and for each}\ t > 0,$$
$$E\left[\int_0^{t^+}\int_x |f_1(s, x, \cdot)|^2 \tilde{N}_p(dsdx)\right] < \infty\}$$

$F_p{}^{2,loc} = \{f(t, x, \omega);$ f is $\mathcal{F}_t -$ predictable and there exist a sequence of $\mathcal{F}_t-$stopping times σ_n such that $\sigma_n \uparrow \infty$ a.s. and $I_{[0,\sigma_n]}(t) f(t, x, \omega) \in F_p{}^2, n = 1, 2, \ldots\}$.

Definition 2.3.11. *An $\mathcal{F}_t$-adapted stochastic process X_t defined on $(\Omega;\mathcal{F};\mathcal{P})$ is called a semi-martingale if it is expressed as*

$$X_t = X_0 + M_t + A_t + \int_0^{t^+}\int_x f_1(s,x,\cdot)N_p(dsdx) + \int_0^{t^+}\int_x f_2(s,x,\cdot)\tilde{N}_p(dsdx)$$

Where

(i) X_0 is an $\mathcal{F}_0$-measurable random variable.

(ii) M_t is a local martingale.

(iii) A_t is a continuous $\mathcal{F}_t$-adapted process such that a.s. $A_0 = 0$ and $t \mapsto A_t$ is of bounded variation on each finite interval.

(iv) p is an $\mathcal{F}_t$-adapted point process of the class (QL) on some state space $(X, \mathcal{B}_X)$, $f_1 \in F_p$ and $f_2 \in {F_p}^{2,\mathrm{loc}}$ such that $f_1 f_2 = 0$.

Define a d-dimensional semi-martingale $X_t = ({X_t}^1, {X_t}^2, \ldots, {X_t}^d)$ by

$$X_t = X_0 + M_t + A_t + \int_0^{t^+}\int_x f(s,x,\cdot)N_p(dsdx) + \int_0^{t^+}\int_x g(s,x,\cdot)\tilde{N}_p(dsdx)$$

Where $f = (f^1, f^2, \ldots, f^d)$ and $g = (g^1, g^2, \ldots, g^d)$. Then

Theorem 2.3.1. *(Itô's formula). Let F be a function of class C^2 on R^d and $X(t)$ a $d-$dimentional semi-martingale given above. Then the stochastic process $F(X(t))$ is also a semi-martingale (with respect to $(\mathcal{F}_t)_{t\geq 0}$) and the* *following formula holds:*

$$
\begin{aligned}
F(X_t) - F(X_0) = & \sum_{i=1}^{d} \int_0^t F_i{}'(X_s)\, dM^i(s) + \sum_{i=1}^{d} \int_0^t F_i{}'(X_s)\, dA^i(s) \\
& + \frac{1}{2} \sum_{i,j=1}^{d} \int_0^t F_{ij}{}''(X_s)\, d\langle M^i, M^j \rangle(s) \\
& + \int_0^{t^+} \int_X [F(X_{s^-} + f(s,x,\cdot)) - F(X_{s^-})]\, N_p(dsdx) \\
& + \int_0^{t^+} \int_X [F(X_{s^-} + g(s,x,\cdot)) - F(X_{s^-})]\, \tilde{N}_p(dsdx) \\
& + \int_0^{t^+} \int_X \{[F(X_{s^-} + g(s,x,\cdot)) - F(X_{s^-})] \\
& - \sum_{i=1}^{d} g^i(s,x,\cdot) F_i{}'(X_s)\}\, \tilde{N}_p(dsdx).
\end{aligned}
$$

2.4 Ruin theory and risk models

Ruin theory studies an insurer's vulnerability to insolvency based on stochastic models of the insurer's surplus. The most important questions are the time of ruin at which the surplus becomes negative for the first time, the surplus immediately before the time of ruin and the deficit at the time of ruin. In most cases, the principal objective of the classical model and its extensions was to calculate the probability of ultimate ruin.

Ruin theory was first introduced in 1903 by the Swedish actuary Filip Lundberg [2], then it received a substantial boost with the articles of Powers [3] in 1995 and Gerber and Shiu [4] in 1998, which introduced the expected discounted penalty function, a generalization of the probability of ultimate ruin. This fundamental work was followed by a large number of papers in the ruin literature deriving related quantities in a variety of risk models. The interested reader can read more in Asmussen [5], Embrechts et al. [7], Gerber et al. [16] and Ren [17].

The following is a brief introduction of ruin models that relate to my dissertation.

(1) The Cramér Lundberg model

Gerber, H.U. and Shiu in [4] studied the Cramér Lundberg ruin model. Let u denote the insurer's initial surplus, assume the premium received in a continuous constant rate c, per unit time, and the aggregate claims constitute a compound Poisson process:

$$S(t) = \sum_{j=1}^{N(t)} x_j,$$

where $N(t)$ is a Poisson process with parameter λ, and x_j's are i.i.d with pdf $p(x)$ and cdf $P(x)$. Then the insurer's surplus, $u(t)$, at time t, is modeled by the following stochastic process:

$$u(t) = u + ct - S(t) = u + ct - \sum_{j=1}^{N(t)} x_j.$$

Definition 2.4.1. *The time of ruin is defined to be* $\mathbf{T} = \inf\{\mathbf{t} \,|\, \mathbf{u}(\mathbf{t}) < \mathbf{0}\}$.

As mentioned previously, technical ruin of the insurance company occurs when the surplus becomes negative (or below a given threshold). Therefore, the definition of the infinite time probability of ruin is

$$\psi(u) = \Pr\{T < \infty \,|\, u\}$$

Definition 2.4.2. *The adjustment coefficient is defined as the smallest strictly positive solution (if it exists) of the Lundberg fundamental equation*

$$\lambda + \delta - c\xi = \lambda \hat{p}(\xi) = \lambda \int_0^\infty e^{-\xi x} p(x)\, dx.$$

The main result related to my work is

Theorem 2.4.1. *(Lundburg's asympototic formula)*

$$\psi(u) \sim \frac{c - \lambda \int_0^\infty x p(x)\, dx}{\lambda \int_0^\infty y e^{Ry} p(y)\, dy - c} e^{-Ru},$$

as $u \to \infty$. *Where* $-R$ *is the negative root of Lundberg foundamental equation.*

(2) Powers' Diffusion Model

Powers in [3] studied a diffusion model. Let $u^* \in (0, u_0)$ be the infimum of the set of capitalization levels at which the insurer is considered solvent, $L(t)$ be cumulative incurred losses to time t, $Y(t)$ be cumulative investment income to time t, $P(t)$ be cumulative earned premium to time t, $X(t)$ be cumulative earned losses to time t, $\mathbf{T} = \inf\{\mathbf{t} \,|\, \mathbf{u(t)} \leq \mathbf{u^*}\}$ be the time of insolvency, u_0 be the initial net worth, $u(t)$ be the net worth at time t, $W(t)$ be the interrupted net worth at time t, $b_L(\cdot)$ and $b_Y(\cdot)$ be positive nondecreasing functions. Under the following assumptions

- $P(t) = (1+\pi)L(t)$
- $X(t) = \varepsilon_L L(t) + \varepsilon_p P(t)$
- $dS(t) = g(S(t))dt + H(S(t))[dZ_L(t),\ dZ_Y(t)]^T$
- The process $S(t)$ satisfies the Lipschitz condition.

where

$$S(t) = [L(t), Y(t)]^T$$

$$g(S(t)) = [\lambda u(t), \nu u(t)]^T$$

$$H(S(t)) = \begin{bmatrix} b_L(u(t)) & 0 \\ 0 & b_Y(u(t)) \end{bmatrix}.$$

Then Power proposed a diffusion model

$$du(t) = \alpha u(t)dt + b(u(t))dZ(t)$$

where $Z(t)$ is a standard Brownian motion and

$$\alpha = c_L\lambda + c_Y\nu$$

$$b(u(t)) = \sqrt{c_L^2 b_L^2(u(t)) + c_Y^2 b_Y^2(u(t))}.$$

The main result related to my work are

Theorem 2.4.2. *Define*

$$W(t) = \begin{cases} u(t), & if\ t \leq T \\ \\ 0 & if\ t \geq T. \end{cases}$$

Then the Laplace transform of the probability distribution of T, $\varphi_z(u_0) = E[e^{-zT} \,|u_0]$, for $z > 0$, may be expressed as

$$\varphi_z(u_0) = \frac{\eta_1(+\infty)\eta_2(u_0) - \eta_2(+\infty)\eta_1(u_0)}{\eta_1(+\infty)\eta_2(u^*) - \eta_2(+\infty)\eta_1(u^*)}$$

where $\eta_1(u)$ and $\eta_2(u)$ are two linearly independent solutions of the second order linear differential equation

$$z\varphi_z(u) - \alpha u\varphi_z'(u) - \frac{1}{2}b^2(u)\varphi_z''(u) = 0.$$

and

Corollary 2.4.1. *Let the net worth process be given by $W(t)$. If $b(\cdot)^2$ is concave downward, then the probability of ruin, $\psi(u_0) = Pr\{T < +\infty |\ u_0\}$, is bounded above as follows:*

$$\Psi(u_0) \le \frac{2\frac{u^*}{u_0} + \frac{1}{\alpha}\int_{u_0}^{\infty}\frac{b^2(y)}{y^3}dy}{(1 - \frac{u^*}{u_0})^2}.$$

Remark 2.4.1. *This corollary shows that the decay rate of ruin probability is polynomial. Later in my dissertation, we can show the decay rate is exponential by martingale approach.*

(3) Jiandong Ren's Model

Ren in [17] studied a six dimensional diffusion model. Let $D(t)$ cumulative paid losses to time t, and $R(t)$ be cumulative earned premium to time t. Let $L(t), P(t), Y(t), X(t)$ be as above. Set

$$V(t) = [L(t), D(t), P(t), R(t), Y(t), U(t)]^T$$

$$dZ(t) = [dZ_L(t), dZ_D(t), dZ_R(t), dZ_Y(t)]^T$$

Define

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \lambda \\ \delta & -\delta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1+\pi)\lambda \\ 0 & 0 & \rho & -\rho & 0 & 0 \\ \nu & -\nu & \nu & -\nu & 0 & \nu \\ c_Y\nu & -c_Y\nu & c_Y\nu + c_R\rho & -c_Y\nu - c_R\rho & 0 & c_Y\nu + c_L\lambda + c_P\lambda(1+\pi) \end{bmatrix}$$

$$S = \begin{bmatrix} \sigma_L(\cdot) & 0 & 0 & 0 \\ 0 & \sigma_D(\cdot) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_R(\cdot) & 0 \\ 0 & 0 & 0 & \sigma_Y(\cdot) \\ c_L\sigma_L(\cdot) & 0 & c_R\sigma_R(\cdot) & c_Y\sigma_Y(\cdot) \end{bmatrix}.$$

Then Jingdong's model can be written as

$$dV(t) = AV(t)dt + SdZ(t).$$

His main results are

Theorem 2.4.3. *If let*

$$\gamma_1(t) = \frac{L(t) - D(t) - P(t) - R(t)}{u(t)}$$

and

$$\gamma_2(t) = \frac{P(t) - R(t)}{u(t)}$$

and assume

$\gamma_1(t) \to \gamma_1$ *and* $\gamma_2(t) \to \gamma_2$ *where* γ_1, γ_2 *are constants, if we denote the implied net worth process by* $\hat{u}(t)$ *then*

$$d\hat{u}(t) = \alpha\hat{u}(t)dt + \sigma(\hat{u}(t))dZ(t) \tag{2.4.1}$$

where

$$\alpha = c_Y\nu(1+\gamma_1) + c_L\lambda + c_P\lambda(1+\pi) + c_R\rho\gamma_2$$

and

$$\sigma(\hat{u}(t)) = \sqrt{c_L^2\sigma_L^2(u(t)) + c_Y^2\sigma_Y^2((1+\gamma_1)\hat{u}(t)) + c_R^2\sigma_R^2(\gamma_2\hat{u}(t))}.$$

Theorem 2.4.4. *If* $\sigma_L(\cdot) = \sqrt{\beta_L}$, $\sigma_D(\cdot) = \sqrt{\beta_D}$, $\sigma_R(\cdot) = \sqrt{\beta_R}$, $\sigma_Y(\cdot) = \sqrt{\beta_Y}$ *are constants, then the stochastic differential equations :*

$$dV(t) = AV(t)dt + SdZ(t)$$

posses solution:

$$V(t) = e^{At}\left[C + \int_0^t e^{-A\tau}SdZ(\tau)\right]$$

where $C = V(0) = [0, 0, 0, 0, 0, u_0]^T$, *and*

$$S = \begin{bmatrix} \sqrt{\beta_L} & 0 & 0 & 0 \\ 0 & \sqrt{\beta_D} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\beta_R} & 0 \\ c_L\sqrt{\beta_L} & 0 & c_R\sqrt{\beta_R} & c_Y\sqrt{\beta_Y} \end{bmatrix}.$$

Theorem 2.4.5. *If the ISDs (infinitesimal standard deviation)* σ^* *are proportional to the infinitesimal drifts, then*

$$d\hat{u}(t) = \alpha\hat{u}(t)dt + \sqrt{\beta}(\hat{u}(t))dZ(t)$$

where

$$\alpha = c_Y\nu(1+\gamma_1) + c_L\lambda + c_P\lambda(1+\pi) + c_R\rho\gamma_2$$

and

$$\sqrt{\beta} = \sqrt{c_L^2\beta_L + c_Y^2\beta_Y((1+\gamma_1)^2 + c_R^2\beta_R\gamma_2^2}.$$

The interested reader can read more on these subjects in [24] by Klugman et al. (2004); [25] by Gerber (1979); [26] by Denuit and Charpentier (2004); [27] by Kaas et al. (2001), among others.

2.5 Lanchester equations

This section and the following section are quoted from [30] written by Powers. Because my work on terrorism risk (chapter 4) partially was advised by Powers.

During the First World War F. W. Lanchester described one of the simplest, and most enduring, mathematical attrition models of force-on-force combat in [18] in 1916, which may be described by a system of differential equations of the form

$$dA = -k_1 A^{\alpha_1} D^{\delta_1} dt \tag{2.5.1}$$

$$dD = -k_2 A^{\alpha_2} D^{\delta_2} dt \tag{2.5.2}$$

where $A = A(t) \geq 0$ and $D = D(t) \geq 0$ denote, respectively, the sizes of the attackers and defenders forces at time $t \geq 0; A(0) = A_0$ and $D(0) = D_0$ are known boundary conditions; k_1, k_2 are positive real-valued parameters denoting, respectively, the defender and attacker effective destruction rates; and k_1, k_2 and δ_1, δ_2 are real-valued parameters reflecting the fundamental nature of the combat under study. In his original formulation, Lanchester (1916) considered two cases one for ancient-warfare, in which $\alpha_1 = 1, \delta_1 = 1, \alpha_2 = 1, \delta_2 = 1$, and one for modern-warfare, in which $\alpha_1 = 0, \delta_1 = 1, \alpha_2 = 1, \delta_2 = 0$. The principal conclusion to be drawn from Lanchester's original analysis is that the ratio of the opposing armies' initial forces (i.e., $\frac{D_0}{A_0}$) plays a greater role in modern combat (with unaimed fire). The results are stated as the Lanchester's linear law and square law respectively.

2.6 Ad Hoc models for terrorism risk

Following the terrorist attacks of September 11, 2001, the United States Congress passed the Terrorism Risk Insurance Act (TRIA) of 2002 to "establish a temporary Federal program that provides for a transparent system of shared public and private compensation for insured losses resulting from acts of terrorism". In return for requiring U.S. property-liability insurers to include terrorism coverage in certain critical lines of business, the legislation supplemented private reinsurance coverage for terrorism-related losses through the end of 2005. Two subsequent extensions of TRIA have carved out a far from "temporary" role for the U.S. federal government in financing terrorism risk. As Powers noted in [31], a necessary condition for private insurers and reinsurers to remain in the terrorism-risk market is the industry's confidence that total losses can be forecast with sufficient accuracy.

Major in [29] proposed that the conditional probability of destruction of a target i, given that target i is selected for attack by terrorists, can be expressed as

$$p_i = \exp(-\frac{A_i D_i}{\sqrt{W_i}})(\frac{A_i^2}{A_i^2 + W_i}) \tag{2.6.1}$$

where A_i denotes the size of the forces assigned by the terrorists to attack i, D_i denotes the size of the forces assigned by government (and possibility private security) to defend i, and W_i denotes the value of i as a target (which is assumed to have a square-root relationship to the target's physical presence). In this formulation, the first factor on the right-hand side of equation (2.6.1) represents the probability that the terrorists avoid detection prior to their attack (derived from a simple search model), and the second factor represents the probability that the terrorists are then successful in destroying the target (derived from a dose-response model).

Powers and Shen in [32] replaced the above formula with

$$p_i = \exp(-\frac{A_i^s D_i^s}{V_i^s})(\frac{A_i^c}{A_i^c + D_i^c}) \tag{2.6.2}$$

where V_i denotes the (three-dimensional) physical volume of target i, and $s > 1, c \in (0,1)$ are scale parameters. The biggest conceptual difference between equations (2.6.1) and (2.6.2) is the substitution of a power of D_i for a power of W_i in the denominator of the second factor (representing the terrorists' probability of success in destroying the target once they have avoided detection).

CHAPTER 3

RUIN ON DIFFUSION MODELS

3.1 Ruin on generalized Powers model

In this section, we reinvestigate Corollary 2.1 in [3] by using martingale approach, and obtain a better upper bound on the probability of ruin. Our result shows that the probability of ruin exponentially decay as the initial net worth $u_0 \to \infty$.

Let n be a positive integer. We will use u^* to denote the infimum of the set of capitalization levels at which the insurer is considered solvent. Set

$$\tau_n = \inf\{t \geq 0;\ U_t \in (u^*, n)^c\}$$

be the first time for the net worth process U_t going out of the interval (u^*, n). Set

$$T = \inf\{t \geq 0;\ U_t \leq u^*\}$$

be the time of the insolvency, and

$$\psi(u_0) = Pr_{u_0}\{T < \infty\}$$

be the probability of ruin. These notation will be fixed throughout this chapter. Also, we will keep the assumptions and notation in [3] regarding the stochastic differential equation

$$dU_t = \alpha U_t dt + b(U_t)dZ_t. \tag{3.1.1}$$

Instead of working directly on Powers Model, we will work on the generalized Powers Model:

$$dU_t = \alpha U_t^{\beta} dt + b(U_t)dZ_t, \tag{3.1.2}$$

where $\beta \geq 1$.

Lemma 3.1.1. *Let θ be any positive real number, $\alpha > 0$, $\beta \geq 1$ and $b(x)$, a nonnegative continuous function, defined as in SDE (3.1.2). Set*

$$X_t = U_t - U_0 - \int_0^t \alpha U_s^{\beta}\, ds,$$

and

$$Y_t = exp\left(-\theta X_t - \frac{1}{2}\langle -\theta X\rangle_t\right).$$

Then $X_{t\wedge\tau_n}$ and $Y_{t\wedge\tau_n}$ are L^2-martingales.

Proof. Integrating SDE (3.1.2), we have

$$U_t = U_0 + \int_0^t \alpha U_s^{\beta}\, ds + \int_0^t b(U_s)\, dZ_s. \tag{3.1.3}$$

Then

$$X_t = U_t - U_0 - \int_0^t \alpha U_s^{\beta}\, ds = \int_0^t b(U_s)\, dZ_s$$

is a local martingale, and so

$$\begin{aligned} Y_t &= \exp\left(-\theta X_t - \frac{1}{2}\theta^2 \int_0^t b^2(U_s)\, ds\right) \\ &= \exp\left(-\theta U_t + \theta U_0 + \theta \int_0^t \alpha U_s^{\beta}\, ds - \frac{1}{2}\theta^2 \int_0^t b^2(U_s)\, ds\right) \end{aligned}$$

is also a local martingale. The L^2-norm of $X_{t\wedge\tau_n}$ can be computed as follows:

$$\|X_{t\wedge\tau_n}\|_{L^2}^2 = E\left[\left(\int_0^{t\wedge\tau_n} b(U_s)dZ_s\right)^2\right] = E\left[\int_0^{t\wedge\tau_n} b^2(U_s)ds\right]. \tag{3.1.4}$$

Note that $U_{t\wedge\tau_n}$ is bounded by n and that the function $b(x)$ is continuous. It follows that $b^2(U_s)$ is bounded for $0 \le s \le t\wedge\tau_n$. Hence the integral on the right hand side of (3.1.4) is bounded for each t, and so $X_{t\wedge\tau_n}$ is a L^2-martingale. Next, since $b(U_s)$ is bounded for $0 \le s \le t\wedge\tau_n$, moreover, $t\wedge\tau_n \le t$, we have

$$|X_{t\wedge\tau_n}| = |U_{t\wedge\tau_n} - U_0 - \int_0^{t\wedge\tau_n} \alpha U_s^\beta \, ds| \le |U_{t\wedge\tau_n}| + U_0 + |\int_0^t \alpha U_s^\beta \, ds| \le n + U_0 + \alpha n^\beta t$$

for each t. So $|Y_{t\wedge\tau_n}| \le c(t,n)$, where $c(t,n)$ is a constant depending on t and n. It now follows that $Y_{t\wedge\tau_n}$ is also a L^2-martingale.

Lemma 3.1.2. *Suppose that $b(x)$ is increasing and continuous twice differentiable, and that $g(x) = b^2(x)$ is concave down on $[u^*, \infty)$ and $g'(u^*) > 0$. Then there exists a positive real number $\theta_0 = \min\left\{\frac{2\alpha u^{*\beta}}{g(u^*)}, \ \frac{2\alpha\beta u^{*\beta-1}}{g'(u^*)}\right\}$ such that*

$$K(\theta) := \lim_{n\to\infty} E_{u_0}\left[exp\left(\int_0^{\tau_n} \{\theta\alpha U_s^\beta - \frac{1}{2}\theta^2 b^2(U_s)\ \}ds\right) \mid U_{\tau_n} = u^*\right] \ge 1 \quad (3.1.5)$$

for any $\theta \in [0, \ \theta_0]$.

Proof. Set $h(x) = \alpha x^\beta - \frac{1}{2}\theta g(x)$. Then $h'(x) = \alpha\beta x^{\beta-1} - \frac{1}{2}\theta g'(x)$. Now solve the following inequality system:

$$h'(u^*) \ge 0$$

$$h(u^*) \ge 0.$$

We get the solution: $\theta \in [0, \ \theta_0]$. Since $g(x)$ is concave down on $[u^*, \infty)$ and $\beta \ge 1$, so $h''(x)$ is nonnegative and $h'(x)$ is increasing on $[u^*, \infty)$. Hence for any $\theta \in [0, \ \theta_0]$, we have

$$h'(x) \ge h'(u^*) \ge 0, \ \forall x \ge u^*.$$

It follows that $h(x)$ is increasing on $[u^*, \infty)$. Hence for any $\theta \in [0, \ \theta_0]$, we have

$$h(x) \ge h(u^*) \ge 0, \ \forall x \ge u^*.$$

Now since $U_s \ge u^*$ on $[0, \tau_n]$, hence the integrand

$$\theta\alpha U_s^\beta - \frac{1}{2}\theta^2 b^2(U_s) = \theta\{\alpha U_s^\beta - \frac{1}{2}\theta b^2(U_s)\} = \theta h(U_s) \ge 0, \ \forall \theta \in [0, \ \theta_0].$$

It now follows that

$$\begin{aligned} K(\theta):=&\lim_{n\to\infty} E_{u_0}\left[\exp\left(\int_0^{\tau_n}\left(\theta\alpha U_s^{\beta}-\frac{1}{2}\theta^2 b^2(U_s)\right)ds\right) \mid U_{\tau_n}=u^*\right] \\ \geq& E_{u_0}\left[1 \mid U_{\tau_n}=u^*\right]=1 \end{aligned}$$

for any $\theta \in [0,\ \theta_0]$.

Theorem 3.1.1. *Let $\alpha > 0$, $\beta \geq 1$ and $b(x)$, a nonnegative continuous function, defined as in SDE (3.1.2). Suppose further that $b(x)$ is increasing and continuous twice differentiable, and that $g(x) = b^2(x)$ is concave down on $[u^*, \infty)$, and $g'(u^*) > 0$. Then there exists a positive real number $\theta_0 = \min\left\{\frac{2\alpha u^{*\beta}}{g(u^*)},\ \frac{2\alpha\beta u^{*\beta-1}}{g'(u^*)}\right\}$ such that the probability of ruin*

$$\psi(u_0) \leq exp\left(-\theta(u_0 - u^*)\right) \tag{3.1.6}$$

for any $\theta \in [0,\ \theta_0]$.

Proof. If $\psi(u_0) = 0$, then (3.1.6) holds for any θ. It is sufficient to show (3.1.6) assuming $\psi(u_0) > 0$. It follows from Lemma (3.1.1) that $1 = E[Y_0] = E[Y_{t\wedge\tau_n}]$, for each $t \geq 0$. Hence

$$\lim_{t\to\infty} E[Y_{t\wedge\tau_n}] = 1.$$

On the other hand, it follows from Fatou's lemma that

$$E[Y_{\tau_n}] \leq \lim_{t\to\infty} E[Y_{t\wedge\tau_n}].$$

Therefore

$$E_{u_0}\left[\exp\left(-\theta U_{\tau_n} + \theta U_0 + \int_0^{\tau_n}\{\theta\alpha U_s^{\beta} - \frac{1}{2}\theta^2 b^2(U_s)\ \}ds\right)\right] = EY_{\tau_n} \leq 1.$$

However, since

$$\begin{aligned} &E_{u_0}\left[\exp\left(-\theta U_{\tau_n} + \theta U_0 + \int_0^{\tau_n}\{\theta\alpha U_s^{\beta} - \frac{1}{2}\theta^2 b^2(U_s)\ \}ds\right)\right] \\ =&Pr\{U_{\tau_n} = u^*\}e^{\theta(u_0-u^*)}E_{u_0}\left[\exp\left(\int_0^{\tau_n}\{\theta\alpha U_s^{\beta} - \frac{1}{2}\theta^2 b^2(U_s)\ \}ds\right)\ \middle|\ U_{\tau_n} = u^*\right] \\ &+ Pr\{U_{\tau_n} = n\}e^{\theta(u_0-n)}E_{u_0}\left[\exp\left(\int_0^{\tau_n}\{\theta\alpha U_s^{\beta} - \frac{1}{2}\theta^2 b^2(U_s)\ \}ds\right)\ \middle|\ U_{\tau_n} = n\right], \end{aligned}$$

and the second term is nonegative, we have

$$Pr\{U_{\tau_n} = u^*\}e^{\theta(u_0-u^*)}E_{u_0}\left[\exp\left(\int_0^{\tau_n}\{\theta\alpha U_s^\beta - \frac{1}{2}\theta^2 b^2(U_s)\ \}ds\right)\ \Bigg|\ U_{\tau_n} = u^*\right] \leq 1.$$

By lemma (3.1.2),

$$K(\theta) := \lim_{n\to\infty} E_{u_0}\left[\exp\left(\int_0^{\tau_n}\{\theta\alpha U_s^\beta - \frac{1}{2}\theta^2 b^2(U_s)\ \}ds\right)\ \Bigg|\ U_{\tau_n} = u^*\right] \geq 1,$$

for any $\theta \in [0,\ \theta_0]$.

Therefore we have

$$\psi(u_0) = \lim_{n\to\infty} Pr\{U_{\tau_n} = u^*\} \leq \exp\left(-\theta(u_0 - u^*)\right).$$

The proof is completed.

Remark 3.1.1. *In the case of* $0 < \beta < 1$*, if we assume* $g(\cdot) = b^2(\cdot)$ *is a function of* u_t^β*, then it can be reduced to the above case where* $\beta = 1$*, that is, the probability of ruin* $\psi(u_0)$ *also exponentially decays in the case where* $0 < \beta < 1$*.*

3.2 Laplace transform of PDF of the first exit time

In this section, we introduce a general system of m dimensional stochastic differential equations and use its infinitesimal operator to form a partial differential equation. Then we show that the Laplace transform $E_{u_0}\left[e^{-zT}\right]$ of the probability distribution of ruin time T is the unique solution that satisfies the partial differential equation. Also we discuss under what conditions the solution exists.

We consider the following stochastic differential equations:

$$X_t = X_0 + \int_0^t b(X_s)\ ds + \int_0^t \sigma(X_s)\ dB_s,$$

or namely,

$$X_t^i = X_0^i + \int_0^t b_i(X_s)\, ds + \sum_{j=1}^{m} \int_0^t \sigma_{ij}(X_s)\, dB_s,$$

where $B_t = (B_t^1, B_t^2, ..., B_t^m)^\top$ is a standard m dimensional Brownian Motion, where $\sigma = (\sigma_{ij})_{d\times m}$ is a $d \times m$ matrix. and where $b = (b_1, b_2, ..., b_d)^\top, X_t$ are column vectors.

Let $a = (a_{ij})_{d\times m} = \sigma\sigma^T$ and A be the infinitesimal operator w.r.t the stochastic differential equations above. namely,

$$Af(x) = \frac{1}{2}\sum_{i,j} a_{ij}(x) D_{ij} f(x) + \sum_i b_i(x) D_i f(x),$$

and let $V(x) = E_x\left[e^{-zT}\right]$, where $\mathbf{T} = \inf\{\mathbf{t} \geq \mathbf{0} \,|\, \mathbf{X_t} \notin \mathbf{G}\}$. We will show that $V(x) = E_x\left[e^{-zT}\right]$ is the unique solution that satisfies

(a) $AV(x) - zV(x) = 0, \forall x \in G$.

(b) $V(y) = 1, \forall y \in \partial G$.

Remark 3.2.1. *The definition of T is equivalent to $\mathbf{T'} = \inf\{\mathbf{t} > \mathbf{0} \,|\, \mathbf{X_t} \notin \mathbf{G}\}$ for $\forall x \in G$, since G is open. If $y \in \partial G$, then $P_y(T = 0) = 1$ and $V(y) = 1$ is always true.*

This proof is essentially taken from section 4.6. in [23]. Since the proof for general case in [23] is far more complicated, we put a simplified proof in our case for reader's convenience.

Theorem 3.2.1. *If $U(x)$ satisfies (a), then $M_t = U(X_t)e^{-zt}$ is a local martingale on $[0, T)$.*

Proof: Applying Itô's formula gives

$$\begin{aligned} U(X_t)e^{-zt} - U(X_0) &= \int_0^t e^{-zs} \sum_i b_i(X_s) D_i U(X_s)\, ds - z\int_0^t e^{-zs} U(X_s)\, ds \\ &\quad + \int_0^t e^{-zs} \frac{1}{2}\sum_{i,j} a_{ij}(X_s) D_{ij} U(X_s)\, ds + \textit{local mart.} \\ &= \int_0^t e^{-zs}(AU(X_s) - zU(X_s))\, ds + \textit{local mart.} \end{aligned}$$

for $t < T$. It follows from (a) that $M_t = U(X_t)e^{-zt}$ is a local martingale on $[0, T)$.

Assume G is a bounded connected open set from now on.

Theorem 3.2.2. *If there is a solution satisfying both* (a) *and* (b) *that is bounded, then it must be* $V(x) = E_x\left[e^{-zT}\right]$.

Proof: By Theorem 3.2.1, $M_s = U(X_s)e^{-zs}$ is a local martingale on $[0, T)$. Let $s \nearrow T \wedge t$ and using the bounded convergence theorem gives

$$U(x) = EM_0 = EM_{T\wedge t} = E_x\left[e^{-zT};\ T \leq t\right] + E_x\left[e^{-zt};\ T > t\right].$$

As $t \to \infty$, the first term approaches to $V(x) = E_x[e^{-zT}]$. Since $\{T > t\} \in \mathcal{F}_t$, the definition of conditional expectation and Markov property imply

$$\begin{aligned} E_x\left[U(X_t)e^{-zT}; T > t\right] &= E_x E_x\left[U(X_t)e^{-zT}|\mathcal{F}_t;\ T > t\right] \\ &= E_x\left[U(X_t)e^{-zt}E_x\left[e^{-zT}\right];\ T > t\right]. \end{aligned}$$

For all $y \in G$, since G is a bounded, open connected set, we have

$$E_y\left[e^{-zT}\right] \geq e^{-z}P_y(T \leq 1) \geq \epsilon > 0.$$

Hence replace $E_x\left[e^{-zT}\right]$ by ϵ in the equation above, we have

$$\begin{aligned} E_x\left[|U(X_t)|e^{-zt};\ T > t\right] &\leq \epsilon^{-1}E_x\left[|U(X_t)|e^{-zT};\ T > t\right] \\ &\leq \epsilon^{-1}\|U\|_\infty E_x\left[e^{-zT};\ T > t\right] \to 0. \end{aligned}$$

as $t \to \infty$, by Dominated Convergence Theorem, since $P_x(T < \infty) = 1$. Going back to the first equation in the proof, we have shown the solution must be $V(x)$.

Theorem 3.2.3. *If* $V(x) \in C^2$, *then it satisfies* (a) *in* G.

Proof: The Markov property implies that

$$E_x\left[e^{-zT} \mid \mathcal{F}_{s\wedge T}\right] = e^{-z(s\wedge T)}E_{X_{(s\wedge T)}}[e^{-zT}] = e^{-z(s\wedge T)}V(X_{s\wedge T}).$$

Since the left-hand side is a bounded local martingale on $[0, T)$ and hence is a UI (uniformly integrable) martingale. So is $e^{-z(s\wedge T)}V(X_{s\wedge T})$. Applying Itô's formula to $e^{-z(s\wedge T)}V(X_{s\wedge T})$ gives

$$de^{-z(s\wedge T)}V(X_{s\wedge T}) = [AV(X_{s\wedge T}) - zV(X_{s\wedge T})]\, e^{-z(s\wedge T)}d(s\wedge T) + local\ mart.$$

However, the first term is continuous and locally of bounded variation, it must be zero, that is,

$$\int_0^{t\wedge T} [AV(X_{s\wedge T}) - zV(X_{s\wedge T})]\, e^{-z(s\wedge T)}\ d(s\wedge T) \equiv 0.$$

Since $V(x) \in C^2$, it follows that

$$AV(X_{s\wedge T}) - zV(X_{s\wedge T}) \equiv 0,\ \ P_{X_0}\ a.s.$$

For if it were $\neq 0$ at some point X_0, by continuity, then it would be $> 0\,(< 0)$ on an open ball $D(X_0, r)$ for some $r > 0$. If we choose $s(\omega)$ to be the first exit time from the ball $D(X_0, r)$, then the integral would be positive(or negative), a contradiction.

Theorem 3.2.4. *If G is a bounded connected open set, then $V(x) \in C^2$ hence satisfies* (a)*.*

Proof: Follows from theorem (3.6) in [23].

3.3 Applications

Let

$$\tau_n = \inf\{t \geq 0;\ U_t \in (u^*, n)^c\}$$

be the first time for the net worth process U_t going out of the interval (u^*, n). Let

$$T = \inf\{t \geq 0;\ U_t \leq u^*\}$$

be the time of the insolvency. We apply the theorem (3.2.2) to the following three examples.

Example 3.3.1. ***Powers' one dimensional Diffusion Model.***
Powers proved in [3] that U_t is the homogeneous diffusion process specified by the unique solutions of the SDE

$$dU_t = \alpha U_t dt + b(U_t) dZ_t.$$

It is a one dimensional diffusion model. In Theorem 1 in [3], Powers proved that $\varphi_z(u_0) = E_{u_0}\left[e^{-zT}\right]$ can be expressed as two linear independent solutions of the following ODE:

$$z\varphi_z(u) - \alpha u \varphi\prime_z(u) - \frac{1}{2} b^2(u) \varphi\prime\prime_z(u) = 0 \tag{3.3.1}$$

He referred Darling and Siegert's (1953) proof. However, Applying our Theorem (3.2.2) to Powers' model on the open set $G_n = (u^*, n)$, $E_{u_0}\left[e^{-z\tau_n}\right]$ satisfies (3.3.1). Let n go to infinity, then $\varphi_z(u_0) = \lim_{n\to\infty} E_{u_0}\left[e^{-z\tau_n}\right]$ by Bounded Convergence Theorem. It is not hard to prove the Powers' result about $\varphi_z(u_0)$ on $G = (u^*, \infty)$.

Remark 3.3.1. *For the generalized powers' model, $\varphi_z(u_0) = E_{u_0}\left[e^{-zT}\right]$ can be expressed as two linear independent solutions of the following ODE:*

$$z\varphi_z(u) - \alpha u^\beta \varphi\prime_z(u) - \frac{1}{2} b^2(u) \varphi\prime\prime_z(u) = 0.$$

Example 3.3.2. ***Powers' two dimensional Diffusion Model.***
Powers constructed a two dimensional SDE's Model in [3]:

$$dS_t = g(S_t)dt + H(S_t)\left[d{Z_t}^L,\ d{Z_t}^Y\right]^\top$$

where

$$S_t = [L_t, Y_t]^\top, \quad g(S_t) = [\lambda U_t, \nu U_t]^\top$$
$$H(S_t) = \begin{bmatrix} b_L(U_t) & 0 \\ 0 & b_Y(U_t) \end{bmatrix}.$$

Based on some further assumptions, he successfully converted it into the one dimensional diffusion model in example (3.3.1). However, if apply our Theorem (3.2.2) to this model on the open set $G = \{(x_1, x_2) \in R^2 \mid n > u_0 =$

$c_Lx_1+c_Yx_2 > u^*, x_1 > 0, x_2 > 0\}$, we can conclude that $V(x_1,x_2) = E_{u_0}\left[e^{-z\tau_n}\right]$ satisfies the following partial differential equation

$$zV(x_1,x_2) - \lambda u V_{x_1} - \nu u V_{x_2} - \frac{1}{2}b_L^2(u)V_{x_1x_1} - \frac{1}{2}b_Y^2(u)V_{x_2x_2} = 0 \qquad (3.3.2)$$

on G. Note that $V(x_1,x_2)$ only depends on u_0, not the point (x_1,x_2). So if we put $\varphi_z(u) = V(x_1,x_2)$, where $u = c_Lx_1 + c_Yx_2$, then the equation (3.3.2) implies the equation (3.3.1).

Similarly, we can apply our Theorem (3.2.2) to Ren's multi-dimensional model [17] as well.

Example 3.3.3. ***Ren's six-dimensional Diffusion Model.***
Let $V(u_0) = E_{u_0}\left[e^{-z\tau_n}\right]$, then it satisfies the following partial differential equation:

$$\begin{aligned}
&zV(x) - \lambda x_6 V_{x_1} - \delta(x_1 - x_2)V_{x_2} - \lambda(1+\pi)x_6 V_{x_3} - \rho(x_3 - x_4)V_{x_4} \\
&- v(x_1 - x_2 + x_3 - x_4 + x_6)V_{x_5} - [c_Y v(x_1 - x_2 + x_3 - x_4 + x_6) \\
&+ c_R\rho(x_3 - x_4) - c_P\lambda(1+\pi)x_6 - c_L\lambda x_6]V_{x_6} - \frac{1}{2}(1+c_L^2)\sigma_L^2 V_{11} \\
&- \frac{1}{2}\sigma_D^2 V_{22} - \frac{1}{2}(1+c_R^2)\sigma_R^2 V_{33} - \frac{1}{2}(1+c_Y^2)\sigma_Y^2 V_{44} + c_Lc_R\sigma_L\sigma_R V_{13} \\
&+ c_Lc_Y\sigma_L\sigma_Y V_{14} - c_Yc_R\sigma_Y\sigma_R V_{34} = 0
\end{aligned}$$

on any open bounded domain such that $0 < u^* < u_0 < n$.

Remark 3.3.2. *Note that Ren obtained (2.4.1) by assuming that the ratios $\frac{x_1-x_2+x_3-x_4}{x_6} \to \gamma_1$, $\frac{x_3-x_4}{x_6} \to \gamma_2$, as the time $t \to \infty$. In turn, (2.4.1) holds only for large t in his paper.*

Although our theorem only applies to the bounded domain, it is good enough for industry practices if n is large enough. In some cases, see example 1, the conclusion can be extended to unbounded domains.

CHAPTER 4

TERRORISM RISK

4.1 Stochastic formulation

For terrorism combat, the choice of a Lanchester approach might seem somewhat ill-advised. The conflict is far from deterministic; terrain plays a major role; and the asymmetries of objectives (instilling fear vs. maintaining stability), information (surprise attacks vs. constant vigilance), and weaponry (suicide bombers, airplanes, etc. vs. a more conventional arsenal) are extreme. However, one crucial aspect of terrorist attacks tends to offset many of these apparent difficulties: the fact that such attacks are extremely localized in both space and time. These limitations-to both a small physical domain and a short time duration-tend to homogenize various complex characteristics of the problem, permitting more effective modeling.

Lemma 4.1.1. *If $0 \leq \alpha_1 < 1, 0 \leq \delta_2 < 1$, then the Lanchester equations are equivalent to the following system*

$$d\tilde{A} = -K_1 \tilde{D}^{\alpha} dt \qquad (4.1.1)$$

$$d\tilde{D} = -K_2 \tilde{A}^{\delta} dt \qquad (4.1.2)$$

where $\alpha > 0$ and $\delta > 0$ are constants.

Remark 4.1.1. *The Lanchester equations can be always reduced to a simpler form. If $\alpha_1 = 1$ or $\delta_2 = 1$, then the only difference is that $\tilde{D}^\alpha$ or $\tilde{A}^\delta$ will be replaced by an exponential function form.*

Proof: Divid (2.5.1) by A^{α_1} on both sides, then combine $A^{-\alpha_1}$ with dA, we have

$$dA^{1-\alpha_1} = -k_1(1-\alpha_1)D^{\delta_1}dt.$$

Similarly, we divid (2.5.2) by D^{δ_2} on both sides, then combine $D^{-\delta_2}$ with dD, we have

$$dD^{1-\delta_2} = -k_2(1-\delta_2)A^{\alpha_2}dt$$

Now let $\tilde{A} = A^{1-\alpha_1}$, $\tilde{D} = D^{1-\delta_2}$, $K_1 = k_1(1-\alpha_1)$ and $K_2 = k_2(1-\delta_2)$, then the above two equations can be rewritten as:

$$d\tilde{A} = -K_1\tilde{D}^\alpha dt$$

$$d\tilde{D} = -K_2\tilde{A}^\delta dt$$

where $\alpha = \frac{\delta_1}{1-\delta_2}$ and $\delta = \frac{\alpha_2}{1-\alpha_1}$.

Based on the lemma (4.1.1), We propose the following SDE model

$$d\tilde{A} = -K_1\tilde{D}^\alpha dt + \sigma_1(\tilde{A},\tilde{D})dZ_1(t) \quad (4.1.3)$$

$$d\tilde{D} = -K_2\tilde{A}^\delta dt + \sigma_2(\tilde{A},\tilde{D})dZ_2(t) \quad (4.1.4)$$

on the open set $S = \{(x,y) \in R^2 \mid x > 0, y > 0\}$, where $\sigma_1(\tilde{A},\tilde{D})$, $\sigma_2(\tilde{A},\tilde{D})$ are nonnegative continuous functions, $Z_1(t)$ and $Z_2(t)$ are standard Brownian motions. α and δ are parameters in $(0,\infty)$.

Let

$$T = \inf\{t \geq 0;\ \min\{\tilde{A}(t),\tilde{D}(t)\} \leq 0\}$$

be the first time that the stochastic process $\tilde{A}(t)$ or $\tilde{D}(t)$ exits from domain S and

$$\psi_D = Pr\{T < \infty,\ \tilde{A}(T) \leq 0,\ \tilde{D}(T) > 0\}$$

$$\psi_A = Pr\{T < \infty,\ \tilde{D}(T) \leq 0,\ \tilde{A}(T) > 0\}$$

be the probability that Defender or Attacker will win the combat respectively.

In this chapter, we will study the Laplace transform of the probability distribution of the first passage time T, ruin probability and the asymptotic behavior of the probabilty of target destruction.

4.2 Laplace transform of the PDF of first passage time

Let n be a positive integer. Set

$$D_n = \{(x, y) \in R^2 \mid\ x^2 + y^2 < n^2, x \geq 0, y \geq 0\}$$

$$\tau_n = \inf\{t \geq 0;\ U_t \in (D_n)^c\}$$

be the first time for the process U_t going out of the region D_n.

Theorem 4.2.1. *For the above Lanchester SDEs, $V^n(\tilde{A}_0, \tilde{D}_0) = E_{\tilde{A}_0,\tilde{D}_0}\left[e^{-z\tau_n}\right]$, satisfies the following partial differential equation in D_n:*

$$zV^n(x_1, x_2) + K_1 x_2^{\alpha} V^n_{x_1} + K_2 x_1^{\delta} V^n_{x_2} - \frac{1}{2}\sigma_1^2(x_1, x_2)V^n_{x_1x_1} - \frac{1}{2}\sigma_2^2(x_1, x_2)V^n_{x_2x_2} = 0, \tag{4.2.1}$$

subject to the boundary condition $V^n(y) = 1$ for $\forall y \in \partial D$. Furthermore, if let $V(\tilde{A}_0, \tilde{D}_0) = E_{\tilde{A}_0,\tilde{D}_0}\left[e^{-zT} \mid T < \infty\right]$, then $V^n(\tilde{A}_0, \tilde{D}_0) \to V(\tilde{A}_0, \tilde{D}_0)$ as $n \to \infty$.

Proof: Apply theorem (3.2.2), we have (4.2.1). By Dominated Convergence Theorem, we have $V^n(\tilde{A}_0, \tilde{D}_0) \to V(\tilde{A}_0, \tilde{D}_0)$ as $n \to \infty$.

Corollary 4.2.1. *Let* $u = \sqrt{K_2}x_1 + \sqrt{K_1}x_2$, *if we further assume that* $\sigma_1(x_1, x_2)$ *and* $\sigma_2(x_1, x_2)$ *are functions of* u, *and* $\alpha = \delta = 1$, *then the PDE (4.2.1) implies the folloing ODE:*

$$zV^n(u) + \sqrt{K_1K_2}u(V^n)'(u) - \frac{1}{2}\{K_2\sigma_1^2(u) + K_1\sigma_2^2(u)\}(V^n)''(u) = 0, \quad (4.2.2)$$

subject to the boundary condition above.

Proof: Note that

$$V_{x_1} = (V^n)'(u)\frac{\partial u}{\partial x_1} = \sqrt{K_2}(V^n)'(u),\ V_{x_2} = (V^n)'(u)\frac{\partial u}{\partial x_2} = \sqrt{K_1}(V^n)'(u),$$

$$V^n_{x_1x_1} = K_2(V^n)''(u),\ V^n_{x_2x_2} = K_1(V^n)''(u).$$

Plug in (4.2.1), we have

$$zV^n(u)+K_1\sqrt{K_2}x_2(V^n)'(u)+K_2\sqrt{K_1}x_1(V^n)'(u)-\frac{1}{2}\{K_2\sigma_1^2(u) + K_1\sigma_2^2(u)\}(V^n)''(u) = 0$$

Combination of the second and third terms gives us (4.2.2).

4.3 Ruin is for certain

In this section, it is shown that the ruin is certain almost surely.

We rewrite the above stochastic differential equations as follow:

$$dU_t = GV_tdt + HdZ_t \quad (4.3.1)$$

where

$$U_t = \left[\tilde{A}(t), \tilde{D}(t)\right]^\top,\ \ V_t = \left[\tilde{A}^\delta(t), \tilde{D}^\alpha(t)\right]^\top,\ \ dZ_t = [dZ_1(t),\ dZ_2(t)]^\top$$

$$G = \begin{bmatrix} 0 & -K_1 \\ -K_2 & 0 \end{bmatrix},$$

$$H = \begin{bmatrix} \sigma_1(\cdot) & 0 \\ 0 & \sigma_2(\cdot) \end{bmatrix}.$$

Recall that

$$D_n = \{(x, y) \in R^2 \mid \ x^2 + y^2 < n^2, x \geq 0, y \geq 0\},$$

$$\tau_n = \inf\{t \geq 0;\ U_t \in (D_n)^c\}.$$

Lemma 4.3.1. *Let $\theta = [\theta_1, \theta_2]^\top$, where θ_1, θ_2 are real numbers. Set*

$$X_t = U_t - U_0 - \int_0^t GV_s\, ds,$$

and

$$Y_t = exp(-\theta^\top X_t - \frac{1}{2}\langle -\theta^\top X\rangle_t).$$

Then $Y_{t\wedge\tau_n}$ is a L^2-martingale which implies that $E[Y_{\tau_n}] \leq 1$.

Proof. Integrating SDE (4.3.1), we have

$$U_t = U_0 + \int_0^t GV_s\, ds + \int_0^t H\, dZ_s. \tag{4.3.2}$$

Then

$$X_t = U_t - U_0 - \int_0^t GV_s\, ds = \int_0^t H\, dZ_s$$

and so

$$\begin{aligned} Y_t &= \exp(-\theta^\top X_t - \frac{1}{2}\langle -\theta^\top X\rangle_t) \\ &= \exp(-\theta^\top U_t + \theta^\top U_0 + \int_0^t \theta^\top GV_s\, ds - \frac{1}{2}\int_0^t (\theta^\top H)^\top(\theta^\top H)\, ds) \end{aligned}$$

is a local martingale. The expectation of L^2-norm of $X_{t\wedge\tau_n}$ can be computed as follows:

$$E\|X_{t\wedge\tau_n}\|_{L^2}^2 = ||E\left(\int_0^{t\wedge\tau_n} H\, dZ_s\right)^2||_{L^1} = E\int_0^{t\wedge\tau_n} ||H^\top H||_{L^1}\, ds. \tag{4.3.3}$$

Note that $||U_{t\wedge\tau_n}||_\infty$ is bounded by n and that the function $\sigma_1(x), \sigma_2(x)$ are continuous. It follows that σ_1^2, σ_2^2 are bounded for $0 \leq s \leq t \wedge \tau_n$. Hence the integral on the right hand side of (4.3.3) is bounded for each t.

Next, since $\sigma_1(x), \sigma_2(x)$ are bounded for $0 \le s \le t \wedge \tau_n$, moreover, $t \wedge \tau_n \le t$, we have

$$\begin{aligned}||X_{t\wedge\tau_n}||_\infty &= ||U_{t\wedge\tau_n} - U_0 - \int_0^{t\wedge\tau_n} GV_s\, ds||_\infty \\ &\le ||U_{t\wedge\tau_n}||_\infty + ||U_0||_\infty + ||\int_0^t GV_s\, ds||_\infty \\ &\le n + \max\{\tilde{A}_0, \tilde{D}_0\} + \max\{K_1, K_2\} t n^{\alpha+\delta}\end{aligned}$$

for each t. So $|Y_{t\wedge\tau_n}| \le c(t,n)$, where $c(t,n)$ is a constant depending on t and n. It now follows that $Y_{t\wedge\tau_n}$ is a L^2-martingale.

It then follows that $1 = E[Y_0] = E[Y_{t\wedge\tau_n}]$, for each $t \ge 0$. Hence

$$\lim_{t\to\infty} E[Y_{t\wedge\tau_n}] = 1.$$

On the other hand, it follows from Fatou's lemma that

$$E[Y_{\tau_n}] \le \lim_{t\to\infty} E[Y_{t\wedge\tau_n}] = 1.$$

Theorem 4.3.1. *Let G and H be defined as in SDE (4.3.1). Suppose that $\sigma_1(\cdot)$ and $\sigma_2(\cdot)$ are functions of $\tilde{A}_t^\delta$ and $\tilde{D}_t^\alpha$, denote $g_i(x,y) = {\sigma_i}^2(\cdot), i = 1,\ 2.$ If $|g_i(x,y)/y| \le C,\ \forall\ x,y > 0,\ \ i = 1,\ 2.$ Then we have*
i) $Pr\{||U_{\tau_n}|| = n\} \le exp(-\frac{K_1}{2C}(\delta_n - \max(\tilde{A}_0, \tilde{D}_0)))$ for large n,
ii) $Pr_{\tilde{A}_0,\tilde{D}_0}\{T < \infty\} = 1$,
where $\delta_n = \frac{n-\sqrt{\tilde{A}_0^2+\tilde{D}_0^2}}{\sqrt{2}}$.

Proof. It follows from Lemma (4.3.1) that

$$E_{\tilde{A}_0,\tilde{D}_0}\left[\exp(-\theta^\top U_{\tau_n} + \theta^\top U_0 + \int_0^{\tau_n}\{\theta^\top GV_s - \frac{1}{2}(\theta^\top H)^\top(\theta^\top H)\}\, ds)\right] = E[Y_{\tau_n}] \le 1.$$

Notes that

$$\begin{aligned}1 \ge& E[Y_{\tau_n}] \\ =& Pr\{\tilde{A}_{\tau_n} \le 0\} \cdot r \cdot E[\mathcal{M} \mid \tilde{A}_{\tau_n} \le 0,\ \tilde{D}_{\tau_n} > 0] \\ &+ Pr\{\tilde{D}_{\tau_n} \le 0\} \cdot r \cdot E[\mathcal{M} \mid \tilde{A}_{\tau_n} > 0,\ \tilde{D}_{\tau_n} \le 0\,] \\ &+ Pr\{|U_{\tau_n}| = n\} \cdot r \cdot E[\mathcal{M} \mid |U_{\tau_n}| = n\,]\end{aligned}$$

where $r = e^{\theta^\top U_0}$, and where

$$\mathcal{M} = \exp(-\theta^\top U_{\tau_n} + \int_0^{\tau_n} \{\theta^\top G V_s - \frac{1}{2}(\theta^\top H)^\top(\theta^\top H)\}\, ds)$$

and since all terms are nonegative, we have

$$Pr\{|U_{\tau_n}| = n\} \cdot r \cdot E\left[\mathcal{M} \mid |U_{\tau_n}| = n\right] \leq 1 \tag{4.3.4}$$

Let

$$\begin{aligned} F(x,y) = F(\tilde{A}^\delta, \tilde{D}^\alpha) &= \theta^\top G V_s - \frac{1}{2}(\theta^\top H)^\top(\theta^\top H) \\ &= -\theta_1 K_1 y - \theta_2 K_2 x - \frac{1}{2}\theta_1^2 g_1(x,y) - \frac{1}{2}\theta_2^2 g_2(x,y) \\ &= -\theta_2 K_2 x + y\{-\theta_1 K_1 - \frac{1}{2}\theta_1^2 g_1(x,y)/y - \frac{1}{2}\theta_2^2 g_2(x,y)/y\}, \end{aligned}$$

if we pick $\theta_1 = \theta_2 = -\frac{K_1}{2C}$, then

$$F(x,y) \geq -\theta_2 K_2 x + y\{-\theta_1 K_1 - \frac{1}{2}(\theta_1^2 + \theta_2^2)C\} \geq -\theta_2 K_2 x - \frac{1}{2}\theta_1 K_1 y \geq 0,$$

for any $x \geq 0, y \geq 0$.

Now pick n such that $\delta_n > max(\tilde{A}_0, \tilde{D}_0)$ in D_n. Denote the part of ∂D_n in the 1st quadrant by C_n, let $A_1 = \{C_n | \tilde{A}_{\tau_n} - \tilde{A}_0 > \delta_n\}$, $A_2 = \{C_n | \tilde{D}_{\tau_n} - \tilde{D}_0 > \delta_n\}$, since (4.3.4) holds for any $\theta = [\theta_1, \theta_2]^\top$, especially holds for $\theta_1 = \theta_2 = -\frac{K_1}{2C}$. Hence we have

$$\begin{aligned} K(\theta) := E &\left[\exp(\theta^\top U_0 - \theta^\top U_{\tau_n} + \int_0^{\tau_n} F(x,y)\, ds) \,\middle|\, |U_{\tau_n}| = n\right] \\ = E &\left[1_{A_1}\exp(\theta^\top U_0 - \theta^\top U_{\tau_n} + \int_0^{\tau_n} F(x,y)\, ds) \,\middle|\, |U_{\tau_n}| = n\right] \\ + E &\left[1_{A_2 - A_1}\exp(\theta^\top U_0 - \theta^\top U_{\tau_n} + \int_0^{\tau_n} F(x,y)\, ds) \,\middle|\, |U_{\tau_n}| = n\right]. \\ \geq E &\left[1_{A_1} e^{\frac{K_1}{2C}(\delta_n - \tilde{D}_0)} + 1_{A_2 - A_1} e^{\frac{K_1}{2C}(\delta_n - \tilde{A}_0)} | \; |U_{\tau_n}| = n\right] \\ \geq e^{\frac{K_1}{2C}(\delta_n - \max(\tilde{A}_0, \tilde{D}_0))} &E\left[1_{A_1} + 1_{A_2 - A_1} | \; |U_{\tau_n}| = n\right] \geq e^{\frac{K_1}{2C}(\delta_n - \max(\tilde{A}_0, \tilde{D}_0))}. \end{aligned}$$

Hence

$$Pr\{|U_{\tau_n}| = n\} \leq \frac{1}{K(\theta)} \leq e^{-\frac{K_1}{2C}(\delta_n - \max(\tilde{A}_0, \tilde{D}_0))}$$

for large n, which is part i). Now let $n \to \infty$, we have

$$\lim_{n\to\infty} Pr\{|U_{\tau_n}| = n\} = 0.$$

and since

$$Pr_{\tilde{A}_0,\tilde{D}_0}\{T < \infty\} < Pr_{\tilde{A}_0,\tilde{D}_0}\{|U_{\tau_n}| = n\},\ \forall\, n.$$

Therefore

$$Pr_{\tilde{A}_0,\tilde{D}_0}\{T < \infty\} = 1 - Pr_{\tilde{A}_0,\tilde{D}_0}\{T = \infty\} \geq 1 - \lim_{n\to\infty} Pr\{|U_{\tau_n}| = n\} = 1.$$

Remark 4.3.1. *The above theorem shows that $Pr\{|U_{\tau_n}| = n\}$ exponentially decays, and the ruin probability for terrorism risk is equal to* 1*. That is, the terrorism combat will end within finite time.*

4.4 Asymptotical behavior of ruin probability

In this section, by using martingale approach, we obtain an upper bound on the probability of ruin. Our result shows that the probability of ruin of each side exponentially decay as the initial $\tilde{A}_0$ or $(\tilde{D}_0) \to \infty$.

Theorem 4.4.1. *Let G and H be defined as in SDE (4.3.1). Suppose that $\sigma_1(\cdot)$ and $\sigma_2(\cdot)$ are functions of $\tilde{A}_t^\delta$ and $\tilde{D}_t^\alpha$, denote $g_i(x,y) = {\sigma_i}^2(\cdot), i = 1,\ 2,$then*
1a) If $|g_i(x,y)| \leq Cmin(1,y),\ \forall\ x,y > 0,\ \ i = 1,\ 2,\ \delta > \alpha,$ then there exist $\theta_1 > 0, \theta_2 < 0$, such that

$$\psi_D \leq e^{\frac{-\theta_1\tilde{A}_0}{2} - \theta_2\tilde{D}_0}(1 + e^{\frac{-\theta_1\tilde{A}_0}{2}}) \tag{4.4.1}$$

for large $\tilde{A}_0$ and fixed $\tilde{D}_0 > 0$.
1b)If $|g_i(x,y)| \leq Cmin(1,y),\ \forall\ x,y > 0,\ \ i = 1,\ 2,\ \delta \leq \alpha,$ then there exist $\theta_1 > 0, \theta_2 < 0$, such that

$$\psi_D \leq e^{\frac{-\theta_1\tilde{A}_0}{2} - \theta_2\tilde{D}_0} + e^{-\lambda\tilde{A}_0^{\delta+\frac{\delta}{\alpha}}}. \tag{4.4.2}$$

for large $\tilde{A}_0$ *and fixed* $\tilde{D}_0 > 0$. *Where* $\lambda = \frac{K_2}{CC_0 2^{\delta+\frac{\delta}{\alpha}-1}} > 0$, *and* $C_0 = (\frac{-\theta_2 K_2}{2\theta_1 K_1})^{\frac{-1}{\alpha}}$.
2a) If $|g_i(x,y)| \le Cmin(1,x)$, $\forall\ x,y > 0$, $\ i = 1,\ 2,\ \delta < \alpha$, *then there exist* $\theta_1 < 0, \theta_2 > 0$, *such that*

$$\psi_A \le e^{\frac{-\theta_2 \tilde{D}_0}{2} - \theta_1 \tilde{A}_0}(1 + e^{\frac{-\theta_2 \tilde{D}_0}{2}}) \tag{4.4.3}$$

for large $\tilde{D}_0$ *and fixed* $\tilde{A}_0 > 0$.
2b)If $|g_i(x,y)| \le Cmin(1,y)$, $\forall\ x,y > 0$, $\ i = 1,\ 2,\ \delta \ge \alpha$, *then there exist* $\theta_1 < 0, \theta_2 > 0$, *such that*

$$\psi_A \le e^{\frac{-\theta_2 \tilde{D}_0}{2} - \theta_1 \tilde{A}_0} + e^{-\lambda \tilde{D}_0^{\delta+\frac{\delta}{\alpha}}}. \tag{4.4.4}$$

for large $\tilde{D}_0$ *and fixed* $\tilde{A}_0 > 0$. *Where* $\lambda = \frac{K_1}{CC_0 2^{\alpha+\frac{\alpha}{\delta}-1}} > 0$, *and* $C_0 = (\frac{-\theta_1 K_1}{2\theta_2 K_2})^{\frac{-1}{\delta}}$.

Proof. Part 1). By the above assumption, we have

$$\begin{aligned} F(x,y) &= -\theta_1 K_1 y - \theta_2 K_2 x - \frac{1}{2}\theta_1^2 g_1(x,y) - \frac{1}{2}\theta_2^2 g_2(x,y) \\ &\ge -\theta_1 K_1 y - \theta_2 K_2 x - \frac{1}{2}(\theta_1^2 + \theta_2^2)C. \end{aligned}$$

Denote $L_1(x,y) = -\theta_1 K_1 y - \theta_2 K_2 x$ and $L_2(x,y) = -\theta_1 K_1 y - \theta_2 K_2 x - \frac{1}{2}(\theta_1^2 + \theta_2^2)C$. Then we have $L_2(x,y) \le F(x,y) \le L_1(x,y)$, hence the curve $F(x,y) = 0$ will be governed by the curves $L_1(x,y) = 0$ and $L_2(x,y) = 0$. Pick $\theta_1 > 0, \theta_2 < 0$, we will divide into two cases to prove the theorem.

Case I, assume $\delta > \alpha$.
Let $A_1 = \{(\tilde{A}, \tilde{D}) \mid \tilde{A} = \frac{\tilde{A}_0}{2}, 0 \le \tilde{D} \le \frac{\theta_1 \tilde{A}_0}{-2\theta_2}\}$, $A_2 = \{(\tilde{A}, \tilde{D}) \mid \theta_1 \tilde{A} + \theta_2 \tilde{D} = 0,\ n \ge \tilde{A} \ge \frac{\tilde{A}_0}{2}\}$, $A_3 = \{(\tilde{A}, \tilde{D}) \mid \tilde{A} = n, 0 \le \tilde{D} \le \frac{\theta_1 n}{-\theta_2}\}$, $A_4 = \{(\tilde{A}, \tilde{D}) \mid \tilde{A} \ge \frac{\tilde{A}_0}{2}, \tilde{D} = 0\}$ and Let E_n be the region bounded by A_1, A_2, A_3 and A_4 (See figure 1). Define

$$\nu_n = \inf\{t \ge 0;\ U_t \in (E_n)^c\},$$

$$F^+ := \{(\tilde{A}, \tilde{D}) \mid F(x,y) > 0, \tilde{A} \ge 0, \tilde{D} \ge 0\}.$$

Then $E_n \subseteq F^+$ if $\tilde{A}_0$ is large enough. We will assume $E_n \subseteq F^+$ from now on. A similar argument as Lemma (4.3.1) will yield that $E[Y_{\nu_n}] \le 1$. That is,

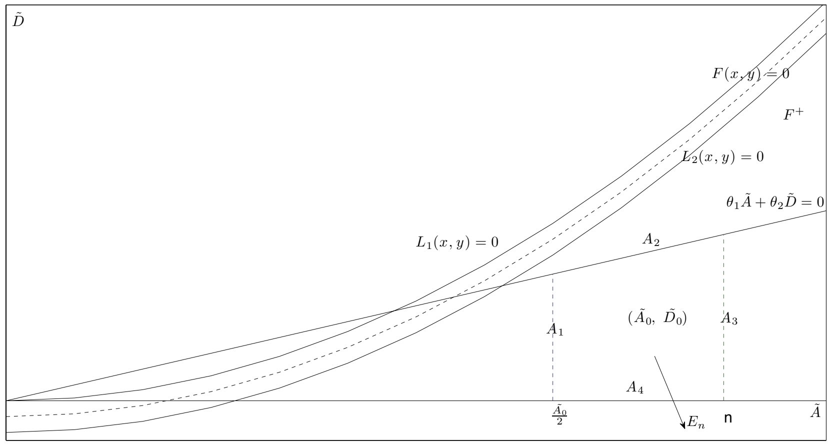

Figure 4.1: Case I—Ruin probability

$$
\begin{aligned}
1 \geq\ & E_{\tilde{A}_0,\tilde{D}_0}\left[\exp(-\theta^\top U_{\nu_n} + \theta^\top U_0 + \int_0^{\nu_n} F(x,y)\ ds)\right] \\
=& Pr\{U_t \text{ hits } A_1\} e^{\theta_1\tilde{A}_0+\theta_2\tilde{D}_0} E\left[\exp(-\theta^\top U_{\nu_n} + \int_0^{\nu_n} F(x,y)\ ds)\ \middle|\ U_t \text{ hits } A_1\right] \\
&+ Pr\{U_t \text{ hits } A_2\} e^{\theta_1\tilde{A}_0+\theta_2\tilde{D}_0} E\left[\exp(-\theta^\top U_{\nu_n} + \int_0^{\nu_n} F(x,y)\ ds)\ \middle|\ U_t \text{ hits } A_2\right] \\
&+ Pr\{U_t \text{ hits } A_3\} e^{\theta_1\tilde{A}_0+\theta_2\tilde{D}_0} E\left[\exp(-\theta^\top U_{\nu_n} + \int_0^{\nu_n} F(x,y)\ ds)\ \middle|\ U_t \text{ hits } A_3\right] \\
&+ Pr\{U_t \text{ hits } A_4\} e^{\theta_1\tilde{A}_0+\theta_2\tilde{D}_0} E\left[\exp(-\theta^\top U_{\nu_n} + \int_0^{\nu_n} F(x,y)\ ds)\ \middle|\ U_t \text{ hits } A_4\right].
\end{aligned}
$$

Notes that

$$
E\left[\exp(-\theta^\top U_{\nu_n} + \int_0^{\nu_n} F(x,y)\ ds)\ \middle|\ U_t \text{ hits } A_1\right] \geq e^{\frac{-\theta_1\tilde{A}_0}{2}},
$$

and

$$
\begin{aligned}
& E\left[\exp(-\theta^\top U_{\nu_n} + \int_0^{\nu_n} F(x,y)\ ds)\ \middle|\ U_t \text{ hits } A_2\right] \\
&= E\left[\exp\int_0^{\nu_n} F(x,y)\ ds\ \middle|\ U_t \text{ hits } A_2\right] \\
&\geq 1.
\end{aligned}
$$

We have

$$Pr\{U_t \text{ hits } A_1\} \leq e^{\frac{-\theta_1 \tilde{A}_0}{2} - \theta_2 \tilde{D}_0},$$

and

$$Pr\{U_t \text{ hits } A_2\} \leq e^{-\theta_1 \tilde{A}_0 - \theta_2 \tilde{D}_0}.$$

Let $n \to \infty$, then the above inequality still holds, we have

$$Pr\{U_t \text{ hits } \hat{A}_2\} \leq e^{-\theta_1 \tilde{A}_0 - \theta_2 \tilde{D}_0},$$

where $\hat{A}_2 = \{(\tilde{A}, \tilde{D}) \mid \theta_1 \tilde{A} + \theta_2 \tilde{D} = 0,\ \tilde{A} \geq \frac{\tilde{A}_0}{2}\}$.
Since U_t has to hits either A_1 or $\hat{A}_2$ first before it hits $\tilde{D}$-axis, hence

$$\psi_D \leq Pr\{U_t \text{ hits } A_1\} + Pr\{U_t \text{ hits } \hat{A}_2\} \leq e^{\frac{-\theta_1 \tilde{A}_0}{2} - \theta_2 \tilde{D}_0}(1 + e^{\frac{-\theta_1 \tilde{A}_0}{2}}). \quad (4.4.5)$$

Case II, assume $\delta \leq \alpha$.
Let E_n denote the region in 1^{st} quadrant surrounded by
$\tilde{A} = \frac{\tilde{A}_0}{2}$, $L_2(x, y) = 0$, $\tilde{A}^2 + \tilde{D}^2 = 1$ and $q = 0$. Let A_1, A_2, A_3, A_4 denote the boundary of E_n corresponding to the four curves (see figure 2). Let ν_n and F^+ be defined as case I, then $E_n \subseteq F^+$.

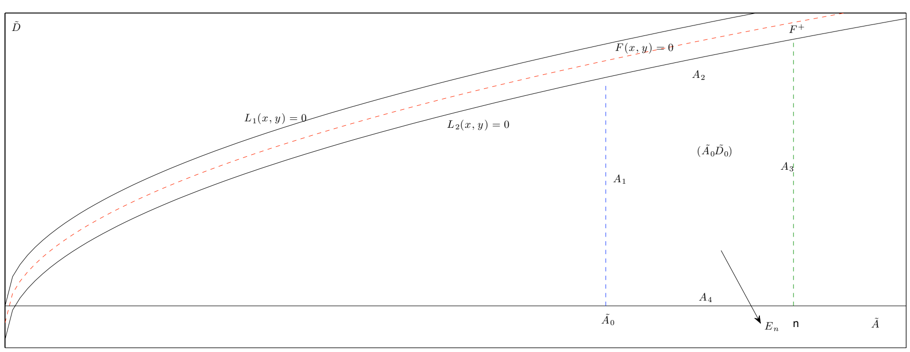

Figure 4.2: Case II—Ruin probability

Similarly to case I, we have

$$Pr\{U_t \text{ hits } A_1\} \leq e^{\frac{-\theta_1 \tilde{A}_0}{2} - \theta_2 \tilde{D}_0},$$

and

$$Pr\{U_t \text{ hits } A_2\}E\left[\exp(\theta^\top U_0 - \theta^\top U_{\nu_n} + \int_0^{\nu_n} F(x,y)\,ds)\;\middle|\; U_t \text{ hits } A_2\right] \leq 1.$$

However, since the above inequality holds for any θ_1 and θ_2, we have

Lemma 4.4.1. *For fixed* $\theta_1,\ \theta_2,\ Pr\{U_t$ *hits* $\hat{A}_2\} \leq e^{-\lambda \tilde{A}_0^{\delta+\frac{\delta}{\alpha}}}$ *for large* $\tilde{A}_0$*, where* $\hat{A}_2 = \{(\tilde{A},\tilde{D}) \mid -\theta_1 K_1 y - \theta_2 K_2 x - \frac{1}{2}(\theta_1^2+\theta_2^2)C = 0,\ \tilde{A} \geq \frac{\tilde{A}_0}{2}\}$.

We will prove this lemma later.

Since U_t has to hits either A_1 or $\hat{A}_2$ first before it hits $\tilde{D}$-axis, hence

$$\psi_D \leq Pr\{U_t \text{ hits } A_1\} + Pr\{U_t \text{ hits } \hat{A}_2\} \leq e^{\frac{-\theta_1\tilde{A}_0}{2} - \theta_2\tilde{D}_0} + e^{-\lambda\tilde{A}_0^{\delta+\frac{\delta}{\alpha}}}. \quad (4.4.6)$$

Now let's complete the proof of the above lemma.
Fixed $\hat{\theta}_1 = -4\lambda\tilde{A}_0^{\delta+\frac{\delta}{\alpha}-1} < 0$ and $\hat{\theta}_2 = \hat{\theta}_1 2^{\frac{\delta}{\alpha}} C_0 \tilde{A}_0^{1-\frac{\delta}{\alpha}} < 0$, then we have
a) $\tilde{D} = (\frac{-\theta_2 K_2 \tilde{A}^\delta - (\theta_1^2+\theta_2^2)C/2}{\theta_1 K_1})^{\frac{1}{\alpha}} \geq \frac{1}{C_0}\tilde{A}^{\frac{\delta}{\alpha}} \geq \frac{1}{C_0}2^{-\frac{\delta}{\alpha}}\tilde{A}_0^{\frac{\delta}{\alpha}}$, on A_2;
b) $\hat{F}(x,y) \geq -\hat{\theta}_1 K_1 y - \hat{\theta}_2 K_2 x - \frac{1}{2}(\hat{\theta}_1^{\,2} + \hat{\theta}_2^{\,2})C \geq -\hat{\theta}_2 K_2 x - \frac{1}{2}(\hat{\theta}_1^{\,2}+\hat{\theta}_2^{\,2})C \geq 0$, on E_n;
c)$-\hat{\theta}_2(\tilde{D}_{\nu_n} - \tilde{D}_0) \geq \hat{\theta}_2(\frac{1}{C_0}2^{\frac{-\delta}{\alpha}}\tilde{A}_0^{\frac{\delta}{\alpha}} - \tilde{D}_0) \geq -\hat{\theta}_1\tilde{A}_0 + \hat{\theta}_2\tilde{D}_0$, on E_n;
d)$-\hat{\theta}_1(\tilde{A}_{\nu_n} - \tilde{A}_0) \geq -\hat{\theta}_1\frac{-\tilde{A}_0}{2}$. on E_n;
for large $\tilde{A}_0$.
Notes that $\hat{F}(x,y)$ denotes the integrand associated with $\hat{\theta}_1, \hat{\theta}_2$. Hence for large $\tilde{A}_0$, we have

$$-\hat{\theta}_2(\tilde{D}_{\nu_n} - \tilde{D}_0) - \hat{\theta}_1(\tilde{A}_{\nu_n} - \tilde{A}_0) \geq -\hat{\theta}_1\frac{\tilde{A}_0}{2} + \hat{\theta}_2\tilde{D}_0 \geq -\hat{\theta}_1\frac{\tilde{A}_0}{4}.$$

The second inequality holds because $-\hat{\theta}_1\frac{\tilde{A}_0}{2}$ dominated the summation

$$-\hat{\theta}_1\frac{\tilde{A}_0}{2} + \hat{\theta}_2\tilde{D}_0.$$

Therefore on E_n, we have

$$Pr\{U_t \text{ hits } A_2\} \leq e^{\hat{\theta}_1\tilde{A}_0/4}.$$

Let $n \to \infty$, we have

$$Pr\{U_t \text{ hits } \hat{A}_2\} \leq e^{\hat{\theta}_1 \tilde{A}_0/4} \leq e^{-\lambda \tilde{A}_0^{\delta+\frac{\delta}{\alpha}}}.$$

The lemma follows.

Proof of Part 2). By symmetry of $\tilde{A}$ and $\tilde{D}$, Part ii) is also true.

Case I: Assume $\delta < \alpha$. Pick $\theta_1 < 0,\ \theta_2 > 0$. The proof of this case is similar to case I in Part i):

Case II: Assume $\delta \geq \alpha$. Pick $\theta_1 < 0,\ \theta_2 > 0$. The proof of this case is similar to case II in Part i).

Remark 4.4.1. *$g_i(x, y)$ can be functions like $C \arctan ay \arctan bx,\ Cy \arctan(ax/y)$, $Cye^{-ay} \arctan bx,\ Cxe^{-ax} \arctan by$ and so on, where $C > 0, a > 0, b > 0$.*

Remark 4.4.2. *If there exists a $0 < \gamma < \delta$ such that $g_i(x, y) \leq C\tilde{A}^\gamma$, or $g_i(x, y) \leq C\tilde{A}^\delta$, but $C < \frac{2\theta_2 K_2}{\theta_1^2+\theta_2^2}$, the theorem still holds. The idea is that $-\theta_2 K_2 \tilde{A}^\delta$ has to dominate $-\theta_2 K_2 \tilde{A}^\delta - \frac{1}{2}(\theta_1^2 g_1(x, y) + \theta_2^2 g_2(x, y))$ for large $\tilde{A} > 0$.*

CHAPTER 5

THE CRAMER LUNDBERG MODEL WITH RISKY INVESTMENTS

In this chapter, we consider the same model as that in [14]. In the case of $\rho := 2a/\sigma^2 > 1$, we provided an upper bound for the ruin probability. In the case of large volatility, i.e. $\rho := 2a/\sigma^2 \leq 1$. We combine a martingale argument and a reduction argument to prove that the ruin probability is equal to 1 without any assumption on the distribution of the claim size as long as it is not identically zero.

5.1 Cramer Lundberg model with risky investments

When an insurance company invests in a risky asset whose price follows a geometric Brownian motion, the risk process is given by

$$X_t = X_0 + \int_0^t aX_s ds + \int_0^t \sigma X_s dW_s + \int_0^t c_s ds - \sum_{j=1}^{N(t)} \xi_j, \tag{5.1.1}$$

or

$$dX_t = (aX_t + c_t)dt + \sigma X_t dW_t - dP_t, \tag{5.1.2}$$

where W_t is the Wiener process (standard Brownian motion), $N(t)$ is a Poisson process with intensity λ, and the claim sizes ξ_i; $i = 1, 2, 3, ...,$ are independent, identically distributed positive random variables, having the density function $p(x)$, with positive mean μ and finite variance. Moreover, we assume that W_t, $N(t)$, ξ_i are independent and the filtration is defined as $\mathcal{F}_t = \sigma\{W_s, N_s, \sum_{i=1}^{N_s} \xi_i, 0 \leq s \leq t\}$. Furthermore, $c_t = c(t, X)$ is a bounded nonnegative $(\mathcal{F}_t)$-adapted process (i.e. $0 \leq c_t \leq c$) such that (5.1.1) has a unique strong solution, see e.g., Chapter 14 [11]. X_0 is the initial capital and $P_t = \sum_{j=1}^{N(t)} \xi_j$. The capital X_t is continuously invested in a risky asset, with relative price increments $dX_t = aX_t dt + \sigma X_t dW_t$, where $a > 0$ and $\sigma > 0$ are the drift and volatility of the returns of the asset.

We will assume that the claim size is bounded by a constant $M > 0$ throughout the entire section. In insurance, M can be understood as the limit or cap of a policy. We will drop this assumption in the next section. Let $T_{u^*} = \inf\{t > 0;\ X_t < u^*\}$ be the first time that $X_t < u^*$, and let

$$\psi_{u^*}(u) = P(T_{u^*} < \infty\ |X_0 = u)$$

be the probability of ruin at level u^*, where $0 \leq u^* < u$. If $u^* = 0$, we denote the probability of ruin by $\psi(u)$. We will discuss the probability of ruin on the Cramér-Lundberg model with investments based on (1) $\rho = 1$ and (2) $\rho < 1$. We first prove the following

Lemma 5.1.1. *Let X_t be a stochastic process that satisfies (5.1.2). If $c_t = c \geq 0$ is a constant for all t and $0 \leq v \leq u$, then*

$$\psi(v) \geq \psi(u).$$

Proof. We first derive a closed form of the strong solution for (5.1.2). Let $Y_t = \exp\{(\frac{\sigma^2}{2} - a)t - \sigma W_t\}$. By Itô's formula [10], $dX_tY_t = X_t dY_t +$

$Y_t dX_t + dX_t dY_t$, and simple calculation yields $dX_t Y_t = dV_t^u$, where $V_t^u = u + \int_0^t Y_s c_s\, ds - \int_0^t Y_s\, dP_s$. Integrating both sides, we have $X_t Y_t = V_t^u$. Hence

$$X_t = Y_t^{-1} V_t^u \tag{5.1.3}$$

is a strong solution of (5.1.1) and (5.1.2) with initial condition $X_0 = u$.

Now suppose $c_t = c \geq 0$ is a constant for all t. Let $Z_t = Y_t^{-1} V_t^v$, then $Z_t \leq X_t$, $\forall\, t \geq 0$, since $0 \leq v \leq u$. Hence

$$\begin{aligned} \psi(u) &= P(X_t < 0, \text{ for some } \ 0 < t < \infty\, | X_0 = u) \\ &\leq P(Z_t < 0, \text{ for some } \ 0 < t < \infty\, | Z_0 = v). \end{aligned}$$

Note that Z_t also satisfies (5.1.2) with initial condition $Z_0 = v$. Hence

$$P(Z_t < 0, \text{ for some } \ 0 < t < \infty\, | Z_0 = v) = \psi(v).$$

Therefore

$$\psi(v) \geq \psi(u).$$

Our main tool is Itô's formula for semimartingales with a jump part. Let $t_1 < t_2 < t_3 < ...$ be the times where the Poisson process $N(t)$ has a jump discontinuity. Then the jump discontinuities for P_t are also at t_i with jump size ξ_i. Following the notations on P. 43 [10], for $t > 0$, and a Borel subset U of R, we let

$$N_p((0,t] \times U) = \sharp\{i; t_i \leq t, \xi_i \in U\}.$$

Then $N_p((0,t] \times U)$ defines a random measure $N_p(dtdx)$ on the Borel σ-algebra on $[0,\infty) \times R$. Note that

$$N_p(dtdx) = \sum_{i=1}^{\infty} \delta_{t_i}(dt)\delta_{\xi_i}(dx), \tag{5.1.4}$$

where δ_{t_i} is the Dirac δ-function centered at t_i (probability measure concentrated at one point t_i). It follows that

$$\int_0^t \int_0^{\infty} f(s,x) N_p(dsdx) = \sum_{i; t_i \leq t} f(t_i, \xi_i), \tag{5.1.5}$$

and therefore

$$\int_0^t \int_0^\infty x N_p(dsdx) = \sum_{i;t_i \le t} \xi_i = P_t. \tag{5.1.6}$$

It is well-known, see e.g. P. 60 and P. 65 [10], that there exists a continuous process $\hat{N}_p((0,t] \times U)$ such that

$$\tilde{N}_p((0,t] \times U) = N_p((0,t] \times U) - \hat{N}_p((0,t] \times U), \tag{5.1.7}$$

is a martingale. In our case

$$\hat{N}_p((0,t] \times U) = E[N_p((0,t] \times U)].$$

$E[N_p((0,t] \times U)]$ defines a measure, $n_p(dtdx)$, called the mean (intensity) measure of $N_p(dtdx)$ and it is given by $n_p(dtdx) = \lambda p(x) dtdx$.

Assume that $c_t = c$ is a constant, then equation (5.1.1) can be written as

$$X_t = X_0 + \int_0^t a X_s ds + \int_0^t \sigma X_s dW_s + ct - \int_0^t \int_0^\infty x N_p(dsdx). \tag{5.1.8}$$

By (5.1.3), equation (5.1.8) has a strong solution for each fixed initial condition (see Chapter 14 in [11]) and it is a semimartingale by Definition 4.1, P. 64 [10].

By (5.1.3) and direct calculation, we have

$$X_{t+s} = \bar{Y}_t^{-1} X_s + \bar{Y}_t^{-1} \int_0^t c \bar{Y}_u du - \bar{Y}_t^{-1} \int_0^t \bar{Y}_u d\bar{P}_u, \tag{5.1.9}$$

where

$$\bar{Y}_t = e^{-(a-\frac{\sigma^2}{2})t - \sigma \bar{W}_t}, \tag{5.1.10}$$

$$\bar{W}_t = W_{t+s} - W_s, \tag{5.1.11}$$

$$\bar{P}_t = P_{t+s} - P_s. \tag{5.1.12}$$

Note that $\bar{W}_t$ and $\bar{P}_t$ are independent of $\{X_v; 0 \le v \le s\}$ and therefore given $\{X_v; 0 \le v \le s\}$, X_{t+s} depends on X_s only. This implies that X_t is a Markov process. Moreover, since $\bar{W}_t = W_{t+s} - W_s$ and W_t have the same distribution, and $\bar{P}_t = P_{t+s} - P_s$ and P_t have the same distribution, we have

$$P(X_{t+s} \in U | X_s = x) = P(X_t \in U | X_0 = x), \tag{5.1.13}$$

for all $t > 0$, and all Borel sets U. Therefore, $X_t, t \geq 0$ is a Markov process with a stationary transition function. By (5.1.3) and the Dominated Convergence Theorem, $X_t, t \geq 0$ is a Feller process (see e.g. P. 52 [6]). Moreover, since the sample paths of X_t are right continuous with left limits, $X_t, t \geq 0$ is a strong Markov process, see e.g. Theorem 3.10 [6].

5.2 An upper bound for ruin probability when $\rho > 1$

From now on, we assume $c_t = c$ throughout the chapter unless otherwise specified. In the following lemma, we first prove that X_t exits from any finite interval $[0, n)$ with probability one. This result will be used in the next three lemmas.

Lemma 5.2.1. *Consider the process X_t on $[0, n)$, where n is a positive integer, and let*

$$\tau_n = \inf\{t \geq 0 : \; X_t \notin [0, n)\}$$

be the first exit time from the interval $[0, n)$. Then τ_n is finite a.s. for any $X_0 = u$.

Proof. Let P_u denote the probability measure given the initial condition $X_0 = u$. Since $\tau_n = 0$ for $u \notin [0, n)$, it is sufficient to consider the case $0 \leq u < n$. Our first step is to show that $P_n(\{X_1 < 0\}) > 0$. By (5.1.3), it is equivalent to show that

$$P\left(\int_0^1 Y_s dP_s - \int_0^1 cY_s ds > L\right) > 0,$$

for any $L > 0$.

Let $\delta > 0$, and consider the event

$$A_\delta = \{\sup_{0 \leq s, s' \leq 1, |s - s'| \leq \delta} |W_s - W_{s'}| < \frac{e^{-\sigma^2/2 - \sigma/2}}{2\sigma}, \sup_{0 \leq s \leq 1} |W_s| \leq \frac{1}{2}\}.$$

By the uniform continuity of the path $(W_t, 0 \le t \le 1)$, there exists $\delta_0 > 0$ such that $P(A_\delta) > 0$, for any $0 < \delta < \delta_0$. We also consider the event

$$A'_\delta = \{0 < s_1 < s_2 < \cdots < s_N < 1, ||\Gamma|| < \delta, N > \frac{L}{\eta e^{-\sigma/2}}, \min_{1 \le i \le N} \xi_i > c\delta + \eta\},$$

where $\Gamma = \{0, s_1, s_2, \cdots, s_N, 1\}$, s_i's are jump times of N_t up to $t = 1$, $N = N_1$, $||\Gamma|| = \max_{i=2,\dots,N}\{s_1, s_i - s_{i-1}, 1 - s_N\}$ denotes the norm of the partition Γ on $[0, 1]$ and $\eta > 0$ is a constant. Since ξ is not identically zero, there exist $\delta_1 > 0$ and $\eta > 0$ such that

$$P(\xi > c\delta_1 + \eta) > 0.$$

Then for all $\delta < \delta_1$, we have

$$P(A'_\delta) > 0.$$

Since $\{W_t, t \ge 0\}$ and $\{N_t, t \ge 0, \xi_i, i = 1, 2, 3...\}$ are independent, A_δ and A'_δ are independent, and therefore $P(A_\delta \cap A'_\delta) > 0$, for all $0 < \delta < \min\{\delta_0, \delta_1\}$. Let $\delta_2 = \min\{\delta_0, \delta_1, e^{-\sigma^2/2-\sigma/2}/(\sigma^2 - 2a), e^{-\sigma^2/2-\sigma/2}\}$. If $0 < \delta < \delta_2$, and $A_\delta \cap A'_\delta$ occurs, then

$$\sup_{0 \le s, s' \le 1, |s - s'| < \delta} |Y_s - Y_{s'}| \le 1,$$

$$\int_0^1 cY_s ds \le c \sum_1^N Y_{s_i}(s_i - s_{i-1}) + 2c,$$

and

$$\inf_{0 \le s \le 1} Y_s \ge e^{-\sigma/2}.$$

Hence

$$\begin{aligned}\int_0^1 Y_s dP_s - \int_0^1 cY_s ds &\ge \sum_1^N Y_{s_i}\xi_i - c\sum_1^N Y_{s_i}(s_i - s_{i-1}) - 2c \\ &\ge \sum_1^N Y_{s_i}(\xi_i - c(s_i - s_{i-1})) - 2c \\ &\ge \sum_1^N Y_{s_i}(\xi_i - c\delta) - 2c \\ &\ge e^{-\sigma/2}\eta N - 2c \ge L - 2c.\end{aligned}$$

Since L is arbitrary, we have thus proved $P_n(\{X_1 < 0\}) \equiv C_1 > 0$. By the Markov property at $X_1, X_2, ...X_k$, we have

$$\begin{aligned}
&P_u(0 \leq X_1 < n, 0 \leq X_2 < n, ..., 0 \leq X_k < n\}) \\
&= E_u[1_{[0,n)}(X_1)1_{[0,n)}(X_2)...1_{[0,n)}(X_k)] \\
&= E_u[E_u[1_{[0,n)}(X_1)1_{[0,n)}(X_2)...1_{[0,n)}(X_k)|X_1, ..., X_{k-1}]] \\
&= E_u[1_{[0,n)}(X_1)1_{[0,n)}(X_2)...1_{[0,n)}(X_{k-1})E_{X_{k-1}}[1_{[0,n)}(X_1)]] \\
&\leq E_u[1_{[0,n)}(X_1)1_{[0,n)}(X_2)...1_{[0,n)}(X_{k-1})E_{X_{k-1}}[1_{[0,\infty)}(X_1)]].
\end{aligned}$$

By the comparison of the initial conditions using (5.1.3) , the above

$$\begin{aligned}
&\leq E_u[1_{[0,n)}(X_1)1_{[0,n)}(X_2)...1_{[0,n)}(X_{k-1})E_n[1_{[0,\infty)}(X_1)]] \\
&\leq (1 - C_1)E_u[1_{[0,n)}(X_1)1_{[0,n)}(X_2)...1_{[0,n)}(X_{k-1})] \\
&\leq (1 - C_1)^k, \forall k,
\end{aligned}$$

by repeating the same argument. This implies $P_u(\cap_{k=1}^{\infty}\{0 \leq X_k < n\}) = 0$ and therefore $P_u(\cup_{k=1}^{\infty}\{X_k \notin [0, n)\}) = 1$. Therefore τ_n is finite P_u-a.s.

Theorem 5.2.1. *Consider the model given by (5.1.8), assume that $\rho > 1$ and $c > \lambda\mu$. Then*

$$\psi(u) \leq \left(\frac{L}{u}\right)^{\rho-1} \quad \forall\, u \geq L,$$

where $L = M(\frac{c}{\lambda\mu})^{\frac{1}{\rho}}((\frac{c}{\lambda\mu})^{\frac{1}{\rho}} - 1)^{-1}$.

Remark 5.2.1. *This theorem shows that the probability of ruin has at least an algebraic decay rate if $2a/\sigma^2 > 1$. In fact, we obtain a slightly stronger result in the proof below:*

$$\psi_L(u) \leq \left(\frac{L}{u}\right)^{\rho-1} \quad \forall\, u \geq L.$$

Proof. Let $F(x) = x^{1-\rho}\phi(x)$, and $\phi(x)$ is a C^∞ function such that $\phi(x) = 1$ for $L - \epsilon < x < n + \epsilon$ and $\phi(x) = 0$ for $x \leq L - 2\epsilon$ or $x \geq n + 2\epsilon$. Here ϵ is chosen so small that $L - 2\epsilon > 0$. The function F is a C^∞ function with compact support $\subset [L-2\epsilon, n+2\epsilon]$. Applying Itô's formula [10], we have

$$\begin{aligned}F(X_t)-F(X_0) &= \int_0^t F'(X_s)(aX_s+c)\,ds+\int_0^t F'(X_s)\sigma X_s dW_s\\&+\frac{1}{2}\int_0^t F''(X_s)\sigma^2 {X_s}^2\,ds\\&+\int_0^t\int_0^M F(X_{s^-}-x)-F(X_{s^-})\,N_p(dsdx).\end{aligned}$$

Note that since F is a C^∞ function with compact support $\subset [L-2\epsilon, n+2\epsilon]$, $\int_0^t F'(X_s)\sigma X_s dW_s$ is a martingale. We consider the process X_t on $[L,n)$, where n is an integer ($>L$), and let

$$\tau_n=\inf\{t>0:\ X_t\notin[L,n)\}$$

be the first exit time from the interval $[L,n)$. Then

$$\begin{aligned}F(X_{t\wedge\tau_n})-F(X_0) &= \int_0^{t\wedge\tau_n}(1-\rho)(X_s)^{-\rho}(aX_s+c)\,ds+\int_0^{t\wedge\tau_n}(1-\rho)(X_s)^{-\rho}\sigma X_s dW_s\\&+\frac{1}{2}\int_0^{t\wedge\tau_n}(1-\rho)(-\rho)(X_s)^{-\rho-1}\sigma^2{X_s}^2\,ds\\&+\int_0^{t\wedge\tau_n}\int_0^M[(X_{s^-}-x)^{1-\rho}-(X_{s^-})^{1-\rho}]\,N_p(dsdx).\end{aligned}$$

Hence

$$\begin{aligned}F(X_{t\wedge\tau_n}) &= F(X_0)+\text{ mart. }+\int_0^{t\wedge\tau_n}(1-\rho)(X_s)^{-\rho}(aX_s+c)\,ds\\&+\frac{1}{2}\int_0^{t\wedge\tau_n}(1-\rho)(-\rho)(X_s)^{-\rho-1}\sigma^2{X_s}^2\,ds\\&+\int_0^{t\wedge\tau_n}\int_0^M[(X_{s^-}-x)^{1-\rho}-(X_{s^-})^{1-\rho}]\,\hat{N}_p(dsdx)\\&\le F(X_0)+\text{ mart. }+\int_0^{t\wedge\tau_n}(1-\rho)(X_{s^-}-M)^{-\rho}(-x)\lambda p(x)ds.\end{aligned}$$

Here, and through-out this chapter, mart. denotes a martingale at time t. The above inequality holds because

$$(X_{s^-}-x)^{1-\rho}-(X_{s^-})^{1-\rho}\le(1-\rho)(X_{s^-}-x)^{-\rho}(-x)\le(1-\rho)(X_{s^-}-M)^{-\rho}(-x),\ \forall X_{s^-}\ge M.$$

Notes that

$$(X_{s^-} - M)^{-\rho} \leq (X_{s^-})^{-\rho} \frac{c}{\lambda\mu}, \ \ \forall\, X_{s^-} \geq L.$$

Hence

$$\begin{aligned} F(X_{t\wedge\tau_n}) &\leq F(X_0) + \text{ mart. } + \int_0^{t\wedge\tau_n} (1-\rho)(X_{s^-} - M)^{-\rho}(-x)\lambda p(x) ds \\ &\leq F(X_0) + \text{ mart. } + \int_0^{t^+} \int_0^M (1-\rho)\frac{c}{\lambda\mu}(X_{s^-})^{-\rho}(-x)\lambda p(x) dx ds \\ &= F(X_0) + \text{ mart.} \end{aligned} \quad (5.2.1)$$

for any $t \geq 0$ and $X_{s^-} > L$. Taking expectation on both sides of the above inequality, and by the Optional Stopping Theorem, we have

$$E[F(X_{\tau_n})] \leq E[F(X_0)]. \quad (5.2.2)$$

Since $\xi_j > 0$ for all $j = 1, 2, \ldots$, we have $X_{\tau_n} = n$ or $X_{\tau_n} < L$. Moreover, since $F(x)$ is decreasing. By Lemma 5.2.1, $P(\tau_n < \infty) = 1$ a.s. Let $t \to \infty$, and by the Dominated Convergence Theorem, we have

$$E[F(X_{\tau_n})] \geq \frac{1}{L^{\rho-1}} P(X_{\tau_n} < L \,|X_0 = u) + \frac{1}{n^{\rho-1}} P(X_{\tau_n} = n \,|X_0 = u).$$

Hence

$$\frac{1}{L^{\rho-1}} P(X_{\tau_n} < L \,|X_0 = u) + \frac{1}{n^{\rho-1}} P(X_{\tau_n} = n \,|X_0 = u) \leq \frac{1}{u^{\rho-1}}\,.$$

Therefore

$$P(X_{\tau_n} < L \,|X_0 = u) \leq \left(\frac{L}{u}\right)^{\rho-1}.$$

Let n go to infinity, we have

$$\psi_L(u) \leq \left(\frac{L}{u}\right)^{\rho-1}.$$

Since $\psi(u) \leq \psi_L(u)$, we have

$$\psi(u) \leq \left(\frac{L}{u}\right)^{\rho-1} \quad \forall\, u \geq L.$$

5.3 Ruin at certain level of $u^* > 0$

By using a martingale argument, we prove that the price of the risky asset will drop below a threshold with probability one for all initial capital u, if $\rho \leq 1$ and the distribution of the claim size has a bounded support.

Lemma 5.3.1. *Consider the model given by (5.1.8) and assume that $\rho < 1$. Then there exists $u^* > 2M$, such that*

$$\psi_{u^*}(u) = 1, \ \ \forall\, u \geq u^*.$$

Proof. Let $F(x) = x^\alpha \phi(x)$, where $0 < \alpha < 1-\rho$, and $\phi(x)$ is a C^∞ function such that $\phi(x) = 1$ for $M - \epsilon < x < n + \epsilon$ and $\phi(x) = 0$ for $x \leq M - 2\epsilon$ or $x \geq n+2\epsilon$. Here ϵ is chosen so small that $M - 2\epsilon > 0$. The function F is a C^∞ function with compact support $\subset [M - 2\epsilon, n + 2\epsilon]$. Applying Itô's formula, we have

$$\begin{aligned} F(X_t) - F(X_0) = &\int_0^t F'(X_s)(aX_s + c)\, ds + \int_0^t F'(X_s)\sigma X_s dW_s \\ &+ \frac{1}{2}\int_0^t F''(X_s)\sigma^2 {X_s}^2\, ds \\ &+ \int_0^t \int_0^M F(X_{s^-} - x) - F(X_{s^-})\, N_p(dsdx). \end{aligned}$$

Note that since F is a C^∞ function with compact support $\subset [M - 2\epsilon, n + 2\epsilon]$, $\int_0^t F'(X_s)\sigma X_s dW_s$ is a martingale. Let $u^* = \max(2M, 2c/\sigma^2(1 - \rho - \alpha))$. We consider the process X_t on $[u^*, n)$, where n is an integer ($> u^*$), and let

$$\tau_n = \inf\{t > 0 : \ X_t \notin [u^*, n)\}$$

be the first exit time from the interval $[u^*, n)$. Then

$$\begin{aligned} F(X_{t\wedge\tau_n}) - F(X_0) = &\int_0^{t\wedge\tau_n} \alpha(X_s)^{\alpha-1}(aX_s + c)\, ds + \int_0^{t\wedge\tau_n} \alpha(X_s)^{\alpha-1}\sigma X_s dW_s \\ &+ \frac{1}{2}\int_0^{t\wedge\tau_n} \alpha(\alpha-1)(X_s)^{\alpha-2}\sigma^2 {X_s}^2\, ds \\ &+ \int_0^{t\wedge\tau_n} \int_0^M (X_{s^-} - x)^\alpha - (X_{s^-})^\alpha\, N_p(dsdx). \end{aligned}$$

Hence

$$
\begin{aligned}
F(X_{t\wedge\tau_n}) &= F(X_0) + \text{mart. } + \int_0^{t\wedge\tau_n} \alpha(X_s)^{\alpha-1}(aX_s + c)\, ds \\
&+ \frac{1}{2}\int_0^{t\wedge\tau_n} \alpha(\alpha-1)(X_s)^{\alpha-2}\sigma^2 {X_s}^2\, ds \\
&+ \int_0^{t\wedge\tau_n}\int_0^M (X_{s^-} - x)^\alpha - (X_{s^-})^\alpha\ \hat{N}_p(dsdx) \\
&\le F(X_0) + \text{ mart. } + \alpha\int_0^{t\wedge\tau_n} (X_s)^\alpha \left(\frac{\sigma^2}{2}(\rho+\alpha-1) + cX_s^{-1}\right) ds \\
&\le F(X_0) + \text{ mart.}
\end{aligned}
$$

$\forall\, t \ge 0$. The above inequality holds because $(X_{s^-} - x)^\alpha \le (X_{s^-})^\alpha,\ \forall X_{s^-} \ge M$. Hence

$$
F(X_{t\wedge\tau_n}) \le F(X_0) + \text{ mart.} \tag{5.3.1}
$$

Taking expectation on both sides of the above inequality, and by the Optional Stopping Theorem, we have

$$
E[F(X_{t\wedge\tau_n})] \le u^\alpha.
$$

By Lemma 5.2.1, $P(\tau_n < \infty) = 1$ a.s. Let $t \to \infty$, and by the Dominated Convergence Theorem, we have

$$
E[F(X_{\tau_n})] \le u^\alpha.
$$

Note that by (5.1.3) with $c_t = c$ for all t, $X_t - X_{t-} \le 0$. Therefore, for $X_0 < n$, if $X_{\tau_n} \ge n$ then $X_{\tau_n} = n$. Since F is increasing in $[M, n)$ and $u^* - M \ge M$, we have

$$
E[F(X_{\tau_n})] \ge (u^* - M)^\alpha P(X_{\tau_n} < u^* \,|X_0 = u) + n^\alpha P(X_{\tau_n} = n \,|X_0 = u).
$$

Hence

$$
(u^* - M)^\alpha P(X_{\tau_n} < u^* \,|X_0 = u) + n^\alpha P(X_{\tau_n} = n \,|X_0 = u) \le u^\alpha.
$$

Therefore

$$
P(X_{\tau_n} = n \,|X_0 = u) \le \left(\frac{u}{n}\right)^\alpha.
$$

Let n go to infinity, we have

$$\psi_{u^*}(u) = 1 - \lim_{n\to\infty} P(X_{\tau_n} = n \,|X_0 = u) \geq 1 - \lim_{n\to\infty} \left(\frac{u}{n}\right)^{\alpha} = 1, \ \ \forall\, u \geq u^*.$$

Lemma 5.3.2. *Consider the model given by (5.1.8) and assume that $\rho = 1$. Then there exists $u^* > 2M + 4$, such that*

$$\psi_{u^*}(u) = 1 \ \ \forall\, u \geq u^*.$$

Proof. Let $F(x) = \phi(x) \ln\ln x$, where $\phi(x)$ is a C^∞ function such that $\phi(x) = 1$ for $M + 4 - \epsilon < x < n + \epsilon$ and $\phi(x) = 0$ for $x \leq M + 4 - 2\epsilon$ or $x \geq n + 2\epsilon$. Here ϵ is chosen so small that $M + 4 - 2\epsilon > M + 3$. The function F is a C^∞ function with compact support $\subset [M + 4 - 2\epsilon, n + 2\epsilon]$. Applying Itô's formula, we have

$$\begin{aligned} F(X_t) - F(X_0) = & \int_0^t F'(X_s)(aX_s + c)\, ds + \int_0^t F'(X_s)\sigma X_s dW_s \\ & + \frac{1}{2}\int_0^t F''(X_s)\sigma^2 {X_s}^2\, ds \\ & + \int_0^t \int_0^M F(X_{s^-} - x) - F(X_{s^-})\, N_p(dsdx). \end{aligned}$$

Note that since F is a C^∞ function with compact support $\subset [M+4-2\epsilon, n+2\epsilon]$, $\int_0^t F'(X_s)\sigma X_s dW_s$ is a martingale. Let $\tilde{u}$ be the solution of $\sigma^2 x = 2c\ln x$, and $u^* = \max(2M + 4, \tilde{u})$. We consider the process X_t on $[u^*, n)$, where n is an integer ($> u^*$), and let

$$\tau_n = \inf\{t > 0 :\ X_t \notin [u^*, n)\}$$

be the first exit time from the interval $[u^*, n)$. Then we have

$$\begin{aligned} F(X_{t\wedge\tau_n}) - F(X_0) = & \int_0^{t\wedge\tau_n} (X_s \ln X_s)^{-1}(aX_s + c)\, ds + \int_0^{t\wedge\tau_n} (X_s \ln X_s)^{-1}\sigma X_s dW_s \\ & + \frac{1}{2}\int_0^{t\wedge\tau_n} (-\ln X_s - 1)(X_s \ln X_s)^{-2}\sigma^2 {X_s}^2\, ds \\ & + \int_0^{t\wedge\tau_n} \int_0^M [\ln\ln(X_{s^-} - x) - \ln\ln X_{s^-}]\, N_p(dsdx). \end{aligned}$$

Hence

$$\begin{aligned}F(X_{t\wedge\tau_n}) = F(X_0) + \text{ mart. } &+ \int_0^{t\wedge\tau_n} (X_s \ln X_s)^{-1}(aX_s + c)\, ds \\ &+ \frac{1}{2}\int_0^{t\wedge\tau_n} (-\ln X_s - 1)(X_s \ln X_s)^{-2}\sigma^2 {X_s}^2\, ds \\ &+ \int_0^{t\wedge\tau_n}\int_0^M [\ln\ln(X_{s^-} - x) - \ln\ln X_{s^-}]\, \hat{N}_p(dsdx) \\ &\leq F(X_0) + \text{ mart. } + \int_0^{t\wedge\tau_n} \left(cX_s^{-1} - \frac{\sigma^2}{2\ln X_s}\right)(\ln X_s)^{-1} ds.\end{aligned}$$

The above inequality holds because $\ln\ln(X_{s^-} - x) \leq \ln\ln X_{s^-},\ \forall X_{s^-} \geq M$. Hence

$$F(X_{t\wedge\tau_n}) \leq F(X_0) + \text{ mart.} \tag{5.3.2}$$

Taking expectation on both sides of the above inequality, and by the Optional Stopping Theorem, we have

$$E[F(X_{t\wedge\tau_n})] \leq \ln\ln u.$$

By Lemma 5.2.1, $P(\tau_n < \infty) = 1$ a.s. Let $t \to \infty$, and by the Dominated Convergence Theorem, we have

$$E[F(X_{\tau_n})] \leq \ln\ln u.$$

Since $F(x)$ is increasing in $(M + 4 - \epsilon, n + \epsilon)$ and $u^* - M \geq M + 4$, we have

$$\begin{aligned}E[F(X_{\tau_n})] \geq \ln\ln(u^* - M)P(X_{\tau_n} < u^* - M\ |X_0 = u) \\ + \ln\ln n P(X_{\tau_n} = n\ |X_0 = u).\end{aligned}$$

Hence

$$\ln\ln(u^* - M)P(X_{\tau_n} < u^* - M\ |X_0 = u) + \ln\ln n P(X_{\tau_n} = n\ |X_0 = u) \leq \ln\ln u.$$

Therefore

$$P(X_{\tau_n} = n\ |X_0 = u) \leq \frac{\ln\ln u}{\ln\ln n}\ .$$

Let n go to infinity, we have

$$\psi_{u^*}(u) = 1 - \lim_{n\to\infty} P(X_{\tau_n} = n\ |X_0 = u) \geq 1 - \lim_{n\to\infty}\frac{\ln\ln u}{\ln\ln n} = 1,\ \ \forall\ u \geq u^*.$$

5.4 Ruin at the level of zero

From the last section, we have proved that the price of the risky asset will drop below a threshold with probability one for all initial capital u, if $\rho \leq 1$ and the distribution of the claim size has a bounded support. In this section, assuming that c_t is a constant c and using a reduction argument, we will prove that the ruin probability is equal to one if $\rho \leq 1$ and the distribution of the claim size has a bounded support. First we prove the following reduction lemma.

Lemma 5.4.1. *(Reduction Lemma) Let $u^* > 0$ be any positive real number and $[0, M], 0 < M < \infty$ be the support of the distribution for ξ_1. Suppose $\psi_{u^*}(u) = 1$, for all $u \geq u^*$. Then*

$$\psi_K(u) = 1, \;\; \forall \; u \geq K = \max(u^* - \frac{M}{2}, 0).$$

Remark 5.4.1. *$u^* > 0$ in the above Lemma is any positive real number, it needs not be the one defined in Lemma 5.3.1 or Lemma 5.3.2.*

Proof. Our first step is to show that for any $0 < C_1 < 1$, there exists a $\beta_0 = \beta_0(M, C_1)$ such that $P\left(X_t \leq u^* + \frac{M}{8}, \; \forall \, 0 \leq t \leq \beta_0 \mid X_0 = u\right) \geq C_1 > 0$, for all $u^* \geq u \geq K$.

Let Y_t, V_t be the same as in Lemma 5.1.1, and $X_t = Y_t^{-1} V_t^u$ the solution of (5.1.8). Define $Z_t{}^{u^*} = Y_t^{-1}\left(u^* + c\int_0^t Y_s \, ds\right)$. Since $dZ_t{}^{u^*} = (aZ_t^{u^*} + c)dt + \sigma Z_t^{u^*} dW_t$, $Z_t{}^{u^*}$ is a diffusion process. By continuity of $Z_t{}^{u^*}$, we have

$$\lim_{\beta \to 0} \sup_{0 \leq s \leq \beta} |Z_s{}^{u^*} - u^*| = 0, a.s.$$

Hence for all $\epsilon > 0$ and all $0 < C_1 < 1$, $\exists \, \beta_0 = \beta_0(\epsilon, C_1) > 0, \; s.t.$

$$P\left(\sup_{0 \leq s \leq \beta_0} |Z_s{}^{u^*} - u^*| < \epsilon\right) \geq C_1 > 0.$$

In particular, choose $\epsilon = \frac{M}{8}$, $\exists \, \beta_0 = \beta_0(M, C_1) > 0, s.t.$

$$P\left(Z_t{}^{u^*} \leq u^* + \frac{M}{8}, \; \forall \, 0 \leq t \leq \beta_0\right) \geq C_1 > 0.$$

Let δ be the time that the first jump occurs. Our next step is to show that there exists $C_2 = C_2(C_1, M) > 0$ such that

$$P\left(X_\delta < K \mid X_0 = u\right) \geq C_2 > 0, \ \forall\, K \leq u \leq u^*.$$

Note that $\forall\, K \leq u \leq u^*$, by (5.1.3) with $c_s = c$, we have ${Z_t}^{u^*} \geq {Z_t}^{u} \geq X_t, \ \forall\, t \geq 0$, and therefore

$$\begin{aligned} &P\left(X_t \leq u^* + \frac{M}{8}, \ \forall\, 0 \leq t \leq \beta_0, \ \delta < \beta_0, \ \xi_1 > \frac{3M}{4} \mid X_0 = u\right) \\ &\geq P\left({Z_t}^{u^*} \leq u^* + \frac{M}{8}, \ \forall\, 0 \leq t \leq \beta_0, \ \delta < \beta_0, \ \xi_1 > \frac{3M}{4}\right). \end{aligned}$$

Since ${Z_t}^{u^*}$ depends on W_t, δ depends on $N(t)$ only, and W_t, $N(t)$ and ξ_i are assumed to be independent processes, the above probability is equal to

$$\begin{aligned} &= P\left({Z_t}^{u^*} \leq u^* + \frac{M}{8}, \ \forall\, 0 \leq t \leq \beta_0\right) P\left(\delta < \beta_0\right) P\left(\xi_1 > \frac{3M}{4}\right) \\ &\geq C_1 P\left(\delta < \beta_0\right) P\left(\xi_1 > \frac{3M}{4}\right) = C_2 > 0, \end{aligned}$$

since $[0, M]$ is the support of the distribution of ξ_1 and therefore $P(\xi_1 > \frac{3M}{4}) > 0$. On the other hand,

$$\begin{aligned} &P\left(X_t \leq u^* + \frac{M}{8}, \ \forall\, 0 \leq t \leq \beta_0, \ \delta < \beta_0, \ \xi_1 > \frac{3M}{4} \mid X_0 = u\right) \\ &\leq P\left(X_t \leq u^* + \frac{M}{8}, \ \forall\, 0 \leq t < \delta, \ \delta < \beta_0, \ \xi_1 > \frac{3M}{4} \mid X_0 = u\right) \\ &\leq P\left(X_\delta \leq u^* + \frac{M}{8} - \frac{3M}{4} = u^* - \frac{5M}{8} < u^* - \frac{M}{2} \leq K \mid X_0 = u\right). \end{aligned}$$

Hence

$$P\left(X_\delta < K \mid X_0 = u\right) \geq C_2 > 0, \ \forall\, K \leq u \leq u^*.$$

Our final step is to show that

$$\psi_K(u) = 1, \ \ \forall\, u \geq K = \max(u^* - \frac{M}{2}, 0).$$

Define

$$T_1 = \begin{cases} \inf\{t > \delta, \ X_t \leq u^*\}, & if\ X_\delta \geq K \\ \\ \infty, & if\ X_\delta < K. \end{cases}$$

Note that the infimum of an empty set is ∞. But by the assumption $\psi_{u^*}(u) = 1$, for all $u \geq u^*$, we have $T_1 = \infty$ if and only if $X_\delta < K$. Let $B = \{X_t \geq K, \forall\, 0 \leq t < \infty\}$. We will apply the strong Markov property at T_1 on B. To this end, we define the shift operator θ_s as follows (see e.g. P. 99 [6]). For a sample path of $X = (X_t, t \geq 0)$, θ_s maps a sample path to a sample path defined by

$$(\theta_s X)_t = X_{s+t}, t \geq 0. \tag{5.4.1}$$

Thus $\theta_s X$ is the path that is obtained by cutting off the part of X before time s and then shift the time so that the time s for X becomes time 0 for the new path $\theta_s X$. For a random time $S(X)$ with values in $[0, \infty]$, we define

$$(\theta_S X)_t = (\theta_{S(X)} X)_t = X_{S(X)+t}, t \geq 0, \text{ if } S(X) < \infty. \tag{5.4.2}$$

We also define the shift operator θ_s which maps a function of path to a function of path. Let $F(X)$ be a function of path. Define

$$(\theta_s F)(X) = F(\theta_s X), \tag{5.4.3}$$

and

$$(\theta_S F)(X) = F(\theta_S X), \text{ if } S(X) < \infty. \tag{5.4.4}$$

Now consider the event B, we have

$$\begin{aligned} P(B|\, X_0 = u^*) &= E[1_B 1_{T_1<\infty} \mid X_0 = u^*] + E[1_B 1_{T_1=\infty} \mid X_0 = u^*] \\ &= E[1_B 1_{T_1<\infty} \mid X_0 = u^*] \\ &= E[1_{T_1<\infty} \theta_{T_1}[1_B] \mid X_0 = u^*], \end{aligned}$$

since if $T_1 < \infty$, then 1_B is invariant under the shift operator θ_{T_1}. In what follows, we denote $E_x[1_B] = E[1_B|\, X_0 = x]$. By the strong Markov property of X_t (see e.g. Theorem 3.11 [6]), we have

$$\begin{aligned} E[1_{T_1<\infty}\theta_{T_1}[1_B] \mid X_0 = u^*] &= E[1_{T_1<\infty}E_{X_{T_1}}[1_B] \mid X_0 = u^*] \\ &\le E\left[1_{T_1<\infty}E_{u^*}[1_B] \mid X_0 = u^*\right] \\ &= E[1_{T_1<\infty} \mid X_0 = u^*]E_{u^*}[1_B] \\ &\le (1 - C_2)E[1_B \mid X_0 = u^*] \\ &= P(B|\ X_0 = u^*)(1 - C_2). \end{aligned}$$

The first inequality holds since $K \le X_{T_1} \le u^*$ on $\{T_1 < \infty\}$. Hence we have

$$P(B|\ X_0 = u^*) \le P(B|\ X_0 = u^*)(1 - C_2).$$

Therefore $P(B|\ X_0 = u^*) = 0$, i.e. $\psi_K(u^*) = 1$. Since $u \le u^*$, by Lemma 5.1.1,

$$\psi_K(u) \ge \psi_K(u^*) = 1.$$

The proof is completed.

Theorem 5.4.1. *Consider the model given by (5.1.8) and assume that* $\rho \le 1$. *Suppose also the jump distribution has support* $[0, M], M > 0$. *Then*

$$\psi(u) = 1, \ \ \forall\ u \ge 0.$$

Proof. By Lemma 5.3.1, Lemma 5.3.2 and the Reduction Lemma 5.4.1, $\psi_{K_1}(u) = 1, \ \ \forall\ u \ge K_1 = \max(u^* - \frac{M}{2}, 0)$. Applying the Reduction Lemma 5.4.1 again, with u^* replaced by K_1, we have

$$\psi_{K_2}(u) = 1, \ \ \forall\ u \ge K_2 = \max(K_1 - M, 0) = \max(u^* - 2\frac{M}{2}, 0).$$

Repeating this argument $N = \lceil \frac{2u^*}{M} \rceil$ times, we have

$$\psi_{K_N}(u) = 1, \ \ \forall\ u \ge K_N = \max(u^* - N\frac{M}{2}, 0) = 0,$$

i.e.,

$$\psi(u) = 1, \ \ \forall\ u \ge 0.$$

We have thus finished the ruin probability problem for the case of $\rho \le 1$, $c_t = c$ and the distribution of the claim size has a bounded support.

Finally, we prove our main theorem:

Theorem 5.4.2. *Let*

$$X_t = X_0 + \int_0^t aX_s ds + \int_0^t \sigma X_s dW_s + \int_0^t c_s ds - \sum_{j=1}^{N(t)} \xi_j, \tag{5.4.5}$$

where W_t is the standard Brownian motion, $a > 0$, $\sigma \geq 0$, $N(t)$ is a Poisson process with intensity λ, and the claim sizes ξ_i; $i = 1, 2, 3, ...$, are independent, identically distributed non-negative random variables, with positive mean and finite variance. We assume that W_t, $N(t)$, ξ_i are independent processes. Let the filtration $\mathcal{F}_t = \sigma\{W_s, N_s, \sum_{i=1}^{N_s} \xi_i; 0 \leq s \leq t\}$. Let $c_t = c(t, X)$ be a bounded nonnegative $(\mathcal{F}_t)$-adapted process. Suppose $\rho := \frac{2a}{\sigma^2} \leq 1$. Then the ruin probability

$$\psi(u) = 1, \ \ \forall\, u \geq 0.$$

Proof. Our first step is to extend Theorem 5.4.1 to the case where the same assumptions hold except that the claim size has an unbounded support.

Let $M > 0$ be a large constant, define

$$\hat{\xi}_i = \begin{cases} \xi_i, & if\ \xi_i \leq M \\ \\ M, & if\ \xi_i > M, \end{cases}$$

and $\hat{P}_t = \sum_{j=1}^{N(t)} \hat{\xi}_j$. Let Y_t, V_t be the same as in Lemma 5.1.1, and $X_t = Y_t^{-1} V_t^u$ be the solution of (5.1.8). Define

$$Z_t = Y_t^{-1} \left(u + c \int_0^t Y_s\, ds - \int_0^t Y_s\, d\hat{P}_s \right),$$

then $Z_t \geq X_t$, $\forall\, t \geq 0$. Hence

$$\psi(u) = P(X_t < 0, \text{ for some } \ 0 < t < \infty \mid X_0 = u) \tag{5.4.6}$$

$$\geq P(Z_t < 0, \text{ for some } \ 0 < t < \infty \mid Z_0 = u). \tag{5.4.7}$$

On the other hand, since $dZ_t = (aZ_t + c)dt + \sigma Z_t dW_t - d\hat{P}_t$, Z_t satisfies (5.1.8) with bounded claim size distribution. Hence, by Theorem 5.4.1,

$$P(Z_t < 0, \text{ for some } \ 0 < t < \infty \mid Z_0 = u) = 1, \ \forall\, u \geq 0.$$

Therefore

$$\psi(u) = 1, \ \ \forall \, u \geq 0.$$

Next we prove the general situation where c_t is bounded but not necessarily a constant.

Let $X_t = Y_t^{-1} V_t^u$ be the solution of (5.1.2) given by (5.1.3). Define

$$Z_t = Y_t^{-1} \left(u + c \int_0^t Y_s \, ds - \int_0^t Y_s \, dP_s \right),$$

where $c_t \leq c$ for all t. Then $Z_t \geq X_t, \ \forall \, t \geq 0$. Hence

$$\begin{aligned} \psi(u) &= P(X_t < 0, \text{ for some } \ 0 < t < \infty \mid X_0 = u) && (5.4.8) \\ &\geq P(Z_t < 0, \text{ for some } \ 0 < t < \infty \mid Z_0 = u). && (5.4.9) \end{aligned}$$

On the other hand, by Ito's formula, $dZ_t = (aZ_t + c)dt + \sigma Z_t dW_t - dP_t$, i.e., Z_t satisfies (5.1.8). Hence, by the result of the first step, we have

$$P(Z_t < 0, \text{ for some } \ 0 < t < \infty \mid Z_0 = u) = 1, \ \forall \, u \geq 0.$$

Therefore

$$\psi(u) = 1, \ \ \forall \, u \geq 0.$$

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