LINE BUNDLES OVER b-HOLOMORPHIC COMPLEX CURVES

A Dissertation
Submitted to
the Temple University Graduate Board

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in Partial Fulfillment
of the Requirements for the Degree of
DOCTOR OF PHILOSOPHY

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by
Paul Nekoranik
May 2001

Abstract

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Professor Gerardo Mendoza, Chair

A compact orientable surface M with boundary admits almost-complex structures J on its compressed tangent bundle ^bTM. We call a surface equipped with such a structure a *b-holomorphic complex curve.* The interior of a b-holomorphic curve is an ordinary non-compact Riemann surface. But the holomorphic structure at the boundary is singular, in the sense that the curve cannot be realized as an embedded submanifold (with boundary) of a larger Riemann surface.

In this dissertation, we study b-holomorphic structures. We discover invariant or characteristic objects associated to b-holomorphic curves, and others associated to holomorphic line bundles over such curves, including a generalized degree. We then use these invariants to prove classification theorems.

We also investigate the existence of constant-curvature connections on these line bundles. In particular, we provide a necessary and sufficient condition for the existence of a hermitian holomorphic b-connection whose curvature is a constant times the volume form (that is, the volume form induced by a given hermitian metric on the base manifold). Such a connection is an absolute minimum for the Yang–Mills functional on the bundle.

Acknowledgements

I wish to thank Gerardo Mendoza, my advisor; with a mighty effort, he turned me into a mathematician. Thanks also to Leon Ehrenpreis, my unofficial guru. From his lectures and his conversation I learned that understanding an idea is better than proving a theorem.

Thanks to Eric Grinberg, Omar Hijab, and Howard Jacobowitz for serving on my examining committee. Thanks to my teachers Shiferaw Berhanu, Omar Hijab, and Simeon Vishik for setting my feet on the geometric path. Thanks to Juan Gil for helping me to find my way in the b world. And thanks to Tong Banh and Cristian Guriţă, with whom I have had many productive and picturesque mathematical conversations over the years.

I wish to thank Robert Gluckstern for encouraging me to become a scientist. Thanks to Michael Brin for introducing me to mathematics; and to Edward Popowski and Dan Storaska for introducing me to physics. Thanks also to Anthony Waskie, who taught me to be unsatisfied.

Thanks to Aaron and Elisa Robertson and Thomas Reimel for their friendship and encouragement. Thanks to my family, especially my parents Emil and Rita, without whose encouragement and support this project could never have been undertaken. Finally, I thank Marjorie Cresta, to whom my debt of gratitude is beyond measure.

For Marjorie

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Introduction

1.1 Totally characteristic operators

1.1.1 Background and definitions

When we think of a differential operator on a manifold M, we usually imagine a linear combination over smooth functions of products of the basic vector fields $\partial_{x_1}, \ldots, \partial_{x_n}$, where $x_1, \ldots, x_n$ form a local chart. When the coefficients of the highest-order products $\partial_{x_1}^{m_1} \cdots \partial_{x_n}^{m_n}$ ($m_1 + \cdots + m_n$ maximal) form a matrix of functions (the *principal symbol*) which is pointwise non-degenerate, we call the operator *elliptic*. An elliptic operator is very nice, in the sense that it has the Fredholm property when thought of as acting between suitable Sobolev spaces, and it has a good pseudodifferential parametrix. See [34]. (Here we are imagining that M is compact and boundaryless.) This is in sharp contrast to the case of the generic differential operator, about which very few general results are known.

When M is a manifold with boundary, it has been possible to ascertain an analogous notion of ellipticity for some classes of operators which are degenerate on the boundary. One such class of operators is that of the *totally characteristic operators*, which are those that can be written near the boundary in the form

$$\sum p_{m_1 \ldots m_n} \partial_{x_1}^{m_1} \cdots \partial_{x_{n-1}}^{m_{n-1}} (x_n \partial_{x_n})^{m_n}$$

where $x_1, \ldots, x_n$ form a local chart (x_n defining the boundary) and the p_{mn} are smooth. Here, the criterion for ellipticity has two parts. One is the usual ellipticity criterion on the interior, which is invertibility of the principal symbol; and the other involves the invertibility of a new "boundary" symbol, called the *conormal symbol*. The elliptic totally characteristic operators have a Fredholm property when thought of as acting between certain weighted Sobolev spaces, and they have good pseudodifferential parametrices. See [12], [14], [21], [24], [25], [26], and [31]; there are many other treatments of this subject as well.

1.1.2 The b category

Let M be a compact Riemannian manifold with boundary for which the geometric operator Δ is totally characteristic. Many authors have focused their attention on such manifolds, operating by way of analogy to compact Riemannian manifolds without boundary. In his book "The Atiyah–Patodi–Singer Index Theorem" [24], Melrose takes the point of view that surfaces with boundary for which Δ is totally characteristic constitute a reasonable category, which he calls (and we will call) the b category. (The b is for "boundary.") In support of this viewpoint, he shows how to prove the APS index theorem by regarding it as the generalization, from the category of compact Riemannian manifolds without boundary to the b category, of the Atiyah–Singer index theorem.

What makes a compact manifold with non-empty boundary a b-manifold is that its metric exhibits a certain behavior near the boundary. Or, we could say that what makes a compact manifold without boundary a b-manifold is that the metric has singularities of a certain sort on a discrete set of points; this is the picture of the blow-down. In this situation, the compact manifold is said to have conical singularities.

1.1.3 The b-holomorphic category

As was stated in Section 1.1.2, a b-manifold is a Riemannian manifold for which the geometric operator Δ is totally characteristic. But a geometric structure (a metric) is not the only kind of structure on a manifold that gives rise to a natural elliptic operator. On a complex-analytic manifold, we have the first-order conformal elliptic operator $\bar{\partial}$ in addition to the second-order geometric elliptic operator Δ (which is equal, up to a constant multiple, to $\star\bar{\partial}\partial$).

So our point of view in this dissertation will be that there's a category, called the b-holomorphic category, which consists of compact complex-analytic manifolds with boundary for which $\bar{\partial}$ is totally characteristic.

1.2 The b-holomorphic category

1.2.1 b-holomorphic complex curves

Let M be a compact manifold with boundary. Then the first-order differential operators that can be written locally as a linear combination over smooth functions of $\partial_{x_1}, \ldots, \partial_{x_n}$

are precisely the smooth sections of TM. We may say that the tangent bundle TM is custom-made so that the smooth sections are of this sort.

It should be possible to construct another bundle ${}^bTM$ whose smooth sections are precisely the first-order totally characteristic operators. And this is possible. The construction we give may be found in [25] or [26], and is a special case of a general method for constructing new bundles from old which can be found in [24].

We define V to be the vector space of all smooth vector fields on M which are tangent to the boundary over points of the boundary. For each $p \in M$, we define $\sim_p$ on V by

$$X \sim_p Y \text{ if } \begin{cases} X_p = Y_p & p \in M^\circ \\ X_p = Y_p \text{ and } d_p X(r) = d_p Y(r) & p \in \partial M, \end{cases}$$

where r is any local defining function for the boundary component of M that contains p. Finally, we take ${}^bT_pM$ to be $V/\sim_p$. Each fiber ${}^bT_pM$ is a vector space, and the union ${}^bTM$ of all the fibers inherits from TM (in a natural way) the structure of a smooth vector bundle over M. ${}^bTM$ is called the *compressed tangent bundle* of M. Its sections are called *compressed vector fields*. There is a natural map $\pi : {}^bTM \longrightarrow TM$, which is a fiberwise isomorphism over the interior. The kernel of π over a boundary point is one-dimensional (it is spanned by $r\partial_r$ where r is a defining function for the boundary); and the range of π over a boundary point p is $T_p\partial M$. It is important to note that $r\partial_r$ is a *nonzero* element of ${}^bT_pM$, even if $p \in \partial M$.

We proceed now to define the *compressed cotangent bundle* ${}^bT^*M = {}^b\Lambda^1 M$ and the *compressed exterior algebra* ${}^b\Lambda^n M$ in the usual way. These smooth vector bundles over M are associated to analysis involving totally characteristic operators.

From now on, we take the real dimension of M to be 2.

Definition 1.1. A compact smooth surface with nonempty boundary, equipped with a smooth anti-involution J on ${}^bTM$ (or on ${}^bT^*M$), is called a *b-holomorphic complex curve*.

Note that J extends to $\mathbb{C}\,{}^bTM$ (or to $\mathbb{C}\,{}^bT^*M$) and induces splittings

$$\mathbb{C}\,{}^bTM = {}^bT^{1,0}M \oplus {}^bT^{0,1}M$$
$$\mathbb{C}\,{}^bT^*M = {}^b\Lambda^{1,0}M \oplus {}^b\Lambda^{0,1}M,$$

just as in the ordinary (non-b) situation. The factors are smooth complex vector bundles over M. The second splitting is produced as follows. Since $J^2 = -1$, the eigenvalues

of J are $\pm i$. Each eigenspace has complex dimension 1. We define ${}^b\Lambda^{1,0}M$ to be the smooth subbundle of $\mathbb{C}\,{}^bT^*M$ whose fibers are the i eigenspaces of J, and ${}^b\Lambda^{0,1}M$ to be the smooth subbundle of $\mathbb{C}\,{}^bT^*M$ whose fibers are the $-i$ eigenspaces of J. The first splitting is produced by dualization.

When we need to write projection maps associated to these splittings, we will call them $\pi_{1,0}$ and $\pi_{0,1}$. And, for $\nu \in \mathbb{C}\,{}^bT^*M$, we will write $\pi_{1,0}\nu = \nu^{1,0}$, etc.

Over the interior of M, J is an ordinary almost-complex structure. And since we are in real dimension 2, the almost-complex structure is automatically integrable. That is, M° has a complex structure. So M° is a noncompact Riemann surface. Define $\bar{\partial} = \pi_{0,1} \circ d$. Over M°, this is the usual $\bar{\partial}$ operator associated to the Riemann surface M°. The b-symbol of $\bar{\partial}$,

$$ {}^b\sigma(\bar{\partial}) : {}^bT_p^*M \longrightarrow \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, {}^b\Lambda_p^{0,1}M), $$

is given by ${}^b\sigma(\bar{\partial})(\nu) = (1 \mapsto i\nu^{0,1})$. This is clearly an isomorphism of real vector spaces. Since the b-symbol is an isomorphism, we may say that $\bar{\partial}$ is b-elliptic. In other words, $\bar{\partial}$ is an elliptic operator of totally characteristic type.

Definition 1.2. Two b-holomorphic complex curves M, N are called *equivalent* if there exists a C^∞ diffeomorphism $M \longrightarrow N$ which respects J (carries J_M to J_N).

This is the same as saying that $M \simeq N$ if there exists a C^∞ diffeomorphism $M \longrightarrow N$ which is holomorphic on M°.

Note 1.3. One usually defines a b-manifold to be a manifold with a smoothly varying positive-definite quadratic form on the fibers of ${}^bTM$. This is equivalent to saying that the Laplace operator is totally characteristic. But in practice, some sort of extra condition is always imposed. For example, in Melrose's book [24], an *exact b-metric* is taken to be one for which there exist special coordinates at the boundary in which the Laplacian has a particularly nice form.

Here, we have defined a b-holomorphic complex curve to be a manifold with a smoothly varying almost-complex structure J on the fibers of ${}^bTM$. This is equivalent to saying that the $\bar{\partial}$ operator is totally characteristic. But in practice, we too will impose an extra condition. Just as in the definition of an exact b-metric, we will assume the existence of special coordinates at the boundary in which the $\bar{\partial}$ operator has a particularly nice form. This existence of these special coordinates is crucial to the type of analysis that will be used

throughout this paper. This extra condition will be formulated precisely in Section 2.4; and we will show that there is a large supply of b-holomorphic curves that satisfy the condition. Whether there are others that do not satisfy the condition, I do not know.

1.2.2 Line bundles and connections

For basic definitions and theory regarding complex differential geometry, see [15], [19], and [20].

Definition 1.4. A *holomorphic line bundle* over a b-holomorphic complex curve M is a C^∞ complex line bundle E over M equipped with a first-order linear differential operator

$$\bar{\partial}_E : C^\infty(M; E) \longrightarrow C^\infty(M; E \otimes {}^b\Lambda^{0,1}M)$$

whose principal b-symbol

$${}^b\sigma(\bar{\partial}_E) : {}^bT_p^*M \longrightarrow \mathrm{Hom}_{\mathbb{C}}(E_p; E_p \otimes {}^b\Lambda_p^{0,1}M)$$

is given by

$${}^b\sigma(\bar{\partial}_E)(\nu) = (e \mapsto ie \otimes \nu^{0,1}).$$

Note that this definition is equivalent to the usual formula $\bar{\partial}_E(fe) = e \otimes \bar{\partial}f + f\bar{\partial}_E e$ for functions f on M and sections e of E.

When we speak of line bundles over a b-holomorphic complex curve, we will always assume them to be holomorphic line bundles in the sense of the above definition.

Definition 1.5. A b-*connection* on a line bundle E over a b-holomorphic complex curve M is a first-order linear differential operator

$$\nabla_E : C^\infty(M; E) \longrightarrow C^\infty(M; E \otimes \mathbb{C}\,{}^bT^*M)$$

whose principal b-symbol

$${}^b\sigma(\nabla_E) : {}^bT_p^*M \longrightarrow \mathrm{Hom}_{\mathbb{C}}(E_p; E_p \otimes \mathbb{C}\,{}^bT_p^*M)$$

is given by

$${}^b\sigma(\nabla_E)(\nu) = (e \mapsto ie \otimes \nu).$$

Note that this definition is equivalent to the usual formula $\nabla_E(fe) = e \otimes df + f\nabla_E e$ for functions f on M and sections e of E.

Definition 1.6. A *b*-connection ∇_E on a line bundle E over a *b*-holomorphic complex curve M is called *holomorphic* if $\pi_{0,1} \circ \nabla_E = \bar{\partial}_E$.

Note 1.7. In the future, we will write $\bar{\partial}$ and ∇ instead of $\bar{\partial}_E$ and ∇_E.

1.2.3   Metrics

Let M be a *b*-holomorphic complex curve and E be a line bundle over M. We will be concerned with hermitian metrics on M and on E, as will be explained in Section 1.3.3; but we include the relevant definitions here so they will be easier to find.

Definition 1.8. A *hermitian metric on M* is a hermitian metric on $T^{1,0}M^\circ = ({}^bT^{1,0}M){\downarrow}M^\circ$ such that the norm of any local smooth frame for ${}^bT^{1,0}M$ defined up to the boundary extends to a local defining function for the boundary.

Note 1.9. A hermitian metric on M induces a Riemannian metric on M° in the usual way.

Definition 1.10. A *geometric b-holomorphic complex curve* is a *b*-holomorphic complex curve M equipped with a hermitian metric such that, with respect to the induced Riemannian metric, $\text{vol}(M) = 1$.

Definition 1.11. Let E be a line bundle over the *b*-holomorphic complex curve M. A *hermitian metric on E* is just a smooth hermitian metric on the C^∞ line bundle $E{\downarrow}M$, in the usual sense.

Definition 1.12. Let M be a *b*-holomorphic complex curve and E be a line bundle over M equipped with a hermitian metric. A *b*-connection ∇ on $E{\downarrow}M$ is called *compatible with the metric of E* if the Leibnitz-type formula

$$d\langle e_1, e_2 \rangle = \langle \nabla e_1, e_2 \rangle + \langle e_1, \nabla e_2 \rangle$$

holds for all smooth sections e_1, e_2 of E.

A *b*-connection that is compatible with some hermitian metric on E will be called *hermitian*.

1.3 Objectives, by analogy to compact Riemann surfaces

1.3.1 Classification of surfaces

There is, to my knowledge, no easily stated full classification of compact Riemann surfaces. But the desire for such a classification (up to biholomorphic equivalence) is quite natural and motivates many results in the general theory; for example, the uniformization theorem, the Weierstrass gap theorem, the Noether gap theorem, and various results on Weierstrass points and automorphism groups. See [5], [10], and [32].

For each type of b-holomorphic complex curve we examine, the first goal will be to find a classification and to compute (if possible) the automorphism groups.

1.3.2 Classification of holomorphic line bundles

The set of holomorphic line bundles over a given compact Riemann surface or b-holomorphic complex curve forms a group. The operation is tensor product, and inversion is dualization. The strongest possible type of classification of holomorphic line bundles over the surface (up to holomorphic isomorphism) is therefore a homomorphism of this group onto a *prima facie* simpler group, such that the kernel is the subgroup of all bundles equivalent to the trivial bundle. For this would furnish a isomorphism from the group of all equivalence classes of holomorphic line bundles to the simpler group. This is what we will call a classification of holomorphic line bundles. (For some ideas on holomorphic vector bundles over a compact Riemann surface, see [11].)

For each type of b-holomorphic complex curve we examine, the second goal will be to find a classification of holomorphic line bundles in this sense.

1.3.3 Connections of constant curvature

In 1965, Narasimhan and Seshadri [27] proved a theorem on the existence of special connections on a given holomorphic vector bundle over a compact Riemann surface. See also Donaldson's proof in [9]. In the simplest case of a line bundle (for which the result was known earlier), the theorem may be phrased as follows:

Theorem 1.13 (Narasimhan and Seshadri). *Let M be a compact Riemann surface with hermitian metric normalized to unit volume. Let E be a holomorphic line bundle over*

M. Then there exists a unique hermitian holomorphic connection on E whose curvature is a constant times the volume form. The constant is necessarily $-2\pi i \cdot \text{degree}(E)$.

Here, $\text{degree}(E)$ is a C^∞ bundle invariant, defined as the integral over M of the first Chern class of E.

For each type of b-holomorphic complex curve we examine, the third and final goal will be to identify a bundle invariant analogous to the degree and then to prove a result analogous to the theorem of Narasimhan and Seshadri.

Note 1.14. There is a bit of a puzzle regarding what kind of metrics on M to treat as the "natural" ones in the case of a b-holomorphic complex curve. One obvious choice would be to assume that the metric is a non-degenerate inner product on the fibers of $^bT^{1,0}M$. In this case, the induced Riemannian metric would be a b-metric, and the Laplacian would be totally characteristic. But with a Riemannian b-metric, the volume of M is infinite. So with this kind of hermitian metric for M, it is not clear how to make sense of an analog to the theorem of Narasimhan and Seshadri, which assumes M to have unit volume.

The definition we gave in Section 1.2.3 assumes that the norm of a smooth frame for $^bT^{1,0}M$ vanishes to first order at ∂M (instead of being nonvanishing). This forces the volume of M to be finite, so that an analog for the theorem of Narasimhan and Seshadri becomes sensible.

1.4 Techniques

By restricting attention to manifolds of one complex dimension and to vector bundles of complex rank 1, we place ourselves in the simplest possible situation. And we make an important gain in doing so, because locally M will look like a patch in $\mathbb{C}$, and a section of our complex line bundle will look like a function of one complex variable. We therefore avoid the considerable complications associated to several complex variables and to higher-rank holomorphic vector bundles. This allows us to focus on the global geometric properties of the b-holomorphic curves and bundles without too many technical distractions. The main techniques come from the theory of functions of one complex variable, along with some results on Riemann surfaces thrown in.

In this simplified setting, it is not necessary to make explicit use of the general techniques for analysis of totally characteristic operators, although many features of the general

theory can be recognized in our results. But it may be expected that generalizations along various different lines might be achieved by making use of the general techniques. For example, one could study higher-dimensional manifolds, higher-rank bundles, and even singularities of different types at the boundary.

$$\overline{\mathbf{2}}$$

The Collar

Before proceeding toward the three objectives mentioned in Section 1.3, we need to gain some general understanding of the semi-global situation (near a boundary circle) and the pseudo-global situation (on the blow-down). In this chapter, we examine the semi-global picture; and in the next, we examine the pseudo-global picture. Then we will be ready to engage in the main line of inquiry, starting in Chapter 4.

Definition 2.1. A *collar* is a C^∞ surface with boundary, diffeomorphic to $[0,1) \times S^1$, equipped with a smooth anti-involution J on ${}^bTM$.

Clearly, a neighborhood of a boundary circle of any b-holomorphic complex curve is a collar.

2.1 The C^∞ structure

Here I summarize some useful facts about the compressed tangent and cotangent bundles of a collar. (Our references are [24], [25], and [26].) $X \equiv x\partial_x$ is a well-defined global section of ${}^bTM\big|_{\partial M}$. Here x is any global defining function for the boundary.

We have the natural map π from ${}^bTM$ to TM. Over the boundary, it degenerates by one dimension, taking ${}^bTM\big|_{\partial M}$ to $T\partial M$. The kernel is $N =$ the span of X. So we have the natural exact sequence

$$0 \longrightarrow N \longrightarrow {}^bTM\big|_{\partial M} \longrightarrow T\partial M \longrightarrow 0.$$

Dualizing, we find the natural exact sequence

$$0 \longrightarrow T^*\partial M \longrightarrow {}^bT^*M\big|_{\partial M} \longrightarrow N^* \longrightarrow 0.$$

Here N^* is the (abstract) dual to N. N^* has a natural global frame: $X^* =$ the dual section to X. Among those sections of ${}^bT^*M\big|_{\partial M}$ which are mapped to X^*, there is a distinguished set: those of the form $\frac{dx}{x}$ where x is again a global defining function for the boundary.

(Every $\frac{dx}{x}$ *is* mapped to X^*, clearly.) Any two of these special pre-images of X^* differ by an element of $T^*\partial M$ which is exact.

From the sequence, we see that there's a natural isomorphism

$$^bT^*M\big|_{\partial M} \text{ modulo } T^*\partial M \longrightarrow N^*.$$

If we choose an element ξ of the pre-image of X^*, this choice determines an isomorphism

$$N^* \oplus T^*\partial M \longrightarrow {}^bT^*M\big|_{\partial M}$$

by $(n, \eta) \mapsto \langle n, X\rangle \xi + \eta$. This yields up a way of projecting a compressed 1-form lying over the boundary to 1-form on the boundary, depending on the choice of ξ.

Everything extends nicely to complexifications.

2.2   Orientation

At this point, we must pause to establish a convention regarding the orientation of b-holomorphic complex curves as it relates to integration. Let M be a collar. Then M° is an ordinary Riemann surface. As such, it has a natural orientation induced by the complex structure: namely, $d\xi \wedge d\eta$ is a local oriented 2-form if $\xi + i\eta$ is a local holomorphic chart. This stipulation is coherent. That is, it distinguishes a ray in each fiber of $\Lambda^2 M$ in a continuous way.

The orientation extends to the boundary. That is, M is oriented as a C^∞ surface with boundary. Therefore ∂M (a circle) inherits an orientation from M. But from now on, whenever we speak of ∂M as an oriented manifold, and in particular when we integrate a 1-form over ∂M, we will take ∂M to have the orientation *opposite* to the one it inherits from M. Furthermore, whenever we speak of an angular coordinate y on (or near) ∂M, we will assume that dy is oriented (with this "opposite" orientation for ∂M); that is, that $\int_{\partial M} dy = +2\pi$.

We will point out the reason for this choice at the appropriate time. But it will be important to keep in mind that, with this orientation convention, Stokes's theorem must read

$$\int_M d\eta = -\int_{\partial M} \eta.$$

2.3   The holomorphic structure

Theorem 2.2. *There's a (complex) de Rham cohomology class η of ∂M which is defined by the holomorphic structure of the collar. We can therefore define the complex collar invariant c by*

$$1/c = \int_{\partial M} \eta.$$

Construction. Let x be a global defining function for the boundary. Let $\xi = \frac{dx}{x}$ so that ξ maps to X^*. As just noted, this defines projections

$$\pi_{T^*\partial M} \text{ and } \pi_{N^*}.$$

${}^b\Lambda^{1,0}M\big|_{\partial M}$ is a complex line bundle over a circle. So it has a global frame. Choose a global frame κ. Define

$$\eta = \frac{1}{2\pi i}\left[\frac{\pi_{T^*\partial M}(\kappa)}{\langle\pi_{N^*}(\kappa), X\rangle}\right].$$

Or, we can do it in coordinates. Let x be a global defining function for the boundary, and y a local angular coordinate. If $\kappa = \alpha\frac{dx}{x} + i\beta\,dy$ is a compressed 1,0-form lying over the boundary, then $\eta = \frac{1}{2\pi}\left[\frac{\beta}{\alpha}dy\right]$.   □

Proof of invariance under a change of x. Let $x, \tilde{x}$ be two defining functions for the boundary. Then $\tilde{x}/x$ is a positive smooth function. Define f by the relation $\tilde{x}/x = e^f$. So

$$\frac{d\tilde{x}}{\tilde{x}} - \frac{dx}{x} = d\log\tilde{x} - d\log x = d(\log\tilde{x} - \log x) = d\log(\tilde{x}/x) = df.$$

Now choose $\kappa = \alpha\frac{dx}{x} + i\beta\,dy$. Then

$$\kappa = \alpha\left(\frac{d\tilde{x}}{\tilde{x}} - df\right) + i\beta\,dy$$

$$= \alpha\left(\frac{d\tilde{x}}{\tilde{x}} - \tilde{x}f_{\tilde{x}}\frac{d\tilde{x}}{\tilde{x}} - f_y\,dy\right) + i\beta\,dy$$

$$= \alpha(1 - \tilde{x}f_{\tilde{x}})\frac{d\tilde{x}}{\tilde{x}} + i(\beta + i\alpha f_y)\,dy.$$

So over ∂M,

$$\kappa = \alpha\frac{d\tilde{x}}{\tilde{x}} + i(\beta + i\alpha f_y)\,dy.$$

So

$$\widetilde{\eta} = \frac{1}{2\pi} \left[\frac{\beta + i\alpha f_y}{\alpha} \, dy \right]$$

$$= \frac{1}{2\pi} \left[\frac{\beta}{\alpha} \, dy + i f_y \, dy \right]$$

$$= \frac{1}{2\pi} \left[\frac{\beta}{\alpha} \, dy + d_{\partial M}(if) \right]$$

$$= \frac{1}{2\pi} \left[\frac{\beta}{\alpha} \, dy \right]$$

$$= \eta.$$

□

Proof of invariance under a change of frame. Let $\widetilde{\kappa}$ be a different C^∞ frame for ${}^b\Lambda^{1,0}M$ over ∂M. Then $\widetilde{\kappa} = g\kappa$ for some nonvanishing function g. If we write $\kappa = \alpha \frac{dx}{x} + i\beta \, dy$, then $\widetilde{\kappa} = g\alpha \frac{dx}{x} + ig\beta \, dy$. So

$$\widetilde{\eta} = \frac{1}{2\pi} \left[\frac{g\beta}{g\alpha} \, dy \right]$$

$$= \frac{1}{2\pi} \left[\frac{\beta}{\alpha} \, dy \right]$$

$$= \eta.$$

□

Note that c may also be defined by

$$c = \frac{1}{\frac{1}{2\pi} \int_{\partial M} \frac{1}{q} \, dy}$$

where $x\partial_x + iq\partial_y$ is a semi-global frame for ${}^b T^{0,1}M$.

Theorem 2.3. *The real part of c is positive.*

Proof. Choose a smooth defining function x for the boundary and an angular coordinate y. Then dx points inward, and dy (regarded as a form on the boundary) is oriented opposite to the orientation of ∂M induced by the orientation of M. So $dx \wedge dy$ is an oriented 2-form on M near the boundary. So $\frac{dx}{x} \wedge dy$ is oriented on M.

Now, let $\frac{dx}{x} + i\beta \, dy$ be a non-vanishing $(1,0)$-form over ∂M. Write $\beta = A + iB$. Then our $(1,0)$-form may be written as

$$\left(\frac{dx}{x} - B \, dy \right) + i(A \, dy).$$

Therefore

$$\left(\frac{dx}{x} - B\,dy\right) \wedge (A\,dy) = A\frac{dx}{x} \wedge dy$$

is oriented. Thus A, being the quotient of two oriented 2-forms, is a positive function. So

$$\Re(1/c) = \Re\frac{1}{2\pi}\int_{\partial M} \beta\,dy = \frac{1}{2\pi}\int_{\partial M} A\,dy > 0.$$

Therefore $\Re c > 0$. □

Definition 2.4. A *collar isomorphism* is a diffeomorphism which respects the holomorphic structure of the compressed (co)tangent bundle. We will call two collars equivalent if there's a collar isomorphism from a neighborhood of the boundary of one to a neighborhood of the boundary of the other.

It's clear from the construction that c will be the same for two collars that are equivalent. But we should seek a converse. This is the problem we address in the next section.

2.4   First integrals

M denotes a collar from now on in this chapter.

2.4.1   Existence of first integrals, and classification of collars

In the following two definitions, $c = a + ib$ is a complex constant with $a > 0$.

Definition 2.5. We define the *standard c-collar* $\mathcal{M}_c$ as follows. As a set, $\mathcal{M}_c = [0, 1) \times S^1$, with radial coordinate x and angular coordinate y. The b-holomorphic structure is defined by stipulating that the compressed cotangent bundle of type $(1, 0)$ be spanned by $c\,dx/x + i\,dy$. This means that the compressed tangent bundle of type $(0, 1)$ is spanned by $x\partial_x + ic\partial_y$.

Definition 2.6. We define $\chi_c : D\backslash 0 \longrightarrow (0, 1) \times S^1$ by

$$\chi_c(re^{i\theta}) = \left(r^{1/a}, \theta - (b/a)\log r\right). \tag{2.1}$$

Lemma 2.7. *As a map from $D\backslash 0$ to $\mathcal{M}_c^\circ$, χ_c is a biholomorphism.*

Proof. χ_c is a bijection. The inverse is given by $\chi_c^{-1}(x, y) = x^c e^{iy}$. So

$$\frac{d(\chi_c^{-1})}{\chi_c^{-1}} = c\frac{dx}{x} + i\,dy,$$

which is a $(1, 0)$-form. So $d(\chi_c^{-1})$ itself is of type $(1, 0)$. That is, χ_c^{-1} is holomorphic. □
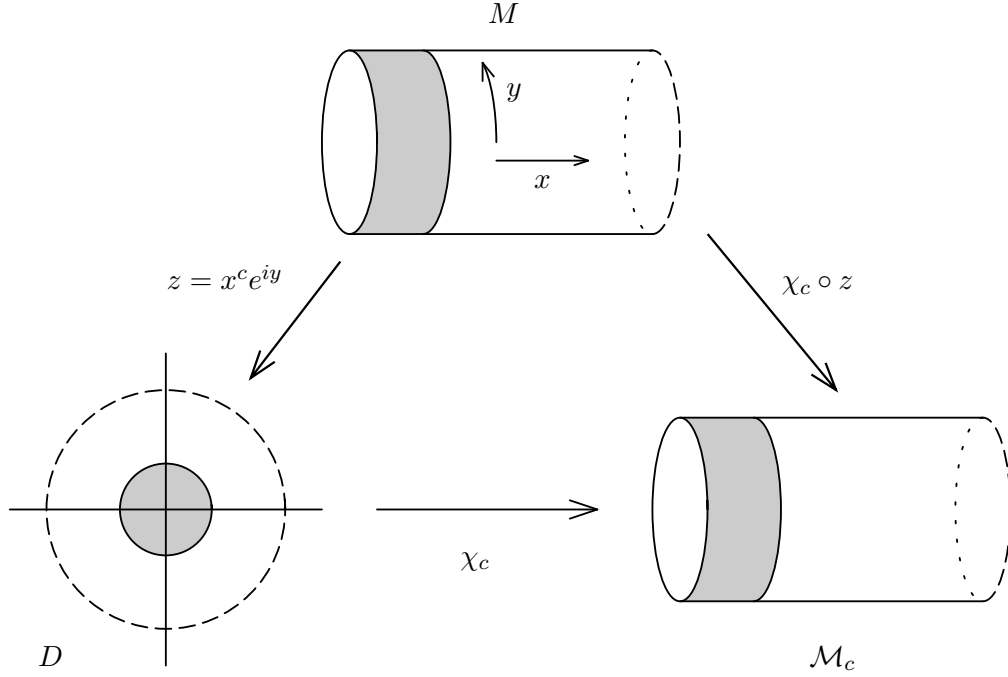
Figure 1: The Collar

We now offer four propositions. We prove their equivalence, but not their truth. It is not clear whether they are true for all collars, or if they are not, in which cases they fail. From now on, we restrict ourselves to the (nontrivial) class of collars for which these propositions are valid. The truth of these propositions is the "extra condition" we spoke of in Section 1.2.1.

Proposition 2.8. *There's a collar isomorphism φ from M to $\mathcal{M}_{\text{const}}$.*

Proposition 2.9. *There exists a global C^∞ (real) defining function x for the boundary of M and a real C^∞ function y on M minus a cut transversal to ∂M such that:*

(a) y *has a jump discontinuity of 2π across the cut;*

(b) dy *extends as a C^∞ form to all of M and is independent of dx; and*

(c) $(\text{const})\frac{dx}{x} + i\, dy$ *is type $(1,0)$ for some complex constant.*

Proposition 2.10. *There exists a smooth function ϕ on the interior of M minus a cut transversal to ∂M such that:*

(a) $\Im\phi - (\text{const}_1)\Re\phi$ is a C^∞ angular coordinate for M for some real constant;

(b) $e^{(\text{const}_2)\Re\phi}$ is a global C^∞ defining function for the boundary of M for some real constant; and

(c) the differential $d\phi$ is type $(1,0)$.

Proposition 2.11. *There exists a smooth nonvanishing function z on the interior of M such that:*

(a) $\Im\log z - (\text{const}_1)\Re\log z$ is a C^∞ angular coordinate for M for some real constant;

(b) $e^{(\text{const}_2)\Re\log z}$ is a global C^∞ defining function for the boundary of M for some real constant; and

(c) the logarithmic differential dz/z is type $(1,0)$.

Note 2.12. In Propositions 2.8 and 2.9, the constant is necessarily equal to c. In Propositions 2.10 and 2.11, the constants are necessarily $\frac{b}{a}$ and $\frac{1}{a}$ if we write $c(M) = a + ib$.

Proof of equivalence. Refer to Figure 1.

(1) $\implies$ (2): x is projection of φ on the first factor; y is projection of φ on the second factor.

(2) $\implies$ (3): $\phi = c\log x + iy$.

(3) $\implies$ (4): $z = e^\phi$.

(4) $\implies$ (1): $\varphi = \chi_c \circ z$. $\qquad\qquad\square$

A classification of collars now follows trivially from Proposition 2.8. In other words, by assuming the truth of these propositions, we are assuming that the collars that appear as collars of b-holomorphic curves are classified by the collar invariant c.

Theorem 2.13. *Let M_1 and M_2 be two collars. Then M_1 is equivalent to M_2 if and only if $c(M_1) = c(M_2)$. The reason: for any collar M, $M \simeq \mathcal{M}_{c(M)}$.*

Example 2.14. Let $M_1 = M^{1/n}$, with n some positive whole number. Let M_2 be the n-sheeted branched cover of the unit disk associated to $z^{1/n}$. Then $(x, y) \mapsto z = xe^{iny}$ is a biholomorphic blow-down of M_1 to M_2, because

$$\frac{dz}{z} = \frac{dx}{x} + in\,dy.$$

It will be useful to keep in mind the special cases $n = 1$ and $n = 2$ (which means $c = 1$ or $c = 1/2$). They correspond to the blow-up (at zero) of the disk in $\mathbb{C}$ and the blow-up (at the branch point) of the two-sheeted cover of the disk corresponding to the function $\sqrt{z}$.

Remark 2.15. Every complex number with positive real part is realized as c for some collar.

Definition 2.16. A function z of the sort whose existence is asserted in Proposition 2.11 will be called a *first integral* for (the holomorphic structure of) M.

It should be recorded here that many attempts were made to either prove the existence of first integrals, or find a counterexample. All attempts have been unsuccessful. And although some positive information was gained about the coordinate changes that are possible in general, the results are somewhat weak, and in the present discussion, they are inconsequential. We therefore omit the details.

Remark 2.17. If z is a first integral for M, then dz/z is a smooth frame for ${}^b\Lambda^{1,0}M$. This is because with x and y defined as in the proof of equivalence of the four propositions,

$$\frac{dz}{z} = c\frac{dx}{x} + i\,dy,$$

and x, y are C^∞ boundary coordinates for M.

By definition, z is a first integral if and only if z is holomorphic and $z = x^c e^{iy}$ for some C^∞ boundary coordinates x, y. We have also seen that z is a first integral if and only if $\chi_c \circ z : M^\circ \longrightarrow \mathcal{M}_c^\circ$ extends to M as a collar isomorphism.

2.4.2   Uniqueness of first integrals

Lemma 2.18. *A first integral is a biholomorphism from the collar's interior to a punctured neighborhood of zero in $\mathbb{C}$.*

Proof. Write $z = x^c e^{iy}$ where x is a defining function for the boundary and y is an angular coordinate. z is clearly a bijection from the collar's interior to a punctured neighborhood of zero in the complex plane. Since dz/z is type $(1, 0)$, so dz is also type $(1, 0)$. Therefore z is holomorphic. Since dz/z is a frame, and z is nonvanishing, so dz is also a frame. Therefore z is a local biholomorphism. Thus z is a bijective local biholomorphism; that is, a global biholomorphism. $\square$

Technical Lemma 2.19. *Let ζ be a biholomorphism from a neighborhood of zero to a neighborhood of zero which takes zero to zero.*

(a) If $z\zeta'/\zeta$ is constant, then so is ζ/z.

(b) If $z\zeta'/\zeta$ is an analytic function of z^q for some $q \in \mathbb{Z}_{>0}$, then so is ζ/z.

(From now on, ζ' is written when we mean $\partial\zeta/\partial z$.)

Proof of (a). Write $\zeta = c_1 z + c_2 z^2 + \cdots$. Then for some constant a we have $z\zeta' = a\zeta$, or

$$c_1 z + 2c_2 z^2 + 3c_3 z^3 + \cdots = ac_1 z + ac_2 z^2 + ac_3 z^3 + \cdots.$$

Since ζ is injective, $c_1 \neq 0$; so we must have $a = 1$. So the equation becomes

$$c_1 z + 2c_2 z^2 + 3c_3 z^3 + \cdots = c_1 z + c_2 z^2 + c_3 z^3 + \cdots.$$

Identifying power series coefficients yields $nc_n = c_n$ for all $n \geq 1$. So $c_n = 0$ for all $n \geq 2$, and we are left with $\zeta = c_1 z$. $\qquad\square$

Proof of (b). We could use the same method as in the proof of (a), but we'd have to multipy power series and keep track of the coefficients of the product. So we'll use a different method instead. Write

$$\frac{z\zeta'}{\zeta} = a_0 + a_q z^q + a_{2q} z^{2q} + \cdots.$$

Then

$$\frac{\zeta'}{\zeta} = a_0 z^{-1} + a_q z^{q-1} + a_{2q} z^{2q-1} + \cdots.$$

But note that

$$1 = \text{index of } \zeta \text{ on a small loop around zero}$$
$$= \frac{1}{2\pi i} \int \frac{\zeta'}{\zeta} dz$$
$$= \text{residue of } \zeta'/\zeta \text{ at zero}$$
$$= a_0.$$

So actually we have

$$\frac{\zeta'}{\zeta} = z^{-1} + a_q z^{q-1} + a_{2q} z^{2q-1} + \cdots.$$

Let $g = \zeta/z$. Then

$$g'/g = \zeta'/\zeta - z^{-1} = a_q z^{q-1} + a_{2q} z^{2q-1} + \cdots.$$

Now since ζ is injective, g is nonvanishing near zero. So we may choose a logarithm of g there and call it $\log g$. So we have

$$(\log g)' = a_q z^{q-1} + a_{2q} z^{2q-1} + \cdots ,$$

so that

$$\log g = b_0 + b_q z^q + b_{2q} z^{2q} + \cdots ,$$

where $b_{nq} = a_{nq}/nq$ for $n \geq 1$. So $\log g$ is an analytic function composed with the qth power function. So

$$g = \exp \circ \log g = \exp \circ (\text{analytic}) \circ (q\text{th power}) = (\text{analytic}) \circ (q\text{th power}).$$

$\square$

Theorem 2.20. *Let z and ζ be first integrals for a collar. If $c \in \mathbb{Q}$, with $c = p/q$ in reduced form, then ζ/z is an analytic function of z^q. If $c \notin \mathbb{Q}$, then ζ/z is constant.*

Proof. Suppose z and ζ are first integrals. Let x and y be the C^∞ radial and angular coordinates associated to z, and define $\widetilde{x}$ and $\widetilde{y}$ similarly for ζ. Then $z = x^c e^{iy}$ and $\zeta = \widetilde{x}^c e^{i\widetilde{y}}$. So we have two compressed cotangent sections of type $(1,0)$, namely

$$\frac{dz}{z} = c\frac{dx}{x} + i\, dy$$
$$\frac{d\zeta}{\zeta} = c\frac{d\widetilde{x}}{\widetilde{x}} + i\, d\widetilde{y}.$$

Each is a smooth frame for ${}^b\Lambda^{1,0}M$. They are related by

$$\frac{d\zeta}{\zeta} = \frac{\zeta' dz}{\zeta} = \frac{z\zeta'}{\zeta}\frac{dz}{z}.$$

So the transition function $z\zeta'/\zeta$ must be C^∞ up to $x = 0$.

Also, each of z and ζ is a biholomorphism from the collar's interior to a punctured neighborhood of zero in the complex plane (by a lemma above). So, thought of as a function of z, ζ is a biholomorphism from a punctured neighborhood of zero to a punctured neighborhood of zero. But such a function is automatically regular at zero with value 0 there. So, $\zeta(z)$ is actually a biholomorphism from a neighborhood of zero to a neighborhood of zero, taking zero to zero. So ζ is a series in positive powers of z. So the transition function $g(z) = \frac{z\zeta'}{\zeta}$ is a series in non-negative powers of z since ζ was.

We have found that

$$g(x, y) = c_0 + c_1 x^c e^{iy} + c_2 x^{2c} e^{i2y} + \cdots .$$

This is a Fourier series in y each of whose coefficients is a function of x. Since g is C^∞ down to $x = 0$ and x is a defining function for the boundary, each of these coefficients must be C^∞ down to $x = 0$ as a function of x.

Let $n \geq 1$. Then the nth coefficient of g is $c_n x^{nc}$. Now x^{nc} is C^∞ down to $x = 0$ only if $nc \in \mathbb{Z}$. This can only happen if c is rational (write it in reduced form as p/q) and n is a multiple of q. We just have to summarize:

If $c = p/q$ is a rational in reduced form, then $c_n = 0$ unless n is a multiple of q. And if c is not rational, then $c_n = 0$ for all $n \geq 1$. This means that if $c = p/q$ is a rational in reduced form, then $z\zeta'/\zeta$ is an analytic function of q; and if c is not rational, then $z\zeta'/\zeta$ is constant. These properties of $z\zeta'/\zeta$ are carried over to ζ/z by the technical lemma. $\qquad \square$

Converse 2.21. *Let z be a first integral. Then any nonzero complex-constant multiple of z is also a first integral. If $c = p/q$ is a rational in reduced form, then z times any analytic function of z^q with nonzero constant term is also a first integral.*

Proof. The first assertion is obvious. So we focus on the second. Let $\zeta = zg(z)$, with g being some analytic function of z^q with nonzero constant term. Clearly ζ is holomorphic since z was. By the first assertion we can assume WLOG that the constant term of g is 1. Now define

$$x = e^{\frac{q}{p} \Re \log z} = |z|^{q/p}$$

$$y = \Im \log z,$$

then define $\widetilde{x}$ and $\widetilde{y}$ similarly using ζ. Then x and y form C^∞ coordinates for the collar, with $x = 0$ defining the boundary. We have to check that this is also true of $\widetilde{x}$ and $\widetilde{y}$.

Compute

$$\frac{\widetilde{x}}{x} = \frac{|\zeta|^{q/p}}{|z|^{q/p}} = \left|\frac{\zeta}{z}\right|^{q/p} = |g|^{q/p} = (|g|^2)^{q/2p}.$$

We'll work from the inside out to show this is C^∞ at $x = 0$. $z^q = x^p e^{iqy}$ is C^∞. So g, which is an analytic function of z^q, is also C^∞. So $\Re g$ and $\Im g$ are C^∞. So $|g|^2 = (\Re g)^2 + (\Im g)^2$ is C^∞. $|g|^2$ is bounded away from zero at the boundary; and away from zero, the function $(\cdot)^{q/2p}$ is C^∞. So the compostition $(|g|^2)^{q/2p}$ is C^∞. Also, g is a nonzero constant on the boundary by hypothesis. So $(|g|^2)^{q/2p}$ is a strictly positive constant on the boundary. Thus $\widetilde{x}/x$ is C^∞ and nonvanishing at the boundary. So $\widetilde{x}$ is a defining function for the boundary.

Now compute

$$\widetilde{y} - y = \Im \log \zeta - \Im \log z = \Im \log(\zeta/z) = \Im \log g.$$

Now, g is bounded away from zero near the boundary. So we can regard this $\log g$ in a full neighborhood of the boundary, without a cut. Namely,

$$\log g(z) = \int_p^z \frac{g'(\xi)}{g(\xi)} d\xi.$$

Suppose $q = 1$. Then g is analytic in ξ, so $\log g$ is also analytic in ξ. But suppose $q > 1$. Then g is analytic in ξ^q with nonzero constant term. So it must be true that g'/g is analytic in ξ^{q-1} with no constant term. This is a combinatorial fact which can be verified directly. So by the integral representation above, $\log g$ is analytic in z^q, no matter what q is. So it is C^∞ at $x = 0$ and globally defined. It is clearly constant at the boundary. So $\widetilde{y} - y = \Im \log g$ is C^∞ at $x = 0$ and constant there. Thus, since y is a "global" angular coordinate, $\widetilde{y}$ is as well. $\qquad \square$

Corollary 2.22 (uniqueness of first integrals). *Let z be a first integral, and ζ be some other function. If c is not rational, then ζ is a first integral if and only if ζ/z is a nonzero constant. If $c = p/q$ is a rational in reduced form, then ζ is a first integral if and only if ζ/z is an analytic function of z^q with nonzero constant term.*

Remark 2.23. A first integral is, essentially, a pair of C^∞ coordinates for the collar, all packaged together. We require that $x^c e^{iy}$ be holomorphic; this gives us a set of "distinguished" pairs of real coordinates. The situation is analogous to the ordinary punctured disk in $\mathbb{C}$; here, we have a family of special (conformal) coordinate pairs x, y which are distinguished by the fact that $x + iy$ is holomorphic and maps to another punctured neighborhood of zero. Here, multiplication by any analytic function of $x + iy$ with nonzero constant term will define a new conformal pair. But in the situation of the collar, the class of special coordinate pairs is further restricted, because we impose an additional condition beyond conformality: namely, that x be a defining function for the boundary. This can be rephrased by saying that the conformal pair x, y is compatible with the C^∞ structure at the boundary. Thus, the class of allowable change-of-coordinate functions is reduced. Instead of being able to multiply by any analytic function of $x^c e^{iy}$ with nonzero constant term, we are only allowed to multiply by a constant (if c is not rational) or by an analytic function of $x^p e^{iqy}$ with nonzero constant term (if $c = p/q$ is a rational in reduced form). Exactly why

we have considerably more freedom in the choice of a first integral when c is rational than when it is not (though we still have less freedom than in the analogous case of the marked or punctured disk) is not clear.

2.5   Collar automorphisms

Since a first integral is essentially a C^∞ biholomorphism from a collar to the standard c-collar, our knowledge of first integrals can be rephrased as information about collar automorphisms. (Actually, we could have started from an analysis of automorphisms of the standard c-collar and used this to understand the first integrals of an arbitrary collar.) Throughout this section, c is the collar invariant of M.

Lemma 2.24. *Let z be a first integral. Then $\varphi : M \to M$ is a collar automorphism if and only if $z \circ \varphi$ is a first integral.*

Proof of $\implies$. Since z is a first integral, we have the diagram

$$M \xrightarrow{\chi_c \circ z} \mathcal{M}_c,$$

with the arrow representing a collar isomorphism by Remark 2.17. Now since φ is a collar automorphism, we can extend the diagram by

$$M \xrightarrow{\varphi} M \xrightarrow{\chi_c \circ z} \mathcal{M}_c,$$

which can be redrawn as

$$M \xrightarrow{\chi_c \circ (z \circ \varphi)} \mathcal{M}_c,$$

and the long path is still a collar isomorphism. So by Remark 2.17, $z \circ \varphi$ is a first integral. $\square$

Proof of $\impliedby$. This is the same argument, only played in reverse. If $z \circ \varphi$ is a first integral, then we have the diagram

$$M \xrightarrow{\chi_c \circ (z \circ \varphi)} \mathcal{M}_c$$

with the long path being a collar isomorphism. Since z is a first integral, we also have the diagram

$$M \xrightarrow{\chi_c \circ z} \mathcal{M}_c$$

with the long path being a collar isomorphism. Thus the "difference" φ is a collar automorphism $M \longrightarrow M$. $\square$

Theorem 2.25 (collar automorphisms). *Let M be a collar with first integral z. Let φ be a function from M to itself. If c is not rational, then φ is a collar automorphism if and only if it is conjugate via z to multiplication by a nonzero constant: $z \circ \varphi \circ z^{-1} =$ multiplication by A, $A \neq 0$. If $c = p/q$ is a rational in reduced form, then φ is a collar automorphism if and only if it is conjugate via z to multiplication by an analytic function of the qth power with nonzero constant term: $z \circ \varphi \circ z^{-1} =$ multiplication by (analytic)$\circ$(qth power), (analytic)$_0 \neq 0$.*

Proof. By Lemma 2.24, φ is a collar automorphism if and only if $z \circ \varphi$ is a first integral. Suppose c is not rational. By Corollary 2.22, $z \circ \varphi$ is a first integral if and only if $z \circ \varphi \circ z^{-1}$ is multiplication by a nonzero constant. Suppose $c = p/q$ is a rational in reduced form. By Corollary 2.22, $z \circ \varphi$ is a first integral if and only if $z \circ \varphi \circ z^{-1}$ is multiplication by an analytic function of the qth power with nonzero constant term. $\square$

2.6 Independence of collar invariants

The following theorem shows that any combination of appropriate constants c_i can appear as the collar invariants for the collars of a b-holomorphic complex curve.

Theorem 2.26. *Let M be a C^∞ surface with boundary. Label the boundary circles $\mathcal{C}_1, \ldots, \mathcal{C}_n$. Let M_i denote a neighborhood of C_i. Then for any n complex numbers $c_1, \ldots, c_n$ with positive real part, there exists a b-holomorphic structure for M such that $c(M_i) = c_i$.*

Proof. M may be realized as a genus g compact (C^∞) surface with n open disks cut out of it. So we begin with the compact surface of genus g, equipped with a (non-singular) holomorphic structure. Choose n disjoint unit disks D_i on this surface, each with local complex coordinate $z_i = xe^{iy}$ (x and y are ordinary radial coordinates). Excise the closed disk of radius $1/2$ from each of the D_i and throw it away. Call the resulting surface $\widetilde{M}$. For each i we will prepare a collar M_i to graft onto D_i, with $c(M_i) = c_i$, in such a way that the coordinate change function from D_i to M_i is holomorphic.

As a C^∞ manifold, $M_i = [0, 1) \times S^1$. It now suffices to define a C^∞ function ζ_i there with the following properties:

1. The Jacobian of ζ_i (as a map into $\mathbb{R}^2$) is everywhere non-singular.

2. Near $x = 0$, $\zeta_i = x^{c_i} e^{iy}$.

3. For $\frac{1}{2} < x < 1$, $\zeta_i = xe^{iy}$.

Condition 1 says that M_i° is a holomorphic manifold. Condition 2 says that M_i is a collar and that $c(M_i) = c_i$. And Condition 3 says that if we graft M_i onto $\widetilde{M}$ by identifying the point x, y of M_i with the point xe^{iy} of D_i for each x between $\frac{1}{2}$ and 1 and every y, then the change of coordinates from z to ζ is $z \mapsto z$ (which is holomorphic). We proceed with the construction. Fix i. Write $c_i = a + ib$. For brevity we will assume that $a > 1$, the other cases being very similar. From now on, we drop the subscript i.

The strategy is simple. We will pick some function ϕ of x only which is 0 near $x = 0$ and 1 for $x \geq 1/2$, and define $g = (1 - \phi)u + \phi v$, where $u(x) = x^c$ and $v(x) = x$. Then we define $\zeta = ge^{iy}$. Conditions 2 and 3 are satisfied automatically by this construction. So all we have to do is choose ϕ wisely so that the Jacobian of ζ is non-singular. It is easy to compute that this Jacobian is $\frac{1}{2}(|g|^2)'$. So it suffices to show that (for some good choice of ϕ) $|g|^2$ has nonzero derivative.

We define $A = (1 - \phi)x^a$, $B = \phi x$, and $\theta = b \log x$. Then we write $g = Ae^{i\theta} + B$.

Near 0, $x > x^a$. Pick a point x_0 in this region so that $b \log x_0$ is a multiple of 2π: $x_0 = e^{2\pi\ell/b}$ for some ℓ. We're going to make ϕ go from 0 to 1 in a small neighborhood of x_0, so that θ remains close to $2\pi\ell = \theta(x_0)$ there. So for some $\Delta\theta$ (to be determined later) we let $x_L = e^{(2\pi\ell - \Delta\theta)/b}$, $x_R = e^{(2\pi\ell + \Delta\theta)/b}$; then let ϕ be smooth and monotone, 0 to the left of x_L, and 1 to the right of x_R. I claim that if we choose $\Delta\theta$ small enough, and ϕ nice, then $|g|^2$ will be strictly increasing in (x_L, x_R). (It obviously is elsewhere.)

Compute that $|g|^2 = A^2 + B^2 + 2AB \cos\theta$. Making second order approximations for small θ, we find that

$$(|g|^2)' = 2(A + B)(A + B)' - [A'B\theta^2 + AB'\theta^2 - 2AB\theta\theta'].$$

First we look at term I. $(A + B)$ is bounded away from 0 in the interval (x_L, x_R). Also,

$$(A + B)' = ax^{a-1} + \phi + \phi'(x - x^a).$$

The first term of this is bounded away from 0, and the other two are nonnegative (in the interval). So term I is bounded away from 0, independent of ϕ.

Next, we have to see that term II is uniformly small in absolute value there, with a small choice of $\Delta\theta$ (good choice of ϕ). First of all, we have to see how ϕ' is related to $\Delta\theta$. Compute that

$$x_R - x_L = 2e^{2\pi\ell/b} \sinh\frac{\Delta\theta}{b} \sim \left(\frac{2}{b}e^{2\pi\ell/b}\right)\Delta\theta$$

(again approximating for $\Delta\theta$ small). So there's a constant d such that for any small $\Delta\theta$, we can choose the ϕ so that $\phi' \leq d\frac{1}{\Delta\theta}$. Now you compute the three terms in term II, make use of this estimate for ϕ', and see that in each one, we have something which is bounded (in the interval (x_L, x_R)) times either one or two powers of $\Delta\theta$. $\qquad\square$

2.7 Bundles over a collar: c non-rational

When we examine line bundles over a generic b-holomorphic complex curve, we will assume that all the collar invariants are in $(\mathbb{C}\backslash\mathbb{Q}) \cup 1$. This is only a matter of convenience. The reason is that $c = 1$ and $c \notin \mathbb{Q}$ represent the limiting cases $q = 1$ and $q = \infty$ of $c = p/q$. This is meant in more than just a formal sense; that is, the analysis required for collar invariants in $\mathbb{Q}\backslash 1$ is essentially a mixture of what happens for $c = 1$ and for $c \notin \mathbb{Q}$. Time and space restrictions suggest that it would be superfluous to write out the explicit calculations for $c \in \mathbb{Q}\backslash 1$.

In this section, M is a collar with $c \notin \mathbb{Q}$.

We wish to classify the holomorphic line bundles over M, the purpose being to establish some basic results and pave the way for our global analysis of bundles over a b-holomorphic complex curve. Now, every C^∞ complex line bundle E over a collar (or over any surface with non-empty boundary) has a global frame. So we may (whenever we wish) regard E as $M \times \mathbb{C}$ equipped with a smooth section α of ${}^b\Lambda^{0,1}M$, whereby $\bar\partial\mathbf{1} = \mathbf{1} \otimes \alpha$. But it has to be remembered that this $\mathbf{1}$ is not canonical.

One of our most basic tools for dealing with line bundles will be the Cauchy integral formula, which we formulate as follows.

Lemma 2.27 (Cauchy integral formula). *Let α be a compactly supported $(0,1)$-form over $\mathbb{C}$. Define*

$$f(z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\alpha(\xi)\wedge d\xi}{z - \xi}.$$

Then

(a) $\bar\partial f = \alpha$.

(b) In a neighborhood of ∞, f can be written as the convergent power series

$$f(z) = \sum_{n\geq 1}\left[\frac{1}{2\pi i}\int_{\mathbb{C}} \xi^n\alpha(\xi)\wedge \frac{d\xi}{\xi}\right]z^{-n}.$$

(c) *If $\alpha = 0$ in a neighborhood of zero, then in such a neighborhood f can be written as the convergent power series*

$$f(z) = \sum_{n \geq 0} \left[\frac{(-1)^n}{2\pi i} \int_{\mathbb{C}} \xi^{-n} \alpha(\xi) \wedge \frac{d\xi}{\xi} \right] z^n.$$

Proof of (a). Standard. $\square$

Proof of (b). Define $\varphi(z) = f(1/z)$. Since α is compactly supported, f must be analytic in a (punctured) neighborhood of infinity. This says precisely that φ is analytic in a punctured neighborhood of zero; that is, it has a Laurent series. But we can see directly from the formula for f that f approaches 0 at infinity. So φ approaches 0 at 0. So φ is a series in positive powers of z only.

We compute directly that for any $n \geq 1$,

$$\varphi^{(n)}(z) = \frac{n!}{2\pi i} \int \frac{\xi^{n-1} \alpha(\xi) \wedge d\xi}{(1 - z\xi)^n}.$$

So

$$\varphi^{(n)}(0) = \frac{n!}{2\pi i} \int \xi^n \alpha \wedge \frac{d\xi}{\xi}.$$

Thus we have the power series representation (near 0)

$$\varphi(z) = \sum_{n \geq 1} \left[\frac{1}{2\pi i} \int \xi^n \alpha \wedge \frac{d\xi}{\xi} \right] z^n.$$

Therefore

$$f(z) = \varphi(1/z) = \sum_{n \geq 1} \left[\frac{1}{2\pi i} \int \xi^n \alpha \wedge \frac{d\xi}{\xi} \right] z^{-n},$$

which was to be proven. $\square$

Proof of (c). Similar to the proof of (b). $\square$

Remark 2.28 (Taylor's theorem). Let u be a complex-valued C^∞ function in a neighborhood of zero in the complex plane. Then by a combinatorial re-shuffling of Taylor's theorem, we may say that

$$u \sim \sum_{\substack{m \geq 0 \\ n \geq 0}} u_{mn} z^m \bar{z}^n$$

with $u_{mn} = m! n! \partial_z^m \partial_{\bar{z}}^n u(0)$. This only needs a bit of interpreting. First, the derivatives in z and $\bar{z}$ are not meant in the sense of limits of difference quotients in the complex variables;

such limits do not exist unless u happens to be holomorphic or anti-holomorphic. They are meant in the other sense, as linear combinations of ordinary partial derivatives in x and y. That is,

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$$

$$\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y).$$

Second, the asymptotic formula is to be interpreted as meaning: for any $N \geq 0$,

$$\left[u - \sum_{m+n \leq N} u_{mn} z^m \bar{z}^n \right] \div |z|^{N+1}$$

is bounded in a (punctured) neighborhood of zero.

In general, a holomorphic line bundle over a collar does not possess a holomorphic C^∞ global frame. Yet we know that the pull-back of the bundle to the collar's interior does posess a holomorphic C^∞ global frame [11]. So our strategy for understanding these bundles will be to seek a special global holomorphic frame over the interior of the collar, whose behavior at the boundary (with respect to a global C^∞ frame) is known.

Definition 2.29. An nth power of a first integral ($n \geq 1$) is called an *nth integral*. (A zeroth integral is a nonzero complex constant.)

Lemma 2.30. *Any non-zero constant multiple of an nth integral is another nth integral.*

Definition 2.31. An infinite formal series, whose nth entry ($n \geq 0$) is an nth integral (or zero), will be called an *integral sequence*.

Definition 2.32. Two integral sequences shall be called equivalent if their difference, which is a formal power series in any chosen first integral z, has a positive radius of convergence and has its constant term equal to an integral multiple of c. This condition is independent of the first integral (and is, in fact, a relation of equivalence).

Lemma 2.33. *Let $M = [0, \infty)_x \times S_y^1$. Let $c = a + ib$ (with $a > 0$) be a non-rational constant. Let $z = x^c e^{iy}$. Let $\bar{L} = 2a\bar{z}\partial_{\bar{z}} = x\partial_x + ic\partial_y$. If f is a compactly supported C^∞ function on M, and the average value of f on the boundary of M is f_{00}, then there exists a C^∞ solution u to $\bar{L}u = f$ on M° such that u tends to zero at $x = \infty$ and $u - f_{00} \log x$ is bounded at $x = 0$. Such a solution is unique, and will be called the* unique good solution *to $\bar{L}u = f$ on M. Moreover, u can be written in the form*

$$u = h_1 + \phi f_{00} \log x + h_2$$

where h_1 is C^∞ on M, ϕ is a C^∞ cut-off function supported in a neighborhood of ∂M, and h_2 is a C^∞ function on M° which has an asymptotic expansion at the boundary in positive powers of z.

Note 2.34. z is a biholomorphism from M° to the punctured complex plane. So we will shift between these two points of view freely.

Proof of uniqueness. Suppose u and $\widetilde{u}$ are solutions with the required decay properties, and let $v = \widetilde{u} - u$. Then $\bar{L}v = 0$ in M°. So $2a\bar{z}\partial_{\bar{z}}v = 0$ in the punctured z-plane. Because of the decay properties of u and $\widetilde{u}$, we know that v tends to zero at $z = \infty$. Also, v is bounded at $z = 0$, because $v = (\widetilde{u} - f_{00}\log x) - (u - f_{00}\log x)$. So we have a holomorphic function on the punctured z-plane which is bounded at $z = 0$ and which tends to zero at $z = \infty$. The only such function is the zero function. $\qquad\square$

Proof of existence. Choose a cut-off function ϕ which is supported near ∂M, whose variation takes place outside the support of f, and which is a function of x only. Write

$$f \sim \sum_{\substack{m \geq 0 \\ n \in \mathbb{Z}}} f_{mn}x^m e^{iny}.$$

Define $u^1_{mn} = f_{mn}/(m - nc)$ for $m \geq 0$, $n \in \mathbb{Z}$, $(m,n) \neq (0,0)$ and choose a compactly supported C^∞ function u^1 on M such that

$$u^1 \sim \sum_{\substack{m \geq 0 \\ n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} u^1_{mn}x^m e^{iny}.$$

Let u^2 be the Cauchy solution (Lemma 2.27) in the z-plane to $\partial_{\bar{z}}u^2 = -f_{00}(\partial_{\bar{z}}\phi)\log x$. (The datum is compactly supported away from the boundary of M, so we can regard it as a compactly supported C^∞ function in the z-plane.) Let u^3 be the Cauchy solution in the z-plane to

$$\partial_{\bar{z}}u^3 = \frac{1}{2a\bar{z}}(f - f_{00})\phi - \partial_{\bar{z}}u^1.$$

(We must explain why the datum is a C^∞ function of compact support in the z-plane. $(f - f_{00})\phi - \bar{L}u^1$ vanishes to infinite order at the boundary, by construction. So if we push down to the z-plane, this function vanishes to infinite order at $z = 0$. Thus we can divide by $2a\bar{z}$ to get a (compactly supported) C^∞ function in the z-plane. This is our datum.)

Now let $u = f_{00}\phi \log x + u^1 + u^2 + u^3$. u^1 is compactly supported and C^∞ on M. u^2 and u^3, being Cauchy solutions in the z-plane, tend to zero at $z = \infty$ and are C^∞ at $z = 0$ (which means they are bounded at ∂M). Therefore u has the correct decay properties. A direct (trivial) calculation from the construction shows that $\bar{L}u = f$. $\qquad\square$

Proof of "moreover". Since u^3 is C^∞ at $z = 0$ in the z-plane, we may write

$$u^3 \sim \sum_{\substack{m \geq 0 \\ n \geq 0}} u_{mn} z^m \bar{z}^n.$$

But we know that $\partial_{\bar{z}} u^3$ vanishes to infinite order at $z = 0$, by its construction. So we find (see Remark 2.28) that, for $n \geq 1$,

$$u^3_{mn} = m!\, n!\, \partial_z^m \partial_{\bar{z}}^{n-1} (\partial_{\bar{z}} u^3)(0) = 0.$$

So the only terms that survive are those with $n = 0$. So

$$u^3 \sim \sum_{m \geq 0} u^3_{m0} z^m \bar{z}^0 = \sum_{m \geq 0} u^3_{m0} z^m.$$

Now define $h_2 = u^3 - u^3_{00}$. Then h_2 is a C^∞ function on M° which has an asymptotic expansion at the boundary in positive powers of z.

Next define $h_1 = u^1 + u^2 + u^3_{00}$. Then we have

$$u = f_{00}\phi \log x + u^1 + u^2 + u^3$$
$$= h_1 + f_{00}\phi \log x + h_2.$$

h_2 has the correct properties. Now we only need to look at h_1, and show that it is C^∞ on M. u^1 and u^3_{00} are C^∞ on M. So we only need to show that u^2 is C^∞ on M. u^2 is a Cauchy solution in the z-plane, and the datum is zero in a neighborhood of $z = 0$. Thus u^2 may be represented as a convergent power series in non-negative powers of z near $z = 0$. I claim that all the power series coefficients are zero except the zeroth, so that u^2 is constant in a neighborhood of $z = 0$ and therefore C^∞ on M. Proof of this claim:

Let $n \geq 1$, and compute (ignoring multiplicative constants) that, by Lemma 2.27, the nth coefficient is

$$\int_{\mathbb{C}} \xi^{-n} (-f_{00} \log x(\xi)\, \partial_{\bar{\xi}}\phi\, d\bar{\xi}) \wedge \frac{d\xi}{\xi} = \int_{\mathbb{C}} \frac{\log x(\xi)}{\xi^{n+1}} \bar{\partial}\phi \wedge d\xi$$
$$= \int_G \frac{\log x(\xi)}{\xi^{n+1}} \bar{\partial}\phi \wedge d\xi$$

where G is an annular region containing the variation of ϕ whose bounding circles are level curves for x. Integrate by parts to get

$$-\int_G \frac{\bar{\partial} \log x(\xi)}{\xi^{n+1}} \phi \wedge d\xi + \int_C \frac{\log x(\xi)}{\xi^{n+1}} d\xi$$

where C is the inner bounding circle of G. We will show that both terms I and II are zero.

First, II is an integral of the exact form $d(\xi^{-n})$, since $\log x$ is a constant on C. Next we have to simplify and evaluate I. Note that

$$\bar{\partial} \log x = \partial_{\bar{\xi}} \log x \, d\bar{\xi} = \frac{1}{2a\bar{\xi}} \bar{L} \log x \, d\bar{\xi} = d\bar{\xi}/\bar{\xi}.$$

So I is

$$\int_G \frac{\phi}{\xi^n} \frac{d\bar{\xi}}{\bar{\xi}} \wedge \frac{d\xi}{\xi}.$$

This can be rewritten as $\int_G \phi x^{-nc} e^{-iny} \frac{dx}{x} dy$. If we perform the integration in y first, we clearly get zero, because ϕ is a function of x only. The claim is proven. $\qquad\square$

Sub-lemma 2.35. *Let all be as it was in the previous lemma, and suppose we have two functions of the sort that f was, called f and $\tilde{f}$. Suppose $\tilde{f} - f = \bar{L}g$ where g is a C^∞ function defined on a neighborhood of the region where f and $\tilde{f}$ have their support. Then g is holomorphic in an annular region. Let γ be the restriction of g to that annular region, and write γ as the sum of its holomorphic and singular parts in the z-plane, $\gamma = \mathrm{HP}(\gamma) + \mathrm{SP}(\gamma)$. Then $g - \mathrm{HP}(\gamma)$, defined in a neighborhood of $z = 0$ and holomorphic away from $z = 0$, extends holomorphically out to $z = \infty$. Call this extension g'.*

If u and $\tilde{u}$ are the good solutions to $\bar{L}u = f$ and $\bar{L}\tilde{u} = \tilde{f}$, then

$$\tilde{u} - u = g'.$$

Proof. First, note that $(\bar{L}g)_{00} = 0$, because

$$\int_{\partial M} \bar{L}g \, dy = \int_{\partial M} (x\partial_x g + ic\partial_y g) \, dy = \int_{\partial M} d(icg) = 0$$

and icg is a C^∞ function on M. Therefore $\tilde{f}_{00} = f_{00}$.

Now suppose u is the good solution to $\bar{L}u = f$. Then

$$\bar{L}(u + g') = f + \bar{L}g'.$$

Outside the domain of g, g' is holomorphic, and $\widetilde{f} = 0 = f$. Thus $\bar{L}(u + g') = \widetilde{f}$. Inside the domain of g, $g' = g - \mathrm{HP}(\gamma)$. So

$$\bar{L}(u + g') = f + \bar{L}g - \bar{L}\,\mathrm{HP}(\gamma).$$

But the holomorphic part of γ is certainly holomorphic. So

$$\bar{L}(u + g') = f + \bar{L}g = \widetilde{f}.$$

So $u + g'$ is a solution for $\widetilde{f}$.

It remains to see whether this is the *good* solution. At $z = 0$, $u - f_{00} \log x$ is bounded, and g is bounded. Also, $\mathrm{HP}(\gamma)$ is bounded. So

$$u + g' - \widetilde{f}_{00} \log x = u - f_{00} \log x + g - \mathrm{HP}(\gamma)$$

is bounded. Now we go to $z = \infty$. u tends to zero. Near $z = \infty$, $g' = \mathrm{SP}(\gamma)$ and therefore g' tends to zero at $z = \infty$. So $u + g'$ tends to zero. Therefore, $\widetilde{u} = u + g'$ is in fact the good solution to $\bar{L}\widetilde{u} = \widetilde{f}$. $\qquad\square$

Lemma 2.36. *Let M be a collar and E be a holomorphic line bundle over M. Then there exists a frame for E which is C^∞ up to the boundary and is holomorphic everywhere except in a neighborhood of the boundary. Such a frame will be called* holomorphic away from the boundary.

Some notation which will remain fixed in the rest of this section: M is a collar. x, y are the C^∞ coordinates related to some first integral $z = x^c e^{iy}$. $c = a + ib$ is the collar invariant defined earlier. We have the basis

$$
\begin{aligned}
L &= x\partial_x - i\bar{c}\,\partial_y = 2az\partial_z & (1,0) \\
\bar{L} &= x\partial_x + ic\,\partial_y = 2a\bar{z}\partial_{\bar{z}} & (0,1)
\end{aligned}
\tag{2.2}
$$

for $\mathbb{C}^{\,b}TM$. The dual basis is

$$
\begin{aligned}
\lambda &= \frac{1}{2a}\left[c\frac{dx}{x} + i\,dy\right] = \frac{1}{2a}\frac{dz}{z} & (1,0) \\
\bar{\lambda} &= \frac{1}{2a}\left[\bar{c}\frac{dx}{x} - i\,dy\right] = \frac{1}{2a}\frac{d\bar{z}}{\bar{z}} & (0,1).
\end{aligned}
\tag{2.3}
$$

Theorem 2.37. *Let M be a collar and E be a holomorphic line bundle over M. Then to E there corresponds an integral sequence class. It is a bundle invariant.*

Construction. Choose a first integral z for M. Let s be a global C^∞ frame for E which is holomorphic away from the boundary. Let α be the $\bar\partial$ form of E with respect to s. Then α is compactly supported, so we may regard it as a compactly supported compressed $(0,1)$-form on $\mathcal{M}_c(z)$.

Let $f = \langle \alpha, \bar L \rangle$. Then f is a smooth compactly supported function of $\mathcal{M}_c(z)$. So by Lemma 2.33 there is a unique good solution to $\bar L r = -f$. (This is the same as saying that r is the unique good solution to $\bar\partial r = -\alpha$ on $\mathcal{M}_c(z)$.) At the boundary this solution has the form

$$h_1 - f_{00}\phi \log x + h_2$$

where h_1 is C^∞, ϕ is a cut-off function, and h_2 has an asymptotic expansion in positive powers of z. The asymptotic expansion for h_2, plus the constant term $-f_{00}$, is taken to represent the bundle invariant. $\qquad\square$

Proof of invariance under a change of first integral. This proof uses the micro-lemma which says that if $\partial_{\bar\xi} p = q$ (we are running out of letters!), then $\partial_{\bar\xi}(p \circ (D)) = (D) \circ q \circ (D)$, where (D) represents "multiplication by the non-zero constant D." The proof of this micro-lemma is just an exercise in applying the chain rule, so we omit the proof.

Now to the proof of invariance. Let z and $\widetilde z$ be first integrals. Then $\widetilde z = Dz$. We start the construction: s is the frame, α its form. Now, whether computed via z or $\widetilde z$, the $\bar L$ operator (on M°) is the same. So we get the same function $f = \langle \alpha, \bar L \rangle$ on the manifold in either case. The two angular coordinates y and $\widetilde y$ differ by a constant, so averaging f against dy gives the result as averaging f against $d\widetilde y$. Thus z_0 is independent of the first integral.

We proceed to $n \geq 1$. We have to be careful about composing with coordinate functions, though this is blurred in the construction. In the non-tilde case, we are to take $g = f \circ z^{-1}$, find the good solution ρ in the punctured complex plane to $2a\bar\xi\partial_{\bar\xi}\rho = g$, and then let $r = \rho \circ z$.

Let $E = 1/D$. In the tilde case, we are to take $\widetilde g = f \circ \widetilde z^{-1} = f \circ z^{-1} \circ (E) = g \circ (E)$. Now we seek the good solution $\widetilde\rho$ in the punctured complex plane to $2a\bar\xi\partial_{\bar\xi}\widetilde\rho = \widetilde g$; we can

compute by means of the micro-lemma that it must be $\widetilde{\rho} = \rho \circ (E)$. Finally, we take

$$
\begin{aligned}
\widetilde{r} &= \widetilde{\rho} \circ \widetilde{z} \\
&= \rho \circ (E) \circ \widetilde{z} \\
&= \rho \circ (E) \circ (D) \circ z \\
&= \rho \circ z \\
&= r.
\end{aligned}
$$

So all the z_n are independent of the first integral. $\qquad\square$

Proof of invariance under change of frame. Let s and $\widetilde{s}$ be two global C^∞ frames which are holomorphic away from the boundary. Let n be the winding number of $\widetilde{s}/s$ around zero on any small deformation of ∂M. Then I claim that $\widetilde{s}/s$ may be written in the form

$$
\widetilde{s}/s = z^n e^{g - nc\phi \log x}
$$

where g is a C^∞ function on M which is holomorphic away from the boundary and ϕ is a cut-off function which is supported near the boundary.

Proof of this claim: z^n winds n times, and $e^{-nc \log x}$ does not wind. Therefore $z^n e^{-nc\phi \log x}$ winds n times. Furthermore,

$$
\begin{aligned}
z^n e^{-nc\phi \log x} &= e^{n(c \log x + iy) - nc\phi \log x} \\
&= e^{(1-\phi)nc \log x + iny}
\end{aligned}
$$

is C^∞ up to the boundary. Finally, this function is holomorphic away from the boundary since it agrees with z^n there.

So what do we have? We have $\widetilde{s}/s$ being a C^∞ function, holomorphic away from the boundary, and winding n times; and we have $z^n e^{-nc\phi \log x}$ being a C^∞ function, holomorphic away from the boundary, and winding n times. So the quotient is C^∞ up to the boundary and winds not at all. So this quotient is e^g for some g which is C^∞ up to the boundary and holomorphic away from the boundary. We are done proving the claim.

So assume $\widetilde{s} = sz^n e^{g - nc\phi \log x}$, as claimed. Then $\widetilde{\alpha} = \alpha + \bar{\partial}(g - nc\phi \log x)$. So

$$
\begin{aligned}
\widetilde{f} &= f + \bar{L}g - \bar{L}(nc\phi \log x) \\
&= f + \bar{L}g - nc(\bar{L}\phi) \log x - nc\phi,
\end{aligned}
$$

so that

$$
-\widetilde{f} = -f - \bar{L}g + nc(\bar{L}\phi) \log x + nc\phi.
$$

First we show invariance of the zeroth integral. $\widetilde{z}_0 = -f_{00} = -f_{00} - (\bar{L}g)_{00} + nc = -f_{00} + nc$, so that $\widetilde{z}_0 = z_0 + nc$. That is, $\widetilde{z}_0$ and z_0 are equivalent. This depends on the fact that $(\bar{L}g)_{00} = 0$ as proved in Sub-lemma 2.35.

Now we show invariance of the other integrals. So let r be the good z-plane solution to $\bar{L}r = -f$. We seek the good z-plane solution $\widetilde{r}$ to $\bar{L}\widetilde{r} = -\widetilde{f} = -f - \bar{L}g + \bar{L}(nc\phi \log x)$. This can be found by seeking the good solutions to the two $\bar{L}$ problems with data $-f - \bar{L}g$ and $\bar{L}(nc\phi \log x) = nc(\bar{L}\phi) \log x + nc\phi$ and adding, since these two data are C^∞ and compactly supported (so that Lemma 2.33 applies).

The good solution to the first problem is $r - g + \text{HP}(\gamma)$, by Sub-lemma 2.35. I claim that the good solution to the second problem is $nc\phi \log x$ itself. Why? Simply by the definition of a good solution!

Thus $\widetilde{r} = r - g + \text{HP}(\gamma) + nc\phi \log x$. So $\widetilde{h}_1 = h_1 - g + \text{HP}(\gamma)_0$ and $\widetilde{h}_2 = h_2 + \text{HP}(\gamma) - \text{HP}(\gamma)_0$. Thus the expansions in positive powers of z for h_2 and for $\widetilde{h}_2$ differ by an absolutely convergent series (the holomorphic function $\text{HP}(\gamma) - \text{HP}(\gamma)_0$). That is, they define the same class (Definition 2.32). $\square$

Theorem 2.38. *Let E be a bundle over the collar M. Let $z_0, z_1, \ldots$ be an integral sequence which represents the bundle invariant of E. Then there exists a C^∞ frame s and a function $v \sim \sum_{n \geq 1} z_n$ such that*

$$se^{z_0 \log x + v}$$

is a global holomorphic frame for the pull-back of E to the interior of M. (Here x is the defining function for ∂M associated to any first integral.) This frame is unique up to a constant multiple.

Proof of uniqueness. Suppose there were two such. Then their quotient would be C^∞ and holomorphic on the collar. The only such function is a constant. $\square$

Proof of existence. Choose a first integral z. Choose a C^∞ frame $\mathbf{1}$ for E. Let α be the associated $\bar{\partial}$ form. Let $f = \langle \alpha, \bar{L} \rangle$. Let r be the unique good solution to the equation $\bar{L}r = -f$. Then by Lemma 2.33, $r = h_1 - f_{00} \log x + h_2$, with h_1 being C^∞ and h_2 having an asymptotic expansion in positive powers of z. By definition, we know that $z_0 + f_{00} = nc$ for some integer n, and $\sum_{n \geq 1} z_n$ minus the asymptotic series in positive powers of z for h_2 is absolutely convergent near $z = 0$. Call this analytic function F. Now define a new C^∞

frame s by

$$s = \mathbf{1}e^{h_1 + iny}.$$

I claim that $se^{z_0 \log x + v}$ is a global holomorphic frame over the interior. Why so? We only have to compute that

$$
\begin{aligned}
se^{z_0 \log x + v} &= \mathbf{1}e^{h_1 + iny + z_0 \log x + v} \\
&= \mathbf{1}e^{h_1 - f_{00} \log x + h_2 + (z_0 + f_{00}) \log x + iny + (v - h_2)} \\
&= \mathbf{1}e^{h_1 - f_{00} \log x + h_2 + nc \log x + iny + F} \\
&= \mathbf{1}e^{h_1 - f_{00} \log x + h_2} z^n e^F.
\end{aligned}
$$

The first piece is the global holomorphic frame we started with. The other two pieces are nonvanishing holomorphic functions. So the whole thing is a global holomoporphic frame. $\square$

Theorem 2.39. *E posesses a global holomorphic frame which is C^∞ up to the boundary if and only if the integral sequence class of E is the zero class.*

Proof. If the integral sequence class is represented by the zero sequence, then Theorem 2.38 guarantees the existence of a global holomorphic frame which is C^∞ up to the boundary. Conversely, if there is a global holomorphic frame which is C^∞ up to the boundary, then the integral sequence computed with respect to this frame is the zero sequence. $\square$

Definition 2.40. Two holomorphic line bundles over a collar are called *equivalent* if there exists a C^∞ isomorphism between them which preserves the holomorphic structures.

Corollary 2.41. *Two holomorphic line bundles over a collar are equivalent if and only if their integral sequence classes are the same.*

Proof. It suffices to observe that the map which takes a bundle to its integral sequence class is a homomorphism from the group of bundles to the additive group of integral sequence classes. This is a triviality. $\square$

Theorem 2.42. *Let M be a collar. Then for every integral sequence $z_0, z_1, \ldots$, there is a bundle E over M whose class this sequence represents.*

Construction. WLOG take M to be $\mathcal{M}_c$ where c is an arbitrary complex constant with strictly postive real part. Choose an integral sequence. As a set we take E to be $M \times \mathbb{C}$. Choose a function v of compact support on M which has the asymptotic development

$$v \sim \sum_{n \geq 1} z_n.$$

The existence of such a function is guaranteed by a variant of Borel's theorem. Then define the holomorphic structure of E by stipulating that

$$\mathbf{1} e^{z_0 \log x + v}$$

be holomorphic. This defines the correct sort of holomorphic structure, because the $\bar{\partial}$ form with respect to $\mathbf{1}$ is $-\bar{L}(z_0 \log x + v)\bar{\lambda} = -(z_0 + C^\infty)\bar{\lambda}$. ($\bar{L}v$ is C^∞ because it is asymptotic to the zero sequence in powers of z at the boundary.) Thus the $\bar{\partial}$ form with respect to a C^∞ frame is a compressed form. $\qquad\square$

Computation of invariant sequence. Since v is compactly supported, we may take $\mathbf{1}$ as our C^∞ frame which is holomorphic away from the boundary. The $\bar{\partial}$ form with respect to this frame is $\alpha = -\bar{\partial}z_0 \log x + v$. So $f = \langle \alpha, \bar{L} \rangle = -(z_0 + \bar{L}v)$. So $-f_{00} = z_0$ is the zeroth invariant. Also, one solution to $\bar{L}r = -f$ is $z_0 \log x + v$. But a moment's inspection of the definitions shows that this is, in fact, the good solution. So the asymptotic series in positive powers of z for v represents the invariant integral sequence class for E. $\qquad\square$

We have fully classified the holomorphic line bundles over a collar. That is, we have an isomorphism from equivalence classes of such bundles to a simpler group (namely, the additive group of integral sequence classes). We have proven the well-definedness, the preservation of group structure, the injectivity, and the surjectivity. So we may now move on.

$$\overline{\textbf{3}}$$

The Blow-down

As a C^∞ surface, a b-holomorphic complex curve M looks like the blow-up of a compact surface without boundary. We would like to see whether M, regarded together with its holomorphic structure, looks like some sort of a blow-up of a compact Riemann surface. In this chapter, we construct that compact Riemann surface, the *blow-down* $\check{M}$ of M.

Lemma 3.1. *Let D be the open unit disk, and $K = [0,1) \times S^1$. Define the projection map $\pi_0 : K \longrightarrow D$ by $\pi_0(r, \theta) = re^{i\theta}$.*

1. *The topology of D is the topology induced from K via the map π_0. That is, a set in D is open if and only if its pre-image under π_0 is open in K.*

2. *For any $a > 0$ and $b \in \mathbb{R}$, the map $re^{i\theta} \mapsto r^a e^{i(\theta + b \log r)}$ is a homeomorphism from D to D.*

Proof of the first assertion. First note that $\pi_0\big|_{K^\circ}$ is a homeomorphism onto $D \setminus 0$. So if B is a subset of $D \setminus 0$, then B is open if and only if $\pi_0^{-1}(B)$ is open. So let B be a subset of D containing 0. If B is open, then $\pi_0^{-1}(B)$ is open by the continuity of π_0 (which is obvious, if you consider sequences). So assume instead that $\pi_0^{-1}(B)$ is open in K. Then $\pi_0^{-1}(B)$ is an open neighborhood of ∂K in K.

For each $p \in B$, we need to produce an open neighborhood of p contained in B. First, suppose $p \neq 0$. Then let H be an open neighborhood of $\pi_0^{-1}(p)$ contained in K°. Then $\pi_0(H)$ is an open set contained in B, and which contains p.

So all we need to do is produce an open neighborhood of 0 contained in B. $\pi_0^{-1}(B)$ is a an open set in K (by hypothesis) which contains ∂K. So by compactness of S^1, we can choose an $\epsilon > 0$ such that

$$\{(r, \theta) \in K : r < \epsilon\}$$

is contained in $\pi_0^{-1}(B)$. Therefore

$$\{\xi \in D : |\xi| < \epsilon\}$$
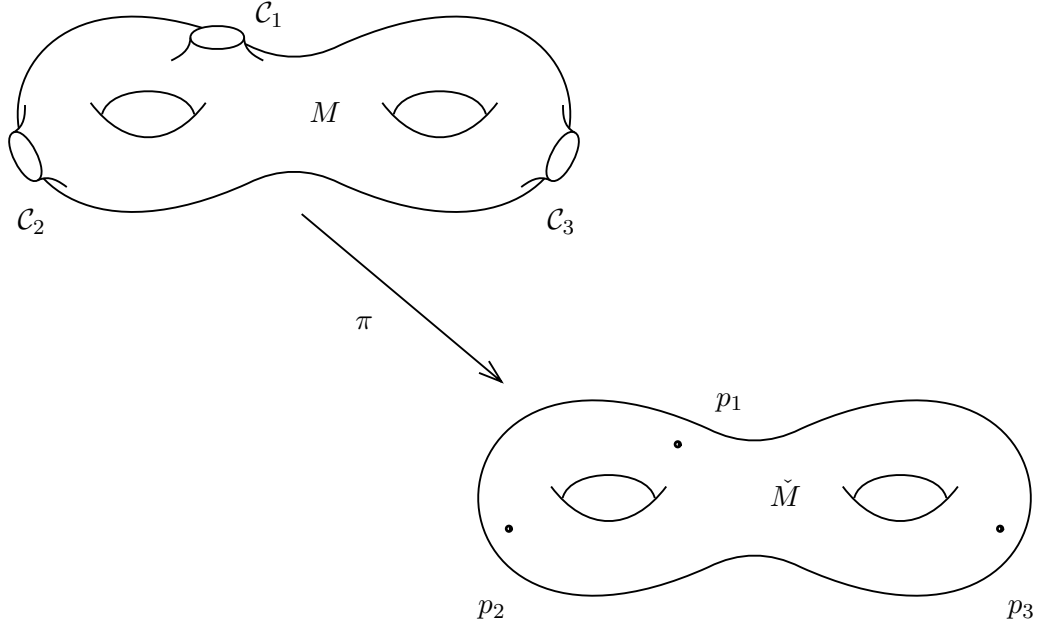
Figure 2: The Blow-down

is in B. So there's an open neighborhood of 0 contained in B. This completes the verification that B is open, based on the assumption that $\pi_0^{-1}(B)$ is open. $\qquad\square$

Proof of the second assertion. The map is a bijection, and its restriction to the punctured disk is a homeomorphism $D \setminus 0$ to $D \setminus 0$. So all that remains is to check continuity at 0 of the map and its inverse. This is obvious by sequences. $\qquad\square$

Theorem 3.2. *Let M be a b-holomorphic complex curve. Define g and k by stipulating that M be diffeomorphic to a genus g compact surface with k open disks removed. Then there exists a natural compact Riemann surface $\check{M}$ of genus g, with k distinguished points $p_1, \ldots, p_k$, and a natural surjection $\pi : M \longrightarrow \check{M}$, such that*

  1. $\pi(\partial M) = \{p_1, \ldots, p_k\}$ *and*

  2. $\pi\big|_{M^\circ} : M^\circ \longrightarrow \check{M} \setminus \{p_1, \ldots, p_k\}$ *is a biholomorphism.*

Construction. As a set,
$$\check{M} = M^\circ \cup \{p_1, \ldots, p_k\}$$

where the p_i are formal adjuncts.

Next we construct π. Label the boundary circles of M $\mathcal{C}_1, \ldots, \mathcal{C}_k$. Then define the surjective map $\pi : M \longrightarrow \check{M}$ by

$$\pi(p) = \begin{cases} p & p \in M^\circ \\ p_i & p \in \mathcal{C}_i. \end{cases}$$

Now we can stipulate that a set in $\check{M}$ is open if and only if its pre-image under π is open. This is a topology. Since $\pi\big|_{M^\circ} : M^\circ \longrightarrow \check{M} \backslash \{p_1, \ldots, p_k\}$ is bijective, it is a homeomorphism.

We claim that with this topology, $\check{M}$ is more than a topological space; it is a topological manifold. Since $\check{M} \backslash \{p_1, \ldots, p_k\} \equiv M^\circ$ is a (boundaryless) topological manifold of dimension 2, every point of $\check{M} \backslash \{p_1, \ldots, p_k\}$ has a neighborhood which is homeomorphic to the unit disk D. So we only need to show that p_i has a neighborhood which is homeomorphic to D. To do this, choose a neighborhood of $\mathcal{C}_i$ in M which has a homeomorphism φ to $K = [0, 1) \times S^1$. Then define $\check{\varphi} = \pi_0 \circ \varphi \circ \pi^{-1}$ as in the diagram.

$$
\begin{array}{ccc}
\mathrm{neigh}(\mathcal{C}_i) & \xrightarrow{\ \pi\ } & \mathrm{neigh}(p_i) \\
{\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\check{\varphi}} \\
K & \xrightarrow[\pi_0]{} & D
\end{array}
$$

φ is clearly a bijection. By part 1 of Lemma 3.1, the diagram's top and bottom rows are perfectly analogous, in that the maps π and π_0 induce topologies in the direction of their arrows; and the left-hand vertical link between them (φ) is a homeomorphism. Therefore the induced right-hand vertical link between the rows ($\check{\varphi}$) must also be a homeomorphism. (This argument can be made more explicit, but it is perhaps more clear how part 1 of the lemma enters into the picture if we think in these less technical terms.)

Now we have a topological manifold $\check{M}$ and a homeomorphism

$$\pi : M^\circ \longrightarrow \check{M} \setminus \{p_1, \ldots, p_k\}.$$

We may therefore push the holomorphic structure of M° forward to $\check{M} \setminus \{p_1, \ldots, p_k\}$. All that remains is to extend this holomorphic structure to all of $\check{M}$. To extend the structure, we only need to produce a homeomorphism from a neighborhood of p_i to the unit disk D, in such a way that the restriction to the punctured neighborhood of p_i is a biholomorphism to $D \setminus 0$.

To do this, choose a first integral z_i for the ith collar of M, and define $\check{z}_i = z_i \circ \pi^{-1}$. To see whether $\check{z}_i$ is a homeomorphism, we need to examine this function in (topological) coordinates. So let $\varphi_i = (x, y)$ be the homeomorphism from $\mathrm{neigh}(\mathcal{C}_i)$ to K defined by z_i; then by our construction,

$$\check{\varphi}_i = \pi_0 \circ \varphi \circ \pi^{-1}$$

is a topological chart near p_i. So we are examining

$$\check{z}_i \circ \check{\varphi}_i^{-1} = z_i \circ \pi^{-1} \circ \pi \circ \varphi^{-1} \circ \pi_0^{-1}$$
$$= z_i \circ \varphi^{-1} \circ \pi_0^{-1},$$

which is clearly the map

$$re^{i\theta} \mapsto r^c e^{i\theta} = r^a e^{i(\theta + b \log r)}.$$

By part 2 of Lemma 3.1, this is a homeomorphism from D to D. So z_i is a homeomorphism from $\mathrm{neigh}(p_i)$ to D. And finally, z_i is a biholomorphism from the interior of a neighborhood of $\mathcal{C}_i$ to a punctured neighborhood of 0; so $\check{z}_i = z_i \circ \pi^{-1}$ is a biholomorphism from a punctured neighborhood of p_i to a punctured neighborhood of 0. So stipulating that $\check{z}_i$ be a holomorphic chart near p_i, we have extended the holomorphic atlas of $\check{M} \setminus \{p_1, \ldots, p_k\}$ to an atlas for $\check{M}$.

It only remains to note that $\check{M}$ is a compact surface of genus g. This is a consequence of the way our projection map π and topological charts near p_i were defined. $\qquad\square$

Proof of naturality. The only choice we made was in the specification of the holomorphic structure of $\check{M}$: a choice of first integral z_i for the ith collar of M. So we need to show that a different choice $\widetilde{z}_i$ results in the same holomorphic structure. This is immediate, because

$$\check{\widetilde{z}}_i \circ \check{z}_i^{-1} = \widetilde{z}_i \circ \pi^{-1} \circ (z_i \circ \pi^{-1})^{-1}$$
$$= \widetilde{z}_i \circ z_i^{-1}.$$

By Lemma 2.18, this is a biholomorphism from a disk to a disk. $\qquad\square$

Definition 3.3. Let M be a b-holomorphic complex curve. Let g be the genus of $\check{M}$ and k be the number of boundary circles of M. When $g = 0$ and $k = 1$, M is called a *cup*. When $g = 0$ and $k = 2$, M is called a *pipe*.

4

The Cup: $c = 1$

Definition 4.1. A biholomorphism from a cup's interior to the complex plane is called an *interior coordinate* for the cup. Interior coordinates always exist by Theorem 3.2.

As a C^∞ surface, a cup is equivalent to the closed unit disk. So a cup is equivalent (as a cup) to the closed unit disk, equipped with a certain kind of holomorphic structure which is singular at the boundary. Exactly what kind of structure would this be? The following theorem, valid for arbitrary c, answers this question. We will not use this theorem in anything that follows; it is intended only to give a concrete picture of the b-holomorphic structure of a cup.

Theorem 4.2. *Let c be a complex number with strictly positive real part. Let φ be a C^∞ diffeomorphism from D to $\mathbb{C}$, such that in a neighborhood of ∂D, φ has the form*

$$re^{i\theta} \mapsto \psi\left([1-r]^c e^{-i\theta}\right),$$

where ψ is a biholomorphism from a punctured disk of small radius to a punctured neighborhood of ∞. Then the pullback to D via φ of the holomorphic structure of $\mathbb{C}$ defines a holomorphic structure for D with respect to which the boundary is a collar invariant c. Moreover, any cup with collar invariant c can be represented in this way.

Proof that such a structure is of type b. We have a holomorphic structure for D. So the question is, does it extend to a b-holomorphic structure on $\bar{D}$? Please refer to Figure 3.

Define $x = 1 - r$ near the boundary of D. This is a defining function for the boundary. Next define $y = -\theta$. This is an angular coordinate near the boundary. Then define $z = x^c e^{iy}$. I claim that z is holomorphic in a neighborhood of the boundary. How do we check this? Because the holomorphic structure on the disk was pulled back via φ from the structure of $\mathbb{C}$, and the boundary of the disk maps to ∞ under φ, we have take the composition $z \circ \varphi^{-1}$ and see whether this is holomorphic near ∞. To see that it is holomorphic, first note that φ^{-1} takes $\psi([1-r]^c e^{-i\theta})$ to $re^{i\theta}$. So $z \circ \varphi^{-1}$ takes $\psi([1-r]^c e^{-i\theta})$ to $[1-r]^c e^{-i\theta}$.
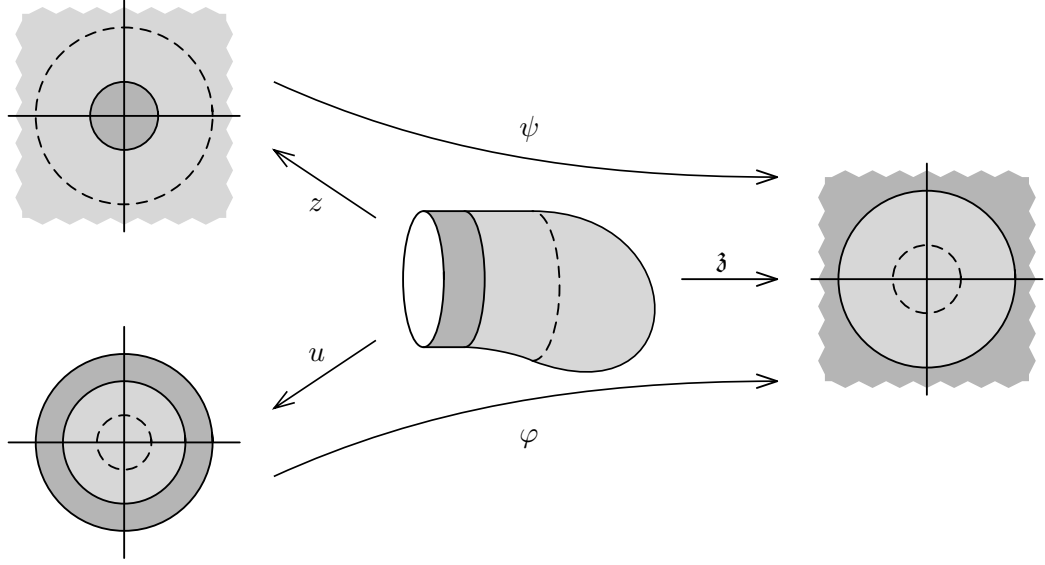
Figure 3: The Cup

That is, $z \circ \varphi^{-1} = \psi^{-1}$, which is holomorphic near ∞. Now that we have z holomorphic, we can compute the $(1,0)$-form

$$\frac{dz}{z} = c\frac{dx}{x} + i\,dy$$

over the interior. This is a compressed form. So the holomorphic structure of the interior induces a b-holomorphic structure at the boundary. We also see that $\eta = (1/2\pi c)\,dy$, so

$$\int_{\partial M} \eta = 1/c.$$

That is, the collar invariant is c. $\hfill\square$

Proof that every cup can be represented this way. Let M be a cup with collar invariant c. Choose a first integral z and let x, y be the corresponding C^∞ coordinates near the boundary. Let u be a C^∞ diffeomorphism from M to $\bar{D}$ (with polar components called r, θ) such that near the boundary of the cup, $x = 1 - r$ and $y = -\theta$. Then let $\mathfrak{z}$ be an interior coordinate.

Under u, $\bar{D}$ becomes a diffeomorphic copy of the cup. So we can pull back the holomorphic structure of the cup to $\bar{D}$ via u^{-1}. Then we have $\bar{D}$ as a biholomorphic copy of M. But since $\mathfrak{z}$ is a biholomorphism over the interior of the cup, this is the same as pulling back the structure of $\mathbb{C}$ to D via $\mathfrak{z} \circ u^{-1}$. We call this function φ.

It only remains to check that φ has the form claimed. If we take a point $re^{i\theta}$ near the boundary of D, and write $[1-r]^c e^{-i\theta}$, this corresponds to going up to the cup via u^{-1} and then down to the small disk via z. So if we take this point in the small disk and apply $\psi = \mathfrak{z} \circ z^{-1}$, it amounts to starting in D and applying

$$\mathfrak{z} \circ z^{-1} \circ z \circ u^{-1} = \mathfrak{z} \circ u^{-1} = \varphi.$$

We have shown that φ has the form

$$re^{i\theta} \mapsto \psi\left([1-r]^c e^{-i\theta}\right)$$

near the boundary of D. Since z and $\mathfrak{z}$ are biholomorphisms where they are defined, ψ is a biholomorphism from the punctured small disk to a punctured neighborhood of infinity. $\square$

4.1 Classification of cups

Theorem 4.3. *Let M be a cup with $c = 1$. Then for any interior coordinate $\mathfrak{z}$, $1/\mathfrak{z}$ is a first integral.*

Proof. Choose a first integral z. Then $\mathfrak{z} \circ z^{-1}$ is a biholomorphism from a punctured neighborhood of 0 to a punctured neighborhood of ∞. So $(1/\mathfrak{z}) \circ z^{-1}$ is a biholomorphism from a neighborhood of 0 to a neighborhood of 0, and takes 0 to 0. So we may write

$$(1/\mathfrak{z}) \circ z^{-1}(\xi) = 0 + c_1\xi + c_2\xi^2 + \cdots$$

(with c_1 nonzero), or

$$1/\mathfrak{z} = c_1 z + c_2 z^2 + \cdots.$$

So

$$(1/\mathfrak{z}) \div z = c_1 + c_2 z + \cdots.$$

That is, $(1/\mathfrak{z}) \div z$ is an analytic function of z^1 with nonzero constant term; and we have $c = 1/1$. So $1/\mathfrak{z}$ is a first integral, by Corollary 2.22. $\square$

Theorem 4.4. *Any two cups having $c = 1$ are isomorphic.*

Proof. Let M_1 and M_2 be two such cups. Choose interior coordinates $\mathfrak{z}_1$ and $\mathfrak{z}_2$, and let $z_1 = 1/\mathfrak{z}_1$ and $z_2 = 1/\mathfrak{z}_2$ be the corresponding first integrals. Define $\varphi = \mathfrak{z}_2^{-1} \circ \mathfrak{z}_1$. This is a

biholomorphism from M_1° to M_2°. We only need to show that it extends to a diffeomorphism from M_1 to M_2. To see this, just write φ near the boundary as

$$\varphi = (1/z_2)^{-1} \circ (1/z_1)$$
$$= (\text{inversion} \circ z_2)^{-1} \circ \text{inversion} \circ z_1$$
$$= z_2^{-1} \circ z_1$$
$$= (\chi_1 \circ z_2)^{-1} \circ (\chi_1 \circ z_1).$$

But by Remark 2.17, $\chi_1 \circ z_1$ is an isomorphism from the collar of M_1 to the standard 1-collar, and likewise $\chi_1 \circ z_2$ is an isomorphism from the collar of M_2 to the standard 1-collar. Thus the composition, which is φ, is an isomorphism from the collar of M_1 to the collar of M_2. $\qquad\square$

Since a cup with $c = 1$ is essentially the Riemann sphere with the point at ∞ distinguished (and blown up), it stands to reason that the automorphism group of such a cup should be essentially the automorphism group of the sphere minus the point at infinity; that is, of $\mathbb{C}$. This is the content of the next theorem.

Theorem 4.5. *Let M be a cup with $c = 1$. Then $\mathrm{Aut}(M) \simeq \mathrm{Aut}(\mathbb{C})$, via the map*

$$\varphi \mapsto \mathfrak{z} \circ \varphi \circ \mathfrak{z}^{-1},$$

where $\mathfrak{z}$ is any fixed interior coordinate for M.

Proof. It is obvious that this map is a well-defined injective homomorphism. So we only need to check that it is surjective. So let φ be an automorpism of $\mathbb{C}$. We need to show that $\mathfrak{z}^{-1} \circ \varphi \circ \mathfrak{z}$ is an automorphism of M. It's certainly an automorphism of M°. So we have to check that it's also a collar automorphism. To do this, we use Theorem 2.25.

So let $z = 1/\mathfrak{z}$ be the first integral associated to the interior coordinate $\mathfrak{z}$. And compute

$$z \circ (\mathfrak{z}^{-1} \circ \varphi \circ \mathfrak{z}) \circ z^{-1} = \text{inversion} \circ \varphi \circ \text{inversion}.$$

But since φ is an automorphism of the plane, it has the form $\varphi(\xi) = a\xi + b$ with $a \neq 0$. So $\text{inversion} \circ \varphi \circ \text{inversion}(\xi) = \xi/(a + b\xi)$. It only remains to note that $1/(a + b\xi)$ is an analytic function with nonzero constant term; for then Theorem 2.25 tells us that $\mathfrak{z}^{-1} \circ \varphi \circ \mathfrak{z}$ is a collar automorphism. $\qquad\square$

4.2 Bundles over a cup

Definition 4.6. A *marked cup* is a cup M with a point $p_0 \in M^\circ$ distinguished.

Theorem 4.7. *To each holomorphic line bundle E over a marked cup (M, p_0) there corresponds an integral sequence on M. (See Definitions 2.29 and 2.31 and Lemma 2.30.)*

Construction. Choose an interior coordinate $\mathfrak{z}$ for M which takes p_0 to 0 and let $z = 1/\mathfrak{z}$ be the corresponding first integral. (As usual, we will write $z = xe^{iy}$.) Choose a global C^∞ frame s for E which is holomorphic away from the boundary. Let α be the corresponding $\bar{\partial}$ form. Define $f = \langle \alpha, \bar{L} \rangle$. Then define

$$f_{mm} = \frac{1}{2\pi} \frac{1}{m!} \int_{\partial M} \partial_x^m f e^{-imy} \, dy.$$

Then the mth entry in the integral sequence is defined to be

$$z_m = -f_{mm} z^m.$$

$\square$

Proof of invariance under change of frame. Let $\widetilde{s}/s = e^u$. Then $\widetilde{\alpha} - \alpha = \bar{\partial} u$. So $\widetilde{f} - f = \bar{L} u$. So $\widetilde{f}_{mm} - f_{mm} = (\bar{L} u)_{mm}$. To prove that $(\bar{L} u)_{mm} = 0$, we will need to use the commutation relation

$$\partial_x^m \bar{L} - \bar{L} \partial_x^m = m \partial_x^m.$$

Now we can compute that $(\bar{L} u)_{mm}$ is equal to a constant times

$$\int_{\partial M} (\partial_x^m \bar{L} u) e^{-imy} \, dy = \int_{\partial M} (\bar{L} \partial_x^m u + m \partial_x^m u) e^{-imy} \, dy$$
$$= i \int_{\partial M} \partial_y \partial_x^m u \cdot e^{-imy} \, dy + m \int_{\partial M} \partial_x^m u \cdot e^{-imy} \, dy$$
$$= 0$$

by integration by parts. $\square$

Proof of invariance under change of interior coordinate. Let $\widetilde{\mathfrak{z}}$ be another interior coordinate which takes p_0 to 0. Then $\widetilde{\mathfrak{z}} = (1/a)\mathfrak{z}$ for some nonzero constant a, which we will write

as $a = re^{i\theta}$. So $\widetilde{z} = re^{i\theta} z$. This tells us that

$$\widetilde{x} = rx$$

$$\widetilde{y} = y + \theta$$

$$\partial_{\widetilde{x}} = \frac{1}{r} \partial_x$$

$$d\widetilde{y} = dy$$

$$\widetilde{f} = f.$$

The reason for $\widetilde{f} = f$ is that $\bar{L} = \bar{z} \partial_{\bar{z}}$ doesn't change when z is multiplied by a constant.

Now compute

$$\widetilde{f}_{mm} = \frac{1}{2\pi} \frac{1}{m!} \int_{\partial M} \partial_{\widetilde{x}}^m f e^{-im\widetilde{y}} \, d\widetilde{y} = \frac{1}{2\pi} \frac{1}{m!} \int_{\partial M} r^{-m} \partial_x^m f e^{-imy} e^{-im\theta} \, dy = a^{-m} f_{mm}.$$

So $\widetilde{f}_{mm} \widetilde{z}^m = a^{-m} f_{mm} a^m z^m = f_{mm} z^m.$ $\qquad\square$

Theorem 4.8. *Let E be a holomorphic line bundle over the marked cup (M, p_0). Let $z_0, z_1, \ldots$ be its invariant integral sequence. Then there exists a holomorphic global frame for the pull-back of E to M° of the form*

$$se^{u \log x}$$

where s is a global C^∞ frame for E and $u \sim \sum_{n \geq 0} z_n$ at the boundary of M. Here, x is a defining function for ∂M associated to a first integral z which is $1/\mathfrak{z}$ for some interior coordinate $\mathfrak{z}$ which respects the marking $(\mathfrak{z}(p_0) = 0)$.

Proof. Let $\mathfrak{z}$ be an interior coordinate that respects the marking, and let $z = 1/\mathfrak{z}$ be the associated first integral. Write $z = xe^{iy}$. Choose a global C^∞ frame s of E which is holomorphic away from the boundary. let α be the corresponding $\bar{\partial}$ form. Let $f = \langle \alpha, \bar{L} \rangle$. Write

$$f \sim \sum_{\substack{m \geq 0 \\ n \in \mathbb{Z}}} f_{mn} x^m e^{iny}.$$

Then define

$$u_{mn}^1 = \begin{cases} -\dfrac{f_{mn}}{m-n} & n \neq m \\ 0 & n = m \end{cases}$$

and

$$u_{mn}^2 = \begin{cases} 0 & n \neq m \\ -f_{mn} & n = m. \end{cases}$$

Let u^1 and u^2 be C^∞ functions on M which are supported near ∂M and which have asymptotic expansions at the boundary

$$u^1 \sim \sum_{\substack{m \geq 0 \\ n \in \mathbb{Z}}} u_{mn}^1 x^m e^{iny}$$

$$u^2 \sim \sum_{\substack{m \geq 0 \\ n \in \mathbb{Z}}} u_{mn}^2 x^m e^{iny}.$$

Then $-f - \bar{L}(u^1 + u^2 \log x)$ vanishes to infinite order at ∂M, and is supported near ∂M. So we may push it down (via z) to a function on the z-sphere which vanishes to infinite order at $z = 0$. There, we can solve

$$\bar{L}u^3 = -f - \bar{L}(u^1 + u^2 \log x)$$

by means of the Cauchy integral formula, as we have done before. (See Lemma 2.27.) The solution is smooth on the z-sphere; therefore it may be lifted to M as a globally defined smooth function. Now define

$$u = u^1 + u^2 \log x + u^3.$$

Then by our construction, $\bar{L}u = -f$. And because of this fact, the frame

$$se^u$$

is holomorphic in the interior. This is seen by computing

$$\bar{\partial}(se^u) = se^u \otimes (\alpha + \bar{\partial}u) = se^u \otimes (f + \bar{L}u)\bar{\lambda} = 0.$$

All that remains is some re-labeling. The new s will be the old s times $e^{u^1 + u^3}$, and the new u is u^2. $\qquad \square$

Theorem 4.9. *Let E be a holomorphic line bundle over a cup M. Let $p_0, \widetilde{p}_0$ be points of M°. Let z_n be the integral sequence associated to E over (M, p_0) and $\widetilde{z}_n$ the integral sequence associated to E over $(M, \widetilde{p}_0)$.*

If $z_n = 0$ for all n, then $\widetilde{z}_n = 0$ for all n.

Proof. Choose first integrals associated to the two markings, z and $\widetilde{z}$. Using Theorem 4.8, with z, construct the global holomorphic frame $se^{u\log x}$ over M°. Since all the z_n are zero, u vanishes to infinite order at $x = 0$. Therefore this frame is actually C^{∞} up to the boundary.

Now, switch to $\widetilde{z}$, but use the above-mentioned C^{∞} frame to construct the $\widetilde{f}_{mm}$. Since that frame is holomorphic, $\widetilde{\alpha} = 0$, so that $\widetilde{f} = 0$. So all the $\widetilde{f}_{mm}$ are zero. So the $\widetilde{z}_m$ are zero. $\square$

Corollary 4.10. *Let E be a holomorphic line bundle over a cup M. Then there exists a global holomorphic C^{∞} frame for E if and only if the each invariant integral sequence for E (corresponding to each interior point p_0) is zero.*

Theorem 4.11. *Choose an interior point p_0 for the cup M. Then the map*

$$E \mapsto \text{the integral sequence for } E \text{ with respect to } p_0$$

is an isomorphism from the group of equivalence classes of holomorphic line bundles over M to the additive group of integral sequences on M.

Proof. Regarded as a map defined on the group of holomorphic line bundles over M, this is clearly a homomorphism. Now suppose E and F are equivalent. Then $E^* \otimes F$ has a global holomorphic C^{∞} frame. So

$$z_n(F) - z_n(E) = z_n(E^* \otimes F) = 0$$

by Corollary 4.10. So $z_n(E) = z_n(F)$ for all n. Thus, the map is well defined on equivalence classes of bundles. Again by the corollary, if $z_n(E) = 0$ for all n, then E belongs to the trivial class. Thus the map is injective.

Finally, we must prove surjectivity. So choose an integral sequence $f_{mm}z^m$. Then there's a C^{∞} function f on M, supported near ∂M, such that

$$f \sim \sum_{m \geq 0} f_{mm}z^m$$

at the boundary. Let $\mathbf{1}$ be the standard frame for the trivial C^{∞} line bundle over M, and define a complex structure for this bundle by stipulating that $\mathbf{1}e^{f\log x}$ is holomorphic in the interior. This being the case, we can compute that the $\bar{\partial}$ form with respect to the C^{∞} frame $\mathbf{1}$ (which is holomorphic away from the boundary, because it agrees with the stipulated holomorphic frame where f is zero) is

$$-\bar{\partial}(f\log x) \sim -\bar{L}\left(\sum f_{mm}z^m\log x\right)\bar{\lambda} \sim -\left(\sum f_{mm}z^m\right)\bar{\lambda}.$$

Since this is a smooth compressed 1-form, we have the correct sort of holomorphic structure for the bundle. And by the construction, the invariant mth integral is $f_{mm}z^m$, which was chosen arbitrarily. □

So we have classified the bundles in the case of the cup with $c = 1$. That is, we have found an isomorphism from the group of equivalence classes of holomorphic line bundles over such a cup to the *prima facie* simpler group of integral sequences. However, this treatment is less satisfactory than we would like, because the isomorphism appears not to be natural; it depends on the choice of a distinguished point p_0 in the cup's interior. In fact, the different z_n are not "uncoupled": if you change the marking point from p_0 to $\widetilde{p}^0$, then $\widetilde{z}_n$ will depend not just on z_n, but perhaps on several of the z_k. In order to identify the correct invariant object in the present case, it would be necessary to understand exactly how the z_n vary with a change of the marking point (or how the "asymptotic function" $\sum f_{mm}z^m \log x$ is independent of the marking point). But this seems difficult to do in the case $c = 1$. (We will see later that we do have a natural classification of bundles over a cup with $c \notin \mathbb{Q}$.)

However, in the treatment of connections which follows (in Section 4.4), it will be convenient to know, at least, that not every $\widetilde{z}_n$ depends on every z_k. Specifically, we have the

Theorem 4.12. *Let z_n and $\widetilde{z}_n$ be the integral sequences corresponding to two different markings for M, and let E denote an arbitrary holomorphic line bundle over M. Then $z_0(E) = \widetilde{z}_0(E)$. Furthermore, if $z_n(E) = 0$ for all $n \geq 1$, then $\widetilde{z}_n(E) = 0$ for all $n \geq 1$.*

Proof of the first assertion. Let p_0 and $\widetilde{p}_0$ be the two marking points. Let $\mathfrak{z}$ be an interior coordinate which respects the first marking. Define $b = -\mathfrak{z}(\widetilde{p}_0)$. Then $\widetilde{\mathfrak{z}} = \mathfrak{z} + b$ is an interior coordinate which respects the second marking. Let z and $\widetilde{z}$ be the corresponding first integrals. Then $\widetilde{z} = z/(1 + bz)$. Using this relation, we may compute that

$$\widetilde{\lambda} = \frac{d\widetilde{z}}{\widetilde{z}} = \left(\frac{1}{1 + bz}\right)\frac{dz}{z} = \left(\frac{1}{1 + bz}\right)\lambda.$$

This allows us to say that

$$\overline{\widetilde{\lambda}} = \left(\frac{1}{1 + \bar{b}\bar{z}}\right)\bar{\lambda}.$$

We will also need to note that

$$\widetilde{y} = y - \arg(1 + bz).$$

Choose a global C^∞ frame s for E which is holomorphic away from the boundary. Let α be the $\bar{\partial}$ form and write $\alpha = f\bar{\lambda}$, as usual. We also write $\alpha = \widetilde{f}\widetilde{\bar{\lambda}}$. Then, using the relation between $\bar{\lambda}$ and $\widetilde{\bar{\lambda}}$ mentioned above, it is easy to see that

$$\widetilde{f} = (1 + \bar{b}\bar{z})f.$$

So when we restrict to the boundary, $\widetilde{f}$ and f agree. Also, $\widetilde{y}$ agrees with y on the boundary, by the formula above. Therefore, the formula

$$z_0 = \frac{-1}{2\pi} \int_{\partial M} f \, dy$$

is independent of the choice of the marking point. That is, $\widetilde{z}_0 = z_0$. $\qquad \square$

Proof of the second assertion. Let $\mathfrak{z}, \widetilde{\mathfrak{z}}, z, \widetilde{z}$ be as above. Calculating in a similar fashion as above, we can compute that

$$\bar{\lambda} = \left(\frac{1}{1 - \bar{b}\widetilde{\bar{z}}} \right) \widetilde{\bar{\lambda}}.$$

Now suppose $z_n = 0$ for all $n \geq 1$. Then Theorem 4.8 guarantees the existence of a holomorphic frame over the interior of the form

$$se^{\phi z_0 \log x}$$

where s is a global C^∞ frame and ϕ is a cut-off function supported near ∂M. We will use the frame s to compute the $\widetilde{z}_n$. Since we will be working near the boundary, we will ignore the ϕ. Using the fact that $\bar{\partial}(se^{z_0 \log x}) = 0$, we compute that the $\bar{\partial}$ form is

$$\alpha = -\bar{\partial}(z_0 \log x) = -\bar{L}(z_0 \log x)\bar{\lambda} = -z_0\bar{\lambda}.$$

We re-express this as

$$\alpha = -z_0/(1 - \bar{b}\widetilde{\bar{z}})\widetilde{\bar{\lambda}},$$

so that we immediately see that

$$\widetilde{f} = -z_0/(1 - \bar{b}\widetilde{\bar{z}}).$$

This is clearly an analytic function of $\widetilde{\bar{z}}$ near $\widetilde{\bar{z}} = 0$, so its asymptotic expansion in terms of powers of $\widetilde{x}$ and powers of $e^{i\widetilde{y}}$ contains only terms of the form

$$\widetilde{x}^m e^{-im\widetilde{y}}.$$

There are no terms of the form $\widetilde{x}^m e^{im\widetilde{y}}$ for $m > 0$. That is, all the $\widetilde{f}_{mm}$, for $m \geq 1$, are zero. This says that $\widetilde{z}_m = 0$ for all $m \geq 1$. $\qquad \square$

4.3   Examples of bundles

The prototypical example of a holomorphic line bundle over a compact Riemann surface is the holomorphic tangent bundle $T^{1,0}M$. So we should expect that the b-holomorphic tangent bundle $^bT^{1,0}M$ is a basic example of a holomorphic line bundle over a cup. This is so; and moreover, we can compute the bundle invariants explicitly for this bundle.

Theorem 4.13. *Let M be a cup with $c = 1$. Then $^bT^{1,0}M$ is a holomorphic line bundle over M. Furthermore, $z_0 = 1$, and $z_n = 0$ for all $n \geq 1$, when computed with respect to any marking.*

Proof. Choose a marking. Choose an interior coordinate $\mathfrak{z}$ that respects that marking, and let $z = 1/\mathfrak{z}$ be the associated first integral. Then

$$\partial_\mathfrak{z}/\partial_z = \partial z/\partial \mathfrak{z}$$
$$= \partial/\partial\mathfrak{z}(\mathfrak{z}^{-1})$$
$$= -\mathfrak{z}^{-2}$$
$$= -z^2,$$

or

$$\partial_\mathfrak{z} = z(-z\partial_z).$$

The point of this calculation is that we would like to glue the holomorphic interior frame $\partial_\mathfrak{z}$ together with the C^∞ boundary frame $-z\partial_z$ to produce a global C^∞ frame. But the winding number of the quotient $-z\partial_z/\partial_\mathfrak{z} = z^{-1}$ around $z = 0$ on a small deformation of ∂M is clearly -1, so the glueing is not possible. But if we take the C^∞ boundary frame $-e^{iy}z\partial_z$ instead, the winding number of the quotient becomes zero, so the glueing becomes possible. Therefore there is a global C^∞ frame which equals $\partial_\mathfrak{z}$ in the interior and equals $-e^{iy}z\partial_z$ near the boundary. Call it s.

Near the boundary,

$$\partial_\mathfrak{z} = z(-z\partial_z) = x(-e^{iy}z\partial_z) = se^{\log x}.$$

Therefore the $\bar{\partial}$ form with respect to s is

$$-\bar{\partial}(\log x) = -\bar{L}(\log x)\bar{\lambda} = -\bar{\lambda}$$

near the boundary, which is a smooth section of $^b\Lambda^{0,1}M$. Thus the bundle is a holomorphic line bundle over the b-holomorphic curve M (in our sense). We still need to compute the

invariants. For this, just compare the holomorphic interior frame $\partial_{\mathfrak{z}}$ to the global C^∞ frame s. In the interior, they agree. Near the boundary, $\partial_{\mathfrak{z}} = se^{\log x}$. Therefore

$$\partial_{\mathfrak{z}} = \widetilde{s}e^{\phi \log x}$$

for some other global C^∞ frame $\widetilde{s}$ and some cut-off function ϕ. In other words, $z_0 = 1$ and all the other z_n are zero. $\qquad\square$

We conclude this section by examining a natural class of examples of bundles over a cup M, namely, lifts of bundles over $\check{M} =$ the Riemann sphere. (Actually, when we treat bundles over a generic b-holomorphic complex curve, we will prove that there is a bundle on $\check{M}$ of which ${}^bT^{1,0}M$ is the lift; so properly speaking, we have already studied one lifted bundle.) To begin, we need to characterize bundles over the sphere. It is well known that bundles over a sphere are characterized by their degree, which may be defined as the integral of the first Chern class of the bundle. We take the Chern class to be defined as $i/2\pi$ times the curvature of any connection on the bundle. See [15].

Theorem 4.14. *Let M be the Riemann sphere. Let $q \in M$ and z be a holomorphic chart centered at q.*

Let E be a holomorphic line bundle over M. Then for $\mathrm{degree}(E) = d$, it is necessary and sufficient that there exist a local holomorphic frame τ for E near q and a holomorphic frame σ for $E{\downarrow}(M\backslash q)$ such that $\sigma = \tau z^d$.

Proof of sufficiency. We will use σ, τ as a system of local holomorphic frames. Define a metric on E such that $|\tau|^2 = 1$. Our strategy is to use the induced connection to compute the degree.

The curvature of the induced connection is $\bar{\partial}\partial \log h$, where h is defined as $|\sigma|^2$ or $|\tau|^2$ depending on the patch. Although h is not well defined as a function, $\bar{\partial}\partial \log h$ is well defined as a 2-form.

Let B_R be the ball around q of radius R. We will need the formula $|\sigma|^2 = |\tau z^d|^2 =$

$|z|^{2d} = r^{2d}$, where we have written $z = re^{i\theta}$. Now we can compute

$$\int_M \bar{\partial}\partial \log h = \lim_{R \to 0} \int_{B_R^c} \bar{\partial}\partial \log |\sigma|^2$$

$$= \lim_{R \to 0} \int_{B_R^c} d\partial \log |\sigma|^2$$

$$= \lim_{R \to 0} \int_{\partial B_R^c} \partial \log |\sigma|^2$$

$$= -\lim_{R \to 0} \int_{\partial B_R} \partial \log r^{2d}$$

$$= -2d \lim_{R \to 0} \int_{\partial B_R} \partial \log r.$$

But we can compute

$$\partial \log r = \frac{1}{2}(r\partial r - i\partial_\theta) \log r \left(\frac{dr}{r} + i\,d\theta \right) = \frac{1}{2}\left(\frac{dr}{r} + i\,dr \right).$$

So we can continue the calculation as

$$-d \lim_{R \to 0} \int_{\partial B_R} i\,d\theta = -2\pi i d.$$

So the degree of E is $i/2\pi$ times that integral; in other words, d. $\qquad\square$

Proof of necessity. Choose a local holomorphic frame $\widetilde{\tau}$ for E near q. Then choose a C^∞ frame $\widetilde{s}$ for $E\lfloor(M\backslash q)$. I claim that the winding number of $\widetilde{s}/\widetilde{\tau}$ on a small oriented loop around q is d. Therefore the winding number of $g = z^{-d}\widetilde{s}/\widetilde{\tau}$, defined on a punctured neighborhood of q, is zero. So we may write $g = e^\gamma$ for some smooth function γ defined in a punctured neighborhood of q.

Let ϕ be a cut-off function supported near q. Then define $s = \widetilde{s}e^{-\phi\gamma}$. This is clearly another C^∞ frame for $E\lfloor(M\backslash q)$. And, we have

$$s = \widetilde{s}e^{-\phi\gamma}$$

$$= \widetilde{\tau}z^d g e^{-\phi\gamma}$$

$$= \widetilde{\tau}z^d e^{\gamma-\phi\gamma}$$

$$= \widetilde{\tau}z^d$$

near q.

Let α be the $\bar{\partial}$ form of E with respect to s. Since s is holomorphic near q, α is zero near q; so α is a smooth $(0,1)$-form on M. Let f be a C^∞ solution on M to the

equation $\bar{\partial}f = -\alpha$. Now define $\sigma = se^f$ and $\tau = \widetilde{\tau}e^f$. Then σ is a holomorphic frame for $E{\downarrow}(M\backslash q)$. Furthermore, since α was zero near q, f is holomorphic near q; so τ is another local holomorphic frame near q. Finally,

$$\sigma = se^f = \widetilde{\tau}z^d e^f = \tau z^d.$$

So it only remains to prove the claim. Let $h = \widetilde{s}/\widetilde{\tau}$. Define connections $\nabla_{\widetilde{s}}$ and $\nabla_{\widetilde{\tau}}$ by $\nabla_{\widetilde{s}}\widetilde{s} = 0$ and $\nabla_{\widetilde{\tau}}\widetilde{\tau} = 0$. Then define $\nabla = (1 - \phi)\nabla_{\widetilde{s}} + \phi\nabla_{\widetilde{\tau}}$. To compute the connection form with respect to $\widetilde{s}$, observe that

$$\nabla\widetilde{s} = \nabla_{\widetilde{\tau}}\widetilde{s}$$
$$= \nabla_{\widetilde{\tau}}h\widetilde{\tau}$$
$$= dh\widetilde{\tau}$$
$$= (dh/h)\widetilde{s}.$$

So the connection form is $\omega = dh/h$. Now we can compute that

$$d = \mathrm{degree}(E)$$
$$= \frac{i}{2\pi} \lim_{R \to 0} \int_{B_R^c(q)} d\omega$$
$$= \frac{i}{2\pi} \lim_{R \to 0} \int_{\partial B_R^c(q)} \omega$$
$$= \frac{1}{2\pi i} \lim_{R \to 0} \int_{\partial B_R(q)} \frac{dh}{h},$$

which is the winding number of h around zero on a small oriented loop around q. $\qquad\square$

We want to show that we can pull back a holomorphic line bundle over S^2 to a holomorphic line bundle over M. Think of S^2 as the compactified plane $\mathbb{C} \cup \infty$, and let E be a holomorphic line bundle over S^2.

Choose an interior coordinate $\mathfrak{z}$ for M, and let $z = 1/\mathfrak{z}$ be the associated first integral. Then we may define $\pi : M \longrightarrow S^2$ by

$$\pi(p) = \begin{cases} \mathfrak{z}(p) & p \in M^\circ \\ \infty & p \in \partial M. \end{cases}$$

(This construction is essentially the same as that given in Theorem 3.2, particularized to the case of a cup.) π is clearly a diffeomorphism from M° to $\mathbb{C}$. But in coordinates, π looks

like

$$x, y \mapsto x e^{iy}$$

near ∂M. Since this is smooth at $x = 0$, π is a smooth map from M to S^2. Thus we may use π to lift the smooth line bundle E over S^2 to a smooth line bundle $\widehat{E}$ over M.

Now $\pi\big|_{M^\circ} = \mathfrak{z}$ is a biholomorphism from M° to $\mathbb{C}$, by definition. So we may also use π to lift the holomorphic structure of $E\lfloor\mathbb{C}$ to a holomorphic structure for $\widehat{E}\lfloor M^\circ$. The question is, does this define a holomorphic structure for $\widehat{E}\lfloor M$, in our sense? To prove that it does, it suffices to show that the $\bar\partial$ form (of $\widehat{E}\lfloor M^\circ$) with respect to some frame which is smooth up to the boundary extends to a smooth compressed $(0, 1)$-form.

But this is easy. Choose a local holomorphic frame for E near ∞. This frame is smooth in a neighborhood of ∞. So it lifts to a local smooth frame for $\widehat{E}$, defined in a neighborhood of the boundary. It is holomorphic in M°, so the $\bar\partial$ form is zero. This certainly extends as the zero form to a neighborhood of ∂M in M.

We have shown that a holomorphic line bundle over S^2 lifts (in a natural way) to a holomorphic line bundle over M.

Theorem 4.15. *Let E be a holomorphic line bundle over S^2, and let $\widehat{E}$ be its lift to a holomorphic line bundle over M. Then*

$$z_n(\widehat{E}) = \begin{cases} \mathrm{degree}(E) & n = 0 \\ 0 & n \geq 1. \end{cases}$$

Proof. Choose an interior coordinate $\mathfrak{z}$ for M, and let $z = 1/\mathfrak{z}$ be the associated first integral. Regard S^2 as $\mathbb{C} \cup \infty$. Let d be the degree of E. Then there exists a local holomorphic frame τ for E near ∞, and a holomorphic frame σ for $E\lfloor\mathbb{C}$, such that $\sigma = \tau z^d$ in a punctured neighborhood of ∞. Let ϕ be a cut-off function on M supported near ∂M. Then define

$$s = \sigma / x^{d\phi}.$$

This is a smooth frame for $\widehat{E}\lfloor M^\circ$. Away from the boundary, $s = \sigma$, and so is holomorphic. Near the boundary,

$$s = \frac{\tau z^d}{x^d}$$
$$= \tau e^{idy}.$$

Since τ is smooth at $\infty \in S^2$, τ is also smooth at ∂M on M. e^{idy} is also smooth at ∂M. So s is smooth up to the boundary.

So we have a global C^∞ frame s for $\widehat{E}$. Furthermore

$$\sigma = sx^{d\phi} = se^{d\phi \log x}$$

is a global holomorphic frame for $\widehat{E}$ over the interior of M. A comparison of this formula with Theorem 4.8 now shows that $z_0 = d$, and all the other z_n are zero. $\qquad\square$

Corollary 4.16. *Let F be a holomorphic line bundle over M. Then F is isomorphic to $\widehat{E}$ for some holomorphic line bundle E over S^2 if and only if $z_0(F) \in \mathbb{Z}$ and $z_n(F) = 0$ for all $n \geq 1$.*

4.4   *b*-connections: *c* arbitrary

Before we proceed with the search for an analog of the theorem of Narasimhan and Seshadri, we must examine b-connections in some generality. In this section, M is a cup with arbitrary collar invariant c.

Theorem 4.17. *Let E be a holomorphic line bundle over M. Then every hermitian metric on E determines a unique hermitian holomorphic b-connection.*

Proof of uniqueness. Choose a global C^∞ frame s. Define the real C^∞ function p by $\langle s, s \rangle = e^p$. Let α be the $\bar{\partial}$ form with respect to s. Suppose we have a hermitian holomorphic b-connection. Let ω be the connection form with respect to s. Then holomorphicity and hermitianity translate into

$$\omega^{0,1} = \alpha$$

and

$$\omega + \bar{\omega} = dp.$$

The first equation says that the $(0,1)$ piece of the connection form is determined (unique). We claim that the second equation says that the $(1,0)$ piece is also determined. Why? We can re-write the second equation as

$$(\omega^{1,0} + \bar{\alpha}) + (\bar{\omega}^{1,0} + \alpha) = \partial p + \bar{\partial} p,$$

which implies

$$\omega^{1,0} + \bar{\alpha} = \partial p.$$

Thus $\omega^{1,0}$ is determined:

$$\omega^{1,0} = \partial p - \bar{\alpha}.$$

$\square$

Proof of existence. Choose a global C^∞ frame s. Define the real C^∞ function p by $\langle s, s \rangle = e^p$. Let α be the $\bar{\partial}$ form with respect to s. Then define

$$\omega = \partial p - \bar{\alpha} + \alpha, \tag{4.1}$$

a compressed 1-form, and let ∇ be the b-connection whose connection form this is. Then ∇ is clearly holomorphic, because the $(0,1)$ piece of ω is α. And we compute

$$\omega + \bar{\omega} = dp$$

so that ∇ is also compatible with the metric. $\square$

Definition 4.18. A b-connection is called *non-singular* if its connection form with respect to a smooth frame is a non-singular 1-form. A b-connection is called *smoothly curved* if its curvature is a non-singular 2-form.

Theorem 4.19. *Let E be a holomorphic line bundle over M. Then the space of smoothly curved hermitian holomorphic b-connections on E is an affine space, with the underlying vector space being the space of real C^∞ functions on M which are constant on ∂M.*

Proof. First, we will show that one such b-connection exists. So choose a global C^∞ frame s which is holomorphic away from ∂M, and let α be the corresponding $\bar{\partial}$ form. What we seek is a real C^∞ function p on M such that

$$\partial \alpha - \bar{\partial}\bar{\alpha} + \bar{\partial}\partial p \tag{4.2}$$

is non-singular as a 2-form. (By differentiating (4.1), we see that this is the curvature of the hermitian holomorphic b-connection induced by the metric defined by $\langle s, s \rangle = e^p$.) Write $\alpha = f\bar{\lambda}$ near the boundary. Then our curvature 2-form may be written as

$$\partial(f\bar{\lambda}) - \bar{\partial}(\bar{f}\lambda) + \bar{\partial}(Lp\lambda) = (Lf + \bar{L}\bar{f} - \bar{L}Lp)\lambda \wedge \bar{\lambda}. \tag{4.3}$$

We have used the fact that $\partial\bar{\lambda} = \bar{\partial}\lambda = 0$, which is true because λ and $\bar{\lambda}$ are locally exact. For example, $\lambda = (1/2a)d(c\log x + iy)$.

It may be computed that $\lambda \wedge \bar{\lambda} = -(i/2a)\frac{dx}{x} \wedge dy$, so for our 2-form to be non-singular we must have $\bar{L}Lp = Lf + \bar{L}\bar{f}$ on the boundary. We can compute that

$$\bar{L}Lp = xp_x + x^2 p_{xx} - 2bxp_{xy} + |c|^2 p_{yy}.$$

So we need to choose the smooth function p so that

$$|c|^2 p_{yy} = -i\bar{c}f_y + ic\bar{f}_y,$$

or

$$p_{yy} = 2\Im(f/c)_y.$$

We will now produce such a p. Let

$$H = \Im(z_0/c) = \frac{-1}{2\pi} \int_{\partial M} \Im(f/c)\,dy.$$

Since the average value of $\Im(f/c) + H$ over ∂M is zero, we may choose a smooth function φ on ∂M such that

$$\varphi_y = \Im(f/c) + H.$$

Finally, let p be a real C^∞ function on M which agrees with 2φ on the boundary. Then p has the desired property.

It only remains to check that the quotient of a metric which defines a smoothly curved hermitian holomorphic b-connection by another such metric has the form e^u where u is constant on the boundary, and that modifying one such metric by multiplying by e^u with u constant on the boundary produces another. This is a triviality. $\qquad \square$

Definition 4.20. Let E be a holomorphic line bundle over a cup M. Let c be the collar invariant and z_0 the invariant zeroth integral of the bundle. Then we may define a new bundle invariant

$$\gamma = \frac{\Re z_0}{\Re c}.$$

Theorem 4.21. *Let E be a holomorphic line bundle over a cup M. Then for any smoothly curved hermitian holomorphic b-connection ∇,*

$$\int_M R(\nabla) = -2\pi i\gamma.$$

Proof. Let s be a global C^∞ frame for E. Let α be the corresponding $\bar\partial$ form. Then let p be a real C^∞ function on M such that the metric defined by $\langle s, s \rangle = e^p$ induces a smoothly curved (hermitian) holomorphic b-connection ∇. Then the connection form with respect to s is

$$\omega = \alpha - \bar\alpha + \partial p,$$

as we have seen in (4.1). We wish to compute the integral over M of $R(\nabla) = d\omega$. We know that $d\omega$ is a non-singular 2-form on M, but ω might be singular. So to compute the integral over M, we will take the limit of the integral over the region $x \geq x_0$ as x_0 tends to zero. Each of these integrals, for $x_0 > 0$, may be computed via Stokes's theorem. So we will find

$$\int_M R(\nabla) = \int_M d\omega$$
$$= -\lim_{x_0 \to 0} \int_{x=x_0} \omega.$$

(The minus sign appears because of our orientation convention; see Section 2.2.) So we have to examine the form of ω on circles of constant x near the boundary.

Write $\alpha = f\bar\lambda$. Then we compute, using the formula for ω above, that

$$2a\omega = f(2a\bar\lambda) - \bar{f}(2a\lambda) + Lp(2a\lambda)$$
$$= f\left(\bar{c}\frac{dx}{x} - i\,dy\right) + (Lp - \bar{f})\left(c\frac{dx}{x} + i\,dy\right)$$
$$= (\bar{c}f - c\bar{f} + cLp)\frac{dx}{x} + (-if - i\bar{f} + iLp)dy \qquad (4.4)$$
$$= |c|^2([f/c] - [\bar{f}/\bar{c}] + [1/\bar{c}](xp_x - i\bar{c}p_y))\frac{dx}{x} - i(f + \bar{f} - (xp_x - i\bar{c}p_y))dy$$
$$= |c|^2(i[2\Im(f/c) - p_y] + (1/\bar{c})xp_x)\frac{dx}{x} - i(2\Re f - xp_x + i\bar{c}p_y)dy.$$

So we can compute that

$$\int_{x=x_0} \omega = \frac{-i}{a}\int_{x=x_0}\Re f\,dy + \frac{ix_0}{2a}\int_{x=x_0} p_x\,dy,$$

or

$$-\int_{x=x_0}\omega = \frac{i}{a}\int_{x=x_0}\Re f\,dy - \frac{ix_0}{2a}\int_{x=x_0}p_x\,dy.$$

All the dx terms of the integrand disappear when we pull back to the circle of constant x, and the term involving p_y clearly integrates to zero. Since $p_x\,dy$ defines a smooth 1-form on M, the second term is killed by the coefficient x_0 in the limit. So the only thing that

survives in the limit is

$$\frac{i}{a} \int_{\partial M} \Re f \, dy = \frac{-2\pi i}{a} \Re \left(\frac{-1}{2\pi} \int_{\partial M} f \, dy \right)$$

$$= -2\pi i \, \Re z_0 / \Re c$$

$$= -2\pi i \gamma.$$

□

Theorem 4.22. *If z_0/c is real, then every smoothly curved hermitian holomorphic b-connection is actually non-singular. Otherwise, no such b-connection is non-singular.*

Proof. Differentiate the last formula for $2a\omega$ in (4.4). You find that

$$2a \, d\omega = |c|^2 (i[2\Im(f/c)_y - p_{yy}] + (1/\bar{c})xp_{xy})dy \wedge \frac{dx}{x}$$

plus something non-singular. We know that this form is non-singular, so we must have

$$p_{yy} = 2\Im(f/c)_y$$

on ∂M. Thus $p_y = 2\Im(f/c) + H$ for some constant H. If we average both sides over the boundary (with respect to the measure dy), then we find that H is -2 times the average of $\Im(f/c)$ over the boundary, or

$$H = 2\Im(z_0/c).$$

Now we can finish the argument:

The connection form is non-singular iff $p_y = 2\Im(f/c)$. (This is by inspecting the last formula for $2a\omega$ in (4.4).) But, $p_y = 2\Im(f/c)$ iff $H = 0$. (By the above remarks.) And, since $H = 2\Im(z_0/c)$, $H = 0$ iff z_0/c is real. □

Note 4.23. In the analysis of lifting bundles from a sphere to a cup with $c = 1$ (Section 4.3), we saw that the quantity z_0 is a "generalized degree." We will see later that for a cup with $c \notin \mathbb{Q}$, the generalized degree is actually z_0/c. When z_0/c is real, it is equal to γ, as an elementary calculation will show. When c is itself real, we see that γ is equal to the real part of this generalized degree. So our formula for the integral of the curvature would seem to agree with the usual formula

$$-2\pi i \cdot \text{degree}$$

(for bundles over a compact surface) as closely as possible. It cannot agree precisely, because the integral of the curvature of a hermitian connection must always be purely imaginary; but our generalized degree is not necessarily real.

4.5   Connections of constant curvature

Here we take M to be a geometric cup with $c = 1$.

First we must establish some notation. Whenever we speak of the Laplacian Δ in the plane, we mean the positive one. With this in mind, we have the formula $\bar{\partial}\partial = (i/2) \star \Delta$. Choose an interior coordinate $\mathfrak{z}$ and let $z = 1/\mathfrak{z}$ be the associated first integral. Then we have $x, y, L, \bar{L}, \lambda, \bar{\lambda}$ as in (2.2) and (2.3). Define

$$\mathcal{L} = \bar{L}L = \begin{pmatrix} x\partial_x & \partial_y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x\partial_x \\ \partial_y \end{pmatrix} \tag{4.5}$$

and

$$\mathcal{T} = \begin{pmatrix} \frac{dx}{x} & dy \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{dx}{x} \\ dy \end{pmatrix}.$$

Define $h = \langle L, L \rangle$. Then h vanishes to second order on the boundary. We have the formulas

$$g = \tfrac{h}{4}\mathcal{T}$$
$$\Delta = \tfrac{4}{h}\mathcal{L} \tag{4.6}$$
$$\mathrm{vol} = \tfrac{h}{4}\tfrac{dx}{x} \wedge dy.$$

(These are taken to be valid near the boundary only, of course.) Note that the volume form is a smooth 2-form which vanishes to first order on the boundary.

In trying to construct a holomorphic section of a bundle in the case $c = 1$, we found that eventually it boiled down to solving the $\bar{\partial}$ problem. So we made use of the Cauchy integral formula. Here, we will be trying to construct a connection with special properties, and we will see that it boils down to solving the Laplace equation. So we will be using the standard integral formula for solving the Laplace equation in the plane. This we present as

Lemma 4.24. *Let f be a compactly supported C^∞ function in $\mathbb{C}$, and define Q to be the integral of f with respect to Lebesgue measure. Define*

$$u(x) = \frac{1}{2\pi} \int_{\mathbb{C}} \log |x - y| \, f(y) \, dy$$

where dy represents Lebesgue measure in the variable y. Then $\Delta u = f$, and

$$u(x) - \frac{Q}{2\pi} \log |x| \to 0$$

as $x \to \infty$.

Proof. The fact that $\Delta u = f$ is so well known that we do not need to prove it here. (See, for example, [2].) We only need to prove the assertion about the behavior of the solution u at infinity.

So, let R be a positive real number such that the support of f is contained in the ball of radius R (which we will call B). Also, let A be the integral of $|f|$ with respect to Lebesgue measure. Compute that

$$u(x) - \frac{Q}{2\pi} \log |x| = \frac{1}{2\pi} \int_{\mathbb{C}} \log |x - y| \, f(y) \, dy - \frac{1}{2\pi} \int_{\mathbb{C}} \log |x| \, f(y) \, dy$$

$$= \frac{1}{2\pi} \int_{B} (\log |x - y| - \log |x|) f(y) \, dy$$

$$= \frac{1}{2\pi} \int_{B} \log \left(\frac{|x - y|}{|x|} \right) f(y) \, dy$$

$$= \frac{1}{2\pi} \int_{B} \log |1 - y/x| \, f(y) \, dy,$$

where the quotient is taken in the sense of complex numbers. So

$$2\pi \, |u(x) - (Q/2\pi) \log |x| \, | \leq \int_{B} |\log |1 - y/x| \, | \cdot |f(y)| \, dy.$$

Choose $\varepsilon > 0$. We will find a δ such that $|x| > \delta$ implies that $|\log |1 - y/x| \, | < \varepsilon/A$ for all $y \in B$; then we'll be done.

So define

$$\delta = \frac{R}{1 - e^{-\varepsilon/A}}.$$

Suppose $|x| > \delta$. Then we have

$$1 - \frac{R}{|x|} > e^{-\varepsilon/A}.$$

So

$$\min_{y \in B} |1 - y/x| = 1 - R/|x| > e^{-\varepsilon/A},$$

or

$$\log |1 - y/x| > -\varepsilon/A$$

for all $y \in B$.

Also, $\cosh \varepsilon/A \geq 1$. So $e^{\varepsilon/A} - 1 \geq 1 - e^{-\varepsilon/A}$. So if $|x| > \delta$, it is also true that

$$|x| > \frac{R}{e^{\varepsilon/A} - 1}.$$

This leads to

$$1 + \frac{R}{|x|} < e^{\varepsilon/A}.$$

Therefore

$$\max_{y \in B} |1 - y/x| = 1 + R/|x| < e^{\varepsilon/A},$$

or

$$\log |1 - y/x| < \varepsilon/A$$

for all $y \in B$.

Thus we have

$$-\varepsilon/A < \log |1 - y/x| < \varepsilon/A$$

or

$$|\log |1 - y/x|| < \varepsilon/A$$

for all $y \in B$. □

Corollary 4.25. *Let f be a compactly supported function in $\mathbb{C}$ whose integral is zero, and define u as in the lemma. Then u and f are smooth functions on the sphere $\mathbb{C} \cup \infty$, and $\Delta u = f$ there.*

The following lemma applies to all cups, not just those with $c = 1$.

Technical Lemma 4.26. *Let u be a smooth function on the interior of M. Suppose xu_x extends as a continuous function on M, and $\bar{\partial}\partial u$ extends as a continuous non-singular 2-form on M. Then*

$$\int_M \bar{\partial}\partial u = \frac{-i}{2a} \int_{\partial M} (xu_x) \, dy.$$

Proof. Since $\bar{\partial}\partial u$ is continuous on M, it is integrable. So we have

$$\int_M \bar{\partial}\partial u = \lim_{x_0 \to 0} \int_{x \geq x_0} \bar{\partial}\partial u$$

$$= \lim_{x_0 \to 0} \int_{x \geq x_0} d\partial u$$

$$= -\lim_{x_0 \to 0} \int_{x = x_0} (Lu)\lambda$$

$$= \frac{-1}{2a} \lim_{x_0 \to 0} \int_{x = x_0} (xu_x - i\bar{c}u_y) \left(c\frac{dx}{x} + i \, dy \right)$$

$$= \frac{-i}{2a} \lim_{x_0 \to 0} \int_{x = x_0} (xu_x - i\bar{c}u_y) dy$$

$$= \frac{-i}{2a} \lim_{x_0 \to 0} \int_{x = x_0} xu_x \, dy$$

$$= \frac{-i}{2a} \int_{\partial M} (xu_x) \, dy.$$

□

The minus sign appears in Stokes's formula because of our orientation convention.

Note 4.27. Using the relation $\bar{\partial}\partial = (i/2) \star \Delta$, we can re-formulate the conclusion of this technical lemma as

$$\int_M \Delta u \cdot \text{vol} = \frac{-1}{a} \int_{\partial M} (xu_x)\, dy. \tag{4.7}$$

From now on, we will assume that M is a geometric cup with $c = 1$.

Proposition 4.28. *Fix an interior coordinate $\mathfrak{z}$ for M, and let $z = 1/\mathfrak{z}$ be the associated first integral. Write $z = xe^{iy}$ as usual. Then there exist real smooth functions u^1 and u^2 on M such that*

1. *u^2 is supported near the boundary; and at the boundary,*

$$u^2 \sim 1 - \sum_{\substack{m \geq 1 \\ n = \pm m}} \frac{\pi}{4m} h_{mn} x^m e^{iny}.$$

2. *Setting $V = u^1 + u^2 \log x$, we have $\Delta V = -2\pi$.*

Proof. Choose a cut-off function ϕ supported near the boundary. Define u^0 to be any real smooth solution to $\Delta u^0 = -2\pi$ on the interior of M. Such a solution exists by the ellipticity of the Laplacian over the interior. Next, let u^1 and u^2 be real smooth functions on M, supported near the boundary, such that

$$u^1 \sim -\frac{\pi}{2} \sum_{\substack{m \geq 1 \\ n \neq \pm m}} \frac{h_{mn}}{m^2 - n^2} x^m e^{iny}$$

$$u^2 \sim 1 - \frac{\pi}{2} \sum_{\substack{m \geq 1 \\ n = \pm m}} \frac{h_{mn}}{2m} x^m e^{iny}$$

at the boundary. Then let u^3 be a real smooth function on M such that

$$\Delta u^3 = -2\pi - \Delta[(1 - \phi)u^0 + u^1 + u^2 \log x].$$

(The claim is that such a function exists.) Then

$$\Delta[(1 - \phi)u^0 + u^1 + u^2 \log x + u^3] = -2\pi;$$

and $(1 - \phi)u^0, u^1, u^3$ are smooth on M. Relabel the sum of these three functions as u^1, and leave u^2 (which is also smooth on M) labeled as it is. We're done, as long as we prove the claim.

To prove the claim, we first take note of three facts about the datum

$$-2\pi - \Delta[(1 - \phi)u^0 + u^1 + u^2 \log x].$$

First, it is supported near the boundary. That's because $-2\pi - \Delta[(1 - \phi)u^0]$ is supported near the boundary by the construction of u^0; and $\Delta(u^1 + u^2 \log x)$ is supported near the boundary since u^1 and u^2 were, by their construction.

Second, the datum vanishes to infinite order on ∂M. To see this, write it as

$$\left(-\frac{\pi}{2}h - \mathcal{L}[(1 - \phi)u^0 + u^1 + u^2 \log x]\right)\frac{4}{h}.$$

Since $(1 - \phi)u^0$ is zero in a neighborhood of the boundary, we only need to show that

$$-\frac{\pi}{2}h - \mathcal{L}(u^1 + u^2 \log x)$$

vanishes to infinite order on the boundary. This can easily be seen by using the definition

$$\mathcal{L} = (x\partial_x)^2 + \partial_y^2$$

and the asymptotic formulas for u^1, u^2, and h. When this calculation is made, it must be remembered that h vanishes on the boundary, so that

$$h \sim \sum_{\substack{m \geq 1 \\ n \in \mathbb{Z}}} h_{mn}x^m e^{iny}$$

(the $m = 0$ terms do not appear).

Third, the integral over M of the datum times the volume form is zero. To see this, just compute the integral of each of the five functions -2π, $\Delta[(1 - \phi)u^0]$, Δu^1, $\Delta[(u^2 - \phi) \log x]$, and $\phi \log x$. The first piece gives -2π, obviously. The second, third, and fourth pieces give zero by Note 4.27. And finally, the fifth piece gives

$$\int_M \Delta(\phi \log x) \cdot \text{vol} = -\int_{\partial M} x\partial_x(\phi \log x)\, dy$$

$$= -\int_{\partial M} dy$$

$$= -2\pi$$

by Note 4.27. A subtraction, and the full integral is zero.

Now we want to transfer this problem to the z plane. The equation

$$\Delta u^3 = -2\pi - \Delta[(1-\phi)u^0 + u^1 + u^2 \log x]$$

may be re-written as

$$\mathcal{L}u^3 = \frac{h}{4}(-2\pi - \Delta[(1-\phi)u^0 + u^1 + u^2 \log x]).$$

But we know that

$$\mathcal{L} = \bar{L}L = 2\bar{z}\partial_{\bar{z}}2z\partial_z = 4x^2\partial_{\bar{z}}\partial_z = x^2\Delta_z.$$

So we can re-write the equation again as

$$\Delta_z u^3 = \frac{h}{4x^2}(-2\pi - \Delta[(1-\phi)u^0 + u^1 + u^2 \log x]).$$

By facts 1 and 2, the datum is compactly supported and smooth in the z plane (in fact, it vanishes to infinite order at the origin). And we can use fact 3, along with the formula

$$\begin{aligned} \text{vol} &= \frac{h}{4}\frac{dx}{x} \wedge dy \\ &= \frac{h}{4x^2}x\,dx \wedge dy \\ &= \frac{h}{4x^2}d\mu(z), \end{aligned}$$

to conclude that

$$\int_{\mathbb{C}} \frac{h}{4x^2}(-2\pi - \Delta[(1-\phi)u^0 + u^1 + u^2 \log x])\,d\mu(z) = 0.$$

So by Lemma 4.24 (and its corollary), there exists solution which is smooth on the z sphere. Therefore its lift is smooth on M. $\square$

Theorem 4.29. *Let M be a geometric cup with $c = 1$. Mark an interior point, and fix an interior coordinate $\mathfrak{z}$ which respects the marking. Let $z = 1/\mathfrak{z}$ be the associated first integral. Then define $h = \langle L, L \rangle$. Express h as an asymptotic sum at the boundary:*

$$h \sim \sum_{\substack{m \geq 0 \\ n \in \mathbb{Z}}} h_{mn}x^m e^{iny}.$$

Let E be a holomorphic line bundle over M. Let z_n be the invariant integral sequence of E with respect to the chosen marking of M. Define $\gamma = \Re z_0$ as usual. Then for there

to exist a hermitian holomorphic b-connection on E whose curvature is $-2\pi i\gamma \cdot \mathrm{vol}$, *it is necessary and sufficient that*

$$z_n = -\frac{\pi\gamma}{2n}h_{nn}z^n$$

for all $n \geq 1$.

Proof of necessity. Choose a smooth frame s. Let α be the $\bar{\partial}$ form with respect to s, and write $\alpha = f\bar{\lambda}$. Then write f as an asymptotic series in the usual way, so that $z_n = -f_{nn}z^n$. We need to show that

$$f_{nn} = \frac{\pi\gamma}{2n}h_{nn}$$

for $n \geq 1$.

To do this, let $\langle s, s \rangle = e^p$ be the metric associated to the special connection, and start with the equation

$$\partial\alpha - \bar{\partial}\bar{\alpha} + \bar{\partial}\partial p = -2\pi i\gamma \cdot \mathrm{vol}.$$

(In (4.2), we saw that the left hand side is the curvature.) As in (4.3), the left hand side can be re-written as

$$(Lf + \bar{L}\bar{f} - \bar{L}Lp)\lambda \wedge \bar{\lambda} = (Lf + \bar{L}\bar{f} - \bar{L}Lp)(-i/2)\frac{dx}{x} \wedge dy,$$

and the right hand side can be re-written by (4.6) as

$$-2\pi i\gamma\frac{h}{4}\frac{dx}{x} \wedge dy = \frac{-\pi i\gamma h}{2}\frac{dx}{x} \wedge dy.$$

This gives the equation

$$Lf + \bar{L}\bar{f} - \bar{L}Lp = \pi\gamma h.$$

Now we will work with the asymptotic formulas for f and h. On the left, we have

$$(x\partial_x - i\partial_y)\sum f_{mn}x^m e^{iny} + (x\partial_x + i\partial_y)\sum \bar{f}_{mn}x^m e^{iny} - ([x\partial_x]^2 + \partial_y^2)\sum p_{mn}x^m e^{iny}$$
$$= \sum(m+n)f_{mn}x^m e^{iny} + \sum(m-n)(f_{m,-n})\bar{}x^m e^{iny} - \sum(m^2-n^2)p_{mn}x^m e^{iny},$$

and on the right we have

$$\pi\gamma\sum h_{mn}x^m e^{iny}.$$

Now examine the $m = n$ terms, for $n \geq 1$. We find that

$$2nf_{nn} = \pi\gamma h_{nn}$$

or

$$f_{nn} = \frac{\pi\gamma}{2n} h_{nn},$$

which was to be proved. □

Proof of sufficiency. As per Theorem 4.8, choose a smooth frame s for E and a smooth function u on E which is supported near the boundary, in such a way that

$$se^{u \log x}$$

is a holomorphic frame over the interior and

$$u \sim \sum_{n \geq 0} z_n.$$

Define a metric for E by stipulating that

$$\langle s, s \rangle = e^{2(\gamma V - (\Re u) \log x)}$$

where V is the function whose existence is asserted in Proposition 4.28. We claim that the function $p = 2(\gamma V - (\Re u) \log x)$ is smooth on M, so the metric is smooth and non-degenerate.

Now compute the curvature of the induced connection ∇. The $\bar{\partial}$ form with respect to s is $\alpha = -\bar{\partial}(u \log x)$. So

$$
\begin{aligned}
R(\nabla) &= \partial \alpha - \bar{\partial}\bar{\alpha} + \bar{\partial}\partial p \\
&= -\partial\bar{\partial}(u \log x) + \bar{\partial}\partial(\bar{u} \log x) + 2\bar{\partial}\partial(\gamma V - (\Re u) \log x) \\
&= \bar{\partial}\partial(u \log x) + \bar{\partial}\partial(\bar{u} \log x) + 2\gamma\bar{\partial}\partial V - \bar{\partial}\partial((2\Re u) \log x) \\
&= 2\gamma\bar{\partial}\partial V \\
&= 2\gamma(i/2)\Delta V \cdot \mathrm{vol} \\
&= -2\pi i \gamma \cdot \mathrm{vol}.
\end{aligned}
$$

All that remains is to prove the claim. Write $V = u^1 + u^2 \log x$ as in Proposition 4.28. Then

$$p = 2(\gamma u^1 + \gamma u^2 \log x - (\Re u) \log x).$$

So it suffices to prove that $\gamma u^2 - (\Re u)$ vanishes to infinite order at the boundary. But by the asymptotic formula for u^2,

$$
\begin{aligned}
\gamma u^2 &\sim \gamma \left(1 - \sum_{\substack{m \geq 1 \\ n = \pm m}} \frac{\pi h_{mn}}{4m} x^m e^{iny} \right) \\
&= \gamma - \sum_{\substack{m \geq 1 \\ n = \pm m}} \frac{\pi \gamma h_{mn}}{4m} x^m e^{iny} \\
&= \gamma - \sum_{m \geq 1} \frac{\pi \gamma h_{mm}}{4m} z^m - \sum_{m \geq 1} \frac{\pi \gamma h_{m,-m}}{4m} \bar{z}^m \\
&= \gamma - \sum_{m \geq 1} \frac{\pi \gamma h_{mm}}{4m} z^m - \sum_{m \geq 1} \frac{\pi \gamma \bar{h}_{mm}}{4m} \bar{z}^m \\
&= \gamma - 2 \Re \left(\sum_{m \geq 1} \frac{\pi \gamma h_{mm}}{4m} z^m \right) \\
&= \Re \left(z_0 - \sum_{m \geq 1} \frac{\pi \gamma h_{mm}}{2m} z^m \right).
\end{aligned}
$$

By hypothesis, the mth term of the sum is equal to z_m. Thus

$$
\gamma u^2 \sim \Re \sum_{n \geq 0} z_n,
$$

which is the same as the asymptotic expansion for $\Re u$. So our function does vanish to infinite order at the boundary. $\qquad \square$

Remarks 4.30. Fix a marking on M, and a compatible first integral z, and define h as in the theorem. Then for each $n \geq 1$, the nth integral

$$
Q_n = \frac{-\pi}{2n} h_{nn} z^n \tag{4.8}
$$

does not depend on the first integral z used in the construction, so long as z respects the marking. (This will be proved in Invariance Theorem 4.32 at the end of this section.) That is, we have here (part of) an invariant integral sequence associated to the marked geometric cup. Since the $z_n(E)$ are also invariant with respect to a change of first integral (though not a change of marking), the condition of Theorem 4.29 (that $z_n(E) = \gamma(E) \cdot Q_n$) is self-consistent, if we think of it as applying to bundles over a marked cup.

But more is true. In fact, if the condition is true (of a certain bundle) for one marking, it is true for any marking. (This will also be proved in the invariance theorem.) Thus the condition is fully self-consistent, and applies to bundles over an unmarked geometric cup, which is what we want. A consequence of this is that the Q_n must transform under a change of the marking in the same way that the integral sequence of a bundle transforms under a change of marking. So we can build a bundle class G over M, which we could call the *metric line bundle class* of the geometric cup M, by stipulating that its invariant integral sequence (with respect to a given marking) be

$$1, Q_1, Q_2, Q_3, \ldots,$$

with the Q_n defined using that same marking.

The advantage of this is that we can now develop a coordinate-free condition on holomorphic line bundles for the existence of a constant-curvature hermitian holomorphic b-connection. The condition of Theorem 4.29 is that the invariant integral sequence of the bundle (with respect to any marking) be equal to γ times the invariant integral sequence of G (with respect to the same marking), except for a purely imaginary discrepancy in the zeroth integral. This needs a bit of further interpretation.

First of all, if we denote by G^γ the holomorphic line bundle class whose invariant integral sequence is γ times the sequence for G, then it seems reasonable to call G^γ the "gammath power of the metric line bundle class." That's because taking a positive integer power of a bundle (which is well defined as the tensor product of that many copies of the bundle) has the effect of multiplying the invariant integral sequence by the power. All we are doing here is generalizing from positive integers to real numbers.

Next, we have to understand the condition of Theorem 4.29 in these terms. The condition is that $z_n(E) = z_n(G^\gamma)$ for all $n \geq 1$, and $z_0(E) - z_0(G^\gamma)$ is purely imaginary. When this is so, it can easily be seen that there is a holomorphic isomorphism over the interior from E to any representative G_0^γ of G^γ, which has the form

$$Ie^{it\phi \log x}$$

where I is a genuine C^∞ diffeomorphism over M from E to G_0^γ, t is some real number, and ϕ is a cut-off function supported near the boundary. That is, we have a holomorphic isomorphism over the interior which only fails to be smooth up to the boundary by virtue of the function $e^{it \log x}$. This function represents a pure twist in the fibres which accelerates

as you approach the boundary of M. You might say, in this case, that the bundles E and G_0^γ are *twist-isomorphic*, and that the classes $[E]$ and G^γ are *twist-equivalent*.

With these remarks in mind, we could re-formulate Theorem 4.29 as follows:

Reformulated Theorem 4.31. *Let M be a geometric cup with $c = 1$, and let G be the metric line bundle class of M. Then a holomorphic line bundle E over M posesses a constant-curvature hermitian holomorphic b-connection if and only if $[E]$ is twist-equivalent to $G^{\gamma(E)}$.*

Now that we've seen (in the remarks above) how we can get from Theorem 4.29 to Reformulated Theorem 4.31, we must actually get there. That means, we should prove the invariance theorem, construct the metric line bundle class and its real powers, define twist isomorphism, and finally prove the re-formulated theorem using all this.

Invariance Theorem 4.32. *Let M be a geometric cup with $c = 1$.*

1. *Fix a marking on M. Choose an interior coordinate $\mathfrak{z}$ which respects the marking, and let $z = 1/\mathfrak{z}$ be the associated first integral. Then for each $n \geq 1$, the formula*

$$Q_n = \frac{-\pi}{2n} h_{nn} z^n$$

   is independent of the choice of interior coordinate $\mathfrak{z}$.

2. *If a holomorphic line bundle satisfies the condition*

$$z_n(E) = \gamma(E) \cdot Q_n \qquad \text{for all } n \geq 1$$

   for one choice of marking, then it satisfies this condition for every choice of marking.

Proof of the first assertion. Let $\mathfrak{z}, \widetilde{\mathfrak{z}}$ be two interior coordinates which respect the marking. Then $\widetilde{\mathfrak{z}} = (1/a)\mathfrak{z}$ for some nonzero complex number $a = re^{i\theta}$. Let $z, \widetilde{z}$ be the associated first integrals. Then

$$\widetilde{z} = az$$
$$\widetilde{x} = rx$$
$$\widetilde{y} = y + \theta.$$

Therefore $\partial_{\widetilde{z}} = (1/a)\partial_z$, and $\widetilde{L} = 2\widetilde{z}\partial_{\widetilde{z}} = 2az(1/a)\partial_z = 2z\partial_z = L$. Therefore, $\widetilde{h} = h$. So all we have to do in order to compute the $\widetilde{h}_{nn}$ is to write h as a series in $\widetilde{x}^m e^{in\widetilde{y}}$, as follows:

$$h = \sum h_{mn} x^m e^{iny}$$
$$= \sum h_{mn} (\widetilde{x}/r)^m e^{in(\widetilde{y}-\theta)}$$
$$= \sum \frac{h_{mn}}{r^m e^{in\theta}} \widetilde{x}^m e^{in\widetilde{y}}.$$

Therefore

$$\widetilde{h}_{nn} = \frac{h_{nn}}{r^n e^{in\theta}} = (1/a^n)h_{nn}.$$

Now it's easy to see that

$$\widetilde{Q}_n = \frac{-\pi}{2n}\widetilde{h}_{nn}\widetilde{z}^n$$
$$= \frac{-\pi}{2n}(1/a^n)h_{nn}a^n z^n$$
$$= \frac{-\pi}{2n}h_{nn}z^n$$
$$= Q_n.$$

$\square$

Proof of the second assertion. If we were in posession of a formula which showed explicitly how the invariant integral sequence of a bundle transforms under a change of the marking, then we would merely have to show that the γQ_n transform the same way under a change of marking. This would be best. But since we don't have that formula, we have to argue more abstractly, in a boot-strapping sort of way. The idea is that Theorem 4.29 states the equivalence of two conditions on a bundle over a marked cup. The first is the condition on the invariant integral sequence. The second is the existence of a constant-curvature hermitian holomorphic b-connection. But since the second condition is independent of the marking, so must be the first.

Explicitly, we may argue as follows. Suppose the condition on the integral sequence is satisfied with respect to a certain marking p_0. Then by Theorem 4.29, there exists a constant-curvature hermitian holomorphic b-connection. Now choose any other marking $\widetilde{p}_0$. Then, again by the theorem, the condition on the integral sequence is satisfied with respect to this other marking. $\square$

Definition 4.33 (metric line bundle class). Let M be a geometric cup with $c = 1$. Mark an interior point, and define G to be the holomorphic line bundle class whose invariant

integral sequence with respect to this marking is

$$z_n(G) = \begin{cases} 1 & n = 0 \\ Q_n & n \geq 1, \end{cases}$$

where the Q_n are of course defined as in Invariance Theorem 4.32 using the same marking. Then G is called the *metric line bundle class* of the geometric cup M.

Proof of the soundness of the definition. Note that any representative G_0 of G satisfies the condition expressed in part 2 of Invariance Theorem 4.32 with respect to this marking (by its construction). So by the invariance theorem, G_0 satisfies the same condition with respect to every other marking. That is, if we had chosen any other marking and defined a class the same way with respect to that marking, we would have ended up with the same class as that defined by means of the original marking. □

Note 4.34. As defined here, G is an isomorphism class of holomorphic line bundles. For our purposes, this is sufficient. But it might be possible to achieve a more coordinate-free way of constructing a specific representative of this class, perhaps by modifying the holomorphic structure of the holomorphic compressed tangent bundle (which we know has zeroth integral 1 and all others 0). We leave this point unresolved.

Definition 4.35 (real power of a line bundle class). Let M be a cup, and $[E]$ be a holomorphic line bundle class over M. Choose any real number γ. We will define a new class, the γth power $[E]^\gamma$ of $[E]$, by constructing a representative E^γ as follows: as a C^∞ object, E^γ is the trivial bundle. Now let s be a global C^∞ frame for E, and α the $\bar\partial$ form for E with respect to s. Then stipulate that the $\bar\partial$ form for E^γ with respect to the global C^∞ frame $\mathbf{1}$ is $\gamma \cdot \alpha$. Finally, set $[E]^\gamma = [E^\gamma]$.

Proof of the soundness of the definition. First, suppose we had performed the construction with $\widetilde{s} = se^u$ instead of with s. Then $\widetilde{\alpha} = \alpha + \bar\partial u$. So the $\bar\partial$ form of $\widetilde{E}^\gamma$ with respect to the global C^∞ frame $\mathbf{1}$ is $\gamma\alpha + \gamma\bar\partial u$. Now define a C^∞ isomorphism Φ from E^γ to $\widetilde{E}^\gamma$ by

$\Phi \mathbf{1} = \mathbf{1}e^{-\gamma u}$. I claim that this is a holomorphic map. This is so because, in $\widetilde{E}^{\gamma}$,

$$\bar{\partial}(\Phi \mathbf{1}) = \bar{\partial}(\mathbf{1}e^{-\gamma u})$$
$$= \bar{\partial}\mathbf{1} \cdot e^{-\gamma u} + \mathbf{1}\bar{\partial}e^{-\gamma u}$$
$$= \mathbf{1}(\gamma\alpha + \gamma\bar{\partial}u)e^{-\gamma u} + \mathbf{1}(-\gamma\bar{\partial}u)e^{-\gamma u}$$
$$= \mathbf{1}e^{-\gamma u}(\gamma\alpha + \gamma\bar{\partial}u - \gamma\bar{\partial}u)$$
$$= \Phi\mathbf{1}(\gamma\alpha),$$

so the the $\bar{\partial}$ form of $\widetilde{E}^{\gamma}$ with respect to the frame $\Phi\mathbf{1}$ is equal to the $\bar{\partial}$ form of E^{γ} with respect to the frame $\mathbf{1}$. Thus E^{γ} and $\widetilde{E}^{\gamma}$ are equivalent, and so the class $[E]^{\gamma}$ does not depend on the choice of frame for E.

We must show that the class $[E]^{\gamma}$ is also independent of the choice of the representative E. So let $\widetilde{E}$ be another bundle equivalent to E, and construct $\widetilde{E}^{\gamma}$; we must show that this is isomorphic to E^{γ}. So let s be a global C^{∞} frame for E, and let Φ be a holomorphic isomorphism from E to $\widetilde{E}$. So $\widetilde{s} = \Phi s$ is a global C^{∞} frame for $\widetilde{E}$. Let α be the $\bar{\partial}$ form of E with respect to s. Then $\widetilde{\alpha} = \alpha$ is also the $\bar{\partial}$ form for $\widetilde{E}$ with respect to $\widetilde{s}$, since Φ is a holomorphic map. So both E^{γ} and $\widetilde{E}^{\gamma}$ (constructed by means of s and $\widetilde{s}$) are the same bundle: namely, the trivial C^{∞} bundle with holomorphic structure defined by $\bar{\partial}\mathbf{1} = \mathbf{1} \otimes \alpha$. $\qquad\square$

Remark 4.36. If $t \in \{1, 2, 3, \ldots\}$, then

$$\underbrace{E \otimes \cdots \otimes E}_{t \text{ times}} \in [E]^{t}.$$

(This justifies the name.) The reason is that tensor product becomes addition at the level of $\bar{\partial}$ forms.

Definition 4.37. Let M be a cup, and let E, F be holomorphic line bundles over M. Then E is *twist-isomorphic* to F if there exists a C^{∞} isomorphism $I : E \longrightarrow F$ such that the map

$$Ie^{it\phi \log x}$$

is holomorphic over the interior, for some real number t and some cut-off function ϕ supported near the boundary of M.

Remarks 4.38. As was mentioned in Remarks 4.30, E is twist-isomorphic to F if and only if $z_n(E) = z_n(F)$ for all $n \geq 1$ and $z_0(E) - z_0(F)$ is purely imaginary; the invariant

integral sequences both computed with respect to any chosen marking. From this, it is easy to see that twist-isomorphism is a relation of equivalence on the set of holomorphic line bundles over M. And since "isomorphic" implies "twist-isomorphic," this relation of equivalence extends to the set of isomorphy classes of holomorphic line bundles over M; here, the relation will be called *twist-equivalence*.

Now we are ready to prove Reformulated Theorem 4.31, using the original theorem (Theorem 4.29).

Proof of necessity. Suppose E has a constant-curvature hermitian holomorphic b-connection. Then by Theorem 4.21, the curvature must be $-2\pi i\gamma \cdot \mathrm{vol}$. Choose a marking p_0 and a compatible interior coordinate $\mathfrak{z}$. Let $z = 1/\mathfrak{z}$. Then by Theorem 4.29,

$$z_n(E, p_0) = \gamma(E) \cdot Q_n(p_0)$$

for all $n \geq 1$. So the invariant integral of E with respect to p_0 equals the invariant integral sequence of $G^{\gamma(E)}$ with respect to p_0, except for the discrepancy

$$z_0(E) - z_0(G^{\gamma(E)}) = z_0(E) - \gamma(E),$$

which is purely imaginary. So $[E]$ is twist-equivalent to $G^{\gamma(E)}$. $\qquad\square$

Proof of sufficiency. Suppose $[E]$ is twist-equivalent to $G^{\gamma(E)}$. Choose a marking p_0. Then for all $n \geq 1$,

$$\begin{aligned} z_n(E, p_0) &= z_n(G^{\gamma(E)}, p_0) \\ &= \gamma(E) \cdot z_n(G, p_0) \\ &= \gamma(E) \cdot Q_n(p_0). \end{aligned}$$

So, by Theorem 4.29, E has a constant-curvature hermitian holomorphic b-connection. $\quad\square$

Theorem 4.39. *Let M be a cup with $c = 1$. Then any line bundle class over M having $z_0 = 1$ and $z_1 = 0$ may be realized as the metric line bundle class associated to some geometric structure for M.*

Proof. Choose an appropriate line bundle class and a marking on M. Let z_n denote the invariant nth integral for this class with respect to the chosen marking. Then the condition for this class to be the metric line bundle class is $z_n = Q_n = -(\pi/2n)h_{nn}z^n$, or

$$h_{nn} = \frac{-2n}{\pi}\frac{z_n}{z^n},$$

for all $n \geq 1$. (The class already satisfies the necessary condition $z_0 = 1$.) So we only need to find an h which has a part of its asymptotic sequence at the boundary specified by this formula. According to a theorem of Borel (or actually, a variant), it is always possible to find a smooth function with the total asymptotic sequence specified. So we are free (under the condition) to stipulate that h vanishes (to first order) at the boundary, since there is no condition on the h_{0n}. Also, since $z_1 = 0$, it must be that $h_{11} = 0$. So we are also free to stipulate that h vanishes to second order at the boundary. Thus $h = \langle L, L \rangle$ indeed defines a metric for M. We can adjust h smoothly away from the boundary so as to make the volume form integrate to 1, so our metric will be normal. And since the above condition is satisfied, the metric line bundle class induced by this metric is the chosen class. $\qquad \square$

Let us take a moment now to discuss the significance of constant-curvature connections. (This discussion is valid for arbitrary c.) On any manifold M, the ordinary d operator can be regarded as the natural connection on the trivial complex line bundle $M \times \mathbb{C}$, and $d \circ d = 0$ is then the curvature. So if we are seeking a connection with which to do analysis on some non-trivial bundle over a manifold, it makes sense to seek one which is somehow of minimal curvature. This would be a choice which brings us closest to euclidean flatness. One way to measure the size of the curvature of a connection is by means of the Yang–Mills functional (see [1]), which may be reasonably defined in the b-holomorphic context as follows:

Definition 4.40. Let M be a geometric cup, and E a holomorphic line bundle over M. Then the *Yang–Mills functional* on smoothly curved hermitian holomorphic b-connections on E is defined by

$$\mathcal{YM}(\nabla) = \int_M \|R(\nabla)\|^2 \cdot \text{vol}.$$

The following theorem may now be regarded as a justification for our interest in connections of constant curvature.

Theorem 4.41. *A hermitian holomorphic b-connection ∇_0 on E whose curvature is $\text{const} \cdot \text{vol}$ is an absolute minimum for the Yang–Mills functional for E.*

Proof. Suppose $R(\nabla_0) = \text{const} \cdot \text{vol}$. We know that $\int_M \text{vol} = 1$ and $\int_M R(\nabla_0) = -2\pi i \gamma$, so

we must have $R(\nabla_0) = -2\pi i \gamma \cdot \text{vol}$. And so,

$$\mathcal{YM}(\nabla_0) = \int_M \|R(\nabla_0)\|^2 \cdot \text{vol}$$

$$= \int_M \| -2\pi i \gamma \cdot \text{vol} \|^2 \cdot \text{vol}$$

$$= \int_M | -2\pi\gamma|^2 \cdot \text{vol}$$

$$= | -2\pi\gamma|^2$$

$$= 4\pi^2\gamma^2.$$

Now let ∇ be any smoothly curved hermitian holomorphic b-connection. Write $R(\nabla) = f \cdot \text{vol}$. Then

$$\mathcal{YM}(\nabla) = \int_M \|f \cdot \text{vol}\|^2 \cdot \text{vol}$$

$$= \int_M |f|^2 \cdot \text{vol}$$

$$\geq \left| \int_M f \cdot \text{vol} \right|^2$$

$$= \left| \int_M R(\nabla) \right|^2$$

$$= | -2\pi i \gamma|^2$$

$$= 4\pi^2\gamma^2.$$

So the value of the Yang–Mills functional on any smoothly curved hermitian holomorphic b-connection is at least as great as the value of the Yang–Mills functional on ∇_0. $\qquad\square$

$$\overline{}$$

5

The Cup: c non-rational

In this chapter, M is a cup with $c \notin \mathbb{Q}$.

5.1 Classification of cups

As in the case $c = 1$, we call a biholomorphism from M° to $\mathbb{C}$ an *interior coordinate*. Interior coordinates exist by Theorem 3.2.

We saw in Chapter 4 that there is essentially only one cup with $c = 1$. In this section, we will show that there are many inequivalent cups whose collar invariant is a given number $c \notin \mathbb{Q}$.

Lemma 5.1. *Let M be a cup. Choose a first integral z for the collar of M. Then there exists a unique interior coordinate $\mathfrak{z}$ for M such that the Laurent series at zero for $\psi = \mathfrak{z} \circ z^{-1}$ has residue 1 and constant term 0.*

From now on, when we do calculations using a first integral and an interior coordinate, we will assume that they are paired in this fashion.

Proof of the lemma. Choose a first integral z. Suppose $\widetilde{\mathfrak{z}}$ is an arbitrary interior coordinate. Then $\widetilde{\mathfrak{z}} \circ z^{-1}$ is a biholomorphism from a punctured neighborhood of zero to a punctured neighborhood of infinity. Thus it is of the form

$$\widetilde{\mathfrak{z}} \circ z^{-1}(\xi) = c_{-1}\xi^{-1} + c_0 + c_1\xi + c_2\xi^2 + \cdots$$

with $c_{-1} \neq 0$. Now, $\mathfrak{z}$ is another interior coordinate if and only if $\mathfrak{z} \circ \widetilde{\mathfrak{z}}^{-1}$ is an automorphism of the plane; that is, if and only if

$$\mathfrak{z} = a\widetilde{\mathfrak{z}} + b$$

for some $a, b \in \mathbb{C}$, $a \neq 0$. In this case, we can write

$$\mathfrak{z} \circ z^{-1}(\xi) = a\widetilde{\mathfrak{z}} \circ z^{-1}(\xi) + b = ac_{-1}\xi^{-1} + (ac_0 + b) + ac_1\xi + ac_2\xi^2 + \cdots.$$

The system of equations

$$ac_{-1} = 1$$

$$ac_0 + b = 0$$

has a unique solution in $a \neq 0, b$. So there's a unique interior coordinate $\mathfrak{z}$ such that $\mathfrak{z} \circ z^{-1}$ has residue equal to 1 and constant term equal to 0. $\qquad\square$

Lemma 5.2. *Suppose z and $\mathfrak{z}$ are paired and that $\widetilde{z}$ and $\widetilde{\mathfrak{z}}$ are paired as well. If we write $\widetilde{z} = az$ (which we can do by Corollary 2.22), then $\widetilde{\mathfrak{z}} = (1/a)\mathfrak{z}$.*

Proof. Suppose $\widetilde{z} = az$. Then $\widetilde{z}^{-1}(\xi) = z^{-1}(\xi/a)$. Now define $Q = \frac{1}{a}\mathfrak{z}$. Then Q is an interior coordinate. And we compute that

$$Q \circ \widetilde{z}^{-1} = \frac{1}{a}\mathfrak{z} \circ z^{-1}(\frac{1}{a}\xi).$$

But we know that

$$\mathfrak{z} \circ z^{-1}(\xi) = \xi^{-1} + 0 + c_1\xi + c_2\xi^2 + \cdots,$$

so we find that

$$Q \circ \widetilde{z}^{-1}(\xi) = \xi^{-1} + 0 + \frac{c_1}{a^2}\xi + \frac{c_2}{a^3}\xi^2 + \cdots.$$

That is, Q and $\widetilde{z}$ are paired. So $\widetilde{\mathfrak{z}} = Q = \frac{1}{a}\mathfrak{z}$ by uniqueness of the pairing. $\qquad\square$

Theorem 5.3. *Let M be a cup. Then there's a biholomorphism β from a neighborhood of zero to a neighborhood of zero in the complex plane, taking 0 to 0, associated to each choice of a first integral for the collar of the cup. If $\widetilde{z} = az$, then $\widetilde{\beta} = (a) \circ \beta \circ (a^{-1})$. So to the cup itself corresponds a class of biholomorphisms from a neighborhood of zero to a neighborhood of zero, taking 0 to 0, under the relation of equivalence "conjugacy via multiplication by a nonzero constant."*

Construction. Choose a first integral z. Let $\mathfrak{z}$ be the induced interior coordinate. Then define

$$\beta = 1/\mathfrak{z} \circ z^{-1}.$$

$\qquad\square$

Proof of dependence on the first integral. Let $\widetilde{z} = az$ (so that $\widetilde{z}^{-1} = z^{-1} \circ (\frac{1}{a})$), and let $\widetilde{\mathfrak{z}}$ be the induced interior coordinate. Then by the lemma, $\widetilde{\mathfrak{z}} = \frac{1}{a}\mathfrak{z}$. So we find

$$\widetilde{\beta} = \text{inversion} \circ \widetilde{\mathfrak{z}} \circ \widetilde{z}^{-1} = \text{inversion} \circ \frac{1}{a}\mathfrak{z} \circ z^{-1} \circ (a^{-1}) = (a) \circ \text{inversion} \circ \mathfrak{z} \circ z^{-1} \circ (a^{-1}) = (a) \circ \beta \circ (a^{-1}).$$

$\qquad\square$

Note 5.4. It is evident that any representative of this class may be realized as $\mathfrak{z} \circ z^{-1}$ for an appropriate choice of first integral z (and its associated interior coordinate $\mathfrak{z}$).

Remark 5.5. For any $c \in \mathbb{C}$ with $\Re c > 0$ and any biholomorphism β from a neighborhood of zero to a neighborhood of zero which takes 0 to 0, there's a cup whose invariants these are. This is a direct consequence of Theorem 4.2.

Theorem 5.6. *Let M_1 and M_2 be two cups whose collar invariants are not rational. Then $M_1 \simeq M_2$ if and only if $c(M_1) = c(M_2)$ and $\beta(M_1) = \beta(M_2)$.*

Proof. Suppose $M_1 \simeq M_2$. Then the collars are equivalent. So the collar invariants must be equal. Now choose $z, \mathfrak{z}$ to be a paired first integral and interior coordinate for M_2 and let φ be a C^∞ biholomorphism from M_1 to M_2. Then $z \circ \varphi$ and $\mathfrak{z} \circ \varphi$ are a paired first integral and interior coordinate for M_1. We see directly that

$$\beta(M_1) = 1/(\mathfrak{z} \circ \varphi) \circ (z \circ \varphi)^{-1} = 1/\mathfrak{z} \circ z^{-1} = \beta(M_2).$$

Now suppose that c and β are the invariants for both M_1 and M_2. Choose a representative function for β and call it still β. Now choose a pair $z, \mathfrak{z}$ for M_1 and pair $\widetilde{z}, \widetilde{\mathfrak{z}}$ for M_2 such that

$$1/\mathfrak{z} \circ z^{-1} = \beta = 1/\widetilde{\mathfrak{z}} \circ \widetilde{z}^{-1}.$$

Define $\varphi = \widetilde{\mathfrak{z}}^{-1} \circ \mathfrak{z}$, a biholomorphism from the interior of M_1 to the interior of M_2. We need to prove that this function extends as a C^∞ function from M_1 to M_2. But near the boundary, we can write

$$\widetilde{z} \circ \varphi \circ z^{-1} = \widetilde{z} \circ \widetilde{\mathfrak{z}}^{-1} \circ \text{inversion} \circ \text{inversion} \circ \mathfrak{z} \circ z^{-1} = \beta^{-1} \circ \beta = \text{identity} .$$

This says that

$$\varphi = \widetilde{z}^{-1} \circ z = \widetilde{z}^{-1} \circ \chi_c^{-1} \circ \chi_c \circ z = (\chi_c \circ \widetilde{z})^{-1} \circ (\chi_c \circ z).$$

Since z and $\widetilde{z}$ are first integrals for M_1 and M_2, $\chi_c \circ \widetilde{z}$ and $\chi_c \circ z$ are biholomorphisms from M_2 to $\mathcal{M}_c$ and from M_1 to $\mathcal{M}_c$ respectively which are C^∞ up to the boundary. This is by Remark 2.17. Thus φ is also C^∞ up to the boundary. $\qquad\square$

We can say something about cup automorphisms. But first a comment about the power series of β. Since for a paired $z, \mathfrak{z}$ we have

$$\mathfrak{z} \circ z^{-1}(\xi) = \xi^{-1} + 0 + c_1\xi + c_2\xi^2 + \cdots ,$$

It must be that

$$\beta(\xi) = \xi + 0 + \widetilde{c}_3 \xi^3 + \widetilde{c}_4 \xi^4 + \cdots.$$

That is, the first three power series coefficients of the analytic function β are necessarily $0, 1, 0$. The meaning of this is unclear.

Definition 5.7. There's a cup invariant $\ell \in \{0, 1, 2, 3, \ldots\}$ which can be derived from β as follows. Let c_k be the power series coefficients for some representative of β. Let $K = \{k \geq 2 : c_{k+1} \neq 0\}$. Since $\widetilde{\beta}(\xi) = (a) \circ \beta \circ (a^{-1})$, this set K is independent of the representative used for β. So we can define

$$\ell = \begin{cases} 0 & K = \emptyset \\ \gcd(K) & K \neq \emptyset. \end{cases}$$

Technical Lemma 5.8. *Let M be a cup. Let β be (a representative function for) the cup invariant. Let a be a nonzero complex number. Then $(a) \circ \beta = \beta \circ (a)$ if and only if a is an ℓth root of unity.*

Proof. The case $\ell = 0$ is vacuous, so we assume $\ell \geq 1$. Suppose $(a) \circ \beta = \beta \circ (a)$. We can write $\beta(\xi) = \xi + c_3 \xi^3 = c_4 \xi^4 + \cdots$, so this equation becomes

$$a\xi + ac_3\xi^3 + ac_4\xi^4 + \cdots = a\xi + a^3 c_3 \xi^3 + a^4 c_4 \xi^4 + \cdots.$$

So for each $n \geq 3$ such that $c_n \neq 0$, we must have $a = a^n$, or $a^{n-1} = 1$. This just says that $a^k = 1$ for every $k \in K$. Since ℓ is the gcd of K, we can write ℓ as a linear combination of elements of K. Thus

$$a^\ell = a^{m_1 k_1 + \cdots + m_r k_r} = (a^{k_1})^{m_1} \cdots (a^{k_r})^{m_r} = 1,$$

and we find a to be an ℓth root of unity.

The reverse direction is similar. Suppose a is an ℓth root of unity. Then $a^k = 1$ for every $k \in K$, since ℓ divides each element of K. So for each $n \geq 3$ such that $c_n \neq 0$, we have $a = a^n$. So

$$(a) \circ \beta(\xi) = a\xi + ac_3\xi^3 + ac_4\xi^4 + \cdots$$
$$= a\xi + a^3 c_3 \xi^3 + a^4 c_4 \xi^4 + \cdots$$
$$= \beta \circ (a)(\xi).$$

$\square$

Theorem 5.9. *The automorphism group of M is isomorphic to the multiplicative group of ℓth roots of unity.*

Proof. Choose a first integral z, and let $\mathfrak{z}$ be the associated interior coordinate. Let $\beta = 1/\mathfrak{z} \circ z^{-1}$, and define ℓ as above. We will prove that the map

$$a \mapsto \mathfrak{z}^{-1} \circ (a) \circ \mathfrak{z}$$

defines an isomorphism from the multiplicative group of ℓth roots of unity to the automorphism group of M.

First, choose an ℓth root of unity a. (So $1/a$ also is an ℓth root of unity.) Define $\varphi = \mathfrak{z}^{-1} \circ (a) \circ \mathfrak{z}$. This is clearly an automorphism of the interior of M. So we only have to check that it's also a collar automorphism. So compute

$$
\begin{aligned}
z \circ \varphi \circ z^{-1} &= z \circ \mathfrak{z}^{-1} \circ (a) \circ \mathfrak{z} \circ z^{-1} \\
&= z \circ \mathfrak{z}^{-1} \circ (a) \circ \text{inversion} \circ \text{inversion} \circ \mathfrak{z} \circ z^{-1} \\
&= z \circ \mathfrak{z}^{-1} \circ \text{inversion} \circ (1/a) \circ \text{inversion} \circ \mathfrak{z} \circ z^{-1} \\
&= \beta^{-1} \circ (1/a) \circ \beta.
\end{aligned}
$$

By Technical Lemma 5.8, $(1/a) \circ \beta = \beta \circ (1/a)$ since $1/a$ is an ℓth root of unity. So $z \circ \varphi \circ z^{-1} = (1/a)$, or $z \circ \varphi = (1/a)z$, which is a first integral. Therefore φ is a collar automorphism. We already know it's an interior automorphism; so φ is actually a cup automorphism. So our map is at least well defined.

Injectivity is fairly evident.

Next, surjectivity. Choose an arbitrary cup automorphism φ. Then $z \circ \varphi$ and $\mathfrak{z} \circ \varphi$ are a new paired first integral and interior coordinate, with $z \circ \varphi = (1/a)z$ and $\mathfrak{z} \circ \varphi = a\mathfrak{z}$ by Lemma 5.2. Therefore

$$
\begin{aligned}
\beta &= \text{inversion} \circ \mathfrak{z} \circ z^{-1} \\
&= \text{inversion} \circ (\mathfrak{z} \circ \varphi) \circ (z \circ \varphi)^{-1} \\
&= \text{inversion} \circ (a)\mathfrak{z} \circ z^{-1} \circ (a) \\
&= (1/a) \circ \text{inversion} \circ \mathfrak{z} \circ z^{-1} \circ (a) \\
&= (1/a) \circ \beta \circ (a).
\end{aligned}
$$

That is, $(a) \circ \beta = \beta \circ (a)$. By Technical Lemma 5.8, a must be an ℓth root of unity. And from above, $\varphi = \mathfrak{z}^{-1} \circ (a) \circ \mathfrak{z}$. Thus our map is surjective.

the collar (z is defined)

1 is holomorphic ($\alpha = 0$)

$\phi = 1$

$\phi = 0$

supp ϕ

supp $(1 - \phi)$

supp $\bar{\partial}\phi$

$\mathfrak{z} = \infty$ $\mathfrak{z} = 0$
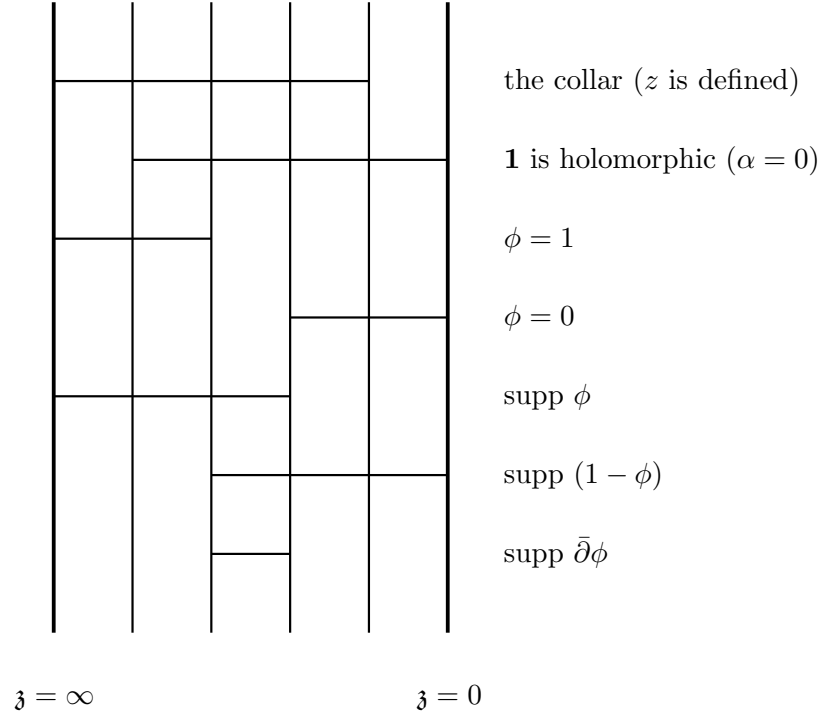
Figure 4: Domains

We only need to check that the map respects the group structure. But this is obvious, because

$$(\mathfrak{z}^{-1} \circ (a) \circ \mathfrak{z}) \circ (\mathfrak{z}^{-1} \circ (b) \circ \mathfrak{z}) = \mathfrak{z}^{-1} \circ (ab) \circ \mathfrak{z}.$$

□

5.2   Bundles over a cup

Theorem 5.10. *To each holomorphic line bundle E over a cup M there corresponds an integral sequence on M. This sequence is a bundle invariant. Moreover, the class it represents is the invariant integral sequence class for the pull-back of E to the collar of M, which was defined in Theorem 2.37.*

Construction. Choose a first integral z and let $\mathfrak{z}$ be the induced interior coordinate. Choose a global C^∞ frame **1** which is holomorphic away from the boundary. We assume without any loss of generality that the domain of holomorphicity of **1** overlaps the domain of z in

an annulus (we can actually make the section holomorphic as close to the boundary as we wish). Choose a cut-off function ϕ which is 1 near the boundary, 0 in the interior, and whose variation takes place only within that annulus. (Please refer to Figure 4.) Let α be the $\bar{\partial}$ form for E with respect to the frame $\mathbf{1}$.

Since α is supported within the domain of z, we may regard α as a compactly supported $(0,1)$-form on $\mathcal{M}_c(z)$. Define $f = \langle \alpha, \bar{L} \rangle$. Let $r = h_1 - f_{00} \log x + h_2$ be the unique good solution to $\bar{\partial} r = -\alpha$ on $\mathcal{M}_c(z)$. (Refer to Section 2.7 for the terminology.) Then let u be the Cauchy solution to

$$\bar{\partial} u = -r \bar{\partial} \phi$$

in the $\mathfrak{z}$ plane. Then, lifting this solution up to M, we see that u is holomorphic near ∂M and tends to zero at ∂M. So u is an analytic function of z near $z = 0$. We now take our integral sequence to be the asymptotic expansion of h_2 in positive powers of z plus the (convergent) power series for u in positive powers of z, with constant term equal to $-f_{00}$. We may note at this point that this integral sequence represents the invariant class for the pull-back of E to the collar. $\qquad \square$

Proof of invariance under change of first integral and interior coordinate. α does not depend on the coordinates used. So f also does not. So z_0 does not.

In Section 2.7, we saw that r is independent of z (and certainly does not depend on the $\mathfrak{z}$). So we only need to show that u is independent of $\mathfrak{z}$. But this is obvious, because u may be defined as the unique solution *on the cup's interior* to $\bar{\partial} u = -r \bar{\partial} \phi$ which has certain growth properties. The r, the ϕ, and the growth conditions are independent of the $\mathfrak{z}$. $\qquad \square$

Proof of invariance under change of cut-off function. Let ϕ and $\widetilde{\phi}$ be two appropriate cut-off functions. α does not depend on ϕ, so z_0 also does not. r does not depend on ϕ either. So we get the same r with ϕ and with $\widetilde{\phi}$. Then if we proceed with the construction, we have u being the $\mathfrak{z}$-plane Cauchy solution to $\bar{\partial} u = -r \bar{\partial} \phi$ and $\widetilde{u}$ being the $\mathfrak{z}$-plane Cauchy solution to $\bar{\partial} \widetilde{u} = -r \bar{\partial} \widetilde{\phi}$. Let $v = \widetilde{u} - u$. Then v must be the $\mathfrak{z}$-plane Cauchy solution to $\bar{\partial} v = -r \bar{\partial} (\widetilde{\phi} - \phi)$. v is analytic at $\mathfrak{z} = \infty$ and has value zero there. We can compute the coefficient of $\mathfrak{z}^{-n}$ $(n \geq 1)$ in the power series for v at $\mathfrak{z} = \infty$ using the Cauchy formula (ignoring multiplicative constants). It is

$$\int_{\mathbb{C}} \mathfrak{z}^n r \bar{\partial} (\widetilde{\phi} - \phi) \frac{d\mathfrak{z}}{\mathfrak{z}}.$$

Choose an annular region G in the $\mathfrak{z}$ plane which contains the support of $\widetilde{\phi} - \phi$. Then we have

$$\int_G \mathfrak{z}^n r \bar{\partial}(\widetilde{\phi} - \phi) \frac{d\mathfrak{z}}{\mathfrak{z}} = \int_G d(r(\widetilde{\phi} - \phi)d(\mathfrak{z}^n)) = \int_{\partial G} r(\widetilde{\phi} - \phi)d(\mathfrak{z}^n) = 0.$$

This calculation depends upon the fact that r is holomorphic on the support of $\widetilde{\phi} - \phi$, as can be seen from the figure. We conclude that $\widetilde{u} - u$ is constant near $\mathfrak{z} = \infty$, so that the asymptotic expansions for $h_2 + u$ and $h_2 + \widetilde{u}$ are the same. $\qquad\square$

Proof of invariance under change of frame. Let $s, \widetilde{s}$ be C^∞ frames which are holomorphic away from the boundary. Write $\widetilde{s}/s = e^g$. We can do this, because the quotient can not wind around zero on a deformation of ∂M and still be non-vanishing on the cup. Then $\widetilde{\alpha} - \alpha = \bar{\partial}g$. We take the same $z, \mathfrak{z}$ for both calculations.

Note that $\widetilde{f} - f = \bar{L}g$. So $\widetilde{z}_0 - z_0 = -(\bar{L}g)_{00} = 0$ as we have seen in the proof of Theorem 2.37.

Now to $n \geq 1$. We would like to take the same cut-off function for both calculations, so that the only difference is between the two $\bar{\partial}$ forms $\alpha, \widetilde{\alpha}$. But the choice of cut-off function depends on the choice of the frame, in particular on the support of the $\bar{\partial}$ form. So in general, we will have to use two different cut-off functions ϕ and $\widetilde{\phi}$, each chosen appropriate to the corresponding frame. However, I claim that it is actually enough to show that the asymptotic series for $\widetilde{h}_2 + \widetilde{u}$ equals that for $h_2 + u$ under the assumption that the *same* cut-off function ϕ_0 (which is chosen to be appropriate to both frames) is used in each calculation. The reason is that this assertion, along with the invariance under a change of cut-off function we have just proved above, allows us to leap-frog:

$$(h_2 + u)[\widetilde{s}, \widetilde{\phi}] = (h_2 + u)[\widetilde{s}, \phi_0] = (h_2 + u)[s, \phi_0] = (h_2 + u)[s, \phi].$$

So we proceed WLOG to choose a single cut-off function ϕ which is appropriate to both frames.

Before we begin, note that g is holomorphic in an annular region on the collar which contains the region where ϕ takes its variation. Let γ be the restriction of g to that region.

One frame is s; the other is $\widetilde{s} = se^g$. Let α be the $\bar{\partial}$-form with respect to s; then $\widetilde{\alpha} = \alpha + \bar{\partial}g$. Let $f = \langle \alpha, \bar{L} \rangle$. Then $\widetilde{f} = f + \bar{L}g$. Let r be the good z-plane solution to $\bar{L}r = -f$. Then $\widetilde{r} = r - g + \mathrm{HP}_z(\gamma)$ by Sub-lemma 2.35. Finally, let u be the $\mathfrak{z}$-plane Cauchy solution to $\bar{\partial}u = -r\bar{\partial}\phi$. I claim that $\widetilde{u} = u + (\phi - 1)g - \phi\,\mathrm{HP}_z(\gamma)$.

Proof of this claim: first, note that this function is well defined and C^∞ on the $\mathfrak{z}$ plane. That's because u is well defined on the $\mathfrak{z}$-plane, g is well defined on the $\mathfrak{z}$ plane, and $\mathrm{HP}_z(\gamma)$ is well defined on the collar (where ϕ has its domain). Next, observe that u tends to zero at $\mathfrak{z} = \infty$. That's because u tends to zero, $\phi - 1 = 0$, and $\mathrm{HP}_z(\gamma)$ tends to zero at $\mathfrak{z} = \infty$. Finally, compute that

$$\bar{\partial}\tilde{u} = \bar{\partial}u + (\bar{\partial}\phi)g + (\phi - 1)\bar{\partial}g - (\bar{\partial}\phi)\,\mathrm{HP}_z(\gamma) - \phi\bar{\partial}\,\mathrm{HP}_z(\gamma)$$
$$= -r\bar{\partial}\phi + (\bar{\partial}\phi)g - (\bar{\partial}\phi)\,\mathrm{HP}_z(\gamma)$$
$$= -(r - g + \mathrm{HP}_z(\gamma))\bar{\partial}\phi$$
$$= -\tilde{r}\bar{\partial}\phi.$$

We have a solution which has the right properties to be the Cauchy solution.

Now let's examine

$$(\tilde{h}_2 + \tilde{u}) - (h_2 + u) = (\tilde{u} - u) + (\tilde{h}_2 - h_2) = ((\phi - 1)g - \phi\,\mathrm{HP}_z(\gamma)) + \mathrm{HP}_z(\gamma) = (1 - \phi)(-g + \mathrm{HP}_z(\gamma)).$$

(We know that $\tilde{h}_2 - h_2 = \mathrm{HP}_z(\gamma)$ for the following reason: $\tilde{r} - r = -g + \mathrm{HP}_z(\gamma)$; and $\tilde{h}_2 - h_2$ is that part of $\tilde{r} - r$ which has an expansion in powers of z at the boundary.) This is zero near the boundary. Therefore the asymptotic series for $\tilde{h}_2 + \tilde{u}$ and for $h_2 + u$ must be the same. $\qquad\square$

Theorem 5.11. *Let E be a holomorphic line bundle over the cup M. Let $z_0, z_1, \ldots$ be its invariant integral sequence. Then there exists a holomorphic global C^∞ frame for the pull-back of E to the interior of M of the form*

$$se^{v_1 + v_2}$$

where s is a global C^∞ frame for E, $v_1 = z_0 \log x$ near the boundary of M, and $v_2 \sim \sum_{n \geq 1} z_n$ at the boundary of M.

Proof. Choose a paired first integral z and interior coordinate $\mathfrak{z}$ for M. Choose a global C^∞ frame $\mathbf{1}$ for E. Let α be the $\bar{\partial}$ form with respect to this frame. Define $f = \langle \alpha, \bar{L} \rangle$. Let r be the unique good solution to $\bar{L}r = -f$ on $\mathcal{M}_c(z)$. (Then $\bar{\partial}r = -\alpha$.) Write $r = h_1 - f_{00} \log x + h_2$. By definition, $-f_{00} = z_0$. So actually $r = h_1 + z_0 \log x + h_2$.

Next choose a cut-off function ϕ as in the construction of the invariant sequence and let u be the Cauchy solution in the $\mathfrak{z}$ plane to $\bar{\partial}u = -r\bar{\partial}\phi$. Then I claim that the interior frame

$$\mathbf{1}e^{\phi r + u} = \mathbf{1}e^{\phi h_1 + \phi z_0 \log x + \phi h_2 + u}$$

is holomorphic. Why? Because the $\bar{\partial}$ form with respect to this frame is

$$\alpha + \bar{\partial}(\phi r + u) = \alpha + r\bar{\partial}\phi + \phi\bar{\partial}r + \bar{\partial}u$$
$$= \alpha + r\bar{\partial}\phi - \alpha - r\bar{\partial}\phi$$
$$= 0.$$

Now we simply define

$$s = \mathbf{1}e^{\phi h_1}$$
$$v_1 = \phi z_0 \log x$$
$$v_2 = \phi h_2 + u.$$

$\square$

Theorem 5.12. *Let E be a holomorphic line bundle over the cup M. Let $z_0, z_1, \ldots$ be its invariant integral sequence. Then there exists a holomorphic global C^∞ frame for E if and only if $z_n = 0$ for every n.*

Proof. If there is a holomorphic global C^∞ frame, then we may use this frame in the construction of the invariant integral sequence. But the $\bar{\partial}$ form α with respect to this frame is zero. In particular, the average value of $\langle \alpha, \bar{L} \rangle$ over ∂M is zero; this says that $z_0 = 0$. Now for $n \geq 1$. Since $\alpha = 0$, the construction will show that r and u are zero. Thus the asymptotic expansion at the boundary for $h_2 + u$ in positive powers of a first integral z is identically zero. Thus all the z_n are zero.

On the other hand, if all the z_n are zero, then Theorem 5.11 gives us a global holomorphic frame over the interior which has the form se^{v_2}, with s a global C^∞ frame for E, and v_2 vanishing to infinite order at the boundary of M. This is a holomorphic global C^∞ frame for E. $\square$

Corollary 5.13. *Two holomorphic line bundles over a cup are equivalent (in the sense that there is a C^∞ bundle isomorphism from one to the other which preserves the holomorphic structure) if and only if their integral sequences are the same.*

Proof. If two bundles are equivalent, then clearly their integral sequences must be the same. To prove the converse, we have to note that at the level of $\bar{\partial}$ forms, tensor multiplication of two bundles (and their frames) becomes addition and dualization becomes additive inversion; and that addition and additive inversion carry over (by the construction) from $\bar{\partial}$-forms to integral sequences. Thus if you take the tensor product of two bundles, the new integral

sequence is the sum of the old ones; and if you take the dual of a bundle, the new integral sequence is minus the old one.

With this in mind, how does the argument go? E and F are equivalent if and only if $E^* \otimes F$ has a holomorphic global C^∞ frame. By Theorem 5.12, this is so if and only if the integral sequence for $E^* \otimes F$ is zero. By the remarks in the above paragraph, this is so if and only if the integral sequence for E equals the integral sequence for F. $\qquad \square$

Theorem 5.14. *Let M be a cup. Then any integral sequence on M may be realized as the invariant integral sequence for a holomorphic line bundle over M.*

Construction. Choose a paired first integral z and interior coordinate $\mathfrak{z}$ for M. Choose an integral sequence. As a set we take E to be $M \times \mathbb{C}$. Choose a function v_1 supported in the collar of M which equals $z_0 \log x$ near the boundary. Choose a function v_2 supported in the collar of M which has the asymptotic development

$$v \sim \sum_{n \geq 1} z_n.$$

The existence of such a function is guaranteed by a variant of Borel's theorem. Then define the holomorphic structure of E by stipulating that

$$\mathbf{1}e^{v_1+v_2}$$

be holomorphic. Note that this defines the correct sort of a holomorphic structure, for the same reason as outlined in the proof of Theorem 2.42. $\qquad \square$

Computation of invariant sequence. Since v_1 and v_2 are compactly supported, we may take $\mathbf{1}$ as our C^∞ frame which is holomorphic away from the boundary. The $\bar{\partial}$ form with respect to this frame is $\alpha = -\bar{\partial}(v_1 + v_2)$. So $f = \langle \alpha, \bar{L} \rangle = -\bar{L}(v_1 + v_2)$. So one solution to $\bar{L}r = -f$ is $v_1 + v_2$. But a moment's inspection of the definitions shows that this is, in fact, the good solution. So $r = v_1 + v_2$; this means that $h_2 = v_2$.

Now we have to compute u. For this, we take the $\mathfrak{z}$-plane Cauchy solution to

$$\bar{\partial}u = -r\bar{\partial}\phi,$$

where ϕ is chosen as in the construction of the invariant integral sequence. But $r = 0$ in the support of $\bar{\partial}\phi$; so $u = 0$. Thus our integral sequence is the sequence for h_2 alone; that is, the one we chose (arbitrarily) at the beginning. It's no trouble to check that the zeroth term is also as chosen. $\qquad \square$

We have completed a classification of the holomorphic line bundles over a cup M with $c \notin \mathbb{Q}$. That is, we have found an isomorphism from the group of equivalence classes of holomorphic line bundles over such an M to a *prima facie* simpler group, namely the additive group of integral sequences on M. The isomorphism is natural.

5.3   Examples of bundles

First, we examine ${}^{b}T^{1,0}M$.

Theorem 5.15. *Let M be a cup with $c \notin \mathbb{Q}$. Then ${}^{b}T^{1,0}M$ is a holomorphic line bundle over M. Furthermore, $z_0 = c$; and the rest of the z_n depend upon the cup invariant β.*

Proof. Choose a first integral z and let $\mathfrak{z}$ be the associated interior coordinate. We start with the global holomorphic frame $\partial_{\mathfrak{z}}$ over the interior. Our strategy is to compare this frame to a global C^{∞} frame over M (taking the quotient); from this, we will see that the bundle is a holomorphic line bundle over M, and we will also be able to see what the invariants are.

To do this, we need to construct a global C^{∞} frame for this bundle. We have the C^{∞} frame $z\partial_z$ defined near the boundary and the holomorphic frame $\partial_{\mathfrak{z}}$ defined in the interior. We would like to glue these together (by taking a pointwise linear combination) to get a global C^{∞} frame.

This is possible if and only if the winding number of the quotient $z\partial_z \div \partial_{\mathfrak{z}}$ around $z = 0$ on a small deformation of ∂M is zero. We need to check whether this condition is satisfied. The quotient is

$$\frac{z\partial_z}{\partial_{\mathfrak{z}}} = z\frac{\partial\mathfrak{z}}{\partial z}$$

$$= z \cdot \frac{\partial}{\partial z}(z^{-1} + 0 + a_1 z + a_2 z^2 + \cdots)$$

$$= z \cdot (-z^{-2} + 0 + a_1 + 2a_2 z + \cdots)$$

$$= -z^{-1} + 0 + a_1 z + 2a_2 z^2 + \cdots .$$

A small deformation of ∂M is a small counterclockwise circuit of $z = 0$ in the z plane. Here, the term $-z^{-1}$ dominates; and the winding number is seen to be -1.

So we cannot glue as we wished. But if we take our C^{∞} boundary frame to be $e^{iy}z\partial_z$ instead of $z\partial_z$, the winding number will become zero. Therefore, there exists a global C^{∞} frame s for ${}^{b}T^{1,0}M$ which equals $\partial_{\mathfrak{z}}$ in the interior and equals $e^{iy}z\partial_z$ near the boundary. Note that s is holomorphic away from the boundary.

Now we can compute the quotient. In the interior, it is smooth. Near the boundary, it is

$$\frac{\partial_{\mathfrak{z}}}{s} = \frac{\partial_{\mathfrak{z}}}{e^{iy}z\partial_z}$$
$$= \frac{e^{-iy}}{z} \cdot \frac{1}{\partial_{\mathfrak{z}}/\partial z}$$
$$= \frac{x^c}{z^2\partial_{\mathfrak{z}}/\partial z}$$
$$= e^{c\log x - \log(z^2\partial_{\mathfrak{z}}/\partial z)}.$$

We are justified in writing $z^2\partial_{\mathfrak{z}}/\partial z$ as the exponential of a logarithm because this function, near the boundary, is (analytic and) non-vanishing:

$$z^2\partial_{\mathfrak{z}}/\partial z = z^2(-z^{-2} + 0 + a_1 + 2a_2 z + \cdots) = -1 + 0 + a_1 z^2 + 2a_2 z^3 + \cdots.$$

Therefore the term $\log(z^2\partial_{\mathfrak{z}}/\partial z)$ is a series in non-negative powers of z.

We have computed that the global holomorphic frame $\partial_{\mathfrak{z}}$ may be written near ∂M as

$$se^{c\log x - \log(z^2\partial_{\mathfrak{z}}/\partial z)}.$$

The $\bar{\partial}$ form near the boundary with respect to s is

$$-\bar{\partial}[c\log x - \log(z^2\partial_{\mathfrak{z}}/\partial z)] = -\bar{L}[c\log x - \log(z^2\partial_{\mathfrak{z}}/\partial z)]\bar{\lambda} = -c\bar{\lambda},$$

since $\bar{L}\log x = 1$ and $\log(z^2\partial_{\mathfrak{z}}/\partial z)$ is analytic. Since this is a smooth section of ${}^b\Lambda^{0,1}M$, our bundle is a holomorphic line bundle (in our sense) over the b-holomorphic complex curve M.

Finally, since s is holomorphic away from the boundary, we can read the invariants directly from the interior holomorphic frame

$$se^{c\log x - \log(z^2\partial_{\mathfrak{z}}/\partial z)}.$$

We see that $z_0 = c$, and the other z_n are given by the entries in the expansion of the analytic function $-\log(z^2\partial_{\mathfrak{z}}/\partial z)$ in powers of z. Since $\beta = 1/\mathfrak{z} \circ z^{-1}$, this function clearly depends on β. □

We conclude this section by examining lifts of bundles from $\check{M} = S^2$. Unfortunately, there is no *natural* way to lift a bundle from the sphere to M, as there was in the situation of $c = 1$. That's because there is no surjective smooth holomorphic map from M to the

sphere. The reason is that a holomorphic map must be representable as a power series in a first integral z near ∂M; but the only such series which is also smooth is the constant series, since $z = x^c e^{iy}$ and c is not rational. So the map is constant near ∂M, and therefore constant on M. In particular, the blow-down map $\pi : M \longrightarrow \check{M}$ fails to be smooth at the boundary of M.

Nevertheless, π is continuous, and it is smooth and holomorphic in the interior of M. So given a line bundle E over the sphere $\check{M}$, we can lift it to a continuous line bundle $\widehat{E}$ over M. We can also lift the smooth and holomorphic structures of E to smooth and holomorphic structures for the pull-back of $\widehat{E}$ to M°. All that remains is to extend the smooth structure of $\widehat{E} \lfloor M^\circ$ to a smooth structure for $\widehat{E} \lfloor M$, in such a way that the holomorphic structure of $\widehat{E} \lfloor M^\circ$ makes $\widehat{E}$ a holomorphic line bundle over M in our sense.

This requires a choice. So choose a local holomorphic frame ν of E near q, and let $\widehat{\nu}$ be its lift to a local frame of $\widehat{E}$. We now stipulate that a local section of $\widehat{E}$ near the boundary is smooth if and only if its quotient by $\widehat{\nu}$ is smooth. The resulting bundle will be called $\widehat{E}_\nu$, the "lift of E by means of ν." Note that this is a holomorphic line bundle in our sense, because the $\bar{\partial}$ form with respect to the local C^∞ frame $\widehat{\nu}$ is zero, which is smooth as a compressed $(0, 1)$-form.

Theorem 5.16. *Let M be a cup, and let $\pi : M \longrightarrow \check{M}$ be the blow-down. Let $q \in \check{M}$ be the distinguished point. Let E be a holomorphic line bundle over $\check{M}$. Let τ be a local holomorphic frame for E near q, and σ be a global holomorphic frame for E over $\check{M} \backslash q$, such that $\sigma = \tau z^d$ near q, where z is any first integral of M (pushed down to $\check{M}$, so that it becomes a local analytic chart). Such frames exist by Theorem 4.14. Here d is the degree of E.*

Choose a local holomorphic frame ν for E near q, and let $\widehat{E}_\nu$ be the lift of E to M by means of ν. Let z_n be the invariant integral sequence of $\widehat{E}_\nu$. Then

$$\frac{z_0}{c} = d,$$

and for $n \geq 1$, z_n equals the nth entry in the power series expansion of the holomorphic function $\log(\tau/\nu)$ in terms of a first integral for M.

Proof. Let z be a first integral. Then z, regarded as a function on $\check{M}$ by means of the projection π, is a local analytic coordinate centered at q. Define $f = \log(\tau/\nu)$, so that

$\tau = \nu e^f$. Then near q,

$$\sigma = \tau z^d$$
$$= \nu e^f x^{dc} e^{idy}$$
$$= \nu e^{f_0 + idy} e^{dc \log x + [f - f_0]}.$$

Choose a cut-off function ϕ supported near the boundary. Let $\widehat{\nu}$ and $\widehat{\sigma}$ be the lifts of ν and σ. Then define

$$s = \widehat{\sigma} e^{-\phi(dc \log x + [f - f_0])}.$$

This is a frame over M°. Away from the boundary, it agrees with $\widehat{\sigma}$, and is therefore holomorphic. Near the boundary, it equals

$$\widehat{\nu} e^{f_0 + idy},$$

and is therefore smooth up to the boundary. That is, s is a global C^∞ frame for $\widehat{E}_\nu$ which is holomorphic away from the boundary. Furthermore, we have the holomorphic interior frame $\widehat{\sigma}$ which equals

$$s e^{\phi(dc \log x + [f - f_0])}$$

near ∂M. The invariants can now be read off from this formula. $\qquad\square$

Corollary 5.17. *Let F be a holomorphic line bundle over M. Then F is isomorphic to $\widehat{E}_\nu$ for some holomorphic line bundle E over S^2 and some local holomorphic frame ν for E if and only if $z_0(F)/c \in \mathbb{Z}$ and $\sum_{n \geq 1} z_n(F)$ has a positive radius of convergence, when expressed as a series in positive powers of a first integral for M.*

5.4   Connections of constant curvature

In this section, we take M to be a geometric cup (see Definition 1.10) with $c \notin \mathbb{Q}$. Choose a first integral z. Then we have $x, y, L, \bar{L}, \lambda, \bar{\lambda}$ as in (2.2) and (2.3). Define

$$\mathcal{L} = \bar{L}L = \begin{pmatrix} x\partial_x & \partial_y \end{pmatrix} \begin{pmatrix} 1 & -b \\ -b & |c|^2 \end{pmatrix} \begin{pmatrix} x\partial_x \\ \partial_y \end{pmatrix} \tag{5.1}$$

and

$$\mathcal{T} = (a^{-2}) \begin{pmatrix} \frac{dx}{x} & dy \end{pmatrix} \begin{pmatrix} |c|^2 & b \\ b & 1 \end{pmatrix} \begin{pmatrix} \frac{dx}{x} \\ dy \end{pmatrix}.$$

Define $h = \langle L, L \rangle$. Then h vanishes to second order on the boundary. We have the formulas

$$g = \tfrac{h}{4}\mathcal{T}$$
$$\Delta = \tfrac{4}{h}\mathcal{L} \tag{5.2}$$
$$\mathrm{vol} = \tfrac{h}{4a}\tfrac{dx}{x} \wedge dy.$$

(These are taken to be valid near the boundary only, of course.) Note that the volume form is a smooth 2-form which vanishes to first order on the boundary.

We have $\bar{\partial}\partial = (i/2) \star \Delta$, as before. Also, we will be using Lemma 4.24 on solving the Laplace equation in the plane, and Technical Lemma 4.26 as well, which is valid for a cup with any c (not just $c = 1$).

Proposition 5.18. *There exists a real smooth function V on M° that satisfies $\Delta V = -2\pi/a$ and such that V can be written as $V = \phi \log x + V^1 + V^2$ with*

1. *ϕ a cut-off function supported near ∂M;*

2. *x is a smooth defining function for the boundary arising from some first integral $z = x^c e^{iy}$;*

3. *V^1 a smooth function on M; and*

4. *V^2 a smooth function on M° which is asymptotically equal at ∂M to the real part of a series in non-negative powers of z.*

Such a function is unique up to an additive constant.

Proof of uniqueness. Suppose V and $\widetilde{V}$ are two such functions, and let $f = \widetilde{V} - V$. Then f is harmonic. Also, near the boundary of M,

$$f = (\log \widetilde{x} - \log x) + (\widetilde{V}^1 - V^1) + (\widetilde{V}^2 - V^2).$$

Since $c \notin \mathbb{Q}$ we know that $\widetilde{z} = \mathrm{const} \cdot z$, which implies that $\widetilde{x} = \mathrm{const} \cdot x$. Therefore $\log \widetilde{x} - \log x$ is constant. Also, each of the four "V" functions on the right hand side is bounded in a neighborhood of the boundary of M. So f is bounded and harmonic on M°, which means that M is bounded and harmonic in the $\mathfrak{z}$ plane (where $\mathfrak{z}$ is any interior coordinate for M). Thus f is everywhere constant. $\qquad\square$

Proof of existence. Choose a first integral z and let $\mathfrak{z}$ be its associated interior coordinate. Choose a cut-off function ϕ supported near the boundary and taking its variation in the collar (the region where z and $\mathfrak{z}$ are both defined). Let $h = \langle L, L \rangle$, and define h_{mn} in the usual way.

We begin the construction. Let u^0 be any real smooth solution to $\Delta u^0 = -2\pi/a$ on the interior of M. Such a solution exists by the ellipticity of the Laplacian over the interior. Next, let u^1 be a real smooth function on M, supported near the boundary, such that

$$u^1 \sim \frac{-\pi}{2a} \sum_{\substack{m \geq 1 \\ n \in \mathbb{Z}}} \frac{h_{mn}}{m^2 - 2ibmn - |c|^2 n^2} x^m e^{iny}$$

at the boundary. Next, we need to solve

$$\Delta u^2 = -\frac{2\pi}{a} - \Delta[\phi \log x + (1 - \phi)u^0 + u^1],$$

which we can write (in the z plane) as

$$\Delta_z u^2 = \frac{h}{4a^2 |z|^2} \left(-\frac{2\pi}{a} - \Delta[\phi \log x + (1 - \phi)u^0 + u^1] \right).$$

To re-write the equation in this form, we have used the definition of L as $2az\partial_z$, and the general fact that if $\langle \partial_z, \partial_z \rangle = F$, then $\Delta = (1/F)\Delta_z$.

The construction of the u^0 and the u^1 was geared to making this equation solvable in the z plane via the usual kernel. That means that the datum should be smooth and compactly supported in the z plane. We argue that it is. First, note that $\Delta(\phi \log x)$ is supported in an annulus in the z plane, since $\log x$ is harmonic. Next, note that

$$-\frac{2\pi}{a} - \Delta[(1 - \phi)u^0]$$

is supported near the boundary. Finally, we have to check that Δu^1 agrees with

$$-\frac{2\pi}{a} - \Delta[(1 - \phi)u^0]$$

to infinite order at the boundary. But since $(1 - \phi)u^0$ is zero near the boundary, this amounts to showing that Δu^1 agrees with the constant function $-2\pi/a$ to infinite order at the boundary. To see this, you can check directly from the definition of $\mathcal{L}$ (5.1) and the asymptotic formula for u^1 that

$$\mathcal{L}u^1 \sim \frac{-\pi}{2a} h$$

at the boundary; so by (5.2)

$$\Delta u^1 = \frac{4}{h}\mathcal{L}u^1$$
$$\sim -\frac{4}{h}\frac{\pi}{2a}h$$
$$= -\frac{2\pi}{a}$$

at the boundary.

Thus, the full datum vanishes to infinite order at the boundary; or, in the z plane, the full datum vanishes to infinite order at $z = 0$ (hence is smooth there); and it is compactly supported. So we can indeed solve by means of the kernel. The solution is smooth in z. This (real) solution will be "asymptotically harmonic" at $z = 0$, which implies that there's an asymptotic development

$$u^2 \sim \Re \sum_{n\geq 0} a_n z^n.$$

Since we cannot lift a function defined on all of the z plane (such as u^2) to the manifold M, we are obliged to cut it off first. So our final task is to solve

$$\Delta u^3 = -\frac{2\pi}{a} - \Delta[\phi \log x + (1 - \phi)u^0 + u^1 + \phi u^2].$$

Let $\mathfrak{z}$ be the interior coordinate associated to z. Let $F = \langle \partial_{\mathfrak{z}}, \partial_{\mathfrak{z}} \rangle$. Then we can re-write this equation as

$$\Delta_{\mathfrak{z}} u^3 = F\left(-\frac{2\pi}{a} - \Delta[\phi \log x + (1 - \phi)u^0 + u^1 + \phi u^2]\right).$$

Since the previous datum was supported near ∂M, this new datum is supported in an annulus. So we may solve by means of the kernel. Then we set

$$V^1 = (1 - \phi)u^0 + u^1$$
$$V^2 = \phi u^2 + u^3.$$

By our construction, we have

$$\Delta(\phi \log x + V^1 + V^2) = -2\pi/a.$$

V^1 is smooth on M. $V^2 - u^3 = \phi u^2$ is asymptotic to the real part of a series in non-negative powers of z at ∂M. So it suffices to show that u^3 is equal to the real part of a series in positive powers of z near ∂M.

To show this, we claim that

$$\int_{\mathbb{C}} \text{datum}_3 \, d\mu(\mathfrak{z}) = 0.$$

Then, the solution u^3 is a real smooth function which is harmonic in a neighborhood of ∞ in the $\mathfrak{z}$ plane, and which tends to zero at $\mathfrak{z} = \infty$ by Lemma 4.24. Thus we may regard u^3 as a real smooth function which is harmonic in a punctured neighborhood of zero in the z plane, and tends to zero at $z = 0$. So u^3 is actually smooth in a (full) neighborhood of zero in the z plane. Since it is real, smooth, and harmonic in a neighborhood of $z = 0$ and tends to zero at $z = 0$, it may be expressed as

$$u^3 = \Re \sum_{n \geq 1} b_n z^n.$$

We are therefore done, as long as we prove the claim about the integral of the datum.

Proof of the claim: Write $\mathfrak{z} = \xi + i\eta$. Then $\mathrm{vol} = F\, d\xi \wedge d\eta = F\, d\mu(\mathfrak{z})$. So we may compute

$$\int_{\mathbb{C}} \mathrm{datum}_3\, d\mu(\mathfrak{z}) = \int_{\mathbb{C}} F\left(-\frac{2\pi}{a} - \Delta[\phi \log x + (1-\phi)u^0 + u^1 + \phi u^2]\right)(1/F) \cdot \mathrm{vol}$$

$$= \int_M \left(-\frac{2\pi}{a} - \Delta[\phi \log x + (1-\phi)u^0 + u^1 + \phi u^2]\right) \cdot \mathrm{vol}.$$

The first piece obviously integrates to $-2\pi/a$. The second piece integrates to $2\pi/a$ by Technical Lemma 4.26. The third, fourth, and fifth pieces integrate to zero by the technical lemma. So the full integral is zero. The claim is proved. $\qquad\square$

Note 5.19. The decomposition of any such V as $\phi \log x + V^1 + V^2$ is unique, except for the shifting of a term of the form $\mathrm{const} \cdot \psi$ between V^1 and V^2, where ψ is a cut-off function supported near ∂M. Also, we have seen that the V itself is unique up to an additive constant. Therefore the positive power entries in the asymptotic expansion of V^2 at ∂M are independent of the V and also of the decomposition $V = \phi \log x + V^1 + V^2$. We will call this invariant thing f. That is, we define the formal series of nth integrals f (over n positive) by the relation

$$V^2 \sim \mathrm{const} + \Re f$$

at the boundary.

Theorem 5.20. *Let M be a geometric cup with $c \notin \mathbb{Q}$. Let E be a holomorphic line bundle over M. Then for there to exist a hermitian holomorphic b-connection on E whose curvature is $-2\pi i \gamma \cdot \mathrm{vol}$, it is necessary and sufficient that the invariant integral sequence of E satisfy the equation*

$$\sum_{n \geq 1} z_n(E) = [\Re z_0(E)]f.$$

Proof of necessity. As per Theorem 5.11, let $se^{u^1+u^2}$ be a holomorphic frame over the interior, where s is a global smooth frame over M, $u^1 = z_0 \log x$ near the boundary, and

$$u^2 \sim \sum_{n \geq 1} z_n$$

at the boundary. Then the $\bar{\partial}$ form with respect to s is $\alpha = -\bar{\partial}(u^1 + u^2)$.

Now let $\langle s, s \rangle = e^p$ denote the metric associated to the special connection. Start with the equation curvature $= -2\pi i\gamma \cdot \text{vol}$, which by (4.2) can be written

$$\partial\alpha - \bar{\partial}\bar{\alpha} + \bar{\partial}\partial p = -\frac{2\pi i\Re z_0}{a} \cdot \text{vol}.$$

Using $\alpha = -\bar{\partial}(u^1 + u^2)$, we can re-write the left hand side as

$$\bar{\partial}\partial[2\Re(u^1 + u^2) + p].$$

And using $\bar{\partial}\partial V = -\frac{\pi i}{a} \cdot \text{vol}$, we can re-write the right hand side as

$$\bar{\partial}\partial[2(\Re z_0) \cdot V].$$

Using this, we can interpret the equation as saying that

$$\Re(u^1 + u^2) + p/2 - (\Re z_0) \cdot V$$

is harmonic. Since p and V are real, we can re-write this as

$$\Re[u^1 + u^2 + p/2 - z_0 V].$$

Now, using the decomposition $V = z_0\phi \log x + V^1 + V^2$, we can re-write this as

$$\Re[(u^1 - z_0\phi \log x) + (p/2 - z_0 V^1) + (u^2 - z_0 V^2)],$$

or using the asymptotic relations $u^2 \sim \sum_{n \geq 1} z_n$ and $V^2 \sim \text{const} + \Re f$, as

$$\Re\left[(u^1 - z_0\phi \log x) + (p/2 - z_0 V^1) + \left(\sum_{n \geq 1} z_n - (\Re z_0)f\right) + \text{const}\right].$$

This first parenthetic term vanishes near the boundary. The second is smooth up to the boundary. And the third is an asymptotic series in non-negative powers of z. So all terms are bounded, and we have a bounded harmonic function on M° (which is equivalent to the plane). So this function is constant.

Now, the terms in the asymptotic expansion of the smooth second piece (terms like $x^m e^{iny}$) cannot cancel the terms in the asymptotic expansion of the non-smooth third piece (terms like $x^{mc} e^{iny}$). So all non-constant terms in each expansion must be zero. In particular,

$$\sum_{n \geq 1} z_n = (\Re z_0) f.$$

$\square$

Proof of sufficiency. Choose a global smooth frame s for E, a cut-off function ϕ supported near the boundary, and a smooth function u^2 on M° which has the asymptotic expansion

$$u^2 \sim \sum_{n \geq 1} z_n$$

at the boundary, in such a way that

$$s e^{z_0 \phi \log x + u^2}$$

is holomorphic over the interior of M. Let V be the function whose existence is asserted in Proposition 5.18. Now define a hermitian metric for E by stipulating that

$$\langle s, s \rangle = e^{2\Re(z_0 V - z_0 \phi \log x - u^2)}.$$

We claim that the function $p = 2\Re(z_0 V - z_0 \phi \log x - u^2)$ is smooth up to the boundary of M, so that this defines a smooth metric on E.

We now compute the curvature. Note that $\alpha = -\bar{\partial}(z_0 \phi \log x + u^2)$. So

$$
\begin{aligned}
R(\nabla) &= \partial \alpha - \bar{\partial}\bar{\alpha} + \partial\bar{\partial}p \\
&= -\partial\bar{\partial}(z_0 \phi \log x + u^2) + \bar{\partial}\partial(\bar{z}_0 \phi \log x + \bar{u}^2) + \bar{\partial}\partial[2\Re(z_0 V - z_0 \phi \log x - u^2)] \\
&= \bar{\partial}\partial(z_0 \phi \log x + u^2) + \bar{\partial}\partial(\bar{z}_0 \phi \log x + \bar{u}^2) + 2(\Re z_0)\bar{\partial}\partial V - \bar{\partial}\partial[2\Re(z_0 \phi \log x + u^2)] \\
&= 2(\Re z_0)(i/2)\Delta V \cdot \mathrm{vol} \\
&= 2(\Re z_0)(-\pi i/a) \cdot \mathrm{vol} \\
&= -2\pi i \gamma \cdot \mathrm{vol}\,.
\end{aligned}
$$

It only remains to prove the claim. We have to show that

$$\Re(z_0 V - z_0 \phi \log x - u^2)$$

is smooth up to the boundary. Write $V = z_0\psi \log x + V^1 + V^2$ as in the construction of V; and so our function is

$$\Re[(z_0\psi \log x - z_0\phi \log x) + (z_0V^1) + (z_0V^2 - u^2)].$$

The first piece, $\Re(z_0\psi \log x - z_0\phi \log x)$, vanishes near the boundary, and so is smooth. The second piece, $(\Re z_0)V^1$, is smooth since V^1 is smooth by its construction. And the third piece we can write (in asymptotic form) as

$$\Re\left[(\Re z_0)f - \sum_{n\geq 1} z_n\right] + \text{const}.$$

This asymptotic series is the constant series. So the third piece is constant to infinite order at ∂M, and is therefore smooth. So the claim is proven. $\square$

Definition 5.21. Let M be a geometric cup with $c \notin \mathbb{Q}$. Let G be the holomorphic line bundle class whose invariant zeroth integral is 1 and whose invariant nth integral is the nth entry in the formal power series f. Then G is called the *metric line bundle class* of the geometric cup M.

Corollary 5.22. *Let M be a geometric cup with $c \notin \mathbb{Q}$, and let G be the metric line bundle class of M. Then a holomorphic line bundle E over M posesses a constant-curvature hermitian holomorphic b-connection if and only if $[E]$ is twist-equivalent to $G^{\Re z_0(E)}$.*

6

Compact Riemann Surfaces — an interlude

Our analysis of the cup made strong use of the fact that the blow-down was the Riemann sphere. In particular, we used precise formulas for solving the $\bar\partial$ problem and the Laplace equation on the sphere.

When we turn to the generic b-holomorphic complex curve, we will need to have some sort of precise information about solutions to the $\bar\partial$ problem and the Laplace equation on the blow-down, which is a generic compact Riemann surface. This chapter therefore studies the $\bar\partial$ problem on a generic compact Riemann surface, and has nothing to do with anything of type b. We will find sufficiently precise information on the solutions to allow us to proceed with our analysis of the generic b-holomorphic complex curve in a fashion analogous to our treatment of the cup.

In Section 7.5 we will discuss the complications that arise in solving the Laplace equation on a generic compact Riemann surface, and we leave the issue partially unresolved.

The references for this chapter are [5], [10], and [32].

6.1 Background (part one)

Definition 6.1. Let M be a compact Riemann surface. We define the genus g of M to be the topological "number of handles" that must be attached to a sphere in order to get (a surface homeomorphic to) M.

From now on, we assume that M is a compact Riemann surface of genus $g \geq 1$.

Theorem 6.2. $\dim H_1(M, \mathbb{Z}) = 2g$.

This is a standard theorem of elementary algebraic topology.

Theorem 6.3. $\dim H^1_{dR}(M, \mathbb{C}) = 2g$.

This theorem is proved by constructing a differential co-cycle η_i for each homological generator c_i, in such a way that

$$\int_{c_i} \eta_j = \text{the intersection number between } c_i \text{ and } c_j.$$

The pairing is non-degenerate.

Lemma 6.4. *Let ω be a smooth 2-form on M. Then there exists a smooth function f on M for which $\bar{\partial}\partial f = \omega$ if and only if $\int_M \omega = 0$. Such a solution is unique up to an additive constant. If ω is purely imaginary, then f may be chosen purely real.*

Theorem 6.5. *There exist natural isomorphisms*

1. *$H_{dR}^1 \simeq H_{\partial}^{1,0} \oplus H_{\bar{\partial}}^{0,1}$; and*

2. *$H_{\partial}^{1,0} \simeq H_{\bar{\partial}}^{0,1}$.*

Proof of the first assertion. We define the map to be

$$[\alpha] \mapsto (\, [\alpha^{1,0}], [\alpha^{0,1}]\,).$$

This is clearly a well-defined homomorphism. To prove injectivity, let α be a closed 1-form such that $\alpha^{1,0} = \partial f$ and $\alpha^{0,1} = \bar{\partial}g$. Then

$$0 = d\alpha$$
$$= \bar{\partial}\partial f + \partial\bar{\partial}g$$
$$= \bar{\partial}\partial(f - g).$$

So $f - g$ is harmonic. So it must be a constant. (We have used the "uniqueness" part of Lemma 6.4 here.) So we have $\alpha^{1,0} = \partial f$ and $\alpha^{0,1} = \bar{\partial}g = \bar{\partial}(f + \text{const}) = \bar{\partial}f$. So $\alpha = df$.

To prove surjectivity, let p and q be smooth 1-forms of type $(1,0)$ and $(0,1)$. It suffices to find a function f such that $(p + \partial f) + q$ is closed. For then,

$$[p + \partial f + q] \mapsto (\, [p + \partial f], [q]\,) = (\, [p], [q]\,)$$

which is our chosen element of the range. So we need to solve the equation $d(p + \partial f) + dq = 0$ for a smooth f. Rewrite the equation as

$$\bar{\partial}\partial f = -d(p + q).$$

There exists a solution by Lemma 6.4. $\qquad\square$

Proof of the second assertion. The map is conjugation. This is clearly an isomorphism, because of the formula

$$\text{conjugate of } \partial f = \bar{\partial}\bar{f}.$$

$\square$

Corollary 6.6. $\dim H_{\bar{\partial}}^{0,1} M = g$.

Weierstrass Gap Theorem. *Let $p \in M$. Then there exist g numbers*

$$1 = n_1 < \cdots < n_g < 2g$$

such that: there does not exist a meromorphic function on M whose only singularity is a pole at p of order n if and only if $n = n_i$ for some i.

These g numbers are called the *Weierstrass gaps at p*. The theorem is a consequence of the Riemann–Roch formula.

6.2   A surface with one marked point

Definition 6.7. Let M be a compact Riemann surface of genus $g \geq 1$ with one point p marked. For any $s \in \mathbb{Z}_{\geq 0}$, we define *Assertion s* to be the following: For every smooth $(0,1)$-form α on M, there exists a smooth function $f = u_1 + u_2$ on $M \backslash p$ such that

1. $\bar{\partial} f = \alpha$ on $M \backslash p$;

2. u_1 is smooth on M; and

3. u_2 is holomorphic in a punctured neighborhood of p, and p is a pole of order at most s.

When Assertion s is true, we call the number s a *pole number* for this marked surface.

Clearly the set of pole numbers, if non-empty, must have the form $\{s_0, s_0+1, s_0+2, \dots\}$ for some number s_0. Our objective is to compute the minimal pole number s_0 from the Weierstrass gaps at p.

Proposition 6.8. *Let z be a holomorphic chart centered at p and ϕ be a cut-off function supported near p. For each $n > 0$, define*

$$\eta^n = (\bar{\partial}\phi)z^{-n}$$

Then the g smooth $(0,1)$*-forms*

$$\eta^{n_1}, \ldots, \eta^{n_g}$$

form a basis for $H_{\bar{\partial}}^{0,1} M$*, where* $n_1, \ldots, n_g$ *are the Weierstrass gaps at* p*.*

Proof. Suppose $a_1 \eta^{n_1} + \cdots + a_g \eta^{n_g} = \bar{\partial} f$ for some smooth function f on M. Then define

$$h = \phi \left(a_1 z^{-n_1} + \cdots + a_g z^{-n_g} \right) - f.$$

Then h is smooth on $M \backslash p$, and holomorphic there. Further, f is holomorphic near p, and the other piece is meromorphic there, the only singularity being a pole at p, of order

$$\max_{i=1 \ldots g} \left(1 - \delta_{(a_i, 0)} \right) n_i.$$

This number is either a Weierstrass gap (if at least one of the a_i is nonzero) or zero (if all the a_i are zero). Since it cannot be a Weierstrass gap, it must be zero, and so all the a_i are zero. So the η^{n_i} are independent in the sense of $\bar{\partial}$ cohomology. Since there are g of them, they span $H_{\bar{\partial}}^{0,1} M$. □

Theorem 6.9. *The set of pole numbers is non-empty, and the minimal pole number is the maximal Weierstrass gap at* p*.*

Proof. First we show that the maximal Weierstrass gap is a pole number. So let α be an arbitrary smooth $(0,1)$-form on M. Now choose numbers $a_1, \ldots, a_g$ such that

$$\alpha \sim a_1 \eta^{n_1} + \cdots + a_g \eta^{n_g}.$$

Then choose a smooth function u_1 on M such that

$$\alpha - a_1 \eta^{n_1} - \cdots - a_g \eta^{n_g} = \bar{\partial} u_1.$$

Finally, define

$$u_2 = \phi \left(a_1 z^{-n_1} + \cdots + a_g z^{-n_g} \right).$$

If we take $f = u_1 + u_2$, then $\bar{\partial} f = \alpha$ on $M \backslash p$; u_1 is smooth on M; and u_2 is smooth on $M \backslash p$, holomorphic in a punctured neighborhood of p, and has a pole of at most n_g at p. Thus Assertion n_g is true.

We have shown that the maximal Weierstrass gap n_g is a pole number. It only remains to show that there is no lesser pole number.

So suppose Assertion m is true, for some $m < n_g$. Then for any smooth $(0,1)$-form α, we have $\bar{\partial}u_1 + \bar{\partial}u_2 = \alpha$ with u_1 smooth on M, u_2 smooth on $M \backslash p$, u_2 holomorphic in a punctured neighborhood of p, with p a pole of order at most m. in particular, this is true with $\alpha = \eta^{n_g}$. Thus we find that

$$\bar{\partial}(u_1 + u_2 - \phi z^{-n_g}) = 0$$

on $M \backslash p$. And it is clear that the singularity of this function at p is a pole of order n_g. So we have a meromorphic function on M whose only singularity is a pole at p of order n_g. But this contradicts the fact that n_g is a Weierstrass gap. $\qquad \square$

6.3   Background (part two)

Divisors on a manifold of dimension 0 or 1 are merely finite formal products of points of the manifold and their formal inverses. The divisors obviously form a group under multiplication. The unit is called 1. The divisors are partially ordered by pointwise comparison of exponent. The degree of a divisor is defined as the sum of all the exponents occurring in an expression for the divisor. An integral divisor is one which has no negative powers in its unique reduced expression.

Let f be a meromorphic function on a Riemann surface, and let the order of f at q be the number n such that $f = z^n g$ near q, where z is a local coordinate centered at q and g is analytic and non-vanishing. The divisor (f) of f is defined as

$$(f) = \prod_{q \in M} q^{\operatorname{ord}_q f}.$$

The polar divisor $[f]$ of f is defined as

$$[f] = \prod_{q \in M} q^{\max(-\operatorname{ord}_q f, 0)}.$$

We will refer to the germ of the principal part (singular part) of f by the symbol $\langle f \rangle$.

Noether Gap Theorem. *Let M be a compact Riemann surface with $g \geq 1$. Let $q_1, q_2, \ldots$ be a sequence of points on M. Define the sequence of divisors on M*

$$D_0 = 1$$
$$D_1 = q_1$$
$$D_2 = q_1 q_2$$
$$D_3 = q_1 q_2 q_3$$

etc. We formulate "Statement j" as follows: there exists a meromorphic function f on M such that $(f) \geq D_j^{-1}$ but $(f) \not\geq D_{j-1}^{-1}$.

There are g integers

$$1 = n_1 < n_2 < \cdots < n_g < 2g$$

such that Statement j is false if and only if j is in this list.

Remark 6.10 (Statement j). Let $\widetilde{q}_1, \ldots, \widetilde{q}_k$ be the *distinct* entries of the list $q_1, \ldots, q_j$, with $\widetilde{q}_k = q_j$. Then write $q_1 \cdots q_j$ as $\widetilde{q}_1^{m_1} \cdots \widetilde{q}_k^{m_k}$. Then Statement j can be equivalently formulated as follows: there exists a meromorphic function f whose only singuarities are poles at the $\widetilde{q}_i$, with the order of the pole at $\widetilde{q}_k$ being precisely m_k, and the order of the pole at any other $\widetilde{q}_i$ $(i \neq k)$ being at most m_i.

Note 6.11. For a given infinite sequence of points of M, the set of numbers $\{\text{gaps} \leq j\}$ depends only on the first j entries of the sequence. Thus, if we are given a finite sequence of j points, it is sensible to ask whether i is a gap, for any $i \leq j$.

We will refer to the *Noether gap scheme* of a marked surface. By this we mean the set of all the Noether gaps for all sequences (of length $2g - 1$) involving only the marked points.

6.4 A surface with several marked points

Definition 6.12. Now let M be a compact Riemann surface of genus $g \geq 1$ with points $p_1, \ldots, p_k$ marked. For any $s \in \mathbb{Z}_{\geq 0}^k$, we define *Assertion s* to be the following: For every smooth $(0, 1)$-form α on M, there exists a smooth function $f = u_1 + u_2$ on $M \backslash \{p_1, \ldots, p_k\}$ such that

1. $\bar{\partial} f = \alpha$ on $M \backslash \{p_1, \ldots, p_k\}$;

2. u_1 is smooth on M; and

3. u_2 is holomorphic in a punctured neighborhood of $\{p_1, \ldots, p_k\}$, and p_i is a pole of order at most s_i $(i = 1 \ldots k)$.

When Assertion s is true, we call the vector s a *pole vector* for this marked surface.

By the arguments in Section 6.2, for each $i = 1 \ldots k$ there exists a pole vector s with $s_i < 2g$ and $s_r = 0$ when $r \neq i$. Thus pole vectors $< (2g, \ldots, 2g)$ exist. (We refer to the standard partial ordering on $\mathbb{Z}_{\geq 0}^k$ given by entrywise comparison.)

We will see that, for any pole vector s, there's another pole vector $s' \leq s$ with $s' <$ $(2g, \ldots, 2g)$. Therefore the set of all minimal pole vectors is finite, and every one of them is $< (2g, \ldots, 2g)$. In order to (try to) compute the minimal pole vectors, we will make use of the Noether gap theorem.

We ask the following questions.

Question 6.13. *Is the set of all minimal pole vectors determined by the Noether gap scheme?*

Question 6.14. *Is the set of all minimal pole vectors computable from the Noether gap scheme?*

In the case of $k = 1$, the Noether gap scheme is the set of Weierstrass gaps at p_1, and the set of minimal pole vectors is just the minimal pole number. Therefore the results of Section 6.2 show that these questions are both answered in the affirmative in the case $k = 1$. So we now turn our attention to the case $k > 1$. We will first address Question 6.13. We will come back to Question 6.14 in Section 6.9.

6.5   Noether gaps and polar divisors

Definition 6.15. Let P be the set of integral divisors on $\{p_1, \ldots, p_k\}$. We define the operation $+$ on P by

$$p_1^{n_1} \cdots p_k^{n_k} + p_1^{m_1} \cdots p_k^{m_k} = p_1^{\max(n_1, m_1)} \cdots p_k^{\max(n_k, m_k)}.$$

The empty sum is defined to be 1.

The operation is commutative and associative, as is clear from the definition.

Definition 6.16. We define the map $t : P \longrightarrow \{N, Y\}$ by

$$t(\mathfrak{p}) = \begin{cases} N & \text{if there does not exist a meromorphic function on } M \text{ whose polar divisor is } \mathfrak{p} \\ Y & \text{if there does exist a meromorphic function on } M \text{ whose polar divisor is } \mathfrak{p}. \end{cases}$$

Note that $t(1) = Y$. This is because the constant functions are meromorphic and have no poles. Also, $t(p_i) = N$ for any i. This is because no meromorphic function can have only a single simple pole and no other singularities.

Proposition 6.17. $t^{-1}(Y)$ *is closed under addition.*

Proof. Suppose $t(\mathfrak{f}) = t(\mathfrak{g}) = Y$ for some $\mathfrak{f}, \mathfrak{g} \in P$. If $\mathfrak{f} = \mathfrak{g} = 1$, then the conclusion follows trivially, for $t(\mathfrak{f} + \mathfrak{g}) = t(1 + 1) = t(1) = Y$. So assume from now on that not both $\mathfrak{f}$ and $\mathfrak{g}$ are 1.

Let $p_{i_1}, \ldots, p_{i_\ell}$ $(i_1 < \cdots < i_\ell, \ell \leq k)$ be the points of $\{p_1, \ldots, p_k\}$ which occur in the reduced expression of either $\mathfrak{f}$ or $\mathfrak{g}$ with a nonzero exponent. Write

$$\mathfrak{f} = p_{i_1}^{n_1} \cdots p_{i_\ell}^{n_\ell}$$
$$\mathfrak{g} = p_{i_1}^{m_1} \cdots p_{i_\ell}^{m_\ell}$$

and define $M_r = \max(n_r, m_r)$. (Thus we know that $M_r > 0$ for all $r = 1 \ldots \ell$.) Then

$$\mathfrak{f} + \mathfrak{g} = p_{i_1}^{M_1} \cdots p_{i_\ell}^{M_\ell}.$$

We need to prove that there exists a meromorphic function on M whose polar divisor this is. So let f and g be meromorphic functions such that $[f] = \mathfrak{f}$ and $[g] = \mathfrak{g}$. It suffices to prove that there exist numbers a, b such that $[af + bg] = \mathfrak{f} + \mathfrak{g}$. Write the germs of the principal parts of f and g as

$$\langle f \rangle = f_1^1 z_{i_1}^{-1} + \cdots + f_{M_1}^1 z_{i_1}^{-M_1}$$
$$+$$
$$\cdots$$
$$+$$
$$f_1^\ell z_{i_\ell}^{-1} + \cdots + f_{M_\ell}^\ell z_{i_\ell}^{-M_\ell}$$

and

$$\langle g \rangle = g_1^1 z_{i_1}^{-1} + \cdots + g_{M_1}^1 z_{i_1}^{-M_1}$$
$$+$$
$$\cdots$$
$$+$$
$$g_1^\ell z_{i_\ell}^{-1} + \cdots + g_{M_\ell}^\ell z_{i_\ell}^{-M_\ell}.$$

We need to find a, b such that the ℓ numbers

$$\text{coeff. of } z_{i_1}^{-M_1} = a f_{M_1}^1 + b g_{M_1}^1$$

$$.$$
$$.$$
$$.$$

$$\text{coeff. of } z_{i_\ell}^{-M_\ell} = a f_{M_\ell}^\ell + b g_{M_\ell}^\ell$$

are all non-zero. So define the two ℓ-vectors

$$F = (f_{M_1}^1, \ldots, f_{M_\ell}^\ell)$$

$$G = (g_{M_1}^1, \ldots, g_{M_\ell}^\ell).$$

We know that, for each $r = 1 \ldots \ell$, either F_r or G_r is nonzero. We want to prove that there exist a, b such that every entry of $aF + bG$ is nonzero. So suppose not. That is, suppose that span$\{F, G\}$ is contained in the union of subspaces of $\mathbb{R}^\ell$

$$\bigcup_{r=1}^{\ell} \{x_r = 0\}.$$

Then it is contained in one of them. That is,

$$\text{span}\{F, G\} \subset \{x_r = 0\}$$

for some $r \in \{1, \ldots, \ell\}$. Therefore, the rth entries of both F and G are zero. This cannot be. $\qquad\square$

Theorem 6.18. Let $p_{i_1}^{n_1} \cdots p_{i_\ell}^{n_\ell}$ ($\ell \leq k$) be an element of P, where the p_{i_r} are distinct and ordered so that $i_1 < \cdots < i_\ell$. Let $n = n_1 + \cdots + n_\ell$ be its degree. Then for $t(p_{i_1}^{n_1} \cdots p_{i_\ell}^{n_\ell}) =$ N, it is necessary and sufficient that n be a Noether gap for at least one of the possible permutations of the finite sequence

$$p_{i_1}, \ldots, p_{i_1}, \ldots \ldots, p_{i_\ell}, \ldots, p_{i_\ell}.$$

(In this sequence, p_{i_r} appears n_r times.)

Proof of sufficiency. Let n be a gap for a permutation in which p_{i_r} is the last (rightmost) entry. This means that there does not exist any meromorphic function on M whose polar divisor is $p_{i_1}^{m_1} \cdots p_{i_\ell}^{m_\ell}$ with $m_r = n_r$ and $m_s \leq n_s$ ($s \neq r$). In particular, there does not exist a meromorphic function on M whose polar divisor is $p_{i_1}^{n_1} \cdots p_{i_\ell}^{n_\ell}$. That is, $t(p_{i_1}^{n_1} \cdots p_{i_\ell}^{n_\ell}) =$ N. $\qquad\square$

Proof of necessity. Suppose n is not a gap for any of those permuted sequences. Then for each $r = 1 \ldots \ell$, n is not a gap for a sequence whose last entry is p_{i_r}. This may be restated as follows: there exists a meromorphic function f_r on M whose polar divisor is $\mathfrak{p}_r = p_{i_1}^{m_1^r} \cdots p_{i_\ell}^{m_\ell^r}$ with $m_r^r = n_r$ and $m_s^r \le n_s$ ($s \ne r$).

We have $t(\mathfrak{p}_r) = \mathrm{Y}$. We also have $\mathfrak{p}_1 + \cdots + \mathfrak{p}_\ell = p_{i_1}^{n_1} \cdots p_{i_\ell}^{n_\ell}$. Therefore, $t(p_{i_1}^{n_1} \cdots p_{i_\ell}^{n_\ell}) = \mathrm{Y}$ by Proposition 6.17. $\qquad\qquad\square$

Corollary 6.19. *t is determined on P by the Noether gap scheme. And in particular, $t(\mathfrak{p}) = \mathrm{Y}$ for every $\mathfrak{p} \in P$ of degree at least $2g$.*

Proof. If $n = \mathrm{degree}\,\mathfrak{p} \ge 2g$, then n is not a gap for any sequence. Thus $t(\mathfrak{p}) = \mathrm{Y}$ by Theorem 6.18. If $n = \mathrm{degree}\,\mathfrak{p} < 2g$, then $t(\mathfrak{p})$ depends upon the gaps of sequences of length n, again by the theorem. $\qquad\qquad\square$

So we have established that t (as a function from P to $\{\mathrm{N}, \mathrm{Y}\}$) is computable from the Noether gap scheme; and it is clear that the Noether gap scheme is computable from t, by Remark 6.10. Thus we have shown that Question 6.13 is equivalent to "Is the set of all minimal pole vectors determined by $t : P \longrightarrow \{\mathrm{N}, \mathrm{Y}\}$?" And similarly for Question 6.14.

6.6 η chains

From now on, fix $g \ge 1$ and $k \ge 1$.

For $Q \ge 1$, we define V_Q to be the free complex vector space on the letters

$$\eta_i^n : i \in \{1 \ldots k\}, n \in \{1 \ldots Q\}.$$

Note that we have canonical inclusions $V_Q \subset V_{Q+1}$.

We define an η *chain* to be an element of V_Q for some Q. That is, it is an element of

$$\bigcup_{Q=1,2,\ldots} V_Q,$$

but we will never need to deal with this full union. Whenever we have a finite bunch of η chains, we will deal with them as elements of V_Q for some fixed Q.

The *order* of an η chain C is the smallest Q for which $C \in V_Q$. An η chain C of order $\leq M$ can be thought of as a $k \times M$ matrix c_i^n, via the identification

$$C = c_1^1 \eta_1^1 + \cdots + c_1^M \eta_1^M$$
$$+$$
$$\cdots$$
$$+$$
$$c_k^1 \eta_k^1 + \cdots + c_k^M \eta_k^M.$$

An η *chain collection* is a set of g independent η chains. The *order* of an η chain collection is the maximal order of its chains.

Let $A = {}^1\!A, \ldots, {}^g\!A$ be an η chain collection of order M. Write each member of A as

$${}^j\!A = {}^j a_1^1 \eta_1^1 + \cdots + {}^j a_1^M \eta_1^M$$
$$+$$
$$\cdots$$
$$+$$
$${}^j a_k^1 \eta_k^1 + \cdots + {}^j a_k^M \eta_k^M.$$

For each $i = 1 \ldots k$, we define the *symbol of A at p_i* to be

$$s_i(A) = \max \bigcup_{j=1\ldots g} \left\{ r \in \{1, \ldots, M\} : {}^j a_i^r \neq 0 \right\};$$

that is, the highest η-superscript over the subscript i occurring among all the ${}^j\!A$. Then we define the *symbol of A* to be the k-vector $s(A) = (s_1(A), \ldots, s_k(A))$.

Now let M be again a compact Riemann surface of genus g with k marked points $p_1, \ldots, p_k$. Choose (once and for all) an analytic coordinate z_i centered at p_i, and a cut-off function ϕ_i supported near p_i (for each $i = 1 \ldots k$). Then for each Q we define the homomorphism

$$V_Q \longrightarrow \Lambda^{0,1} M$$

by specifying its value on basis elements:

$$\eta_i^n \mapsto (\bar{\partial}\phi_i) z_i^{-n}.$$

This map is clearly injective.

Thus we may regard any η chain as either an element of some V_Q, or as an element of $\Lambda^{0,1}M$, or as an element of $H^{0,1}_{\bar{\partial}}M$ (by taking the class of the $(0,1)$-form).

Definition 6.20. An η chain collection is called an η *chain basis* if its image in $H^{0,1}_{\bar{\partial}}M$ is an independent set (and therefore, a basis, by Corollary 6.6).

In order to be able to write down our arguments in a reasonable way, we must now make some more definitions. So let f be a smooth function on $M\backslash\{p_1,\ldots,p_k\}$ which is meromorphic in a neighborhood of $\{p_1,\ldots,p_k\}$ with poles only at the p_i; and let C be an η chain. Let M represent the maximal order of the poles of f or the order of C, whichever is appropriate.

Definition 6.21.

$\langle f \rangle$ = the germ of the principal part of the restriction of f to a neighborhood of the p_i

$[f]$ = the polar divisor of the restriction of f to a neighborhood of the p_i.

Note that if $\langle f \rangle = \langle g \rangle$, then $[f] = [g]$.

Definition 6.22. Write
$$C = c_1^1\eta_1^1 + \cdots + c_1^M\eta_1^M$$
$$+$$
$$\ldots$$
$$+$$
$$c_k^1\eta_k^1 + \cdots + c_k^M\eta_k^M.$$
We define the *function associated to C* to be

$$\text{func}\,C = c_1^1\phi_1 z_1^{-1} + \cdots + c_1^M\phi_1 z_1^{-M}$$
$$+$$
$$\ldots$$
$$+$$
$$c_k^1\phi_k z_k^{-1} + \cdots + c_k^M\phi_k z_k^{-M}.$$

$\text{func}\,C$ is always meromorphic in a neighborhood of $\{p_1,\ldots,p_k\}$. Note that $\bar{\partial}\,\text{func}\,C = C$; this is an equation between $(0,1)$-forms on $M\backslash\{p_1,\ldots,p_k\}$.

Definition 6.23. Write

$$\langle f \rangle = f_1^1 z_1^{-1} + \cdots + f_1^M z_1^{-M}$$

$$+$$

$$\ldots$$

$$+$$

$$f_k^1 z_k^{-1} + \cdots + f_k^M z_k^{-M}.$$

The *η chain associated to f* is

$$\text{chain } f = f_1^1 \eta_1^1 + \cdots + f_1^M \eta_1^M$$

$$+$$

$$\ldots$$

$$+$$

$$f_k^1 \eta_k^1 + \cdots + f_k^M \eta_k^M.$$

Note that $\langle \text{func chain } f \rangle = \langle f \rangle$; therefore, $[\text{func chain } f] = [f]$.

Now three brief lemmas having to do with the addition operation on P which was introduced in Definition 6.15. Then we will be able to prove the main result for this section, which will explain our interest in η chains.

Lemma 6.24. *Let $\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2 \in P$. If $\mathfrak{p}_1 \leq \mathfrak{p}$ and $\mathfrak{p}_2 \leq \mathfrak{p}$, then $\mathfrak{p}_1 + \mathfrak{p}_2 \leq \mathfrak{p}$.*

Lemma 6.24 follows from the definition of addition.

Lemma 6.25. *Let C, D be η chains, and let a, b be numbers. Then*

$$[\text{func}(aC + bD)] \leq [\text{func } C] + [\text{func } D].$$

Proof. The exponent of p_i in the divisor on the left hand side is

$$\exp_i[\text{func}(aC + bD)] = \max\{r : ac_i^r + bd_i^r \neq 0\}.$$

The exponent of p_i in the divisor on the right hand side is

$$\exp_i(\, [\text{func } C] + [\text{func } D]\,) = \max\{\max\{r : c_i^r \neq 0\}, \max\{r : d_i^r \neq 0\}\}$$

$$= \max\{r : c_i^r \neq 0 \text{ or } d_i^r \neq 0\}.$$

But if $ac_i^r + bd_i^r \neq 0$, then either $c_i^r \neq 0$ or $d_i^r \neq 0$. That is,

$$\{r : ac_i^r + bd_i^r \neq 0\} \subset \{r : c_i^r \neq 0 \text{ or } d_i^r \neq 0\}.$$

So the exponent of p_i in the divisor on the left is $\leq$ the exponent of p_i in the divisor on the right. That is, the divisor on the left is $\leq$ the divisor on the right. $\qquad\square$

Definition 6.26. We define the map

$$[\,\cdot\,] : \mathbb{Z}_{\geq 0}^k \longrightarrow P$$

by $[(v_1, \ldots, v_k)] = p_1^{v_1} \cdots p_k^{v_k}$. This map establishes a one-to-one correspondence which respects the partial orderings of $\mathbb{Z}_{\geq 0}^k$ and P.

Lemma 6.27. *Let $A = \{\,^1\!A, \ldots, \,^g\!A\}$ be an η chain collection. Then*

$$[s(A)] = [\text{func }^1\!A] + \cdots + [\text{func }^g\!A].$$

Lemma 6.27 follows directly from the definitions of $s(A)$, func, $[\,\cdot\,]$, and $+$.

Theorem 6.28. *For Assertion s to be true, it is necessary and sufficient that there exist an η chain basis whose symbol is less than or equal to s.*

Proof of sufficiency. Suppose there exists an η chain basis $A = \{\,^1\!A, \ldots, \,^g\!A\}$ with $s(A) \leq s$. Let α be a smooth $(0, 1)$-form. Write $\alpha \sim c_1\,^1\!A + \cdots + c_g\,^g\!A$ for some $c_1, \ldots, c_g$. Then let u_1 be a smooth function on M such that

$$\alpha = c_1\,^1\!A + \cdots + c_g\,^g\!A + \bar\partial u_1.$$

Then define

$$u_2 = \text{func}(c_1\,^1\!A + \cdots + c_g\,^g\!A).$$

As has been remarked just after Definition 6.22, this function is smooth on $M \backslash \{p_1, \ldots, p_k\}$, and meromorphic in a neighborhood of $\{p_1, \ldots, p_k\}$, the only poles being the points p_i.

Define $f = u_1 + u_2$. We may now easily compute that

$$\begin{aligned}
\bar\partial f &= \bar\partial u_1 + \bar\partial u_2 \\
&= \alpha - (c_1\,^1\!A + \cdots + c_g\,^g\!A) + \bar\partial \,\text{func}(c_1\,^1\!A + \cdots + c_g\,^g\!A) \\
&= \alpha
\end{aligned}$$

on $M\backslash\{p_1,\ldots,p_k\}$. So it only remains to check that the order of the pole of u_2 at p_i is $\leq s_i$; in other words, that $[u_2] \leq [s]$. We compute that

$$
\begin{aligned}
[u_2] &= [\text{func}(c_1\,{}^1\!A + \cdots + c_g\,{}^g\!A)] \\
&\leq [\text{func}\,{}^1\!A] + \cdots + [\text{func}\,{}^g\!A] \\
&= [s(A)] \\
&\leq [s].
\end{aligned}
$$

$\square$

The first inequality is by Lemma 6.25. The second inequality is because $[\,\cdot\,]$ respects partial orderings.

Proof of necessity. Suppose Assertion s is true. Choose any basis ${}^1\alpha, \ldots, {}^g\alpha$ for $H_{\bar{\partial}}^{0,1}M$. Then, for each $i = 1\ldots g$, choose ${}^i\!f = {}^i u_1 + {}^i u_2$ such that $\bar{\partial}\,{}^i\!f = {}^i\alpha$ as guaranteed by Assertion s (and satisfying all the properties stated there). Then define ${}^i\!A = \text{chain}\,{}^i u_2$.

I claim that ${}^i\!A \sim {}^i\alpha$. Therefore $A = \{\,{}^1\!A, \ldots, {}^g\!A\}$ is an η chain basis. We then compute that

$$
\begin{aligned}
[s(A)] &= [\text{func}\,{}^1\!A] + \cdots + [\text{func}\,{}^g\!A] \\
&= [\text{func chain}\,{}^1 u_2] + \cdots + [\text{func chain}\,{}^g u_2] \\
&= [\,{}^1 u_2] + \cdots + [\,{}^g u_2].
\end{aligned}
$$

(The first equality is by Lemma 6.27.) Assertion s guarantees that $[\,{}^i u_2] \leq [s]$ for every i. So by Lemma 6.24, we find that

$$[s(A)] \leq [s],$$

which is the same as saying that $s(A) \leq s$.

It only remains to prove the claim. For this part we drop the superscript i. Define

$$g = u_1 + u_2 - \text{func chain}\, u_2.$$

Since u_1 is smooth on M and

$$\langle u_2 \rangle = \langle \text{func chain}\, u_2 \rangle,$$

the function g is smooth on M. Also,

$$
\begin{aligned}
\bar{\partial} g &= \bar{\partial} u_1 + \bar{\partial} u_2 - \bar{\partial}\,\text{func chain}\, u_2 \\
&= \bar{\partial} f - \text{chain}\, u_2 \\
&= \alpha - A.
\end{aligned}
$$

That is, α and A are $\bar{\partial}$-cohomologous. The claim is proved, and so is the theorem. $\qquad\square$

Corollary 6.29. *The set of all minimal pole vectors is equal to the set of all minimal η chain basis symbols.*

Proof. Suppose s is a minimal η chain basis symbol. The Assertion s is true. That is, s is a pole vector. Now suppose $s' \leq s$ is another pole vector. Then there exists an η chain basis with symbol $s'' \leq s$. So $s'' = s$. Therefore $s' = s$, and s is a minimal pole vector.

Now suppose s is a minimal pole vector. Then there exists an η chain basis with symbol $s' \leq s$. So Assertion s' is true. So s' is a pole vector. So $s' = s$. Therefore, s is an η chain basis symbol. Now suppose there were another η chain basis symbol $s' \leq s$. Then Assertion s' would be true. So s' would be a pole vector. Therefore $s' = s$, and so s is a minimal η chain basis symbol. $\qquad\square$

Thus we have shown that Question 6.13 is equivalent to "Is the set of all minimal η chain basis symbols determined by $t : P \longrightarrow \{N, Y\}$?" And similarly for Question 6.14. Now it is time to show that by re-formulating the question in this way, we have made some progress.

6.7   Conditions B, N, and W

For future reference: B is for Basis, N is for N, and W is for Weak.

Our strategy is as follows.

1. Formulate a sufficient condition for an η chain collection to be an η chain basis, in such a way that the condition may be checked by "examining" the (finite) table of the values of t on P. Understood weakly, this simply means that the condition has no explicit dependence on anything but the values of t.

2. Prove that among the symbols of η chain collections that satisfy the sufficient condition, there appear all minimal η chain basis symbols.

If we can achieve this, then we will have shown that the answer to Question 6.13 is "yes."

Before getting started on the first item, let us seek a necessary and sufficient condition for an η chain collection to be an η chain basis; then, we will look for a stronger condition which is "easier to check."

Lemma 6.30. *Let C be an η chain. Then $C \sim 0$ if and only if there exists a meromorphic function f on M such that $\langle f \rangle = \langle \operatorname{func} C \rangle$.*

Proof. Suppose f is meromorphic and $\langle f \rangle = \langle \operatorname{func} C \rangle$. Then the function $g = \operatorname{func} C - f$ is smooth. Also, $\bar{\partial} g = \bar{\partial} \operatorname{func} C - \bar{\partial} f = C$; this equation is valid on $M \backslash \{p_1, \ldots, p_k\}$. But since g and C are smooth, the equation is valid on M, and $C \sim 0$.

For the converse, suppose $C \sim 0$, and choose a smooth function g such that $\bar{\partial} g = C$. Then define $f = \operatorname{func} C - g$. We see that $\bar{\partial} f = \bar{\partial} \operatorname{func} C - \bar{\partial} g = C - C = 0$ on $M \backslash \{p_1, \ldots, p_k\}$. We know that $\operatorname{func} C$ is meromorphic near the p_i with poles at the p_i, and g is smooth. Therefore f is meromorphic on M, with $\langle f \rangle = \langle \operatorname{func} C \rangle$. $\qquad\square$

In the "Conditions" we will write, A represents an η chain collection.

Condition B. The existence of a meromorphic function f on M such that

$$\langle f \rangle = \langle \operatorname{func}(c_1 \, {}^1\!A + \cdots + c_g \, {}^g\!A) \rangle$$

implies that $c_1 = \cdots = c_g = 0$.

Theorem 6.31. *For an η chain collection to be an η chain basis, Condition B is necessary and sufficient.*

Proof.

$$A \text{ is an } \eta \text{ chain basis.}$$

$$\Updownarrow$$

$$\text{If } c_1 \, {}^1\!A + \cdots + c_g \, {}^g\!A \sim 0, \text{ then } c_i = 0 \, \forall i.$$

$$\Updownarrow$$

$$\text{If there exists a meromorphic function } f \text{ on } M$$

$$\text{such that } \langle f \rangle = \langle \operatorname{func}(c_1 \, {}^1\!A + \cdots + c_g \, {}^g\!A) \rangle, \text{ then } c_i = 0 \, \forall i.$$

The second equivalence is by Lemma 6.30. $\qquad\square$

Condition B is clearly impossible to check solely from the knowledge of which integral divisors on $\{p_1, \ldots, p_k\}$ occur as polar divisors of meromorphic functions on M (that is, from the knowledge of $t : P \longrightarrow \{\mathrm{N}, \mathrm{Y}\}$). To check Condition B for a given η chain collection would require more, namely the knowledge of which meromorphic function germs on $\{p_1, \ldots, p_k\}$ appear as germs of principal parts of meromorphic functions on M.

We now present a stronger condition, which relies only on t:

Condition N. The existence of a meromorphic function f on M such that

$$[f] = [\text{func}(c_1 \, {}^1\!A + \cdots + c_g \, {}^g\!A)]$$

implies that $c_1 = \cdots = c_g = 0$.

The fact that Condition N $\implies$ Condition B follows directly from the fact that if $\langle f \rangle = \langle g \rangle$, then $[f] = [g]$. We can reformulate this new condition as follows:

Condition N. For every $c \in \mathbb{R}^g \backslash 0$, $t\big([\text{func}(c_1 \, {}^1\!A + \cdots + c_g \, {}^g\!A)] \big) = \text{N}$.

This reformulation explains the name, and shows that the condition relies solely upon the data given by t. We will call an η chain collection that satisfies Condition N a *CN η chain basis*. Note that the object $[\text{func}\, C]$, which is the "polar divisor of the function associated to the η chain C," is easily ascertained by looking at the matrix for C. Specifically: let M be the order of C, and represent C by the $k \times M$ matrix c_i^n. Define

$$n_i(C) = \max\{r : c_i^r \neq 0\}. \tag{6.1}$$

We will call this number the *order of C at p_i*. Then

$$[\text{func}\, C] = p_1^{n_1(C)} \cdots p_k^{n_k(C)}. \tag{6.2}$$

Conjecture 6.32. *For every η chain basis A, there's a CN η chain basis A' with $s(A') \leq s(A)$.*

Consequence 6.33.

$$\{minimal\ \eta\ chain\ basis\ symbols\} = \{minimal\ CN\ \eta\ chain\ basis\ symbols\}.$$

Proof of the consequece. Let s be a minimal η chain basis symbol. Choose an η chain basis A such that $s = s(A)$. Now choose a CN η chain basis A' such that $s(A') \leq s(A)$. Then $s(A') = s(A)$. That is, s is a CN η chain basis symbol. Now suppose $s' \leq s$ is another CN η chain basis symbol. Then $s' = s$. So s is a minimal CN η chain basis symbol.

To prove the reverse inclusion, let s be a minimal CN η chain basis symbol. Then s is certainly an η chain basis symbol. So let $s' \leq s$ be another η chain basis symbol. Choose an η chain basis A' such that $s(A') = s'$. Then there's a CN η chain basis A with $s(A) \leq s(A')$. So we have $s(A) \leq s(A') = s' \leq s$. But A is a CN η chain basis, and s is a minimal CN η chain basis symbol. So $s(A) = s$. Therefore $s' = s$. That is, s is a minimal η chain basis symbol. $\qquad \square$

If we prove Conjecture 6.32, we will have answered Question 6.13 in the affirmative, as is shown in the following diagram:

$$\text{Noether gap scheme}$$

$$\updownarrow$$

$$t : P \to \{\mathrm{N}, \mathrm{Y}\}$$

$$\downarrow$$

$$\{\text{CN } \eta \text{ chain bases}\}$$

$$\downarrow$$

$$\{\text{CN } \eta \text{ chain basis symbols}\}$$

$$\downarrow$$

$$\{\text{minimal CN } \eta \text{ chain basis symbols}\}$$

$$\|$$

$$\{\text{minimal } \eta \text{ chain basis symbols}\}$$

$$\|$$

$$\{\text{minimal pole vectors}\}.$$
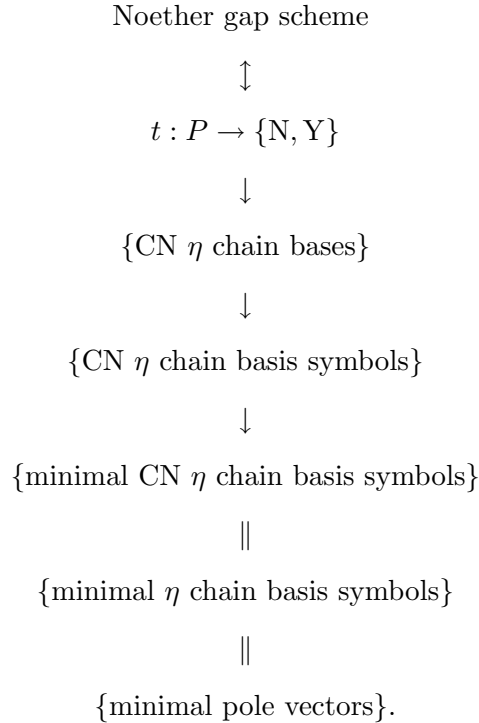
I don't know whether Conjecture 6.32 is true. In the next section, we will verify it in the cases $g = 1$ and $g = 2$ (for arbitrary k). It might be possible to prove Conjecture 6.32 in general using a variation of the following arguments, which lead to a theorem which is weaker than Conjecture 6.32. The first proposition is essentially a re-phrasing of Lemma 6.30.

Proposition 6.34. *Let $\mathfrak{p} \in P$. Then $t(\mathfrak{p}) = \mathrm{Y}$ if and only if there exists an η chain $C \sim 0$ such that $[\mathrm{func}\, C] = \mathfrak{p}$.*

Proof. Suppose $t(\mathfrak{p}) = \mathrm{Y}$. Choose a meromorphic function f on M such that $[f] = \mathfrak{p}$. Then let $C = \mathrm{chain}\, f$. We find $\langle \mathrm{func}\, C \rangle = \langle \mathrm{func}\, \mathrm{chain}\, f \rangle = \langle f \rangle$, and f is meromorphic; therefore by Lemma 6.30, $C \sim 0$. Also, $[\mathrm{func}\, C] = [f] = \mathfrak{p}$.

For the converse, suppose that there exists an η chain $C \sim 0$ with $[\mathrm{func}\, C] = \mathfrak{p}$. By Lemma 6.30, there exists a meromorphic function f on M such that $\langle f \rangle = \langle \mathrm{func}\, C \rangle$. So $[f] = [\mathrm{func}\, C] = \mathfrak{p}$. So $t(\mathfrak{p}) = \mathrm{Y}$. $\square$

Proposition 6.35. *For any η chain C such that $t([\mathrm{func}\, C]) = \mathrm{Y}$, there's an η chain $\widehat{C} \sim 0$ such that $n_i(\widehat{C}) = n_i(C)$ for all $i \in \{1, \ldots, k\}$.*

Proof. This follows directly from the previous proposition, because $[\operatorname{func}\widehat{C}] = [\operatorname{func} C]$ is the same thing as $n_i(\widehat{C}) = n_i(C)\,\forall i$. $\qquad\square$

In this situation $\widehat{C}$ will be called a *shadow* of C.

Proposition 6.36. *Let C be an η chain with $t(\,[\operatorname{func} C]\,) = \mathrm{Y}$. Let $i \in \{1,\ldots,k\}$ be such that $n_i(C) \neq 0$. Then there exists an η chain $C' \sim C$ such that $n_r(C') \leq n_r(C)$ for all $r \in \{1,\ldots,k\}$, with*

$$n_i(C') < n_i(C).$$

Proof. Let $\widehat{C}$ be a shadow of C. Then $\widehat{c}_r^{\,n_r(C)} \neq 0$ for all r such that $n_r(C) \neq 0$. Define

$$C' = C - \frac{c_i^{n_i(C)}}{\widehat{c}_i^{\,n_i(C)}}\widehat{C}.$$

Since $\widehat{C} \sim 0$, $C' \sim C$. Since $n_r(\widehat{C}) \leq n_r(C)\,\forall r$, we have $n_r(C') \leq n_r(C)\,\forall r$. And since $(c')_i^{n_i(C)} = 0$, we have $n_i(C') < n_i(C)$. $\qquad\square$

Proposition 6.37. *For any η chain $C \nsim 0$ such that $t(\,[\operatorname{func} C]\,) = \mathrm{Y}$, there's an η chain $\widetilde{C} \sim C$ such that $n_r(\widetilde{C}) \leq n_r(C)$ for every r, $n_i(\widetilde{C}) < n_i(C)$ for some i, and $t(\,[\operatorname{func}\widetilde{C}]\,) = \mathrm{N}$.*

Proof. Since $C \nsim 0$, there's an $r \in \{1,\ldots,k\}$ such that $n_r(C) \neq 0$. Choose the smallest such, and call it i. Then we can choose $C' \sim C$ with $n_r(C') \leq n_r(C)\,\forall r$ and $n_i(C') < n_i(C)$.

We can repeat this procedure to get $C'', C''', \ldots$ for as long as $t(\,[\operatorname{func} C^{(\ell)}]\,) = \mathrm{Y}$. But since one of the n_i is strictly reduced with each iteration, we would end up with $C^{(\ell)} = 0$ if we went on long enough. This is contrary to our hypothesis that $C \nsim 0$. So the process must terminate with $t(\,[\operatorname{func} C^{(\ell)}]\,) = \mathrm{N}$ for some ℓ. We take $\widetilde{C} = C^{(\ell)}$. $\qquad\square$

Formula 6.38. *Let $A = \{\,^1\!A, \ldots, \,^g\!A\}$ be an η chain collection. Then*

$$s_i(A) = \max\{n_i(\,^1\!A), \ldots, n_i(\,^g\!A)\}.$$

Proof.

$$s_i(A) = \max \bigcup_{j=1\ldots g} \{r \in \{1,\ldots,M\} : \,^j\!a_i^r \neq 0\}$$

$$= \max\left\{\max\left\{r \in \{1,\ldots,M\} : \,^1\!a_i^r \neq 0\right\}, \ldots, \max\left\{r \in \{1,\ldots,M\} : \,^g\!a_i^r \neq 0\right\}\right\}$$

$$= \max\{n_i(\,^1\!A), \ldots, n_i(\,^g\!A)\}.$$

$\qquad\square$

Condition W. For every $j \in \{1, \ldots, g\}$, $t(\,[\text{func } ^{j}\!A]\,) = \text{N}$.

Condition W is certainly weaker than Condition N; in fact, it is not strong enough to guarantee that the η chain collection A is an η chain basis. That is, Condition W does not imply Condition B. But we can prove the following

Theorem 6.39. *For every η chain basis A, there exists a CW η chain basis $\widetilde{A}$ with $\widetilde{s}(\widetilde{A}) \leq s(A)$.*

Proof. If $t(\,[\text{func } ^{j}\!A]\,) = \text{N}$, then define $^{j}\!\widetilde{A} = {}^{j}\!A$. If $t(\,[\text{func } ^{j}\!A]\,) = \text{Y}$, then perform the construction of $^{j}\!\widetilde{A}$ from $^{j}\!A$ as in Proposition 6.37. Since $^{j}\!\widetilde{A} \sim {}^{j}\!A$, $\widetilde{A}$ is an η chain basis. And $t(\,[\text{func } ^{j}\!\widetilde{A}]\,) = \text{N}$ for every j; that is, $\widetilde{A}$ satisfies Condition W. So we only need to verify that $s(\widetilde{A}) \leq s(A)$. But since $n_i(\,^{j}\!\widetilde{A}\,) \leq n_i(\,^{j}\!A\,)$ for every i and j, we find that

$$s_i(\widetilde{A}) = \max\{n_i(\,^{1}\!\widetilde{A}\,), \ldots, n_i(\,^{g}\!\widetilde{A}\,)\} \leq \max\{n_i(\,^{1}\!A\,), \ldots, n_i(\,^{g}\!A\,)\} = s_i(A)$$

by Formula 6.38. $\qquad\square$

We can use this theorem to prove the statement we made in Section 6.4.

Corollary 6.40. *Let s be a pole vector. Then there's a pole vector $\widetilde{s} \leq s$ with $\widetilde{s} < (2g, \ldots, 2g)$.*

Proof. Since s is a pole vector, there's an η chain basis A with $s(A) \leq s$ by Theorem 6.28. By Theorem 6.39, there's an η chain basis $\widetilde{A}$ with $s(\widetilde{A}) \leq s(A)$ and $t(\,[\text{func } ^{j}\!\widetilde{A}]\,) = \text{N}$ for all j. So $\widetilde{s} = s(\widetilde{A})$ is a pole vector, and $\widetilde{s} \leq s$. Further, degree[func $^{j}\!\widetilde{A}$] $< 2g$ since t of this divisor is N. So [func $^{j}\!\widetilde{A}$] $< p_1^{2g} \cdots p_k^{2g}$. That is, $n_i(\,^{j}\!\widetilde{A}\,) < 2g$ for every i and j. So

$$s_i(\widetilde{A}) = \max\{n_i(\,^{1}\!\widetilde{A}\,), \ldots, n_i(\,^{g}\!\widetilde{A}\,)\} < 2g.$$

for every i. Therefore $\widetilde{s} = s(\widetilde{A}) < (2g, \ldots, 2g)$. $\qquad\square$

We can also now offer a second conjecture, which together with Theorem 6.39 would imply Conjecture 6.32.

Conjecture 6.41. *Every CW η chain basis satisfies Condition N.*

This can be re-phrased as follows:

Conjecture 6.41. *Together, Condition B and Condition W imply Condition N.*

6.8   Special cases

g = 1, k arbitrary

For any $i \in \{1, \ldots, k\}$, the η chain collection $\{\eta_i^1\}$ satisfies Condition N. So it is a CN η chain basis. Its symbol is the k-vector which has a 1 in position i and zeros elsewhere, which we call e_i. Thus e_i is a pole vector for any i. Each of the e_i is clearly minimal among non-zero k-vectors, and therefore is a minimal pole vector. And since every element of $\mathbb{Z}_{\geq 0}^k$ is comparable to at least one of the e_i, there can be no other minimal pole vectors. Thus

$$\{\text{minimal pole vectors}\} = \{e_i\}_{i=1\ldots k}.$$

Note that Conjecture 6.32 is true in this case.

g = 2, k = 2

We start by making a list of the *prima facie* consistent Noether gap schemes, and computing $t : P \longrightarrow \{\text{N}, \text{Y}\}$ for each of them.

There are 2^3 sequences of length $2g - 1 = 3$ involving p_1 and p_2. So a gap scheme means a choice of Noether gap sequence (either $1, 2$ or $1, 3$) for each of these 8 sequences.

In order to see which schemes are actually consistent, we start with the following list:

sequence	second gap	$t^{-1}(Y)$ contains	$t^{-1}(N)$ contains
$p_1 p_1 p_1$	2	p_1^3	p_1^2
	3	p_1^2	p_1^3
$p_1 p_1 p_2$	2	$p_1 p_2$ or $p_1^2 p_2$	p_1^2
	3	p_1^2	$p_1 p_2$ and $p_1^2 p_2$
$p_1 p_2 p_1$	2	p_1^2 or $p_1^2 p_2$	$p_1 p_2$
	3	$p_1 p_2$	p_1^2 and $p_1^2 p_2$
$p_1 p_2 p_2$	2	p_2^2 or $p_1 p_2^2$	$p_1 p_2$
	3	$p_1 p_2$	p_2^2 and $p_1 p_2^2$
$p_2 p_1 p_1$	2	p_1^2 or $p_1^2 p_2$	$p_1 p_2$
	3	$p_1 p_2$	p_1^2 and $p_1^2 p_2$
$p_2 p_1 p_2$	2	p_2^2 or $p_1 p_2^2$	$p_1 p_2$
	3	$p_1 p_2$	p_2^2 and $p_1 p_2^2$
$p_2 p_2 p_1$	2	$p_1 p_2$ or $p_1 p_2^2$	p_2^2
	3	p_2^2	$p_1 p_2$ and $p_1 p_2^2$
$p_2 p_2 p_2$	2	p_2^3	p_2^2
	3	p_2^2	p_2^3

Label these eight sequences 1 through 8; and let the ordered pair (n, i) represent the assertion that the second gap in sequence n is i. The first thing to notice is that sequences 3 and 5 yield precisely the same alternatives in the Y and N columns. This means that a choice of second gap for sequence 3 determines the second gap for sequence 5. Likewise for sequences 4 and 6. Therefore, a gap scheme can be thought of as a choice of second gap for each of the sequences

$$1, 2, 3, 4, 7, 8.$$

Now, keeping in mind that one or the other alternative (2 or 3) must hold for each sequence, we find the following relations:

$$(1,2) \iff (2,2) \impliedby (3,3) \iff (4,3) \implies (7,2) \iff (8,2)$$
$$(1,3) \iff (2,3) \implies (3,2) \iff (4,2) \impliedby (7,3) \iff (8,3)$$

Therefore, there are only five consistent gap schemes, as listed below:

$$
\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
(1,2) & (1,2) & (1,2) & (1,3) & (1,3) \\
(2,2) & (2,2) & (2,2) & (2,3) & (2,3) \\
(3,3) & (3,2) & (3,2) & (3,2) & (3,2) \\
(4,3) & (4,2) & (4,2) & (4,2) & (4,2) \\
(7,2) & (7,2) & (7,3) & (7,2) & (7,3) \\
(8,2) & (8,2) & (8,3) & (8,2) & (8,3)
\end{array}
$$

For each consistent gap scheme, we can now determine the value of t on each integral divisor of degree $< 2g = 4$, using just the two tables above. The result is

$$
\begin{array}{cccccc}
 & 1 & 2 & 3 & 4 & 5 \\
p_1^3 & Y & Y & Y & N & N \\
p_1^2 & N & N & N & Y & Y \\
p_1^2 p_2 & N & Y & Y & N & N \\
p_1 & N & N & N & N & N \\
p_1 p_2 & Y & N & N & N & N \\
p_2 & N & N & N & N & N \\
p_1 p_2^2 & N & Y & N & Y & N \\
p_2^2 & N & N & Y & N & Y \\
p_2^3 & Y & Y & N & Y & N
\end{array}
$$

The value of t on every other integral divisor is Y.

We will now list (for each gap scheme) all the CN η chain bases of the form $\{\eta_{i_1}^{n_1}, \eta_{i_2}^{n_2}\}$ (which are called η *bases* since each element is a single η and not a (compound) η chain). To do this is easy. First, choose any two distinct monomial elements of P, say $p_{i_1}^{n_1}$ and $p_{i_2}^{n_2}$. Then, $\{\eta_{i_1}^{n_1}, \eta_{i_2}^{n_2}\}$ is a CN η basis if and only if $t(p_{i_1}^{n_1}) = t(p_{i_2}^{n_2}) = t(p_{i_1}^{n_1} + p_{i_2}^{n_2}) = \mathrm{N}$, because

$$
[\mathrm{func}(a\eta_{i_1}^{n_1} + b\eta_{i_2}^{n_2})] = \begin{cases} p_{i_1}^{n_1} & a \neq 0, b = 0 \\ p_{i_2}^{n_2} & a = 0, b \neq 0 \\ p_{i_1}^{n_1} + p_{i_2}^{n_2} & a \neq 0, b \neq 0. \end{cases}
$$

The result is listed below.

$$
\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
\{\eta_1^1, \eta_1^2\} & \{\eta_1^1, \eta_1^2\} & \{\eta_1^1, \eta_1^2\} & \{\eta_1^1, \eta_1^3\} & \{\eta_1^1, \eta_1^3\} \\
\{\eta_1^1, \eta_2^2\} & \{\eta_1^1, \eta_2^1\} & \{\eta_1^1, \eta_2^1\} & \{\eta_1^1, \eta_2^1\} & \{\eta_1^1, \eta_2^1\} \\
\{\eta_1^2, \eta_2^1\} & \{\eta_2^1, \eta_2^2\} & \{\eta_2^1, \eta_2^3\} & \{\eta_2^1, \eta_2^2\} & \{\eta_2^1, \eta_2^3\} \\
\{\eta_2^1, \eta_2^2\}
\end{array}
$$

So we have discovered the following pole vectors (the symbols of the CN η bases listed above):

$$
\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
(2,0) & (2,0) & (2,0) & (3,0) & (3,0) \\
(1,2) & (1,1) & (1,1) & (1,1) & (1,1) \\
(2,1) & (0,2) & (0,3) & (0,2) & (0,3) \\
(0,2)
\end{array}
$$

Claim 6.42. *This is the list of all minimal pole vectors for each Noether gap scheme.*

Proof. For each pole vector of the form $(n,0)$ or $(0,n)$, the condition $t(p_i^n) = $ N ensures that n is the highest Weierstrass gap at p_i. So by the results of Section 6.2, these pole vectors are minimal. The pole vectors $(1,1)$ are minimal, because there cannot be an η basis whose symbol is $(1,0)$ or $(0,1)$ (it would have only one element, but $\dim H_{\bar{\partial}}^{0,1} M = 2$).

So we have shown that all the pole vectors listed are minimal, with the possible exceptions of $(1,2)$ and $(2,1)$ in gap scheme 1. So we will now show that $(1,1)$ is not a pole vector for gap scheme 1, so these pole vectors are in fact minimal. We start by noting that, in scheme 1, $t(p_1 p_2) = $ Y. So by Proposition 6.34, there's an η chain $C \sim 0$ such that $[\text{func } C] = p_1 p_2$. This means that $n_1(C) = n_2(C) = 1$. So $C = a\eta_1^1 + b\eta_2^1$ with a and b both non-zero. Thus η_1^1 and η_2^1 are co-linear. So no η collection $A = \{\,^1A,\,^2A\}$ of the form

$$
^1A = \,^1a_1^1 \eta_1^1 + \,^1a_2^1 \eta_2^1
$$
$$
^2A = \,^2a_1^1 \eta_1^1 + \,^2a_2^1 \eta_2^1
$$

can be an independent set. Therefore no η chain basis has symbol $(1,1)$. So $(1,1)$ is not a pole vector.

Thus every pole vector listed is minimal. And, for each scheme, every element of $\mathbb{Z}_{\geq 0}^2$ is comparable to one of the pole vectors listed; so there can be no other minimal pole vectors. $\square$

Note that Conjecture 6.32 is true in this case.

g = 2, k arbitrary

It follows from the results of Section 6.2 that Conjecture 6.32 is true for g arbitrary, $k = 1$. So in particular, Conjecture 6.32 is true in case $g = 2, k = 1$. We have also shown (above) that Conjecture 6.32 is true in case $g = 2, k = 2$. So we focus here on $g = 2, k > 2$. We will prove that Conjecture 6.32 is true in this case as well, by imitating the arguments for the case $g = 2, k = 2$. So let A be an η collection with the following property: for all CN η chain bases B, $s(B) \not\leq s(A)$. We need to show that A is not an independent set (in the sense of $\bar{\partial}$-cohomology).

First, suppose A is localized at p_i; that is, every η appearing in the η chains ${}^1\!A$, ${}^2\!A$ has subscript i. We know that we have the CN η basis $\{\eta_i^{n_1}, \eta_i^{n_2}\}$ where $n_1 = 1, n_2$ are the Weierstrass gaps at p_i. So by our hypothesis, $n_2 \not\leq s(A)$. Since A is localized at p_i, this forces $s(A) < n_2$. Therefore A is not an η basis, because n_2 is the minimal pole number of the marked surface (M, p_i); this was the result of Section 6.2.

Before proceeding, define $e_{ij} \in \mathbb{Z}_{\geq 0}^k$ to be the vector with entries of 1 in positions i and j and entries of 0 elsewhere. Also, define $f_{ij} \in \mathbb{Z}_{\geq 0}^k$ to be the vector with an entry of 1 in position i, an entry of 2 in position j, and entries of 0 elsewhere.

Now, suppose A is not localized at p_i for any i. Then $s(A)$ has at least two non-zero entries. Let i, j be distinct elements of $\{1, \ldots, k\}$ such that $s_i(A) \neq 0$ and $s_j(A) \neq 0$. For a moment, mentally replace i by 1 and j by 2. If gap scheme 2, 3, 4, or 5 above held for p_i and p_j (thinking of i as 1 and j as 2), then e_{ij} would be a CN η chain basis symbol, and certainly $e_{ij} \leq s(A)$. This is contrary to our hypothesis on A.

Therefore we must be in the case of gap scheme 1. This means that f_{ij} and f_{ji} are CN η chain basis symbols whenever $s_i(A) \neq 0$ and $s_j(A) \neq 0$. So $f_{ij} \not\leq s(A)$ and $f_{ji} \not\leq s(A)$ for any such i, j, by our hypothesis on A. This shows that all the non-zero entries in $s(A)$ are 1. Let $I \subset \{1, \ldots, k\}$ be the set of indices i for which $s_i(A) \neq 0$. Write $I = \{i_1, \ldots, i_\ell\}$. Then we have

1. $|I| \geq 2$;

2. For every $i, j \in I$, the Noether gap scheme with respect to i, j (re-labelled as $1, 2$) is scheme 1; and

3. For every $i \in I$, $s_i(A) = 1$.

By (3), we easily see that

$$^1\!A = {}^1\!a_1\eta^1_{i_1} + \cdots + {}^1\!a_\ell\eta^1_{i_\ell}$$
$$^2\!A = {}^2\!a_1\eta^1_{i_1} + \cdots + {}^2\!a_\ell\eta^1_{i_\ell},$$

with either $^1\!a_{i_r} \neq 0$ or $^2\!a_{i_r} \neq 0$ for each $r \in \{1, \ldots, \ell\}$.

Choose $r, s \in \{1, \ldots, \ell\}$. By (2), $t(p_{i_r}p_{i_s}) = Y$. So there's an η chain $C \sim 0$ such that $[\text{func } C] = p_{i_r}p_{i_s}$. This means that $n_{i_r}(C) = n_{i_s}(C) = 1$, and $n_j(C) = 0$ for $j \notin \{i_r, i_s\}$. So

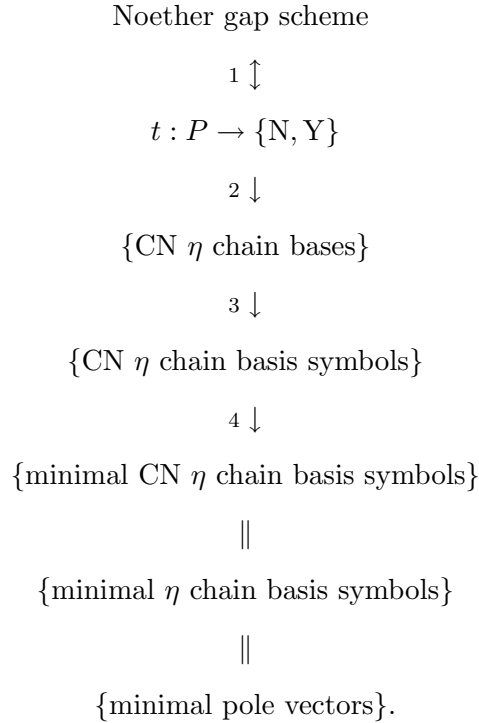$$C = a\eta^1_{i_r} + b\eta^1_{i_s}$$

with both a and b non-zero. So $\eta^1_{i_r}$ and $\eta^1_{i_s}$ are not independent as cohomology classes. Since $\dim H^{0,1}_{\bar\partial} M = 2$, this means that $\eta^1_{i_r}$ and $\eta^1_{i_s}$ are co-linear. Thus $\eta^1_{i_1}, \ldots, \eta^1_{i_\ell}$ are all co-linear. Thus we find that $^1\!A$ and $^2\!A$ are co-linear. That is, A is not an independent set. This ends the proof of Conjecture 6.32 in the case $g = 2$, $k > 2$.

6.9   Computability

Assume, in this section, that Conjecture 6.32 is true. The answer to Question 6.13 is therefore "yes": the set of minimal pole vectors is determined by the Noether gap scheme. Question 6.14 now asks whether one could actually *compute* the (finite) set of minimal pole vectors from the Noether gap scheme, which is a finite set of data. In other words, could we write a computer program which takes as its input g, k, and the Noether gap scheme, and gives as output the list of minimal pole vectors? In the case $k = 1$ we know that the answer is "yes." In this case, the inputs are g and the g Weierstrass gaps; the output is the minimal pole number. The program is very simple: it prints out the highest Weierstrass gap.

We would like to see what must be proved (in addition to Conjecture 6.32) in order to answer this question in the affirmative for arbitrary g and k. So let us take another look at

the following diagram:

$$\text{Noether gap scheme}$$

$$_1 \updownarrow$$

$$t : P \to \{\text{N}, \text{Y}\}$$

$$_2 \downarrow$$

$$\{\text{CN } \eta \text{ chain bases}\}$$

$$_3 \downarrow$$

$$\{\text{CN } \eta \text{ chain basis symbols}\}$$

$$_4 \downarrow$$

$$\{\text{minimal CN } \eta \text{ chain basis symbols}\}$$

$$\|$$

$$\{\text{minimal } \eta \text{ chain basis symbols}\}$$

$$\|$$

$$\{\text{minimal pole vectors}\}.$$

We have labeled the "determines" arrows. We need to see whether (and how) they can be regarded as meaning "computably determines." Note that all the data sets shown are finite, except for $\{\text{CN } \eta \text{ chain bases}\}$.

(1) The results of Section 6.5 clearly show that each of these two determinations (up and down) is computable.

(4) This determination is clearly computable.

(2), (3) The first obstacle here is that the data set $\{\text{CN } \eta \text{ chain bases}\}$ is uncountable. We could try to get around this by specifying a normalization. That is, replace $\{\text{CN } \eta \text{ chain bases}\}$ by $\{\text{normalized CN } \eta \text{ chain bases}\}$, where we call an η chain basis *normalized* if for each j, ${}^j a_r^M = 1$ for some r, where M is the order of ${}^j\!A$. Every η chain basis has a normalized form, and there are only finitely many normalized forms for each η chain basis. And the determinations (2) and (3) are still valid. But it is not in fact clear that there are only finitely many normalized CN η chain bases. So this approach is not so promising.

Our strategy will be to try to leap-frog over the troublesome data set {CN η chain bases}; that is, we will try to get from t to {CN η chain basis symbols} in a computable way.

We define, once and for all, $M = 2g - 1$.

Definition 6.43. Let $c \in \mathbb{R}^g$ and A be an η chain collection of order $\leq M$. Define the $k \times M$ matrix (cA) over $\mathbb{R}$ by

$$(cA)^n_i = c_1 \, {}^1a^n_i + \cdots + c_g \, {}^ga^n_i.$$

Then define

$$|cA|^n_i = \begin{cases} 0 & (cA)^n_i = 0 \\ 1 & (cA)^n_i \neq 0. \end{cases}$$

(cA) is just the matrix representation of the η chain $cA = c_1 \, {}^1A + \cdots + c_g \, {}^gA$. $|cA|$ is called the *binary matrix* of the η chain cA.

Definition 6.44. Let A be an η chain collection of order $\leq M$. We define the *binary matrix set of A* to be

$$\mathcal{M}(A) = \{|cA| : c \in \mathbb{R}^g\}.$$

The set G of all $k \times M$ matrices over $\{0, 1\}$ is finite. So the set H of all subsets of G is also finite. For each A, $\mathcal{M}(A) \subset G$, or $\mathcal{M}(A) \in H$.

Definition 6.45. For each $s < (2g, \ldots, 2g)$,

$$\mathfrak{m}(s) = \{\mathcal{M}(A) : A \text{ is an } \eta \text{ chain collection of symbol } s\}.$$

Note that for any s, $\mathfrak{m}(s)$ is a subset of the finite set H. That is, $\mathfrak{m}(s)$ is finite. So there exists a finite set $\mathcal{A}_s$ of η chain collections of symbol s such that

$$\mathfrak{m}(s) = \{\mathcal{M}(A) : A \in \mathcal{A}_s\}.$$

We call such a set a *sufficient set of symbol s η chain collections*.

Theorem 6.46. *Let $\mathcal{A}_s$ be a sufficient set of of symbol s η chain collections. Then there exists a CN η chain basis of symbol s if and only if some element of $\mathcal{A}_s$ satisfies Condition N.*

Proof. If some element $A \in \mathcal{A}_s$ satisfies Condition N, then A itself is a CN η chain basis of symbol s.

So suppose instead that there exists a CN η chain basis B of symbol s. Choose $A \in \mathcal{A}_s$ such that $\mathcal{M}(A) = \mathcal{M}(B)$. Now choose $c \in \mathbb{R}^g \backslash 0$. Then $|cA| = |\widetilde{c}B|$ for some $\widetilde{c} \in \mathbb{R}^g \backslash 0$. So clearly $[\operatorname{func} cA] = [\operatorname{func} \widetilde{c}B]$. But B satisfies Condition N. Therefore $t([\operatorname{func} \widetilde{c}B]) = \mathrm{N}$. Therefore $t([\operatorname{func} cA]) = \mathrm{N}$. As this is true for any $c \in \mathbb{R}^g \backslash 0$, we have found that A satisfies Condition N. $\qquad\square$

Conjecture 6.47. *There exists an algorithm for constructing a sufficient set of symbol s η chain collections for every $s < (2g, \ldots, 2g)$.*

The truth of this conjecture would evidently reduce our task to computing whether or not Condition N holds for a given η chain collection. We now examine whether this is feasible. Let A be an η chain collection of symbol $< (2g, \ldots, 2g)$.

Definition 6.48. We define the *coefficient set of A* to be

$$\mathcal{C}(A) = \left\{ \, {}^j a_i^n : j \in \{1, \ldots, g\}, i \in \{1, \ldots, k\}, n \in \{1, \ldots, M\} \right\}.$$

Note that $\mathcal{C}(A)$ is a finite set.

Definition 6.49. Let $c \in \mathbb{R}^g \backslash 0$. Then $\alpha \in \mathcal{C}(A)^g$ is called a *relation of c over $\mathcal{C}(A)$* if

$$c_1 \alpha_1 + \cdots + c_g \alpha_g = 0.$$

We define $R^c_{\mathcal{C}(A)}$ to be the set of relations of c over $\mathcal{C}(A)$.

Note that $R^c_{\mathcal{C}(A)}$ is a subset of the finite set $\mathcal{C}(A)^g$.

Definition 6.50. We define the *collection of relation sets over $\mathcal{C}(A)$* to be

$$\mathcal{R}(\mathcal{C}(A)) = \{R^c_{\mathcal{C}(A)} : c \in \mathbb{R}^g \backslash 0\}.$$

Note that $\mathcal{R}(\mathcal{C}(A))$ is a subset of the set of all subsets of the finite set $\mathcal{C}(A)^g$, and is therefore finite. So there exists a finite subset C_A of $\mathbb{R}^g \backslash 0$ such that

$$\mathcal{R}(\mathcal{C}(A)) = \{R^c_{\mathcal{C}(A)} : c \in C_A\}.$$

Such a set C_A will be called a *sufficient set of g-vectors* for A.

Condition N^{C_A}**.** For every $c \in C_A$, $t(\,[\text{func}\,cA]\,) = \text{N}$.

Let C_A be a sufficient set of g-vectors for A. If A satisfies Condition N, then clearly A satisfies Condition N^{C_A}.

Theorem 6.51. *A satisfies Condition N if and only if A satisfies Condition* N^{C_A}.

Proof. We only need to prove one direction. So suppose A satisfies Condition N^{C_A}. Choose $c \in \mathbb{R}^g \backslash 0$. Then there exists $\widetilde{c} \in C_A$ such that $R^{\widetilde{c}}_{\mathcal{C}(A)} = R^c_{\mathcal{C}(A)}$. So for any given $\alpha \in \mathcal{C}(A)^g$, α is a relation of c over $\mathcal{C}(A)$ if and only if α is a relation of $\widetilde{c}$ over $\mathcal{C}(A)$. In particular, take

$$\alpha = (\,{}^1a_i^n, \ldots, {}^ga_i^n)$$

for any i, n. Then $c_1\, {}^1a_i^n + \cdots + c_g\, {}^ga_i^n = 0$ if and only if $\widetilde{c}_1\, {}^1a_i^n + \cdots + \widetilde{c}_g\, {}^ga_i^n = 0$. This shows that $|cA| = |\widetilde{c}A|$. Therefore $[\text{func}\,cA] = [\text{func}\,\widetilde{c}A]$. Since A satisfies Condition N^{C_A}, we know that $t(\,[\text{func}\,\widetilde{c}A]\,) = \text{N}$. Therefore $t(\,[\text{func}\,cA]\,) = \text{N}$. Since $c \in \mathbb{R}^g \backslash 0$ was arbitrary, this shows that A satisfies Condition N. $\square$

Conjecture 6.52. *There exists an algorithm for constructing a sufficient set of g-vectors for any given η chain collection whose symbol is $< (2g, \ldots, 2g)$.*

Let us assume the truth of Conjectures 6.47 and 6.52. We are then ready to show that $\{\text{CN}\ \eta$ chain basis symbols$\}$ is computable from $t : P \longrightarrow \{\text{N}, \text{Y}\}$. The algorithm is as follows:

1. For each $s < (2g, \ldots, 2g)$, construct a sufficient set $\mathcal{A}_s$ of symbol s η chain collections.

2. For each $A \in \mathcal{A}_s$, construct a sufficient set C_A of g-vectors.

3. Check Condition N^{C_A} for each $A \in \mathcal{A}_s$. If Condition N^{C_A} is satisfied for some $A \in \mathcal{A}_s$, then s is a CN η basis symbol. Otherwise, s is not a CN η basis symbol.

We may summarize our efforts up to this point as follows. We have asked two Questions (6.13 and 6.14) and offered four Conjectures (6.32, 6.41, 6.47, and 6.52). Conjecture 6.32 implies that the answer to Question 6.13 is "yes." Conjectures 6.32, 6.47, and 6.52 together imply that the answer to Question 6.14 is "yes." And incidentally, Conjecture 6.41 implies Conjecture 6.32.

7

The Generic *b*-holomorphic Complex Curve

We take M to be a b-holomorphic complex curve. We assume that each collar invariant is in $(\mathbb{C}\backslash\mathbb{Q}) \cup 1$, and that first integrals exist.

7.1   Classification

Theorem 7.1. *Let M and N be b-holomorphic curves. Then for M and N to be equivalent, it is necessary and sufficient that there exist an ordering of the boundary circles of M and N as*

$$\mathcal{C}_1(M), \ldots, \mathcal{C}_k(M)$$
$$\mathcal{C}_1(N), \ldots, \mathcal{C}_k(N)$$

such that

1. *$c_i(M) = c_i(N) \,\forall\, i$; and*

2. *there exist first integrals*

$$z_1^M, \ldots, z_k^M$$
$$z_1^N, \ldots, z_k^N$$

   and a biholomorphism of marked surfaces

$$\check{\varphi} : \check{M} \longrightarrow \check{N}$$

   such that, for every i,

$$\pi_N^{-1} \circ \check{\varphi} \circ \pi_M = (z_i^N)^{-1} \circ z_i^M$$

   on $\mathrm{collar}_i^{\circ}(M)$.

Proof of necessity. Let $\varphi : M \longrightarrow N$ be an isomorphism. Label the boundary circles of M (as you please) as

$$\mathcal{C}_1(M), \ldots, \mathcal{C}_k(M).$$

Then define

$$\mathcal{C}_i(N) = \varphi(\mathcal{C}_i(M)).$$

Since $\varphi : \mathrm{collar}_i(M) \longrightarrow \mathrm{collar}_i(N)$ is an isomorphism, we must have $c_i(M) = c_i(N)$; so the first condition is necessary.

Next, choose first integrals $z_1^N, \ldots, z_k^N$ for N, and define first integrals for M by $z_i^M = z_i^N \circ \varphi$. Now define $\check{\varphi} = \pi_N \circ \check{\varphi} \circ \pi_M^{-1}$ as a biholomorphism

$$\check{M} \backslash \{p_1^M, \ldots, p_k^M\} \longrightarrow \check{N} \backslash \{p_1^N, \ldots, p_k^N\}.$$

Clearly, on $\mathrm{collar}_i^{\circ}(M)$, we have

$$\pi_N^{-1} \circ \check{\varphi} \circ \pi_M = \varphi = (z_i^N)^{-1} \circ z_i^M.$$

So we only have to verify that $\check{\varphi}$ extends as a biholomorphism

$$\check{M} \longrightarrow \check{N}.$$

$\check{\varphi}$ clearly extends to a bijection between these blow-downs. And $\check{\varphi}$ takes p_i^M to p_i^N. We only need to check analyticity at p_i^M. But, in coordinates centered at p_i^M, $\check{\varphi}$ takes the form

$$(z_i^N \circ \pi_N^{-1}) \circ (\pi_N \circ \varphi \circ \pi_M^{-1}) \circ (z_i^M \circ \pi_M^{-1})^{-1} = z_i^N \circ \varphi \circ (z_i^M)^{-1},$$

which is φ written in holomorphic coordinates near $\mathcal{C}_i(M)$. So this function is a biholomorphism from a punctured disk to a punctured disk. So it's a biholomorphism from a disk to a disk.

Therefore the second condition is necessary. $\qquad\square$

Proof of sufficiency. Define $\varphi : M^{\circ} \longrightarrow N^{\circ}$ by $\varphi = \pi_N^{-1} \circ \check{\varphi} \circ \pi_M$. Then φ is a biholomorphism. We need to show that φ extends to a diffeomorphism $M \longrightarrow N$.

Fix i. We have the maps

$$\chi_{c_i(M)} \circ z_i^M : \mathrm{collar}_i^{\circ}(M) \longrightarrow \mathcal{M}_{c_i(M)}^{\circ}$$
$$\chi_{c_i(N)} \circ z_i^N : \mathrm{collar}_i^{\circ}(N) \longrightarrow \mathcal{M}_{c_i(N)}^{\circ}.$$

Since the z_i^M and z_i^N are first integrals, each of these maps extends (by Remark 2.17) to a diffeomorphism

$$\chi_{c_i(M)} \circ z_i^M : \mathrm{collar}_i(M) \longrightarrow \mathcal{M}_{c_i(M)}$$
$$\chi_{c_i(N)} \circ z_i^N : \mathrm{collar}_i(N) \longrightarrow \mathcal{M}_{c_i(N)}.$$

Since we have $c_i(M) = c_i(N) \equiv c_i$, then $\chi_{c_i(M)} = \chi_{c_i(N)} \equiv \chi_{c_i}$. So now we can write φ on collar$_i^{\circ}(M)$ as

$$\varphi = \pi_N^{-1} \circ \check{\varphi} \circ \pi_M$$
$$= (z_i^N)^{-1} \circ z_i^M$$
$$= (z_i^N)^{-1} \circ \chi_{c_i}^{-1} \circ \chi_{c_i} \circ z_i^M$$
$$= (\chi_{c_i} \circ z_i^N)^{-1} \circ (\chi_{c_i} \circ z_i^M).$$

Now it becomes clear that this map collar$_i^{\circ}(M) \longrightarrow$ collar$_i^{\circ}(N)$ extends to a diffeomorphism collar$_i(M) \longrightarrow$ collar$_i(N)$. $\qquad\square$

Let us try to see the equivalence of this condition to the condition for equivalence of cups with $c \notin \mathbb{Q}$ which was given in Chapter 5, namely, that $c(M) = c(N)$ and $\beta(M) = \beta(N)$. Of course, we could at this point simply say

(A) M and N satisfy the particular condition for equivalence of cups

$$\Updownarrow$$

(B) $M \simeq N$

$$\Updownarrow$$

(C) M and N satisfy the general condition for equivalence of cups,

but this is not the point. We wish to see how to go directly from (A) to (C) and back, without passing through (B).

Proof that (C) $\Longrightarrow$ (A). We need to show that $\beta(M) = \beta(N)$. So choose z_M, z_N, and $\check{\varphi} : \check{M} \longrightarrow \check{N}$ as in Theorem 7.1, so that $\pi_N^{-1} \circ \check{\varphi} \circ \pi_M = z_N^{-1} \circ z_M$. Then let $\mathfrak{z}_M$ and $\mathfrak{z}_N$ be the interior coordinates associated to z_M and z_N. Then

$$\mathfrak{z}_N \circ \pi_N^{-1} \circ \check{\varphi} \circ \pi_M \circ \mathfrak{z}_M^{-1}$$

is an automorphism of S^2 which fixes ∞. So it has the form $\xi \mapsto a\xi + b$ for some $a \neq 0$ and some b. Near ∞, this map equals

$$\mathfrak{z}_N \circ z_N^{-1} \circ z_M \circ \mathfrak{z}_M^{-1}.$$

Therefore

$$\mathfrak{z}_N \circ z_N^{-1} = a(\mathfrak{z}_M \circ z_M^{-1}) + b.$$

But both of the functions $\mathfrak{z}_N \circ z_N^{-1}$ and $\mathfrak{z}_M \circ z_M^{-1}$ have residue 1 and constant term 0. So $a = 1$ and $b = 0$. So in fact $\mathfrak{z}_N \circ z_N^{-1} = \mathfrak{z}_M \circ z_M^{-1}$. Thus

$$\begin{aligned}
\beta(M) &= (1/\mathfrak{z}_M \circ z_M^{-1}) \\
&= (1/\mathfrak{z}_N \circ z_N^{-1}) \\
&= \beta(N).
\end{aligned}$$

$\square$

Proof that (A) $\Longrightarrow$ (C). We assume here that $c(M) = c(N)$ and $\beta(M) = \beta(N)$. We need to show the existence of the map $\check{\varphi}$.

Choose a first integral z_M, and let $\mathfrak{z}_M$ be the associated interior coordinate. Then choose z_N such that

$$1/\mathfrak{z}_N \circ z_N^{-1} = 1/\mathfrak{z}_M \circ z_M^{-1},$$

where $\mathfrak{z}_M$ is of course the interior coordinate associated to z_N. (It was remarked in Chapter 5 that the equivalence of $\beta(M)$ and $\beta(N)$ entails this freedom of choice, simply by an appropriate adjustment of the first integral by a multiplicative constant.) Thus $\mathfrak{z}_M \circ z_M^{-1} = \mathfrak{z}_N \circ z_N^{-1}$. Now define $\check{\varphi} : \check{M} \backslash p_1^M \longrightarrow \check{N} \backslash p_1^N$ by

$$\check{\varphi} = \pi_N \circ \mathfrak{z}_N^{-1} \circ \mathfrak{z}_M \circ \pi_M^{-1}.$$

Clearly, this is a biholomorphism. It extends to a bijection $\check{M} \longrightarrow \check{N}$ which takes p_1^M to p_1^N. In coordinates near p_1^M, it looks like

$$(z_N \circ \pi_N^{-1}) \circ (\pi_N \circ \mathfrak{z}_N^{-1} \circ \mathfrak{z}_M \circ \pi_M^{-1}) \circ (z_M \circ \pi_M^{-1})^{-1} = (\mathfrak{z}_N \circ z_N^{-1})^{-1} \circ (\mathfrak{z}_M \circ z_M^{-1}) = \text{identity}\,.$$

This is certainly a diffeomorphism. Therefore, $\check{\varphi} : \check{M} \longrightarrow \check{N}$ is a biholomorphism of marked surfaces.

Finally, by our construction,

$$\begin{aligned}
\pi_N^{-1} \circ \check{\varphi} \circ \pi_M &= \pi_N^{-1} \circ (\pi_N \circ \mathfrak{z}_N^{-1} \circ \mathfrak{z}_M \circ \pi_M^{-1}) \circ \pi_M \\
&= \mathfrak{z}_N^{-1} \circ \mathfrak{z}_M \\
&= z_N^{-1} \circ z_M
\end{aligned}$$

on the collar. $\square$

We may say that there are three joint conditions for the equivalence of two cups M and N:

1. the c are equal

2. the β are equal

3. the blow-downs are equivalent as marked surfaces,

with (3) automatically satisfied (any two once-marked spheres are equivalent). So it might be wondered whether we could formulate three joint conditions for equivalence in the general case, something like:

1. the c_i are equal

2. $\mathcal{C}_i(M)$ is glued to M° in the same fashion as $\mathcal{C}_i(N)$ is glued to N°

3. $\check{M} \simeq \check{N}$.

This seems indeed to be so; but it so happens that our formulation of (2) entails (3). So instead of conditions (1), (2), and (3) jointly necessary and sufficient for equivalence of b-holomorphic complex curves, we merely have (1) and (2) jointly necessary and sufficient (with (3) a consequence of (2)).

It may be possible to split our second condition into distinct conditions for glueing of boundary circles and equivalence of blow-downs (perhaps using the universal cover?), but I have not been able to do it.

7.2   Bundles over a b-holomorphic complex curve

Theorem 7.2. *To each holomorphic line bundle E over M is associated an element κ of $H^{0,1}_{\bar{\partial}}\check{M}$. The map $E \mapsto \kappa(E)$ is a surjective homomorphism from the group of equivalence classes of holomorphic line bundles over M to $H^{0,1}_{\bar{\partial}}\check{M}$.*

Construction. Choose a global C^∞ frame **1** for E which is holomorphic away from ∂M, and let α be the associated $\bar{\partial}$ form. Label the boundary circles of M as $\mathcal{C}_1, \ldots, \mathcal{C}_k$. Now fix $i \in \{1, \ldots, k\}$. Choose a first integral z_i for collar_i, and define $f = \langle \alpha, \bar{L} \rangle$.

First suppose that $c_i = 1$. Define

$$u^1_{mn} = \begin{cases} -\frac{f_{mn}}{m-n} & n \neq m \\ 0 & n = m \end{cases}$$

and

$$u^2_{mn} = \begin{cases} 0 & n \neq m \\ -f_{mn} & n = m. \end{cases}$$

Let u^1 and u^2 be C^∞ functions on M which are supported near $\mathcal{C}_i$ and which have asymptotic expansions at $\mathcal{C}_i$

$$u^1 \sim \sum_{\substack{m \geq 0 \\ n \in \mathbb{Z}}} u^1_{mn} x^m e^{iny}$$

$$u^2 \sim \sum_{\substack{m \geq 0 \\ n \in \mathbb{Z}}} u^2_{mn} x^m e^{iny}.$$

Then let $g_i = u^1 + u^2 \log x$. Note that $\bar{\partial} g_i$ agrees with $-\alpha$ to infinite order on $\mathcal{C}_i$.

Now suppose instead that $c_i \notin \mathbb{Q}$. Choose a cut-off function ϕ_i supported near $\mathcal{C}_i$ and let r be the unique good solution to $\bar{\partial} r = -\alpha$ on $\mathcal{M}_{c_i}(z_i)$, as in Lemma 2.33. Then define $g_i = \phi_i r$. Note that $\bar{\partial} g_i$ agrees with $-\alpha$ near $\mathcal{C}_i$.

Now define $g = g_1 + \cdots + g_k$ on M. This is a C^∞ function in M°, and $\bar{\partial} g$ agrees with $-\alpha$ to infinite order on ∂M. So if we define $\kappa = -\alpha - \bar{\partial} g$, then κ is a smooth $(0, 1)$-form on $\check{M}$. It therefore defines a class in $H^{0,1}_{\bar{\partial}} \check{M}$. □

Proof of invariance under a change of ϕ_i ($c_i \notin \mathbb{Q}$). Perform the construction with ϕ_i and then again with $\widetilde{\phi}_i$. Then $\widetilde{g}_i - g_i = (\widetilde{\phi}_i - \phi_i) r$. So $\widetilde{g} - g = (\widetilde{\phi}_i - \phi_i) r$, which is a smooth function on $\check{M}$. Thus

$$\widetilde{\kappa} - \kappa = -\alpha - \bar{\partial} \widetilde{g} + \alpha + \bar{\partial} g$$
$$= -\bar{\partial}(\widetilde{g} - g),$$

and so $\widetilde{\kappa}$ and κ define the same class. □

Proof of invariance under a change of z_i ($c_i \notin \mathbb{Q}$). Perform the construction with z_i and then again with $\widetilde{z}_i$. As we saw in the construction of the integral sequence class for a bundle over a collar in Chapter 2, $\widetilde{r} = r$. So $\widetilde{g}_i = g_i$. So $\widetilde{g} = g$. So $\widetilde{\kappa} = \kappa$. □

Proof of invariance under a change of u^1 and u^2 ($c_i = 1$). Perform the construction with u^1, u^2, and then with $\widetilde{u}^1, \widetilde{u}^2$. We know that $\widetilde{u}^1$ agrees with u^1 to infinite order at $\mathcal{C}_i$ and $\widetilde{u}^2$ agrees with u^2 to infinite order at $\mathcal{C}_i$. Therefore $\widetilde{g}_i$ agrees with g_i to infinite order at $\mathcal{C}_i$. So $\widetilde{g} - g$ vanishes to infinite order on ∂M. That is, $\widetilde{g} - g$ is a smooth function on $\check{M}$. Thus

$$\widetilde{\kappa} - \kappa = -\alpha - \bar{\partial} \widetilde{g} + \alpha + \bar{\partial} g$$
$$= -\bar{\partial}(\widetilde{g} - g),$$

and so $\widetilde{\kappa}$ and κ define the same class. □

Proof of invariance under a change of z_i ($c_i = 1$). Perform the construction with z_i and then again with $\widetilde{z}_i$. We know that $\bar{\partial}\widetilde{g}_i$ and $\bar{\partial}g_i$ both agree with $-\alpha$ to infinite order at $\mathcal{C}_i$. Therefore $\bar{\partial}(\widetilde{g}_i - g_i)$ vanishes to infinite order at $\mathcal{C}_i$. So $\widetilde{g}_i - g_i$ is asymptotic to a series in non-negative powers of z_i (or $\widetilde{z}_i$) at $\mathcal{C}_i$. So $\widetilde{g} - g$ is smooth on $\check{M}$. So, reasoning as before, $\widetilde{\kappa}$ and κ define the same class. □

Proof of invariance under a change of frame. Let f be a C^∞ function on M which is holomorphic away from ∂M, and define $\widetilde{\mathbf{1}} = \mathbf{1}e^f$. Then $\widetilde{\alpha} - \alpha = \bar{\partial}f$; or $(-\widetilde{\alpha}) - (-\alpha) = -\bar{\partial}f$.

Fix i such that $c_i \notin \mathbb{Q}$, and perform the construction of g_i and $\widetilde{g}_i$. Then by Sub-lemma 2.35, $\widetilde{r} - r = -f$. So $\widetilde{g}_i - g_i = -\phi_i f$.

Now fix i such that $c_i = 1$, and perform the construction of g_i and $\widetilde{g}_i$. Since $\bar{\partial}\widetilde{g}_i$ agrees with $-\widetilde{\alpha}$ to infinite order on $\mathcal{C}_i$ and g_i agrees with $-\alpha$ to infinite order on $\mathcal{C}_i$, $\bar{\partial}(\widetilde{g}_i - g_i)$ agrees with $(-\widetilde{\alpha}) - (-\alpha) = -\bar{\partial}f$ to infinite order on $\mathcal{C}_i$. So $\bar{\partial}([\widetilde{g}_i - g_i] + f)$ vanishes to infinite order at $\mathcal{C}_i$. So $(\widetilde{g}_i - g_i) + f$ is asymptotic to a series in non-negative powers of z_i at $\mathcal{C}_i$. So $(\widetilde{g}_i - g_i) + f$ is smooth near p_i on $\check{M}$.

Now we can compute that

$$\widetilde{\kappa} - \kappa = (-\widetilde{\alpha} - \bar{\partial}\widetilde{g}) - (-\alpha - \bar{\partial}g)$$
$$= (\alpha - \widetilde{\alpha}) + \bar{\partial}(g - \widetilde{g})$$
$$= -\bar{\partial}f + \bar{\partial}(g - \widetilde{g})$$
$$= -\bar{\partial}(\widetilde{g} - g + f),$$

which is a smooth function on $\check{M}$. So $\widetilde{\kappa}$ and κ define the same class. □

Proof of the rest. Since tensor product of bundles becomes addition at the level of $\bar{\partial}$ forms, and dualization becomes additive inversion, it is clear that the map $E \mapsto \kappa(E)$ is a homomorphism. Now suppose E is equivalent to the trivial bundle. Then there's a global C^∞ section which is holomorphic in the interior. The associated $\bar{\partial}$ form is zero. So clearly κ is zero. This means that the kernel of the mapping $E \mapsto \kappa(E)$ contains all bundles equivalent to the trivial one. So $\kappa(E) = \kappa(F)$ whenever E and F are equivalent. So κ is actually a homomorphism from equivalence classes of bundles to $H^{0,1}_{\bar{\partial}}\check{M}$.

All that remains is to prove surjectivity. So choose an element of $H^{0,1}_{\bar{\partial}}\check{M}$. If we choose an η chain basis A for the marked surface $\check{M}$, then our cohomology class can be represented

as $\alpha = b_1 \, {}^1\!A + \cdots + b_g \, {}^g\!A$ for some constants $b_1, \ldots, b_g$. Since α vanishes in a neighborhood of the p_i, we can lift it to a smooth (compressed) $(0,1)$-form on M. The trivial C^∞ bundle, with α as the $\bar\partial$ form with respect to $\mathbf{1}$, has $\kappa = \alpha$. $\qquad\square$

Theorem 7.3. *Let E be a holomorphic line bundle over M. Let A be an η chain basis for $H_{\bar\partial}^{0,1} \check{M}$ and write $\kappa(E) = b_1 \, {}^1\!A + \cdots + b_g \, {}^g\!A$.*

Choose first integrals $z_1, \ldots, z_k$ for the collars of M, cut-off functions ϕ_i supported near $\mathcal{C}_i$, and a global C^∞ frame $\mathbf{1}$ for E which is holomorphic away from the boundary. Define $v = \mathrm{func}(b_1 \, {}^1\!A + \cdots + b_g \, {}^g\!A)$, making use of the chosen first integrals and cut-off functions. Then there exist a smooth function u on M and smooth functions w and f on M°, and complex constants $a_i^0, a_i^1, a_i^2, \ldots$ for each $i = 1 \ldots k$, such that

1. *$\mathbf{1} e^{u+v+w+f}$ is a holomorphic frame for $E \lfloor M^\circ$;*

2. *$w = a_i^0 \log x_i$ near $\mathcal{C}_i$; and*

3. *When $c_i \notin \mathbb{Q}$, $f \sim \sum_{n \geq 1} a_i^n z_i^n$ at $\mathcal{C}_i$; and when $c_i = 1$, $f / \log x_i \sim \sum_{n \geq 1} a_i^n z_i^n$ at $\mathcal{C}_i$.*

Proof. Let α be the $\bar\partial$ form with respect to $\mathbf{1}$. Construct g as in Theorem 7.2. Then we have the smooth $(0,1)$-form $\kappa = -\alpha - \bar\partial g$ on $\check{M}$, and $\kappa \sim b_1 \, {}^1\!A + \cdots + b_g \, {}^g\!A$ in $H_{\bar\partial}^{0,1} \check{M}$. Choose a smooth function h on $\check{M}$ such that $\kappa = b_1 \, {}^1\!A + \cdots + b_g \, {}^g\!A + \bar\partial h$ on $\check{M}$. Then

$$
\begin{aligned}
\bar\partial(v + h) &= \bar\partial \, \mathrm{func}(b_1 \, {}^1\!A + \cdots + b_g \, {}^g\!A) + \bar\partial h \\
&= b_1 \, {}^1\!A + \cdots + b_g \, {}^g\!A + \bar\partial h \\
&= \kappa \\
&= -\alpha - \bar\partial g
\end{aligned}
$$

on $\check{M} \backslash \{p_1, \ldots, p_k\}$. Let $a_i^0 = -(i \Re c_i / \pi) \int_{\mathcal{C}_i} \alpha$, and define $w = \sum_{i=1}^k \phi_i a_i^0 \log x_i$. Then

$$
\begin{aligned}
\bar\partial(v + w + [g + h - w]) &= \bar\partial(v + g + h) \\
&= -\alpha
\end{aligned}
$$

on M°. Thus $\mathbf{1} e^{v+w+[g+h-w]}$ is a holomorphic frame for $E \lfloor M^\circ$, as claimed in (1); and w has the form claimed in (2). So it only remains to show that $g + h - w$ may be written as $u + f$, with u smooth on M and f having the form stated in (3).

Examine the procedure for construction of the g_i. If $c_i = 1$, then near $\mathcal{C}_i$

$$
g_i = (\text{smooth at } \mathcal{C}_i) + a_i^0 \log x_i + (\text{sum in positive powers of } z_i) \log x_i.
$$

And if $c_i \notin \mathbb{Q}$, then near $\mathcal{C}_i$

$$g_i = (\text{smooth at } \mathcal{C}_i) + a_i^0 \log x_i + (\text{sum in positive powers of } z_i).$$

Therefore, near $\mathcal{C}_i$,

$$g - w = \begin{cases} (\text{smooth at } \mathcal{C}_i) + (\text{sum in positive powers of } z_i) \log x_i & c_i = 1 \\ (\text{smooth at } \mathcal{C}_i) + (\text{sum in positive powers of } z_i) & c_i \notin \mathbb{Q}. \end{cases}$$

Now h is a smooth solution to a $\bar{\partial}$ problem on $\check{M}$ whose datum vanishes to infinite order at each of the p_i. Therefore near $\mathcal{C}_i$,

$$h = (\text{sum in non-negative powers of } z_i)$$

$$= \begin{cases} (\text{smooth at } \mathcal{C}_i) & c_i = 1 \\ (\text{constant}) + (\text{sum in positive powers of } z_i) & c_i \notin \mathbb{Q} \end{cases}$$

So we can see that

$$g - w + h = \begin{cases} (\text{smooth at } \mathcal{C}_i) + (\text{sum in positive powers of } z_i) \log x_i & c_i = 1 \\ (\text{smooth at } \mathcal{C}_i) + (\text{sum in positive powers of } z_i) & c_i \notin \mathbb{Q}. \end{cases}$$

Therefore this function can be split into u and f as claimed, with the correct properties. $\quad\square$

As a start toward finding a classification of bundles over M, we ask the question: given that the η chain basis A and the first integrals z_i are fixed for a given M, how unique is the frame constructed in the previous theorem? That is, could the construction lead to two different frames, depending on some choice made in the construction?

Theorem 7.4. *Fix an η chain basis A and first integrals z_i for M. Let E be a holomorphic line bundle over M.*

Suppose $\mathbf{1}e^{u+v+w+f}$ and $\widetilde{\mathbf{1}}e^{\widetilde{u}+\widetilde{v}+\widetilde{w}+\widetilde{f}}$ are holomorphic frames for $E \lfloor M^\circ$ constructed according to the procedure of the previous theorem. Then the quotient of these two frames is a meromorphic function on $\check{M}$ whose poles and zeros occur only at the p_i.

Proof. Call the quotient $g = (\widetilde{\mathbf{1}}/\mathbf{1})e^{(\widetilde{u}-u)+(\widetilde{v}-v)+(\widetilde{w}-w)+(\widetilde{f}-f)}$. Then g is certainly holomorphic and non-vanishing on $\check{M} \backslash \{p_1, \ldots, p_k\}$.

The quotient $\widetilde{\mathbf{1}}/\mathbf{1}$ is C^∞ on M, and therefore bounded. Since u and $\widetilde{u}$ are C^∞ on M, their difference $\widetilde{u} - u$ is also bounded. Since $\kappa(E) \in H_{\bar{\partial}}^{0,1}\check{M}$ is a bundle invariant, and the A

and z_i are fixed, v and $\widetilde{v}$ may differ only by virtue of a different choice of cut-off functions. But even so, it must be that $\widetilde{v} - v = 0$ in a neighborhood of p_i, so that $\widetilde{v} - v$ is bounded. Since f and $\widetilde{f}$ both tend to zero at ∂M, their difference $\widetilde{f} - f$ is bounded. And near $\mathcal{C}_i$, $\widetilde{w} - w$ is equal to a constant times $\log x_i$.

So the quotient g is a bounded function on $\check{M}\backslash\{p_1, \ldots, p_k\}$ times a function that equals $|z_i|$ to a (constant) power near each of the p_i. This implies that g must be meromorphic on $\check{M}$, with poles and zeros only at the p_i. $\qquad\square$

Theorem 7.5. *This non-uniqueness is non-trivial.*

Proof. We give one example. Let M be a pipe, so that $\check{M} = S^2$ with 0 and ∞ marked. Let $z_0 = z$ and $z_\infty = 1/z$. Let E be a holomorphic line bundle over M with $c_0 = c_\infty = 1$. Suppose $\mathbf{1}F$ is a frame for $E{\downarrow}M^\circ$ of the sort we are considering. Let $\widetilde{\mathbf{1}} = \mathbf{1}e^{(1-\phi_0-\phi_\infty)\log x + iy}$. This equals $\mathbf{1}z$ away from the boundary, and so is holomorphic there (since $\mathbf{1}$ was). Then

$$\widetilde{\mathbf{1}}Fe^{\phi_0 \log x_0 - \phi_\infty \log x_\infty} = \widetilde{\mathbf{1}}Fe^{(\phi_0+\phi_\infty)\log x} = \mathbf{1}Fz.$$

This is a different frame from $\mathbf{1}F$, and it is of the same sort. $\qquad\square$

We can now move toward a classification. From now on, a holomorphic frame for $E{\downarrow}M^\circ$ that can be written as $\mathbf{1}e^{u+v+w+f}$ as in the above discussion will be called a *good* frame for E. We assume throughout that M is a fixed b-holomorphic curve with fixed first integrals $z_1, \ldots, z_k$.

Note 7.6. Let us examine the quotient of two good frames, $\widetilde{\mathbf{1}}e^{\widetilde{u}+\widetilde{v}+\widetilde{w}+\widetilde{f}}/\mathbf{1}e^{u+v+w+f}$. For simplicity of notation, we define

$$Q = \widetilde{\mathbf{1}}/\mathbf{1}$$
$$U = \widetilde{u} - u$$
$$V = \widetilde{v} - v$$
$$W = \widetilde{w} - w$$
$$F = \widetilde{f} - f.$$

Then the quotient may be written as $Qe^{U+V+W+F}$, and the properties of the functions

Q, U, V, W, F are as follows (each entry in the list is smooth on M°):

Q non-vanishing; holomorphic away from ∂M; smooth at $\mathcal{C}_i$

U smooth at $\mathcal{C}_i$

V 0 near p_i, or meromorphic near p_i with a pole at p_i

W $= \mathrm{const} \cdot \log x_i$ near p_i

$$F \begin{cases} (\text{asymptotically holomorphic at } p_i \text{ and tending to 0 at } p_i) \cdot \log x_i & c_i = 1 \\ \text{asymptotically holomorphic at } p_i \text{ and tending to 0 at } p_i & c_i \notin \mathbb{Q} \end{cases}$$

(7.1)

Proposition 7.7. *Let m be a meromorphic function on $\check{M}$ whose poles and zeros occur only at the p_i. Write $m = z_i^{d_i} e^{\tau_i + \nu_i}$ near p_i, where d_i is an integer, τ_i is a constant, and ν_i is a holomorphic function with value zero at p_i. (This decomposition is unique up to a choice of τ_i.)*

Then m may be written in the form $Q e^{U+V+W+F}$ by choosing cut-off functions ϕ_i supported near p_i and setting

$$Q = m e^{-\sum \phi_i (c_i d_i \log x_i + \tau_i + \nu_i)}$$

$$U = \begin{cases} \phi_i \tau_i & c_i \notin \mathbb{Q} \\ \phi_i(\tau_i + \nu_i) & c_i = 1 \end{cases}$$

$$V = 0$$

$$W = \sum \phi_i c_i d_i \log x_i$$

$$F = \begin{cases} \phi_i \nu_i & c_i \notin \mathbb{Q} \\ 0 & c_i = 1. \end{cases}$$

This decomposition is unique up to a choice of ϕ_i.

We omit the proof; it is elementary.

Definition 7.8. Let $\{a_i^n\}_{\substack{i=1\dots k \\ n \geq 0}}$ be a group of sequences of complex numbers. We say that $\{a_i^n\} \sim 0$ if $a_i^n = 0 \, \forall n \geq 1$ when $c_i = 1$, and there exists a meromorphic function on $\check{M}$ whose zeros and poles occur only at the p_i, such that

1. When $c_i = 1$, $\mathrm{order}_i(m) = a_i^0 / c_i$; and

2. When $c_i \notin \mathbb{Q}$, $m = (\mathrm{const}) \cdot z_i^{a_i^0/c_i} \cdot e^{\sum_{n \geq 1} a_i^n z_i^n}$.

This is an equivalence relation. We denote the class of $\{a_i^n\}$ by $[a_i^n]$.

Theorem 7.9. *If* $\{a_i^n\}$ *is associated to a good frame for a line bundle which is trivial, then* $[a_i^n] = 0$.

Proof. By hypothesis, there exists a global C^∞ holomorphic frame $\widetilde{\mathbf{1}}$; and there also exists a good frame $\mathbf{1}e^{u+v+w+f}$ with the $\{a_i^n\}$ as the associated sequences. This means that $\mathbf{1}$ is a C^∞ frame that is holomorphic away from the boundary, and

1. $\mathbf{1}e^{u+v+w+f}$ is a holomorphic frame for $E{\lfloor}M^\circ$;

2. $w = a_i^0 \log x_i$ near $\mathcal{C}_i$; and

3. When $c_i \notin \mathbb{Q}$, $f \sim \sum_{n\geq 1} a_i^n z_i^n$ at $\mathcal{C}_i$; and when $c_i = 1$, $f/\log x_i \sim \sum_{n\geq 1} a_i^n z_i^n$ at $\mathcal{C}_i$.

Taking note of the fact that $\kappa(E) = 0$, we find immediately that $v = 0$.

Set $Q = 1/\widetilde{\mathbf{1}}$. Then define $m = Qe^{u+w+f}$ to be the quotient of our two frames. Obviously, $\widetilde{\mathbf{1}}$ is a good frame. So m is meromorphic on $\check{M}$ with zeros and poles only at the p_i, by Theorem 7.4. So by Proposition 7.7 above,

$$Q = me^{-\sum \phi_i(c_i d_i \log x_i + \tau_i + \nu_i)}$$

$$u = \begin{cases} \phi_i \tau_i & c_i \notin \mathbb{Q} \\ \phi_i(\tau_i + \nu_i) & c_i = 1 \end{cases}$$

$$w = \sum \phi_i c_i d_i \log x_i$$

$$f = \begin{cases} \phi_i \nu_i & c_i \notin \mathbb{Q} \\ 0 & c_i = 1. \end{cases}$$

where τ_i and ν_i are defined by the relation $m = z_i^{d_i} e^{\tau_i + \nu_i}$.

Compare these equations to the description of u, w, f just prior. We find that

1. $a_i^n = 0 \, \forall \, n \geq 1$ when $c_i = 1$;

2. $a_i^0 = c_i d_i$; and

3. $\sum_{n\geq 1} a_i^n z_i^n = \nu_i$ at $\mathcal{C}_i$.

So m is the meromorphic function whose existence figures in Definition 7.8, and $\{a_i^n\} \sim 0$. $\qquad\square$

Theorem 7.10. *Let E be a holomorphic line bundle over M. Then for E to possess a global C^∞ holomorphic frame, it is necessary and sufficient that $\kappa(E) = 0$ and $[a_i^n] = 0$.*

Proof. We only need to prove the sufficiency. So choose an η chain basis, and define the b_i and thus the v. Construct the good frame $\mathbf{1}e^{u+v+w+f}$. Since $\kappa(E) = 0$, we know that $v = 0$.

We know that $a_i^n = 0 \,\forall\, n \geq 1$ when $c_i = 1$. We also are guaranteed the existence of a meromorphic function m on $\check{M}$ having the properties stated in Definition 7.8. So $\mathbf{1}e^{u+w+f} \cdot m^{-1}$ is a holomorphic frame for $E\lfloor M^\circ$. We wish to show that this frame is smooth at the boundary. So write m as Qe^{U+W+F} as in Proposition 7.7, where

$$Q = me^{-\sum \phi_i(a_i^0 \log x_i + \tau_i + \nu_i)}$$

$$U = \begin{cases} \phi_i \tau_i & c_i \notin \mathbb{Q} \\ \phi_i(\tau_i + \nu_i) & c_i = 1 \end{cases}$$

$$W = \sum \phi_i a_i^0 \log x_i$$

$$F = \begin{cases} \phi_i \nu_i & c_i \notin \mathbb{Q} \\ 0 & c_i = 1. \end{cases}$$

Then write our holomorphic frame as

$$(\mathbf{1}m^{-1}e^{\sum \phi_i(a_i^0 \log x_i + \tau_i + \nu_i)})e^{(u-U)+(w-W)+(f-F)}.$$

This is smooth at the boundary. $\qquad\square$

Corollary 7.11. *Let E and F be holomorphic line bundles over M. Then for E and F to be equivalent, it is necessary and sufficient that $\kappa(E) = \kappa(F)$ and $[a_i^n](E) = [a_i^n](F)$.*

Theorem 7.12. *For every $\kappa \in H_{\bar{\partial}}^{0,1}\check{M}$ and group of sequences $\{a_i^n\}$, there's a holomorphic line bundle over M whose invariants these are.*

This completes the classification of bundles over an M with fixed first integrals.

7.3 Examples of bundles

We will now study lifts of bundles from $\check{M}$, which are defined by analogy to the case of the cup. (See Sections 4.3 and 5.3.)

Definition 7.13. Let E be a holomorphic line bundle over $\check{M}$. Define $\{i_1, \ldots, i_\ell\} = \{i : c_i \notin \mathbb{Q}\}$. Choose local holomorphic frames ν_{i_r} near p_i. We define the *lift of E by means of* $\nu_{i_1}, \ldots, \nu_{i_\ell}$, which we denote by $\widehat{E}(\nu_{i_1}, \ldots, \nu_{i_\ell})$, as follows. As a topological line bundle, it is the pull back of E from $\check{M}$ to M via the continuous map π. The smooth and holomorphic structures for $E{\downarrow}M^\circ$ are defined via pull back by π as well. For i such that $c_i = 1$, a section is smooth at $\mathcal{C}_i$ if and only if its quotient by the pull-back of any local frame for E near p_i is smooth at $\mathcal{C}_i$; and a section is smooth at $\mathcal{C}_{i_r}$ if and only if its quotient by ν_{i_r} is smooth at $\mathcal{C}_{i_r}$.

Note 7.14. The only questionable part of this definition is the condition for smoothness at $\mathcal{C}_i$ when $c_i = 1$; but by the same arguments as in the case of the cup, this condition is self-consistent.

Before we begin to examine the properties of lifts, we need the following proposition and theorem.

Proposition 7.15. *Let M be a b-holomorphic curve. Then the quantity*

$$d = \sum_{i=1}^k \frac{a_i^0}{c_i}$$

is a bundle invariant.

Proof. If E and F are equivalent as holomorphic line bundles over M, then $[a_i^n](E) \sim [a_i^n](F)$ by Theorem 7.10. By the definition of this equivalence (Definition 7.8), there exists a meromorphic function m on $\check{M}$ with poles and zeros only at the p_i; and the order of m at p_i is $a_i^0(E)/c_i - a_i^0(F)/c_i$. So

$$0 = \sum_{i=1}^k \left(\frac{a_i^0(E)}{c_i} - \frac{a_i^0(F)}{c_i} \right)$$
$$= \sum_{i=1}^k \frac{a_i^0(E)}{c_i} - \sum_{i=1}^k \frac{a_i^0(F)}{c_i}.$$

$\square$

Theorem 7.16. *Let E be a C^∞ line bundle over the marked compact Riemann surface $\check{M}(p_1, \ldots, p_k)$. Then for $\mathrm{degree}(E) = d$ it is necessary and sufficient that there exist integers $d_1, \ldots, d_k$ with $\sum d_i = d$ and local C^∞ frames τ_i near p_i and a C^∞ frame σ for $E{\downarrow}(\check{M} \backslash \{p_1, \ldots, p_k\})$ such that: for each i, the winding number of σ/τ_i around zero on a small oriented loop around p_i is d_i.*

Proof of sufficiency. Define ∇_σ on $E \!\restriction\! (\check{M} \backslash \{p_1, \ldots, p_k\})$ by $\nabla_\sigma \sigma = 0$. Define ∇_{τ_i} on $E \!\restriction\! \text{neigh}(p_i)$ by $\nabla_{\tau_i} \tau_i = 0$. Then choose cut-off functions ϕ_i supported near p_i, and define ∇ on E by

$$\nabla = \sum \phi_i \nabla_{\tau_i} + \left(1 - \sum \phi_i\right) \nabla_\sigma.$$

Define $g_i = \sigma / \tau_i$ near p_i. We compute the connection form ω with respect to σ as follows:

$$\begin{aligned}
\nabla \sigma &= \sum \phi_i \nabla_{\tau_i} \sigma \\
&= \sum \phi_i \nabla_{\tau_i} (g_i \tau_i) \\
&= \sum \phi_i (dg_i) \tau_i \\
&= \sum \phi_i (dg_i / g_i) \sigma.
\end{aligned}$$

So $\omega = \sum \phi_i dg_i / g_i$.

Let $B_R = \cup B_R(p_i)$, and write $\partial B_R = \cup \gamma_R(p_i)$ where $\gamma_R(p_i) = \partial B_R(p_i)$. We have

$$\begin{aligned}
\text{degree}(E) &= (i/2\pi) \lim_{R \to 0} \int_{B_R^c} d\omega \\
&= (i/2\pi) \lim_{R \to 0} \int_{\partial B_R^c} \omega \\
&= (1/2\pi i) \lim_{R \to 0} \int_{\partial B_R} \omega \\
&= (1/2\pi i) \lim_{R \to 0} \int_{\cup \gamma_R(p_i)} \omega \\
&= \lim_{R \to 0} \sum (1/2\pi i) \int_{\gamma_R(p_i)} dg_i / g_i \\
&= \sum d_i \\
&= d.
\end{aligned}$$

$\square$

Proof of necessity. Choose local C^∞ frames τ_i for E near p_i and let σ be a C^∞ frame for $E \!\restriction\! (\check{M} \backslash \{p_1, \ldots, p_k\})$. Define $d_i = $ winding number of σ / τ_i around zero on a small oriented loop around p_i. Now compute the degree of E, just as we did in the proof of sufficiency. We find that $d = \text{degree}(E) = \sum d_i$. $\square$

Now we are ready to turn our attention back to lifts. From now on, M is a b-holomorphic curve. The following theorem shows that the bundle invariant d is a generalization of "degree."

Theorem 7.17. *Let E be a holomorphic line bundle over $\check{M}$, and let $\nu_{i_1}, \ldots, \nu_{i_\ell}$ be local holomorphic frames near $p_{i_1}, \ldots, p_{i_\ell}$. (Here the indices are those for which $c_{i_r} \notin \mathbb{Q}$.) Then*

$$d(\widehat{E}(\nu_{i_1}, \ldots, \nu_{i_\ell})) = \text{degree}(E).$$

Proof. We will construct some frames τ_i and σ for E as in Theorem 7.16. Then we will use these to construct a good frame for $\widehat{E} = \widehat{E}(\nu_{i_1}, \ldots, \nu_{i_\ell})$, from which we will be able to read off the a_i^0. This will yield the desired equality.

So take $\tau_{i_r} = \nu_{i_r}$ for each $r = 1 \ldots \ell$. We are free to choose the τ in this way. We make no stipulation on τ_i when $c_i = 1$. Also, take σ to be holomorphic away from the p_i. Choose first integrals for M and define $d_i = \text{w.n.}(\sigma/\tau_i) \, \forall \, i$. Then $\text{w.n.}(z_i^{-d_i}\sigma/\tau_i) = 0$. So we can write $z_i^{-d_i}\sigma/\tau_i = e^{u_i}$ in a punctured neighborhood of p_i, with u_i smooth. Now define $\widetilde{\sigma} = \sigma e^{-\sum \phi_i u_i}$ where the ϕ_i are cut-off functions supported near the p_i.

Regard τ_i and $\widetilde{\sigma}$ as frames for $\widehat{E}$. Notice that

(a) The τ_i are smooth holomorphic frames for $\widehat{E} \lfloor \text{collar}_i$.

(b) $\widetilde{\sigma}$ is a smooth frame for $E \lfloor M^\circ$ that is holomorphic away from ∂M.

(c) $\widetilde{\sigma} = \tau_i z_i^{d_i}$ near $\mathcal{C}_i$.

Now let α be the $\bar{\partial}$ form of $\widehat{E}$ with respect to $\widetilde{\sigma}$. Since $\widetilde{\sigma}$ is holomorphic near $\mathcal{C}_i$, $\alpha = 0$ in a neighborhood of $\mathcal{C}_i$. So α may be regarded as a smooth $(0,1)$-form on $\check{M}$. Choose an η chain basis A for $H_{\bar{\partial}}^{0,1}\check{M}$ and write $-\alpha \sim b_1 {}^1\!A + \cdots + b_g {}^g\!A$. Then choose a smooth g on $\check{M}$ such that $-\alpha = b_1 {}^1\!A + \cdots + b_g {}^g\!A + \bar{\partial}g$. Let $v = \text{func}(b_1 {}^1\!A + \cdots + b_g {}^g\!A)$. Then $\bar{\partial}(v+g) = -\alpha$. Therefore $\widetilde{\sigma}e^{v+g}$ is a holomorphic frame for $\widehat{E}\lfloor M^\circ$. We need to express $\widetilde{\sigma}e^{v+g}$ in the form of a good frame. So it suffices to write $\widetilde{\sigma}e^g$ in the form $\mathbf{1}e^{u+w+f}$, where $w = \sum \phi_i c_i d_i \log x_i$; for then

$$d(\widehat{E}) = \sum d_i = \text{degree}(E).$$

All this means that it suffices to show that $\widetilde{\sigma}e^{g-w}$ can be written as $\mathbf{1}e^{u+f}$, with

1. $\mathbf{1}$ a global smooth frame for $\widehat{E}$ which is holomorphic away from the boundary;

2. u a smooth function on M; and

3. At $\mathcal{C}_i$,

$$f \sim \begin{cases} \sum_{n \geq 1} a_i^n z_i^n & c_i \notin \mathbb{Q} \\ \left(\sum_{n \geq 1} a_i^n z_i^n \right) \log x_i & c_i = 1. \end{cases}$$

Note that g is smooth on $\check{M}$, so that when $c_i = 1$, g is smooth at $\mathcal{C}_i$, and when $c_i \notin \mathbb{Q}$, $g \sim g_i^0+$ a sum in positive powers of z_i. Define $\mathbf{1} = \widetilde{\sigma}e^{-w}$, $u = g - \sum_{c_i \notin \mathbb{Q}} \phi_i(g - g_i^0)$, and $f = \sum_{c_i \notin \mathbb{Q}} \phi_i(g - g_i^0)$. Clearly $\widetilde{\sigma}e^{g-w} = \mathbf{1}e^{u+f}$. It's easy to see that u and f have the correct properties by our construction; that is, u is smooth on M, and f is zero near $\mathcal{C}_i$ when $c_i = 1$ and is asymptotic to a series in positive powers of z_i at $\mathcal{C}_i$ when $c_i \notin \mathbb{Q}$. Finally, near $\mathcal{C}_i$,

$$\mathbf{1} = \widetilde{\sigma}e^{-w}$$
$$= \tau_i z_i^{d_i} e^{-c_i d_i \log x_i}$$
$$= \tau_i x_i^{c_i d_i} e^{id_i y_i} x_i^{-c_i d_i}$$
$$= \tau_i e^{id_i y_i},$$

which is smooth at $\mathcal{C}_i$. $\qquad\square$

Corollary 7.18. *If $d(E) \notin \mathbb{Z}$, then E is not isomorphic to a lift.*

We finish this discussion by examining the holomorphic line bundle $^b T^{1,0} M$. In particular, we wish to calculate d for this bundle.

Theorem 7.19. $\mathrm{degree}(T^{1,0}\check{M}) = 2 - 2g$.

This theorem is well known, and we omit the proof.

Definition 7.20. From $T^{1,0}\check{M}$, we construct a new holomorphic line bundle $\widetilde{T}^{1,0}\check{M}$ over $\check{M}$ as follows. For $p \in \check{M}$,

$$\text{fiber over } p = \begin{cases} \{\text{germs of holomorphic sections} \\ \text{of } T^{1,0}\check{M} \text{ at } p\} \text{ modulo} \\ \text{equality (to zeroth order) at } p & p \notin \{p_1, \ldots, p_k\} \\ \{\text{germs of holomorphic sections} \\ \text{of } T^{1,0}\check{M} \text{ at } p \text{ which vanish at } p\} \text{ modulo} \\ \text{equality to first order at } p & p \in \{p_1, \ldots, p_k\}. \end{cases}$$

The union of these fibers has the structure of a holomorphic line bundle (defined naturally), and there is a natural map $\pi : \widetilde{T}^{1,0}\check{M} \longrightarrow T^{1,0}\check{M}$ whose restriction to the part lying over the complement of the p_i is an isomorphism.

Theorem 7.21. $\mathrm{degree}(\widetilde{T}^{1,0}\check{M}) = 2 - 2g - k$.

Proof. Choose (local) frames τ_i and σ for $T^{1,0}\check{M}$ as in Theorem 7.16. Then the winding numbers of σ/τ_i, which we call d_i, satisfy $\sum d_i = 2 - 2g$.

σ may be regarded as a C^∞ frame for $\widetilde{T}^{1,0}\check{M}$ over $\check{M}\backslash\{p_1, \ldots, p_k\}$, via the map π. Also, $\widetilde{\tau}_i = z_i\tau_i$ are local C^∞ frames for $\widetilde{T}^{1,0}\check{M}$ near $\mathfrak{p}_i$. Clearly,

$$\widetilde{d}_i = \mathrm{w.\,n.}(\sigma/\widetilde{\tau}_i) = \mathrm{w.\,n.}(z_i^{-1}\sigma/\tau_i) = \mathrm{w.\,n.}(\sigma/\tau_i) - 1 = d_i - 1.$$

So $\widetilde{d} \equiv \sum \widetilde{d}_i = \sum(d_i - 1) = \sum d_i - k = 2 - 2g - k$. And by the Theorem 7.16, $\widetilde{d}$ is the degree of $\widetilde{T}^{1,0}\check{M}$. $\qquad\square$

Theorem 7.22. *If $z_{i_1}, \ldots, z_{i_\ell}$ are first integrals for the collars of M that have non-rational c, then*

$$^bT^{1,0}M \simeq \widehat{\widetilde{T}^{1,0}\check{M}}(z_{i_1}\partial_{z_{i_1}}, \ldots, z_{i_\ell}\partial_{z_{i_\ell}}).$$

This theorem follows from the various definitions, so we omit the proof.

Corollary 7.23. $d(^bT^{1,0}M) = 2 - 2g - k$.

This follows from Theorems 7.17, 7.21, and 7.22. Note that this agrees with the formula $d(^bT^{1,0}M) = 1$ for M a cup, which we computed directly in Sections 4.3 and 5.3.

7.4   *b*-connections

This section parallels the developments of Section 4.4, where we discussed b-connections on bundles over a cup.

Theorem 7.24. *Let E be a holomorphic line bundle over M. Then every hermitian metric on E determines a unique hermitian holomorphic b-connection.*

Proof. This theorem is identical to the one in Section 4.4, and so is the proof. Note that, as before, we have the formula $\omega = \alpha - \bar{\alpha} + \partial p$ for the connection form with respect to s of the hermitian holomorphic b-connection induced by the metric $\langle s, s \rangle = e^p$. Therefore the curvature is $d\omega = \partial\alpha - \bar{\partial}\bar{\alpha} + \bar{\partial}\partial p$. $\qquad\square$

Definition 7.25. A b-connection is called *non-singular* if its connection form with respect to a smooth frame is a non-singular 1-form. A b-connection is called *smoothly curved* if its curvature is a non-singular 2-form.

Theorem 7.26. *Let E be a holomorphic line bundle over M. Then the space of smoothly curved hermitian holomorphic b-connections on E is an affine space, with the underlying vector space being the space of real C^∞ functions on M which are locally constant on ∂M.*

Proof. First, we will show that one such b-connection exists. So choose a global C^∞ frame s which is holomorphic away from the boundary and let α be the corresponding $\bar{\partial}$ form. What we seek is a real C^∞ function p on M such that

$$\partial\alpha - \bar{\partial}\bar{\alpha} + \bar{\partial}\partial p$$

is non-singular as a 2-form. (In the proof of Theorem 7.24, we saw that this is the curvature of the hermitian holomorphic b-connection induced by the metric $\langle s, s \rangle = e^p$.) Choose first integrals $z_1, \ldots, z_k$ for the collars of M. Write $\alpha = f_i \bar{\lambda}_i$ near $\mathcal{C}_i$. Then our curvature 2-form may be written near $\mathcal{C}_i$ as

$$\partial(f_i\bar{\lambda}_i) - \bar{\partial}(\bar{f}_i\lambda_i) + \bar{\partial}(L_i p \lambda_i) = (L_i f_i + \bar{L}_i \bar{f}_i - \bar{L}_i L_i p)\lambda_i \wedge \bar{\lambda}_i.$$

We have used the fact that $\partial\bar{\lambda}_i = \bar{\partial}\lambda_i = 0$, which is true because λ_i and $\bar{\lambda}_i$ are locally exact. For example, $\lambda_i = (1/2a_i)d(c_i \log x_i + iy_i)$.

It may be computed that $\lambda_i \wedge \bar{\lambda}_i = -(i/2a_i)\frac{dx_i}{x_i} \wedge dy_i$, so for our 2-form to be non-singular we must have $\bar{L}_i L_i p = L_i f_i + \bar{L}_i \bar{f}_i$ on $\mathcal{C}_i$, for each i. We can compute that

$$\bar{L}_i L_i p = x_i p_{x_i} + x_i^2 p_{x_i x_i} - 2b_i x_i p_{x_i y_i} + |c_i|^2 p_{y_i y_i}.$$

So we need to choose the smooth function p so that

$$|c_i|^2 p_{y_i y_i} = -i\bar{c}_i(f_i)_{y_i} + ic_i(\bar{f}_i)_{y_i},$$

or

$$p_{y_i y_i} = 2\Im(f_i/c_i)_{y_i}.$$

We will now produce such a p. Let $a_i^0, a_i^1, \ldots$ be the bundle invariants of Section 7.2, computed with respect to our chosen frame s and first integrals $z_1, \ldots, z_k$. Note that, by definition,

$$\begin{aligned}
a_i^0 &= -\frac{a_i i}{\pi} \int_{\mathcal{C}_i} \alpha \\
&= -\frac{a_i i}{\pi} \int_{\mathcal{C}_i} f_i \bar{\lambda}_i \\
&= -\frac{a_i i}{\pi} \int_{\mathcal{C}_i} f_i \frac{1}{2a_i}\left(\bar{c}_i \frac{dx_i}{x_i} - i\,dy_i\right) \\
&= -\frac{1}{2\pi} \int_{\mathcal{C}_i} f_i\,dy_i.
\end{aligned}$$

Let

$$H_i = \Im(a_i^0/c_i) = -\frac{1}{2\pi} \int_{\mathcal{C}_i} \Im(f_i/c_i)\, dy_i.$$

Since the average value of $\Im(f_i/c_i) + H_i$ over $\mathcal{C}_i$ is zero, we may choose a smooth function φ_i on $\mathcal{C}_i$ such that

$$(\varphi_i)_{y_i} = \Im(f_i/c_i) + H_i.$$

Finally, let p be a real C^∞ function on M which agrees with $2\varphi_i$ on $\mathcal{C}_i$ for each i. Then p has the desired property. This shows the existence of a smoothly curved hermitian holomorphic b-connection on E.

Next, let u be a real smooth function on M which is constant on $\mathcal{C}_i$ for each i. The metric above was given by $\langle s, s \rangle = e^p$; define a new metric by $\langle s, s \rangle = e^{p+u}$. Then the new curvature (that is, the curvature of the hermitian holomorphic b-connection induced by the new metric) is $\partial\alpha - \bar{\partial}\bar{\alpha} + \bar{\partial}\partial p + \bar{\partial}\partial u$, which is the old curvature plus $\bar{\partial}\partial u$. Up to a nonzero multiplicative constant, $\bar{\partial}\partial u$ is equal (near $\mathcal{C}_i$) to

$$(\bar{L}_i L_i u)\lambda_i \wedge \bar{\lambda}_i = (x_i u_{x_i} + x_i^2 u_{x_i x_i} - 2b_i x_i u_{x_i y_i} + |c_i|^2 u_{y_i y_i}) \left(\frac{dx_i}{x_i} \wedge dy_i \right),$$

which is smooth up to $\mathcal{C}_i$ as a non-singular 2-form since $u_{y_i y_i}$ vanishes on $\mathcal{C}_i$. So, since the old curvature was non-singular and $\bar{\partial}\partial u$ is non-singular, the new connection is also smoothly curved.

Finally, it only remains to show that the quotient of any two metrics that induce smoothly curved hermitian holomorphic b-connections is of the form e^u, where u is a real smooth function on M which is locally constant on ∂M. This is essentially the same calculation as that of the previous paragraph. $\qquad\square$

Definition 7.27. Let E be a holomorphic line bundle over M. Let c_i be the collar invariants of M and $\kappa, [a_i^n]$ the bundle invariants of E. Then we may define new bundle invariants $\gamma_1, \ldots, \gamma_k \in \mathbb{C}/\mathbb{Z}$ and $\gamma \in \mathbb{C}$ by

$$\gamma_i = \frac{\Re a_i^n}{\Re c_i}$$

$$\gamma = \gamma_1 + \cdots + \gamma_k.$$

We need to check that these definitions make sense; that is, we have to check that the γ_i and γ come out the same if they are computed by means of two sequence groups $\{\tilde{a}_i^n\}$ and $\{a_i^n\}$ that represent the same bundle. When this is the case, we know that there exists

a meromorphic function m on $\check{M}$ with zeros and poles only at the p_i, such that

$$\text{order}_i(m) = \frac{\widetilde{a}_i^0 - a_i^0}{c_i}.$$

Therefore we know that $(\widetilde{a}_i^0 - a_i^0)/c_i$ is an integer for each i, and $\sum(\widetilde{a}_i^0 - a_i^0)/c_i = 0$.

Proof of invariance of γ_i. Since $(\widetilde{a}_i^0 - a_i^0)/c_i$ is real, it is equal to $\Re(\widetilde{a}_i^0 - a_i^0)/\Re c_i$; this is seen by examining similar triangles. Therefore $\widetilde{\gamma}_i - \gamma_i \in \mathbb{Z}$. □

Proof of invariance of γ. Just compute that

$$
\begin{aligned}
\widetilde{\gamma} - \gamma &= \sum_{i=1}^{k} \frac{\Re\widetilde{a}_i^0 - \Re a_i^0}{\Re c_i} \\
&= \sum_{i=1}^{k} \frac{\Re(\widetilde{a}_i^0 - a_i^0)}{\Re c_i} \\
&= \sum_{i=1}^{k} \frac{\widetilde{a}_i^0 - a_i^0}{c_i} \\
&= 0.
\end{aligned}
$$

□

Theorem 7.28. *Let E be a holomorphic line bundle over M. Then for any smoothly curved hermitian holomorphic b-connection ∇,*

$$\int_M R(\nabla) = -2\pi i\gamma.$$

Proof. Let s be a global C^∞ frame for E. Let α be the corresponding $\bar{\partial}$ form. Then let p be a real C^∞ function on M such that the metric defined by $\langle s, s \rangle = e^p$ induces a smoothly curved (hermitian) holomorphic b-connection ∇. Then the connection form with respect to s is

$$\omega = \alpha - \bar{\alpha} + \partial p,$$

as we have seen. We wish to compute the integral over M of $R(\nabla) = d\omega$. We know that $d\omega$ is a non-singular 2-form on M, but ω might be singular. So to compute the integral over M, we will take the limit as $r \to 0$ of the integral over the complement of the region

$$\bigcup_{i=1}^{k} \{x_i \leq r\}.$$

For any fixed $r > 0$, the integral may be computed via Stokes's theorem. So we will find that

$$\int_M R(\nabla) = \int_M d\omega = -\lim_{r \to 0} \sum_{i=1}^{k} \int_{x_i = r} \omega.$$

(The minus sign appears because of our orientation convention.) So we have to examine the form of ω on circles of constant x_i near $\mathcal{C}_i$.

Write $\alpha = f_i \bar{\lambda}_i$ near $\mathcal{C}_i$. Then we compute, using the formula for ω above, that

$$
\begin{aligned}
2a_i \omega &= f_i(2a_i \bar{\lambda}_i) - \bar{f}_i(2a_i \lambda_i) + L_i p(2a_i \lambda_i) \\
&= f_i \left(\bar{c}_i \frac{dx_i}{x_i} - i\, dy_i \right) + (L_i p - \bar{f}_i) \left(c_i \frac{dx_i}{x_i} + i\, dy_i \right) \\
&= (\bar{c}_i f_i - c_i \bar{f}_i + c_i L_i p) \frac{dx_i}{x_i} + (-if_i - i\bar{f}_i + iL_i p) dy_i \\
&= |c_i|^2 ([f_i/c_i] - [\bar{f}_i/\bar{c}_i] + [1/\bar{c}_i](x_i p_{x_i} - i\bar{c}_i p_{y_i})) \frac{dx_i}{x_i} - i(f_i + \bar{f}_i - (x_i p_{x_i} - i\bar{c}_i p_{y_i})) dy_i \\
&= |c_i|^2 (i[2\Im(f_i/c_i) - p_{y_i}] + (1/\bar{c}_i) x_i p_{x_i}) \frac{dx_i}{x_i} - i(2\Re f_i - x_i p_{x_i} + i\bar{c}_i p_{y_i}) dy_i.
\end{aligned}
$$

So we can compute that

$$\int_{x_i = r} \omega = \frac{-i}{a_i} \int_{x_i = r} \Re f_i \, dy_i + \frac{ir}{2a_i} \int_{x_i = r} p_{x_i} \, dy_i,$$

or

$$-\int_{x_i = r} \omega = \frac{i}{a_i} \int_{x_i = r} \Re f_i \, dy_i - \frac{ir}{2a_i} \int_{x_i = r} p_{x_i} \, dy_i.$$

All the dx_i terms of the integrand disappear when we pull back to the circle of constant x_i, and the term involving p_{y_i} clearly integrates to zero. Since $p_{x_i} dy_i$ defines a smooth 1-form on M, the second term is killed by the coefficient r in the limit. So the only thing that survives in the limit is

$$
\begin{aligned}
\frac{i}{a_i} \int_{\mathcal{C}_i} \Re f_i \, dy_i &= \frac{-2\pi i}{a_i} \Re \left(\frac{-1}{2\pi} \int_{\mathcal{C}_i} f_i \, dy_i \right) \\
&= -2\pi i \, \Re a_i^0 / \Re c_i \\
&= -2\pi i \gamma_i.
\end{aligned}
$$

The final step, taking the sum of these integrals over all the $\mathcal{C}_i$, gives us $\sum (-2\pi i \gamma_i) = -2\pi i \gamma$. $\square$

Theorem 7.29. *If a_i^0 / c_i is real for every i, then every smoothly curved hermitian holomorphic b-connection is actually non-singular. Otherwise, no such b-connection is non-singular.*

Proof. Differentiate the last formula for $2a_i\omega$ in the proof of the previous theorem. You find that, near $\mathcal{C}_i$,

$$2a_i\,d\omega = |c_i|^2(i[2\Im(f_i/c_i)_{y_i} - p_{y_iy_i}] + (1/\bar{c}_i)x_ip_{x_iy_i})dy_i \wedge \frac{dx_i}{x_i}$$

plus something non-singular. We know that this form is non-singular because the connection is assumed to be smoothly curved. So we must have

$$p_{y_iy_i} = 2\Im(f_i/c_i)_{y_i}$$

on $\mathcal{C}_i$. Thus $p_{y_i} = 2\Im(f_i/c_i) + H_i$ for some constant H_i. If we average both sides over $\mathcal{C}_i$ (with respect to the measure dy_i), then we find that H_i is -2 times the average of $\Im(f_i/c_i)$ over the boundary, or

$$H_i = 2\Im(a_i^0/c_i).$$

Now we can finish the argument:

The connection form is non-singular iff $p_{y_i} = 2\Im(f_i/c_i)$ for all i. (This is by inspecting the last formula for $2a_i\omega$ in the proof of the previous theorem.) But, $p_{y_i} = 2\Im(f_i/c_i)$ iff $H_i = 0$. (By the above remarks.) And, since $H_i = 2\Im(a_i^0/c_i)$, $H_i = 0$ iff a_i^0/c_i is real. $\square$

Let us compare these results with the analogous results for connections on a bundle over a compact Riemann surface. In that case, the integral of the curvature of any connection is $-2\pi i \cdot \gamma$ where γ is the degree. In the present situation, the generalized degree is

$$d = \frac{a_1^0}{c_1} + \cdots + \frac{a_k^0}{c_k},$$

and the integral of the curvature of any hermitian holomorphic b-connection is $-2\pi i \cdot \gamma$ where

$$\gamma = \frac{\Re a_1^0}{\Re c_1} + \cdots + \frac{\Re a_k^0}{\Re c_k}.$$

So we can say that, in the case of the compact Riemann surface, γ (defined as the number that appears in the formula for the integral of the curvature) is equal to the degree, which is an integer; but in the case of a b-holomorphic complex curve, γ is some sort of a "real version" of the degree, which may be any complex number.

7.5 Connections of constant curvature

In order to prosecute the analyses of this section in full generality, we would need to understand in some detail the spaces of real harmonic functions on the blow-down, both

those without singularities and those with singularities of specified types at the distinguished points. Such an investigation would be appropriate here, and would constitute a second "interlude" on compact Riemann surfaces. But it would be a considerable expenditure of time and ink.

We therefore restrict our attention in the remainder to the case of b-holomorphic complex curves M whose blow-down $\check{M}$ is the sphere. On a sphere, the only harmonic functions are the constants. Note also that, since $H_{\bar{\partial}}^{0,1} S^2 = 0$, the κ bundle invariant disappears. That is, $\kappa(E) = 0$ for every holomorphic line bundle E over M.

From now on, M is a geometric b-holomorphic complex curve whose blow-down $\check{M}$ is the Riemann sphere, and all of whose collar invariants lie in $(\mathbb{C}\backslash\mathbb{Q}) \cup 1$.

Choose first integrals $z_1, \ldots, z_k$. Then we have $x_i, y_i, L_i, \bar{L}_i, \lambda_i, \bar{\lambda}_i$, as in (2.2) and (2.3). Define

$$\mathcal{L}_i = \bar{L}_i L_i = \begin{pmatrix} x_i \partial_{x_i} & \partial_{y_i} \end{pmatrix} \begin{pmatrix} 1 & -b_i \\ -b_i & |c_i|^2 \end{pmatrix} \begin{pmatrix} x_i \partial_{x_i} \\ \partial_{y_i} \end{pmatrix} \tag{7.2}$$

and $h_i = \langle L_i, L_i \rangle$. Then h_i vanishes to second order on $\mathcal{C}_i$. We have the formula

$$\bar{\partial}\partial = \star \frac{2i}{h_i} \mathcal{L}_i \tag{7.3}$$

valid near $\mathcal{C}_i$.

Technical Lemma 7.30. *Let u be a smooth function on the interior of M. Suppose $x_i u_{x_i}$ extends to $\mathcal{C}_i$ as a continuous function for each i, and $\bar{\partial}\partial u$ extends as a continuous non-singular 2-form on M. Then*

$$\int_M \bar{\partial}\partial u = \sum_{i=1}^k \frac{-i}{2\Re c_i} \int_{\mathcal{C}_i} (x_i u_{x_i})\, dy_i.$$

Proof. Since $\bar{\partial}\partial u$ is continuous on M, it is integrable. So we have (with $G_r = \bigcup_{i=1}^{k}\{x_i \leq r\}$)

$$
\begin{aligned}
\int_M \bar{\partial}\partial u &= \lim_{r \to 0} \int_{G_r^c} d\partial u \\
&= \sum_{i=1}^{k} \lim_{r \to 0} (-1) \int_{x_i = r} \partial u \\
&= \sum_{i=1}^{k} \lim_{r \to 0} (-1) \int_{x_i = r} L_i u \lambda_i \\
&= \sum_{i=1}^{k} \lim_{r \to 0} (-1) \int_{x_i = r} (x_i u_{x_i} - i\bar{c}_i u_{y_i}) \tfrac{1}{2\Re c_i} (c_i \tfrac{dx_i}{x_i} + i\, dy_i) \\
&= \sum_{i=1}^{k} \lim_{r \to 0} \frac{-1}{2\Re c_i} \int_{x_i = r} (x_i u_{x_i} - i\bar{c}_i u_{y_i})(i\, dy_i) \\
&= \sum_{i=1}^{k} \frac{-i}{2\Re c_i} \int_{\mathcal{C}_i} (x_i u_{x_i})\, dy_i.
\end{aligned}
$$

$\square$

The minus sign appears in Stokes's formula because of our orientation convention.

Proposition 7.31. *Choose first integrals $z_1, \ldots, z_k$ and cut-off functions $\phi_1, \ldots, \phi_k$ for M. Then for each $j = 1 \ldots k$ there exists a real smooth function V_j on M° that satisfies $\bar{\partial}\partial V_j = -(\pi i / \Re c_j) \cdot \mathrm{vol}$ and such that V_j can be written as a sum of real functions $V_j = \phi_j \log x_j + V_j^1 + V_j^2$ with*

1. *V_j^1 smooth on M; and*

2. *V_j^2 smooth on M°, and at $\mathcal{C}_i$*

$$
V_j^2 \sim \begin{cases} \Re\left(\sum_{n \geq 1} {}^j d_i^n z_i^n\right) \log x_i & c_i = 1 \\ \Re\left(\sum_{n \geq 1} {}^j d_i^n z_i^n\right) & c_i \notin \mathbb{Q}. \end{cases}
$$

Such a function is unique up to an additive constant.

Proof of uniqueness. The difference between two such functions (defined by means of the same first integrals) is a bounded harmonic function on $\check{M} \setminus \{p_1, \ldots, p_k\}$. So the singularities are removable, and the difference is a harmonic function on $\check{M}$; that is, a constant. $\square$

Proof of existence. Let u_j^1 be a real smooth function on M°, supported near ∂M, and having the asymptotic formulas at $\mathcal{C}_i$

$$u_j^1 \sim \begin{cases} \frac{-\pi}{2\Re c_j} \sum\limits_{\substack{m \geq 1 \\ n \neq \pm m}} \frac{(h_i)_{mn}}{m^2 - n^2} x_i^m e^{iny_i} & c_i = 1 \\ \frac{-\pi}{2\Re c_j} \sum\limits_{\substack{m \geq 1 \\ n \in \mathbb{Z}}} \frac{(h_i)_{mn}}{m^2 - 2ib_i mn - |c_i|^2 n^2} x_i^m e^{iny_i} & c_i \notin \mathbb{Q}. \end{cases}$$

(It is easy to see that, since h_i is real, these asymptotic formulas are real; so u_j^1 may be chosen real.) Next, let u_j^2 be a real smooth function on M°, supported near ∂M, and having the asymptotic formulas at $\mathcal{C}_i$

$$u_j^2 \sim \begin{cases} \frac{-\pi}{2\Re c_j} \sum\limits_{\substack{m \geq 1 \\ n = \pm m}} \frac{(h_i)_{mn}}{2m} x_i^m e^{iny_i} \cdot \log x_i & c_i = 1 \\ 0 & c_i \notin \mathbb{Q}. \end{cases}$$

(Again, it is easy to see that, since h_i is real, this asymptotic formulas is real; so u_j^2 may be chosen real.)

We wish now to solve

$$\bar{\partial}\partial u_j^3 = (-[\pi i / \Re c_j] \cdot \text{vol}) - \bar{\partial}\partial(\phi_j \log x_j + u_j^1 + u_j^2).$$

We claim that the datum vanishes to infinite order on ∂M and satisfies $\int_M \text{datum} = 0$. Then the datum will be smooth on $\check{M}$ and integrate to zero there. Therefore, by Lemma 6.4, there exists a real smooth solution u_j^3 on $\check{M}$, and we will have

$$\bar{\partial}\partial(\phi_j \log x_j + u_j^1 + u_j^2 + u_j^3) = -(\pi i / \Re c_j) \cdot \text{vol}.$$

Now we can define

$$V_j^1 = u_j^1 + \left(u_j^3 - \sum_{c_i \notin \mathbb{Q}} \phi_i u_j^3 \right)$$

$$V_j^2 = u_j^2 + \sum_{c_i \notin \mathbb{Q}} \phi_i u_j^3.$$

Define $V_j = \phi_j \log x_j + V_j^1 + V_j^2$. Then $\partial\bar{\partial}V_j = -(\pi i / \Re c_j) \cdot \text{vol}$. To see that V_j^1 has the correct form, first note that u_j^1 is smooth on M. Then, note that the parenthetic term is smooth on $\check{M}$, 0 near $\mathcal{C}_i$ when $c_i \notin \mathbb{Q}$, and smooth at $\mathcal{C}_i$ when $c_i = 1$. So it remains to check that V_j^2 has the correct form. u_j^2 has the right form; and because the "datum" we discussed above vanished to infinite order at each of the p_i, u_j^3 is "asymptotically harmonic" at p_i. So

u_j^3 is asymptotic to the real part of a holomorphic function at p_i. If we transfer the constant terms back to V_j^1 by means of cut-off functions, then V_j^2 will have the correct form.

So all that remains is to prove the two-part claim, that the datum vanishes to infinite order on ∂M, and that it integrates to zero. First, the vanishing. Using (7.3) we can write the datum near $\mathcal{C}_i$ as

$$\text{datum} = \frac{\pi i}{(\Re c_j)h_i} \, \text{vol} \left(-h_i - \frac{2\Re c_j}{\pi} \mathcal{L}_i[\phi_j \log x_j + u_j^1 + u_j^2] \right).$$

Near $\mathcal{C}_i$, $\phi_j \log x_j$ is either 0 or $\log x_j$, and so is annihilated by $\mathcal{L}_i$. So we need to show that $\mathcal{L}_i(\frac{2\Re c_j}{-\pi} u_j^1 + \frac{2\Re c_j}{-\pi} u_j^2)$ agrees with h_i to infinite order at $\mathcal{C}_i$. This is a simple consequence of the asymptotic formulas for u_j^1 and u_j^2 that were used in their definition, and formula (7.2) for $\mathcal{L}_i$.

Finally, the vanishing of the integral of the datum. The first piece clearly integrates to $-\pi i/\Re c_j$. We can compute the integrals of the second, third, and fourth pieces by means of Technical Lemma 7.30. The third and fourth integrate to zero. For the second, note that at $\mathcal{C}_i$,

$$x_i \partial_{x_i}(\phi_j \log x_j) = \begin{cases} 1 & i = j \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, by the technical lemma,

$$\int_M \bar{\partial}\partial(\phi_j \log x_j) = \frac{-i}{2\Re c_j} \int_{\mathcal{C}_j} dy_j$$

$$= -\pi i/\Re c_j.$$

A subtraction, and the full integral of the datum is zero. This proves the claim, and therefore the proposition. $\square$

Theorem 7.32. *Let M be a geometric b-holomorphic complex curve of genus zero, all of whose collar invariants lie in $(\mathbb{C}\backslash\mathbb{Q}) \cup 1$. Label the boundary circles as $\mathcal{C}_1, \ldots, \mathcal{C}_k$, and fix first integrals $z_1, \ldots, z_k$ and cut-off functions $\phi_1, \ldots, \phi_k$ for the collars of M. Let $V_j = \phi_j \log x_j + V_j^1 + V_j^2$ be as in Proposition 7.31.*

Let E be a holomorphic line bundle over M, and let γ be the derived bundle invariant of Definition 7.27. Let $\mathbf{1}e^{u+w+f}$ be a good frame for E in the sense of Theorem 7.3, with $w = \sum_{j=1}^k a_j^0 \phi_j \log x_j$ (this relation defines the a_j^0). The v is absent because $\kappa = 0$.

For there to exist a hermitian holomorphic b-connection on E whose curvature is $-2\pi i\gamma\cdot$ vol, it is necessary and sufficient that

$$\Re f \sim \sum_{j=1}^{k}(\Re a_j^0)V_j^2$$

at ∂M (that is, the difference vanishes to infinite order).

Proof of necessity. Define $s = \mathbf{1}e^u$. Then s is a smooth frame for E. Let $\langle s, s\rangle = e^p$ represent a smooth hermitian metric for E, the curvature of whose induced hermitian holomorphic b-connection is $-2\pi i\gamma \cdot$ vol.

As usual, we can write the curvature as

$$\partial\alpha - \bar\partial\bar\alpha + \bar\partial\partial p$$

where α is the $\bar\partial$ form with respect to s. So we have the equation

$$\partial\alpha - \bar\partial\bar\alpha + \bar\partial\partial p = -2\pi i\gamma \cdot \text{vol}\,.$$

Since se^{w+f} is holomorphic, we know that $\alpha = -\bar\partial(w + f)$. So the left hand side may be re-written as

$$-\partial\bar\partial(w+f) + \bar\partial\partial(\bar w + \bar f) + \bar\partial\partial p = \bar\partial\partial(2\Re w + 2\Re f + p).$$

We now re-write the right hand side as follows: define $V = \sum_{j=1}^{k} 2(\Re a_j^0)V_j$. We see that

$$\begin{aligned}
\bar\partial\partial V &= \sum_{j=1}^{k} 2(\Re a_j^0)\bar\partial\partial V_j \\
&= \sum_{j=1}^{k} 2(\Re a_j^0)\left(\frac{-\pi i}{\Re c_j}\right)\cdot\text{vol} \\
&= -2\pi i\left(\sum_{j=1}^{k}\frac{\Re a_j^0}{\Re c_j}\right)\cdot\text{vol} \\
&= -2\pi i\gamma\cdot\text{vol}\,.
\end{aligned}$$

So our equation becomes

$$\bar\partial\partial(2\Re w + 2\Re f + p - V) = 0.$$

That is, $2\Re w + 2\Re f + p - V$ is harmonic.

If we write

$$V = \sum_{j=1}^{k} 2(\Re a_j^0)\phi_j \log x_j + \sum_{j=1}^{k} 2(\Re a_j^0)V_j^1 + \sum_{j=1}^{k} 2(\Re a_j^0)V_j^2$$

and

$$2\Re w = \sum_{j=1}^{k} 2(\Re a_j^0)\phi_j \log x_j,$$

then our harmonic function can be written as

$$\left(p - \sum_{j=1}^{k} 2(\Re a_j^0)V_j^1 \right) + 2 \left(\Re f - \sum_{j=1}^{k} (\Re a_j^0)V_j^2 \right).$$

Since p and V_j^1 are smooth on M, they are bounded; so the first parenthetic term is bounded. Since f and V_j^2 tend to zero at ∂M, the second parenthetic term is bounded. So we have a real harmonic function on $\check{M} \backslash \{p_1, \ldots, p_k\}$ which is bounded. Thus it is harmonic on $\check{M}$.

Since $\check{M}$ is a sphere, this harmonic function is constant. Therefore $\Re f \sim \sum_{j=1}^{k}(\Re a_j^0)V_j^2$ at ∂M. $\square$

Proof of sufficiency. Define $s = \mathbf{1}e^u$. Then s is a smooth frame for E. Then define $V = \sum_{j=1}^{k} 2(\Re a_j^0)V_j$. Now let $p = V - 2\Re w - 2\Re f$. We are left with two tasks. The first is to show that p is a smooth function on M. The second is to show that, if we define a metric on E by $\langle s, s \rangle = e^p$, then the curvature of the induced hermitian holomorphic b-connection is $-2\pi i \gamma \cdot$ vol.

The first task. Write

$$V = \sum_{j=1}^{k} 2(\Re a_j^0)\phi_j \log x_j + \sum_{j=1}^{k} 2(\Re a_j^0)V_j^1 + \sum_{j=1}^{k} 2(\Re a_j^0)V_j^2$$

and

$$2\Re w = \sum_{j=1}^{k} 2(\Re a_j^0)\phi_j \log x_j.$$

Then

$$V - 2\Re w - 2\Re f = \sum_{j=1}^{k} 2(\Re a_j^0)V_j^1 + \sum_{j=1}^{k} 2(\Re a_j^0)V_j^2 - 2\Re f.$$

Since V_j^1 is smooth on M for each j, and $\Re f \sim \sum_{j=1}^{k}(\Re a_j^0)V_j^2$ at ∂M, we see that p is smooth at ∂M. So p is smooth on M.

The second task. Since se^{w+f} is holomorphic over M°, the $\bar\partial$ form with respect to s is $\alpha = -\bar\partial(w+f)$. Therefore the curvature of our hermitian holomorphic b-connection is

$$
\begin{aligned}
\partial\alpha - \bar\partial\bar\alpha + \bar\partial\partial p &= -\partial\bar\partial(w+f) + \bar\partial\partial(\bar w + \bar f) + \bar\partial\partial p \\
&= \bar\partial\partial(2\Re w + 2\Re f + p) \\
&= \bar\partial\partial V \\
&= \sum_{j=1}^{k} 2(\Re a_j^0)\bar\partial\partial V_j \\
&= \sum_{j=1}^{k} 2(\Re a_j^0)\left(-\frac{\pi i}{\Re c_j}\right) \cdot \mathrm{vol} \\
&= -2\pi i \left(\sum_{j=1}^{k} \frac{\Re a_j^0}{\Re c_j}\right) \cdot \mathrm{vol} \\
&= -2\pi i\gamma \cdot \mathrm{vol}.
\end{aligned}
$$

$\square$

Invariance Theorem 7.33. *Let M be a geometric b-holomorphic complex curve of genus zero, all of whose collar invariants lie in $(\mathbb{C}\backslash\mathbb{Q}) \cup 1$. Let E be a holomorphic line bundle over M. If E satisfies the necessary and sufficient condition of Theorem 7.32 with respect to one choice of first integrals for M and one good frame for E, then E satisfies the condition with respect to every choice of first integrals for M and every good frame for E.*

Proof. Suppose E satisfies the condition with respect to one choice of first integrals and good frame. Then by the sufficiency of the condition, there exists a hermitian holomorphic b-connection of constant curvature for E. So now make a different choice of first integrals and good frame. By the necessity of the condition, E satisfies this condition with respect to the new choice of first integrals and good frame. $\square$

The important consequence of this invariance theorem is that the coefficients in the asymptotic expansion of V_j^2 at ∂M transform (under a change of first integrals) precisely in the same way as the $n \geq 1$ part of the $\{a_i^n\}$ bundle invariant.

Definition 7.34 (metric line bundle classes). Let M be a b-holomorphic complex curve of genus zero, all of whose collar invariants lie in $(\mathbb{C}\backslash\mathbb{Q}) \cup 1$. Label the boundary circles $\mathcal{C}_1, \ldots, \mathcal{C}_k$, and fix $j \in \{1, \ldots, k\}$.

We define the jth *metric line bundle class* G_j of M by constructing a representative, (G_j), as follows. As a smooth bundle, $(G_j) = M \times \mathbb{C}$. Let $\mathbf{1}$ be the canonical smooth frame. Choose first integrals $z_1, \ldots, z_k$ for M, and let $V_j = \phi_j \log x_j + V_j^1 + V_j^2$ as in Proposition 7.31. We then stipulate that $\mathbf{1}e^{w+f}$ be a good frame, where $w = \phi_j \log x_j$ and $\Re f \sim V_j^2$.

Proof of the soundness of the definition. On its face, the definition depends on a choice of first integrals. We need to show that bundles defined by different first integrals are isomorphic.

We can re-phrase the definition of (G_j) as follows: (G_j) is the bundle which has (with respect to the chosen first integrals $z_1, \ldots, z_k$)

$$a_i^0 = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

and a_i^n $(n \geq 1)$ given by the coefficients in the asymptotic expansion of V_j^2 with respect to the same first integrals. But by Invariance Theorem 7.33, these coefficients transform (under a change of first integrals) exactly as the $n \geq 1$ part of the bundle invariant $\{a_i^n\}$ does. And by the definition of a_i^0, it is clear that a_i^0 does not change with a change of first integrals, just as we have here. So the a_i^n of the (G_j) transform correctly, as bundle invariants, and our definition is sound. $\qquad \square$

Definition 7.35 (real power of a line bundle class). Let M be a b-holomorphic complex curve of genus zero, all of whose collar invariants lie in $(\mathbb{C}\backslash\mathbb{Q})\cup 1$. Let $[E]$ be a holomorphic line bundle class over M. Let δ be any real number. We will define a new class $[E]^\delta$, called the δ*th power of* $[E]$, by constructing a representative E^δ as follows: Let $\mathbf{1}_E e^{u+w+f}$ be a good frame for E. As a smooth bundle, we take E^δ to be $M \times \mathbb{C}$; and we stipulate that $\mathbf{1}e^{\delta(u+w+f)}$ be holomorphic. (Here, $\mathbf{1}$ is the canonical frame.) Finally, set $[E]^\delta = [E^\delta]$.

Definition 7.36. Let E and F be two line bundles. We call E and F *twist-equivalent* if there exists a C^∞ isomorphism $\varphi : E \longrightarrow F$ such that

$$\varphi e^{\sum_{j=1}^k it_j \phi_j \log x_j}$$

is holomorphic. Here, t_j are real numbers, ϕ_j are cut-off functions supported near $\mathcal{C}_j$, and $z_j = x_j^{c_j} e^{iy_j}$ are first integrals.

It's easy to see that twist-isomorphism is a relation of equivalence, and that it also defines a relation of equivalence (called *twist-equivalence*) on bundle classes.

Reformulated Theorem 7.37. *Let M be a geometric b-holomorphic complex curve of genus zero, all of whose collar invariants lie in $(\mathbb{C}\backslash\mathbb{Q}) \cup 1$. Label the boundary circles $\mathcal{C}_1, \ldots, \mathcal{C}_k$, and let $G_1, \ldots, G_k$ be the metric line bundle classes of M.*

Let E be a holomorphic line bundle over M with zeroth invariants a_i^0 (with respect to some C^∞ frame) and derived bundle invariant γ. Then for there to exist a hermitian holomorphic connection on E whose curvature is $-2\pi i\gamma \cdot \text{vol}$, it is necessary and sufficient that $[E]$ be twist-equivalent to

$$G_1^{\Re a_1^0} \otimes \cdots \otimes G_k^{\Re a_k^0}.$$

Proof of necessity. Choose first integrals $z_1 \ldots, z_k$ and cut-off functions $\phi_1, \ldots, \phi_k$. Let $V_j = \phi_j \log x_j + V_j^1 + V_j^2$ be as in Proposition 7.31. Let $F_E = \mathbf{1}_E e^{u+w+f}$ be a good frame for E, with $\{a_i^n\}$ the associated sequences (with respect to the chosen first integrals). Then by Theorem 7.32, we must have $\Re f \sim \sum_{j=1}^k (\Re a_j^0) V_j^2$.

Let $(G_1), \ldots, (G_k)$ be representatives for $G_1, \ldots, G_k$ defined by means of the chosen first integrals. Then define the representative (G) for $G = G_1^{\Re a_1^0} \otimes \cdots \otimes G_k^{\Re a_k^0}$ as in the construction of the real powers of bundle classes, using the representatives (G_j). This means that (G) has the good frame

$$F = \mathbf{1}e^{\sum_{j=1}^k (\Re a_j^0)\phi_j \log x_j + f_G}$$

where $\Re f_G \sim \sum_{j=1}^k (\Re a_j^0) V_j^2$.

So now we define the map $\varphi : (G) \longrightarrow E$ by

$$\varphi(F) = F_E e^{-\sum_{j=1}^k i(\Im a_j^0)\phi_j \log x_j}.$$

Then $\varphi(\mathbf{1}) = F_E e^{-\sum_{j=1}^k a_j^0 \phi_j \log x_j - f_G}$, which agrees to infinite order with $\mathbf{1}_E e^u$ at ∂M. So φ is a C^∞ isomorphism. Furthermore, $\varphi e^{\sum_{j=1}^k i(\Im a_j^0)\phi_j \log x_j}$ takes F to F_E, and is therefore holomorphic. This means that φ is a twist-equivalence map. $\qquad\square$

Proof of sufficiency. Choose first integrals $z_1, \ldots, z_k$ for M, and a good frame $\mathbf{1}_E e^{u+w+f}$ for E. Let $\{a_i^n\}$ be the associated sequences.

Let $(G_1), \ldots, (G_k)$ be representatives for $G_1, \ldots, G_k$, constructed by means of our chosen first integrals. This means that (G_j) is $M \times \mathbb{C}$ and has the good frame

$$\mathbf{1}e^{\phi_j \log x_j + f_j},$$

where $\Re f_j \sim V_j^2$ at ∂M. So we have the good frame

$$\mathbf{1}e^{\sum_{j=1}^{k}(\Re a_j^0)\phi_j \log x_j + f_G}$$

for $(G) = (G_1)^{\Re a_1^0} \otimes \cdots \otimes (G_k)^{\Re a_k^0}$ where $\Re f_G \sim \sum_{j=1}^{k}(\Re a_j^0)V_j^2$ at ∂M.

By hypothesis, E is twist-isomorphic to (G). So there exists a C^∞ isomorphism $\varphi :$ $(G) \longrightarrow E$ such that $\varphi e^{\sum_{j=1}^{k} it_j\phi_j \log x_j}$ is holomorphic. Let $\widetilde{\mathbf{1}}_E = \varphi(\mathbf{1})$. Then the image under $\varphi e^{\sum_{j=1}^{k} it_j\phi_j \log x_j}$ of the good frame for (G) is

$$\widetilde{\mathbf{1}}_E e^{\sum_{j=1}^{k}(\Re a_j^0 + it_j)\phi_j \log x_j + f_G},$$

and this is a new good frame for E. So we have two good frames for E:

$$\mathbf{1}_E e^{u+w+f}$$
$$\widetilde{\mathbf{1}}_E e^{\widetilde{w}+\widetilde{f}}$$

where $\widetilde{w} = \sum_{j=1}^{k} \phi_j(\Re a_j^0 + it_j) \log x_j$ and $\Re\widetilde{f} \sim \sum_{j=1}^{k}(\Re a_j^0)V_j^2$. By Theorem 7.4, we know that the quotient

$$\widetilde{\mathbf{1}}_E e^{\widetilde{w}+\widetilde{f}} \div \mathbf{1}_E e^{u+w+f}$$

is meromorphic on $\check{M}$, with poles and zeros only at the distinguished points; and the order of m at p_j is

$$\frac{1}{c_j}[\Re a_j^0 + it_j - (\Re a_j^0 + i\Im a_j^0)] = \frac{i(t_j - \Im a_j^0)}{c_j}.$$

Since $\Re c_j$ is positive, the only way for this to be real is for it to be zero. So m is a holomorphic function on $\check{M}$. So it is a constant. So our two good frames agree up to a multiplicative constant. In particular, $f \sim \widetilde{f}$. Therefore, at the boundary,

$$\Re f \sim \Re\widetilde{f}$$
$$\sim \sum_{j=1}^{k}(\Re a_j^0)V_j^2.$$

So by Theorem 7.32, there exists a hermitian holomorphic b-connection of constant curvature. $\qquad\square$

An immediate consequence of this theorem is that, whenever $\{a_i^n\} \sim 0$, the class $G = G_1^{\Re a_1^0} \otimes \cdots \otimes G_k^{\Re a_k^0}$ is twist-equivalent to the trivial class. The reason is as follows. Let E be a bundle whose associated sequences are the $\{a_i^n\}$. Then E is equivalent to the trivial

bundle. So E has a hermitian holomorphic b-connection whose curvature is zero. So by the theorem, $[E]$ is twist-equivalent to G.

However, it is not obvious why this should be so. We will conclude this section by proving this consequence of the reformulated theorem directly, to make sure there has been no error.

Proof of the consequence. Choose first integrals $z_1, \ldots, z_k$ for M. Let $V_j = \phi_j \log x_j + V_j^1 + V_j^2$ be as usual. Construct the (G_j) and $(G) = (G_1)^{\Re a_1^0} \otimes \cdots \otimes (G_k)^{\Re a_k^0}$ as usual. For (G_j) we have the good frame

$$\mathbf{1} e^{\phi_j \log x_j + f_j}$$

where $\Re f_j \sim V_j^2$. So for (G) we have the good frame

$$\mathbf{1} e^{\sum_{j=1}^k (\Re a_j^0) \phi_j \log x_j + f}$$

where $\Re f \sim \sum_{j=1}^k (\Re a_j^0) V_j^2$.

Since $\{a_i^n\} \sim 0$, there exists a meromorphic function m on $\check{M}$ with zeros and poles only at the p_i; and m can be written (by Proposition 7.7) as $Q e^{U+W+F}$ where

Q is nonvanishing, and smooth at $\mathcal{C}_j$

U is smooth at $\mathcal{C}_j$

$$W = \sum_{j=1}^k a_j^0 \phi_j \log x_j$$

$$F \sim \begin{cases} 0 & c_j = 1 \\ \sum_{n \geq 1} a_j^n z_j^n & c_j \notin \mathbb{Q} \end{cases} \quad \text{at } \mathcal{C}_j.$$

Define the frame H for (G) by

$$H = \mathbf{1} e^{\sum_{j=1}^k (\Re a_j^0) \phi_j \log x_j + f} \cdot m^{-1}.$$

H is clearly holomorphic. So it will suffice to show that

$$H e^{\sum_{j=1}^k i(\Im a_j^0) \phi_j \log x_j}$$

is smooth; for then (G) is twist-isomorphic to the trivial bundle.

Write

$$H e^{\sum_{j=1}^k i(\Im a_j^0) \phi_j \log x_j}$$

as

$$\mathbf{1}e^{\sum_{j=1}^{k}(\Re a_j^0)\phi_j \log x_j + f}e^{\sum_{j=1}^{k} i(\Im a_j^0)\phi_j \log x_j}Qe^{-U-\sum_{j=1}^{k}\phi_j a_j^0 \log x_j - F} = \mathbf{1}Qe^{-U+(f-F)}.$$

It now suffices show that $f \sim F$; for then $\mathbf{1}Qe^{-U}$ is smooth, and so is e^{f-F}. We know that, at $\mathcal{C}_i$,

$$F \sim \begin{cases} 0 & c_1 = 1 \\ \sum_{n\geq 1} a_i^n z_i^n & c_i \notin \mathbb{Q}. \end{cases}$$

So we have to show that f has this same asymptotic expansion at $\mathcal{C}_i$.

Define $V = \sum_{j=1}^{k}(\Re a_j^0)V_j$. Then

$$\bar{\partial}\partial V = \sum_{j=1}^{k}(\Re a_j^0)\bar{\partial}\partial V_j$$

$$= \sum_{j=1}^{k}(\Re a_j^0)\left(\frac{-\pi i}{\Re c_j}\right) \cdot \mathrm{vol}$$

$$= -\pi i \left(\sum_{j=1}^{k}\frac{\Re a_j^0}{\Re c_j}\right) \cdot \mathrm{vol}$$

$$= 0,$$

since the parenthetic term is equal to the sum of the orders of the zeros and poles of m (which is zero). So V is harmonic. We also know that $\Re \log m = \log |m|$ is harmonic. We wish to see that V and $\Re \log m$ agree.

Write $V = \sum_{j=1}^{k}(\Re a_j^0)\phi_j \log x_j + V^1 + V^2$, where V^1 is smooth and

$$V^2 \sim \begin{cases} \Re\left(\sum_{n\geq 1}\left(\sum_{j=1}^{k} {}^j d_i^n\right) z_i^n\right) \log x_i & c_i = 1 \\ \Re\left(\sum_{n\geq 1}\left(\sum_{j=1}^{k} {}^j d_i^n\right) z_i^n\right) & c_i \notin \mathbb{Q}. \end{cases}$$

Then write $\Re \log m = \mathrm{smooth} + \sum_{j=1}^{k}(\Re a_j^0)\phi_j \log x_j + \Re F$, where

$$\Re F \sim \begin{cases} 0 & c_i = 1 \\ \Re \sum_{n\geq 1} a_i^n z_i^n & c_i \notin \mathbb{Q}. \end{cases}$$

So $V - \Re \log m = \mathrm{smooth} + V^2 - \Re F$, which is harmonic on $\check{M}\backslash\{p_1, \ldots, p_k\}$ and bounded there. Therefore this function is constant, and $V^2 \sim \Re F$.

A comparison between the asymptotic formulas for V^2 and $\Re F$ shows that

$$\sum_{j=1}^{k} {}^j d_i^n = \begin{cases} 0 & c_i = 1 \\ a_i^n & c_i \notin \mathbb{Q}. \end{cases}$$

We can now conclude the argument. We already know that

$$\Re f \sim V^2$$

$$\sim \begin{cases} \Re\left(\sum_{n\geq 1}\left(\sum_{j=1}^{k} {}^j d_i^n\right) z_i^n\right) \log x_i & c_i = 1 \\ \Re\left(\sum_{n\geq 1}\left(\sum_{j=1}^{k} {}^j d_i^n\right) z_i^n\right) & c_i \notin \mathbb{Q} \end{cases}$$

$$\sim \begin{cases} 0 & c_i = 1 \\ \Re \sum_{n\geq 1} a_i^n z_i^n & c_i \notin \mathbb{Q}. \end{cases}$$

Therefore

$$f \sim \begin{cases} 0 & c_i = 1 \\ \sum_{n\geq 1} a_i^n z_i^n & c_i \notin \mathbb{Q}. \end{cases}$$

This concludes the proof. $\qquad\square$

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