CANONICAL QUATERNION ALGEBRA OF THE WHITEHEAD LINK COMPLEMENT

A Dissertation
Submitted to
the Temple University Graduate Board

---

in Partial Fulfillment
of the Requirements for the Degree of
DOCTOR OF PHILOSOPHY

---

by
Rebekah Mae Palmer
May 2023

Examining Committee Members:

Professor Matthew Stover, Advisory Chair, Mathematics
Professor David Futer, Mathematics
Professor Samuel Taylor, Mathematics
Professor Ted Chinburg, External Member, University of Pennsylvania, Mathematics

ABSTRACT

Let Γ_M be the fundamental group of a knot or link complement M. The discrete faithful representation of Γ_M into $\mathrm{PSL}_2(\mathbb{C})$ has an associated quaternion algebra. We can extend this notation to other representations, which are encoded by the character variety $X(\Gamma_M)$. The generalization is the canonical quaternion algebra and can be used to find unifying features of irreducible representations, such as the splitting behavior of their associated quaternion algebras. Within this dissertation, we will determine properties of the canonical quaternion algebra for the Whitehead link complement and explore how the algebra can descend to quaternion algebras of the Dehn (d, m)-surgeries thereon.

For Zach

ACKNOWLEDGEMENTS

I want to begin by thanking my advisor, Matthew Stover, for his constant support, patience, and encouragement throughout my time in the program. His commitment to his students' growth is, quite frankly, inspiring. I am so thankful that I've had the privilege of working under his guidance.

I am grateful for the time Dave Futer, Sam Taylor, and Ted Chinburg have spent reading drafts of this dissertation as well as providing feedback and helpful comments. I want to additionally thank Dave Futer and Sam Taylor for their mentorship over many years.

A graduate school is determined by its graduate students. I want to thank Khánh Lê for his friendship, collaboration, and camaraderie which defined my experience at Temple University. I want to thank Thomas Ng for setting the standard of how to tailor departmental culture to achieve one's goals, both mathematically and socially. My thanks to the many grad students who made all of our offices such a delightful atmosphere, including (but certainly not limited to): Irem, Rhiannon, Katherine, DB, Jeremy, Ross, Brandi, Michael, Tim, MeiRose, Rob, Sowmya, Dipika, Yi, and Will. I also want to shout-out to the changing of the guard — Afrina, Nizar, Andrew, Blessing, Adam, Aniruddha, and Elmira: go out and knock their socks off. There are, however, not words that can sufficiently express my gratitude for the years and years of friendship with Rose Kaplan-Kelly, Leah Leiner, and Brandis Whitfield, three of the best people to ever enter Wachman Hall.

The world outside the mathematics community has been instrumental in shaping me and how I think about math. Thank you to Charli Haynes, my very first friend and writing collaborator, who's kept me grounded in the joy of writing and horror movies. Thank you to Saul Wilson, who knows most longitudinally what it took to achieve this academic culmination and who trusted that it would all work out in the end. Thank you to Mikhail Goldenberg, who taught me that mathematics is music. Вот мой балет.

I have so much admiration for the entire sheaf of Wheatleys — Richard, Sue Ellen, Daniel, Nathan, Joel, and Anna. I could write a whole dissertation about how y'all've built me and how grateful I am to be a part of this family. None of you ever expressed doubt about my ability to do this. You've taught me how to face the world with excessive humor, intense caring, and sheer force of will. Also, thanks for handing me *Flatland* when I was twelve. Turns out I really like geometry.

Last and never least, I want to thank my spouse Zachary Palmer for being in my life, through the spectrum of the brightest joys to the deepest struggles. None of this would be possible with you. I love you so colorfully.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

> I would like, if I may, to take you on a strange journey.
>
> *The Rocky Horror Picture Show*

1.1 Setting the stage

The root of our story lies in two invariants of hyperbolic 3-manifolds: character varieties and quaternion algebras. Character varieties are algebraic varieties that parametrize representations of finitely generated groups to algebraic, reductive Lie groups. Quaternion algebras are central simple algebras that can detect properties of 3-manifolds (or -orbifolds) such as arithmeticity. These two concepts meet at $\mathrm{SL}_2(\mathbb{C})$-representations. The interest herein of $\mathrm{SL}_2(\mathbb{C})$ is as the double cover of the orientation-preserving isometry group of hyperbolic 3-space $\mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{PSL}_2(\mathbb{C})$. A hyperbolic 3-manifold (or -orbifold) by definition is realized as the quotient space of $\mathbb{H}^3$ by some discrete subgroup of

$\mathrm{Isom}^+(\mathbb{H}^3)$. A discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$ is called a **Kleinian group**, and we will thus also use the term Kleinian group to refer to a discrete subgroup of $\mathrm{SL}_2(\mathbb{C})$.

1.1.1 First object: character varieties

The scope of this dissertation is some families of hyperbolic 3-manifolds (and -orbifolds) with finitely generated fundamental group. To this end, we will assume that all groups in this dissertation are finitely generated. The fundamental group $\Gamma_M := \pi_1(M)$ of a complete hyperbolic 3-manifold (or -orbifold) M admits a discrete and faithful representation to $\mathrm{PSL}_2(\mathbb{C})$ whose image acts on $\mathbb{H}^3$ such that the quotient under this quotient precisely produces the 3-manifold (or -orbifold). There are, of course, other possible representations (e.g. the trivial representation). Motivated to encode all representations in a single mathematical structure, Culler–Shalen introduced **character varieties** $X(\Gamma_M)$ of hyperbolic 3-manifolds in [13]. They proved that $X(\Gamma_M)$ is an algebraic set where each point (i.e. character) encodes a representation, detailed further in Section 2.1. There may be multiple components, so the component of $X(\Gamma_M)$ which contains the character corresponding to the chosen discrete and faithful representation is called the **canonical component**. The dimension of the canonical component is precisely the number of cusps of the manifold ([35]).

Character varieties have been used to address several questions in hyperbolic geometry; we give a non-exhaustive list of examples. In their seminal paper [13], Culler–Shalen used character varieties to find essential surfaces in certain 3-manifolds. Przytycki–Sikora drew connections to skein algebras to address incompressible surfaces through a topological lens as well as pursue quantum invariants in [38]. Paoluzzi–Porti handled knot symmetries with character varieties in [37]. Another vital application was in Gordon–Luecke's proof in [18] that knots are determined by their complements.

1.1.2 Second object: quaternion algebras

Quaternion algebras enter the picture as an invariant of Kleinian groups. The **trace field** k_Γ of a non-elementary Kleinian group Γ is the field extension of $\mathbb{Q}$ by all of the traces of all of the elements of Γ. The associated algebra $\mathcal{A}(\Gamma)$ is the k_Γ-algebra of all finite sums of elements from Γ. This construction is proven to be a quaternion algebra ([25, Theorem 3.2.1]).

A **quaternion algebra** is a 4-dimensional central simple k-algebra where k is a field with $\operatorname{char} k \neq 2$ for the extent of this dissertation. Two trivial examples are $\mathrm{Mat}_2(\mathbb{R})$ and $\mathrm{Mat}_2(\mathbb{C})$ (trivial in the sense of Brauer groups in Section 3.3.1). The quintessential non-trivial example (in the sense of motivating the concept; see Section 3.1) is Hamilton's quaternions $\mathcal{H}$ — an $\mathbb{R}$-algebra defined as $\mathbb{R} \oplus i\,\mathbb{R} \oplus j\,\mathbb{R} \oplus k\,\mathbb{R}$ where

$$i^2 = j^2 = k^2 = ijk = -1.$$

Hamilton's quaternions become a matrix algebra under a tensor product: $\mathcal{H} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathrm{Mat}_2(\mathbb{C})$. When the tensor product of a quaternion k-algebra and a field extension $F \supset k$ is isomorphic to $\mathrm{Mat}_2(F)$, we say that quaternion algebra **splits** over F. If not, then the quaternion algebra remains a division algebra ([52, 8.3.4]). (Compatibility with notation in other sources causes us to temporarily use "k" as both a field and a generator of $\mathcal{H}$; this is addressed in Section 3.1.)

There is a naturally arising associated quaternion algebra of a hyperbolic 3-manifold (or -orbifold) by considering the Kleinian group which is the image of the fundamental group under a discrete and faithful representation. However, other irreducible representations of the fundamental group Γ_M have images in $\mathrm{PSL}_2(\mathbb{C})$ that are also Kleinian groups and thus also have associated quaternion algebras. The goal here is to expand on the study of the associated quaternion algebras of points on the character variety to an algebra over the field of rational functions of the canonical component (see Section 3.3). This object is dubbed the **canonical quaternion algebra**.

Any question that we may ask of an associated quaternion algebra may be asked of this generalized object, with the added flavor of searching for connection to the geometry of hyperbolic 3-manifolds (and -orbifolds). Chinburg–Reid in [9] used associated quaternion algebras to find closed hyperbolic 3-manifolds whose geodesics are all simple. The quadratic field extensions over which a quaternion algebra split uniquely determine (up to isometry) the quaternion algebra itself ([25, Theorem 7.3.3]), and further provide restrictions on eigenvalues of matrices in the Kleinian group ([25, Lemma 12.2.1]).

Let M be a hyperbolic 3-manifold (or -orbifold). A key difference between the quaternion algebra associated with an irreducible representation and the canonical quaternion algebra is that the underlying field of a canonical quaternion algebra is not a number field. Instead, the canonical quaternion algebra is over the field of rational functions of an affine algebraic variety. When discussing the canonical quaternion algebra "corresponding" to M, there is a subtle distinction between the algebra being defined over the character variety C_M versus being defined over the function field $k(C_M)$. For the sake of this dissertation, we will consider the canonical quaternion algebra denoted $\mathcal{A}_k(C_M)$ over the function field. These technical details are presented in Section 3.3.

1.2 Main results

We will discuss a family of two-bridge links and look at one of its most famous members: the Whitehead link complement W (more details in Section 4.1.1), as see below in Figure 1.1.

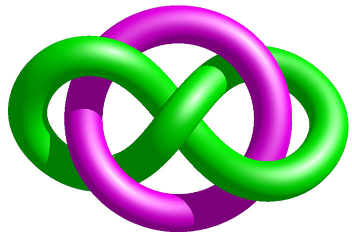


Figure 1.1: Righthanded Whitehead link in lemniscate form

Let C_W denote the canonical component of the $\mathrm{SL}_2(\mathbb{C})$-character variety of the Whitehead link complement. Since the dimension of the canonical component is equal to the number of cusps of a hyperbolic 3-manifold, the variety C_W is a complex surface (i.e. 2-dimensional $\mathbb{C}$-space). A **Dehn surgery point** on C_W is a character that corresponds to a representation that produces a Kleinian group associated with the fundamental group of a manifold (or orbifold) arising from Dehn surgery.

Chapter 4 contains the proofs of the main results listed below. The overarching goal is to explore when (if ever) a canonical quaternion algebra splits over some quadratic extension of $\mathbb{Q}$. The progression of these theorems begins with the Whitehead link complement and performs increasingly specific Dehn surgeries (defined in Section 4.1.2) on this manifold.

Firstly, Theorem A proves that the canonical quaternion algebra of C_W will remain a division algebra under the tensor product of any quadratic field extension of $\mathbb{Q}$. Additionally, Theorem A provides an example criteria for a character whose associated quaternion algebra does split over a given quadratic field extension. Theorem B looks at all possible Dehn (d,m)-surgeries on one component of W to determine that there will always be a character on those subvarieties that splits over the quadratic extension $\mathbb{Q}(i)$. Theorem C specifically concerns $(d,1)$-surgeries and finds a criteria for the failure of a canonical quaternion algebra to split over a quadratic extension. Finally, Theorem D focuses entirely on the properties of Dehn $(-1,1)$-surgery, which produces the figure-eight knot complement.

A character variety can be considered as the vanishing set of some polynomial Ψ with coefficients in $\mathbb{Q}$ (or $\mathbb{C}$). For the Whitehead link complement, the polynomial $\Psi_W \in \mathbb{Q}[x,y,z]$ is given explicitly by Landes in [21] and restated here in Proposition 4.1.1. Furthermore, the component C_W is the vanishing set of every polynomial in the ideal $(\Psi_W) \subset \mathbb{Q}[x,y,z]$. Let $k(C_W)$ denote the function field $\mathrm{Frac}(\mathbb{Q}[x,y,z]/(\Psi_W))$. We can denote the canonical quaternion algebra of C_W as a quaternion algebra over $k(C_W)$ whose generating elements arise from tautological representations (see construction in Section 3.3.2).

The stacked parenthetical notation $k(C_W)(\sqrt{d}) := k(C_W) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{d})$ denotes a quadratic extension of a function field where $d \in \mathbb{Z}$ is squarefree. We will occasionally, for the sake of saving notional space, refer to splitting over $k(C_W)(\sqrt{d})$ as splitting over $\mathbb{Q}(\sqrt{d})$. The first main result of this dissertation in technical language is:

Theorem A. *Let $d \in \mathbb{Q}$. Then $\mathcal{A}_k(W) \otimes k(C_W)(\sqrt{d})$ is a division algebra. In contrast, for all primes $p \equiv 3 \mod 4$, there exists a character $\chi_\rho \in C_W$ such that the associated quaternion algebra A_ρ splits over $\mathbb{Q}(\sqrt{-p})$.*

The surface C_W contains several significant characters, such as the characters of many manifolds (and orbifolds) arising from Dehn surgery on the link components of W. Dehn (d, m)-surgery is a procedure where a torus is glued into the 3-manifold such that a curve of slope d/m on a boundary component now bounds a disc (see Section 4.1.2). We denote this manifold (or orbifold) as W_d^m with the canonical component $C_{W_d^m}$. The appearance of $C_{W_d^m}$ as a subvariety of C_W is the result of the surjection of $\pi_1(W)$ onto $\pi_1(W_d^m)$ (by construction of Dehn surgery in Section 4.1.2):

$$\begin{array}{ccc} \pi_1(W) & \xrightarrow{\rho_{(d,m)}} & \pi_1(W_d^m) \\ & \searrow^{\rho \circ \rho_{(d,m)}} & \downarrow \rho \\ & & \mathrm{SL}_2(\mathbb{C}) \end{array}$$

That is, every $\mathrm{SL}_2(\mathbb{C})$ representation of $\pi_1(W_d^m)$ can be precomposed with this surjection to get a representation of $\pi_1(W)$. Thus, there is an inclusion $C_{W_d^m} \hookrightarrow C_W$. We use this inclusion to prove the following:

Theorem B. *For all but two (d, m)-surgeries, there exists a character $\chi_\rho \in C_{W_d^m}$ such that A_ρ splits over $\mathbb{Q}(i)$ but no subfield of $\mathbb{R}$. The exceptional slopes are $(1, 0)$ and $(2, 0)$.*

Remark 1.2.1. There is technically a third exceptional pair (d, m), namely, $(0, 0)$. However, we will see in Section 4.1.2 why $(0, 0)$ does not qualify for a Dehn surgery.

Remark 1.2.2. As a notational note, the seemingly unprecedented choice of d and m is chosen here due to occurence in the literature of expressing fixed surgeries with d, such as $(d, 0)$-surgeries on a knot complement in [43].

Our next step is to fix one of our surgery coefficients: consider the family of manifolds produced by $(d, 1)$-surgeries. (Note that this is never an orbifold for $d > 0$ because the coefficients are coprime.) These are once-punctured torus bundles of tunnel number 1 (see Section 4.2), whose character varieties have been studied in work such as [4, 5, 48]. Let W_d^1 denote the manifold produced by performing $(d, 1)$-surgery on W, and let E_d denote $C_{W_d^1}$.

Theorem C. *Let $p \in \mathbb{Z}$ be such that $p = (p_1)^2 p_2$ where $p_1, p_2 \in \mathbb{Z}$ and $|p_2|$ is squarefree with $|p_2| \neq 0, 1$. If*

(i) $p < 0$ or

(ii) $p > 4$ such that $p_2 \equiv 7 \mod 8$

then $\mathcal{A}_k(E_p)$ does not split over any quadratic extension of $\mathbb{Q}$.

Our final step takes us to a single yet vital manifold. The figure-eight knot complement is infamous: the only arithmetic knot complement ([40, Theorem

2]); the least volume orientable, cusped hyperbolic manifold (along with its sibling, [6, Theorem 1.1]); the unique 1-cusped hyperbolic 3-manifold with nine or more non-hyperbolic fillings ([17, Theorem 1.7]); and more. Chinburg–Reid–Stover use the figure-eight knot complement as a springboard example for the study of Azumaya algebras arising from knot complements in [10, Theorem 1.7]. In particular, they show that the canonical quaternion algebra of the figure-eight knot complement is a division algebra that splits over $\mathbb{Q}(i)$. The following theorem strengthens their result:

Theorem D. *The canonical quaternion algebra (over the function field) of the figure-eight knot complement splits over* $\mathbb{Q}(i)$ *and no other quadratic extension of* $\mathbb{Q}$*.*

The figure-eight knot complement and the Whitehead link complement share a property called **arithmeticity**, which is directly bound to their associated quaternion algebras. (For extensive details on arithmeticity regarding hyperbolic 3-manifolds, we recommend [25].) One of the intriguing contrasts between Theorem A and Theorem D is that a property dependent on the behavior of the *associated* quaternion algebra seems is not directly dependent on the *canonical* quaternion algebra. As further, there are examples of non-arithmetic 3-manifolds that satisfy the conditions of Theorem C (see Example 4.2.7). There are arithmetic once-punctured torus bundles of tunnel number 1 whose canonical quaternion algebra's splitting behavior is currently unknown. For example, $(5,1)$-surgery produces the sibling manifold to the figure-eight, which is arithmetic; determining the splitting behavior of its canonical quaternion algebra may shed light on this issue, but that is not within the scope of this dissertation.

1.3 Organization

The remainder of this dissertation has been divided into three chapters. Chapter 2 will review character varieties of hyperbolic 3-manifolds (and -orbifolds) as constructed by Culler–Shalen ([13]). The motivating example therein is a family of two-bridge links, whose character varieties are viewed through the lens of Vieta polynomials. Chapter 3 will assemble facts about quaternion algebras in several ways: as algebraic structures, as invariants of hyperbolic 3-manifolds (or -orbifolds), and as invariants of a character variety. Again, we will go through the example of a family of two-bridge links. Chapter 4 turns our attention towards the Whitehead link complement and its surgeries. We will compare how previously studied versions of their character varieties line up with methods as presented in this dissertation. Chapter 4 also completes the proofs of the main results regarding canonical quaternion algebras as presented in Section 1.2.

CHAPTER 2

CHARACTER VARIETIES

Age cannot wither her, nor custom stale her infinite variety.

William Shakespeare,
Antony and Cleopatra (II.ii)

2.1 Origins of the character variety

The dawn of the 20th century saw the birth of representation theory. Frobenius introduced characters for finite groups in a series of papers in 1896 and 1897 (listed in [14]). This construction soon expanded from finite groups to finitely generated groups, including the possibility with infinitely many group relations, throughout the next several decades. In 1983, Culler–Shalen forged a path between hyperbolic manifolds and representation theory ([13], later [12]). We present their construction here from [13, Section 1].

Throughout this dissertation, Γ denotes a finitely generated group. Let $\{\gamma_1, \ldots, \gamma_n\}$ be a generating set of Γ. The set $R(\Gamma) = \text{Hom}(\Gamma, \text{SL}_2(\mathbb{C}))$ can be given the structure of an affine algebraic set over $\mathbb{Q}$ by considering the image $(\rho(\gamma_1), \ldots, \rho(\gamma_n)) \in (\text{SL}_2(\mathbb{C}))^n \subset \mathbb{C}^{4n}$ where $\rho \in R(\Gamma)$. Given a different choice of generators, there is a canonical isomorphism between the two subsets of multi-dimensional complex space obtained in this way ([13, Section 1.4]). There is a one-to-one correspondence between the points of $R(\Gamma)$ and the set of representations of Γ in $\text{SL}_2(\mathbb{C})$. This gives us a natural name for this set.

Definition 2.1.1. $R(\Gamma)$ is called the **representation space** of Γ in $\text{SL}_2(\mathbb{C})$.

Let $\rho, \varphi \in R(\Gamma)$. We call ρ and φ **equivalent** if there exists $g \in \text{SL}_2(\mathbb{C})$ such that $\varphi(C) = g\rho(C)g^{-1}$ for all $C \in \text{SL}_2(\mathbb{C})$. The **character** of such a representation ρ is the function $\chi_\rho : \Gamma \to \mathbb{C}$ defined by

$$\chi_\rho(\gamma) = \text{tr}(\rho(\gamma))$$

If ρ and φ are equivalent, then $\chi_\rho = \chi_\varphi$ because trace is a conjugacy invariant.

For each $\gamma \in \Gamma$, consider the regular function (i.e. function with a finite derivative) $\tau_\gamma : R(\Gamma) \to \mathbb{C}$ defined by evaluating the character of ρ at γ:

$$\tau_\gamma(\rho) = \chi_\rho(\gamma) = \text{tr}(\rho(\gamma))$$

Since trace is a conjugacy invariant, the function τ_γ is constant on equivalence classes of representations. The subring T of the ring of regular functions on

$R(\Gamma)$ generated by $\{\tau_\gamma\}_{\gamma\in\Gamma}$ is finitely generated, as shown in [13, Proposition 1.4.1]. Therefore, we can fix $\gamma_1, \ldots, \gamma_r \in \Gamma$ such that $\{\tau_{\gamma_i}\}_{i=1}^r$ generates T.

Define $t : R(\Gamma) \to \mathbb{C}^r$ by

$$t(\rho) = (\tau_{\gamma_1}(\rho), \ldots, \tau_{\gamma_r}(\rho)) \in \mathbb{C}^r$$

Definition 2.1.2. The $\mathrm{SL}_2(\mathbb{C})$ **character variety** of Γ is $X(\Gamma) = t(R(\Gamma))$.

Every irreducible component of $X(\Gamma)$ containing the character of an irreducible representation is a closed affine algebraic variety by [13, Proposition 1.4.4]. We can additionally induce a rational map $I_\gamma : X(\Gamma) \to \mathbb{C}$ by τ_γ on $R(\Gamma)$:

$$I_\gamma(\chi_\rho) = \chi_\rho(\gamma)$$

Consider known properties of representations and apply them to characters. In general, if a representation ρ has some property $\mathcal{P}$, then we will say that χ_ρ also has property $\mathcal{P}$.

Definition 2.1.3. A representation $\rho \in R(\Gamma)$ is called **reducible** if all the $\rho(\gamma)$ with $\gamma \in \Gamma$ have a common one-dimensional eigenspace. Otherwise, the representation is called **irreducible**.

Definition 2.1.4. A representation $\rho \in R(\Gamma)$ is called **abelian** if its image is an abelian subgroup of $\mathrm{SL}_2(\mathbb{C})$. Otherwise, the representation is called **nonabelian**.

With relevant precision from [13], we now develop further language for use in Section 3.3. An **affine scheme** is a locally ringed space isomorphic to the spectrum of some ring. In general, a **scheme** X is a topological space with a structure sheaf $\mathcal{O}_X$ comprised of affine schemes. Below we construct the character variety $X(\Gamma)$ as a scheme.

Character varieties within the scope of this dissertation will typically be considered as the vanishing set of a family of polynomials; however, there is ambiguity of the field to which the coefficients of these polynomials belong. In general, the character variety $X(\Gamma)$ has an affine coordinate ring $k(C) := \mathbb{Q}[x_1, \ldots, x_r]/\mathcal{V}$, where $\mathcal{V}$ is the ideal of all polynomials that vanish on $X(\Gamma)$ under the identification $x_i = I_{\gamma_i}$. Also in [13], it is shown that $X(\Gamma)$ is defined over $\mathbb{Q}$ in the sense that $\mathcal{V}$ is generated by polynomials in the variables x_i with coefficients in $\mathbb{Q}$. We denote this by $X(\Gamma)_{\mathbb{Q}}$. For a number field $k \subset \mathbb{C}$, let $X(\Gamma)_k$ denote the base change $X(\Gamma)_{\mathbb{Q}} \otimes_{\mathbb{Q}} k$.

Lemma 2.1.5 ([10, Lemma 2.3])**.** *The morphism $R(\Gamma)_k \to X(\Gamma)_k$ is surjective. Suppose that η_C is the generic point of an irreducible curve $C \subset X(\Gamma)_k$. Then there is an irreducible curve $\mathcal{D} \subset R(\Gamma)_k$ such that $t(\eta_{\mathcal{D}}) = \eta_C$ and $t(\mathcal{D}) \subset C$, where $\eta_{\mathcal{D}}$ denotes the generic point of $\mathcal{D}$. The function field $k(\mathcal{D})$ of $\mathcal{D}$ is a finite extension of the function field $k(C)$ of C. Further, there exists* *a representation* $P_C : \Gamma \to \mathrm{SL}_2(k(\mathcal{D}))$ *such that*

$$\chi_{P_C}(\gamma)(\rho) = \chi_\rho(\gamma)$$

for any representation $\rho \in \mathcal{D}$ *and* $\gamma \in \Gamma$. *In other words, evaluating the function* $\chi_{P_C}(\gamma) \in k(\mathcal{D})$ *at the point* ρ *gives the value of the character* χ_ρ *at* γ.

The representation P_C produced by $\eta_{\mathcal{D}}$ is a so-called **tautological representation** $P : \Gamma \to \mathrm{SL}_2(k(\mathcal{D}))$ denoted by

$$P_C(\gamma) = \begin{pmatrix} f_\gamma^{1,1} & f_\gamma^{1,2} \\ f_\gamma^{2,1} & f_\gamma^{2,2} \end{pmatrix}$$

where $f_\gamma^{i,j} \in k(\mathcal{D})$ is the function such that $f_\gamma^{i,j}(\rho)$ is the (i,j)-entry of $\rho(\gamma)$. This tautological representation will arise in the shift from an associated quaternion algebra to a canonical quaternion algebra in Section 3.3.

2.2 Vieta polynomials

We take a moment to address a vital tool to our approach. Many works regarding the $\mathrm{SL}_2(\mathbb{C})$ character variety of cusped hyperbolic manifolds discussed in Chapter 1 rely on recursive functions reproduced independently, particularly as a hybrid of the Fibonacci polynomials and the Chebyshev polynomials (e.g. [5, 7, 8, 49, 50]). This dissertation will use **Vieta polynomials**, which first appeared in [51, Chapter IX, Theorems VI and VII].

Definition 2.2.1. The **Vieta–Fibonacci polynomials** satisfy $V_0(x) = 0$, $V_1(x) = 1$, and the second order recurrence relation

$$V_{n-1}(x) + V_{n+1}(x) = xV_n(x) \tag{2.1}$$

Definition 2.2.2. The **Vieta–Lucas polynomials** satisfy $v_0(x) = 2$, $v_1(x) = x$, and the second order recurrence relation

$$v_{n-1}(x) + v_{n+1}(x) = xv_n(x) \tag{2.2}$$

These recurrence relations are traditionally defined only for $n \geq 2$ by the form, for example, $V_n(x) = xV_{n-1}(x) - V_{n-2}(x)$ in [20, Chapter 47]. However, the symmetry displayed in (2.1) and (2.2) lend themselves to extend to $n \leq -1$ by the form $V_n(x) = xV_{n+1}(x) - V_{n+2}(x)$. With this in mind, we will use the second order recurrence relations in (2.1) and (2.2) to express the Vieta–Fibonacci and Vieta–Lucas polynomials for all $n \in \mathbb{Z}$ with the initial conditions $V_0(x) = 0$, $V_1(x) = 1$, $v_0(x) = 2$, and $v_1(x) = x$.

The Vieta polynomials' relation to other recursive polynomials provides a bond to other work on character varieties. The reparameterization of the Chebyshev polynomials $C_n(x) = 2T_n(x/2)$ and $S_n(x) = U_n(x/2)$ satisfy the shared relation $h_{n+1}(x) = xh_n(x) - h_{n-1}(x)$ with initial conditions $C_0(x) = 2$, $C_1(x) = x$, $S_0(x) = 1$, and $S_1(x) = x$ as listed in [2, Table 22.2]. The Fibonacci polynomials $F_n(x)$ and Lucas polynomials $L_n(x)$ are defined by the shared relation $h_{n+1}(x) = xh_n(x) + h_{n-1}(x)$ with initial conditions $F_0(x) = 2$, $F_1(x) = x$, $L_0(x) = 0$, and $L_1(x) = 1$ ([23], [20, Chapter 31]). In spirit, the Vieta–Fibonacci and Vieta–Lucas polynomials use the reparameterized Chebyshev polynomial relation with the Fibonacci and Lucas polynomial initial conditions. This decision hopefully lends intuition to the notation within this dissertation.

There is a generalization of the Vieta polynomials called the Dickson polynomials of the first and second kinds that were introduced in Dickson and Schur in [15] and [45], respectively. (Note that there is no reference to Vieta polynomials within these works.) The respective polynomials $D_n(x, a)$ and $E_n(x, a)$ satisfy the second order recurrence relation $h_{n+1}(x, a) = x\, h_n(x, a) - a\, h_{n-1}(x, a)$ with initial conditions $D_0(x, a) = 2$, $D_1(x, a) = x$, $E_0(x, a) = 0$, and $E_1(x, a) = 1$. There is extensive work on the Dickson polynomials over finite fields and applications to permutations of integers mod p. For more details, see [31].

Generally, discussion of the Vieta polynomials relies on the choice of ring or field where our parameter x lies. For example, it may be possible to express Vieta polynomials as functional equations:

Lemma 2.2.3. *[20, Section 47.1] When $x = \mu + \mu^{-1}$, the functional equations for $V_n(x)$ and $v_n(x)$ are*

$$V_n(\mu + \mu^{-1}) = \begin{cases} \dfrac{\mu^n - \mu^{-n}}{\mu - \mu^{-1}}, & \mu - \mu^{-1} \neq 0; \\ \mu^{n+1} n, & \mu - \mu^{-1} = 0 \end{cases} \tag{2.3}$$

and

$$v_n(\mu + \mu^{-1}) = \mu^n + \mu^{-n} \tag{2.4}$$

The functional equation (2.4) provides motivation to involve Vieta polynomials in the work of this dissertation. Let $C \in \mathrm{SL}_2(\mathbb{C})$ be a matrix with eigenvalues μ and μ^{-1}. Then $\mathrm{tr}(C) = \mu + \mu^{-1}$ and $\mathrm{tr}(C^n) = \mu^n + \mu^{-n} = v_n(\mathrm{tr}(C))$. We will explore more connections, such as in Lemma 2.2.10. First, however, we list general facts about Vieta polynomials that will assist in later computations. The next several lemmas can be found among others in [20, Chapter 47 Exercises].

Lemma 2.2.4 ([20, Exercises 47])**.** *Vieta–Fibonacci and Vieta–Lucas polynomials admit an index symmetry by negation*

$$V_{-n}(x) = -V_n(x) \quad \text{and} \quad v_{-n}(x) = v_n(x)$$

as well as an argument symmetry by negation

$$V_n(-x) = (-1)^{n+1} V_n(x) \quad \text{and} \quad v_n(-x) = (-1)^n v_n(x)$$

Proof. Both follow from the recurrence relations (2.1) and (2.2), respectively, but we will present the proofs for only $V_n(x)$. Firstly, we have that $V_0(x) = 0$ and $V_{-1}(x) = xV_0(x) - V_1(x) = -1 = -V_1(x)$. Thus, if $V_{-n}(x) = -V_n(x)$ for all $n \le n_0$, then

$$V_{-n_0}(x) = xV_{-n_0+1}(x) - V_{-n_0+2}(x) = -xV_{n_0-1}(x) + V_{n_0-2}(x) = -V_{n_0}(x).$$

To show that $V_n(-x) = (-1)^{n+1}V_n(x)$, it is sufficient to notice that $V_{2n}(-x)$ will be a polynomial with only *odd* powers of x (which are all exactly negated under negation of x) and that $V_{2n+1}(-x)$ will be a polynomial with *even* powers of x (which are all invariant under negation of x). □

Section 2.3 and Section 3.4 extensively use the following identities.

Lemma 2.2.5 ([20, Exercises 47]). *Let $V_n(x)$ be the nth Vieta–Fibonacci polynomial.*

1. $V_n(x)^2 = V_{n-1}(x)V_{n+1}(x) + 1$

2. $V_{r+s}(x) = V_{r+1}(x)V_s(x) - V_r(x)V_{s-1}(x)$

3. $V_{2n}(x) = V_{n+1}(x)V_n(x) - V_n(x)V_{n-1}(x)$

4. $V_{2n+1}(x) = V_{n+1}(x)^2 - V_n(x)^2$

5. $V_{n+1}(x)^2 + V_n(x)^2 = xV_{n+1}(x)V_n(x) + 1$

Proof. We may assume that $n > 0$; the case of $n < 0$ can be deduced using the index symmetry of Lemma 2.2.4. For Property 1, we begin with the base case that $V_0(x)^2 = V_{-1}(x)V_1(x) + 1 = 0$ Assume, then, that for some $n_0 > 0$, the equality $V_n(x)^2 = V_{n-1}(x)V_{n+1}(x) + 1$ holds for all $n < n_0$. Then

$$\begin{aligned}
V_{n_0+1}(x)V_{n_0-1}(x) + 1 &= V_{n_0-1}(x) \cdot (xV_{n_0}(x) - V_{n_0-1}(x)) + 1 \\
&= xV_{n_0}(x)V_{n_0-1}(x) - V_{n_0-1}(x)^2 + 1 \\
&= xV_{n_0}(x)V_{n_0-1}(x) - V_{n_0}(x)V_{n_0-2}(x) \\
&= V_{n_0}(x)(xV_{n_0-1}(x) - V_{n_0-2}(x)) \\
&= V_{n_0}(x)^2
\end{aligned}$$

For Property 2, without loss of generality, we may assume that $r, s > 0$ because $s < 0$ and $r < 0$ can be handled with index symmetry. The property holds for $r + s = 0, 1$, so now we assume that, for some $n_0 > 1$, the equality $V_{r+s}(x) = V_{r+1}(x)V_s(x) - V_r(x)V_{s-1}(x)$ holds for all $r + s < n_0$. We get Property 2 by:

$$\begin{aligned}
V_{r+s+1}(x) &= xV_{r+s}(x) - V_{r+s-1}(x) \\
&= x(V_{r+1}(x)V_s(x) - V_r(x)V_{s-1}(x)) - (V_{r+1}(x)V_{s-1}(x) - V_r(x)V_{s-2}(x)) \\
&= (xV_{r+1}(x)V_s(x) - V_{r+1}(x)V_{s-1}(x)) - (xV_r(x)V_{s-1}(x) - V_r(x)V_{s-2}(x)) \\
&= V_{r+1}(x)V_{s+1}(x) - V_r(x)V_s(x)
\end{aligned}$$

Properties 3 and 4 follows from Property 2 by letting $r = n$ as well as $s = n$ and $s = n + 1$, respectively. Property 5 is the only identity in this lemma not explicitly given in [20], so we verify below using the second order recurrence relation and Property 1:

$$\begin{aligned} V_{n+1}(x)^2 &= V_n(x)V_{n+2}(x) + 1 \\ &= V_n(x)(x\,V_{n+1}(x) - V_n(x)) + 1 \\ &= x\,V_n(x)V_{n+1}(x) - V_n(x)^2 + 1 \end{aligned}$$

Therefore, Property 5 holds. □

Remark 2.2.6. An immensely notable consequence from the above lemma is the fact that Vieta–Fibonacci polynomials always factor nontrivially in $\mathbb{Z}[x]$ with the exception of $n = 0, \pm 1, \pm 2$. Furthermore, $V_{2n}(x)$ is always divisible by x but $V_{2n+1}(x)$ never is.

The next lemma provides an example where one can change an expression from a Vieta–Lucas polynomial to an expression in terms of Vieta–Fibonacci polynomials.

Lemma 2.2.7 ([20, Exercises 47])**.** *Let $V_n(x)$ and $v_n(x)$ be the nth Vieta–Fibonacci and Vieta–Lucas polynomials, respectively. Then*

$$\begin{aligned} v_n(x) &= V_{n+1}(x) - V_{n-1}(x) \\ &= xV_n(x) - 2V_{n-1}(x) \\ &= 2V_{n+1}(x) - xV_n(x) \end{aligned}$$

Proof. To show the first equality, we begin with the base cases that $v_0(x) = 2 = 1 - (-1) = V_1(x) - V_{-1}(x)$ and $v_1(x) = x = x - 0 = V_2(x) - V_0(x)$. As before, we will prove the statements for $n > 2$ with the understanding that symmetry extends this statement to $n < 0$. If we assume that $v_n(x) = V_{n+1}(x) - V_{n-1}(x)$ for all $n \leq n_0$, then

$$\begin{aligned} v_{n_0+1}(x) &= xv_{n_0}(x) - v_{n_0-1}(x) \\ &= x(V_{n_0+1}(x) - V_{n_0-1}(x)) - (V_n(x) - V_{n-2}(x)) \\ &= (xV_{n_0+1}(x) - V_n(x)) - (xV_{n_0-1}(x) - V_{n-2}(x)) \\ &= V_{n_0+2}(x) - V_{n_0}(x) \end{aligned}$$

The second two equalities in the lemma statement follow by the second order recurrence relation of $V_n(x)$. □

There are situations in Section 2.3, Section 3.4, and Chapter 4 that rely on the vanishing sets of Vieta–Fibonacci polynomials. To that end, we find the roots of such polynomials:

Lemma 2.2.8 ([31, Lemma 2.17])**.** *The roots of $V_n(x)$ are $2\cos\frac{j\pi}{n}$ for integers $0 < j < n$.*

Proof. Recall that $V_n(x) = U_{n-1}(x/2)$; that is, the polynomials are the same up to an index shift and scaling the argument of $U_{n-1}(x/2)$ by 2. The roots of $U_n(x)$ are $\cos\frac{j\pi}{n+1}$ for integers $0 < j < n+1$ ([20, Section 41.10]), so by the index shift and doubling the argument, our proof is complete. □

Remark 2.2.9. As a consequence, this lemma emphasizes that $V_{\pm 1}(x)$ has no roots and $V_0(x)$ is identically zero.

One of the connections between Vieta polynomials and linear algebra is the appearance of Vieta polynomials in powers of matrices with determinant 1. Versions of the following lemma appear in several places in various notations, choices of language, and occasional variation of the proof (e.g. [25, Lemma 3.1.3], [49, Lemma 3.1]). We here present this fact about the powers of matrices with Vieta polynomials.

Lemma 2.2.10. *Let $C \in \mathrm{SL}_2(\mathbb{C})$. Then for all $n \in \mathbb{Z}$,*

$$C^n = V_n(\mathrm{tr}(C)) \cdot C - V_{n-1}(\mathrm{tr}(C)) \cdot I_2 \tag{2.5}$$

where I_2 is the 2×2 identity matrix.

Proof. Cayley–Hamilton states that $C^2 = \mathrm{tr}(C) \cdot C - I_2$. For $n > 0$, the proof proceeds by induction. For $n < 0$, repeat the induction using C^{-1} and $-n$. □

This is another way to see that $\mathrm{tr}(C^n) = V_n(\mathrm{tr}(C)) \cdot \mathrm{tr}(C) - V_n(\mathrm{tr}\, C)$ as in Lemma 2.2.7. With these polynomial tools in hand, we now apply them to find the character varieties of some two-bridge links.

2.3 Character varieties of two-bridge links

2.3.1 Two-bridge links (and knots)

A link L in S^3 is a piecewise linear embedding of a disjoint union of copies of S^1 ([39, Definition 0.1]). The 3-manifold $S^3 \setminus L$ is called a **link complement**. A particularly plentiful source of examples is the family of **two-bridge link** complements. These are link complements in the oriented 3-sphere S^3 whose link boundary admits a projection as visualized in Figure 2.1.

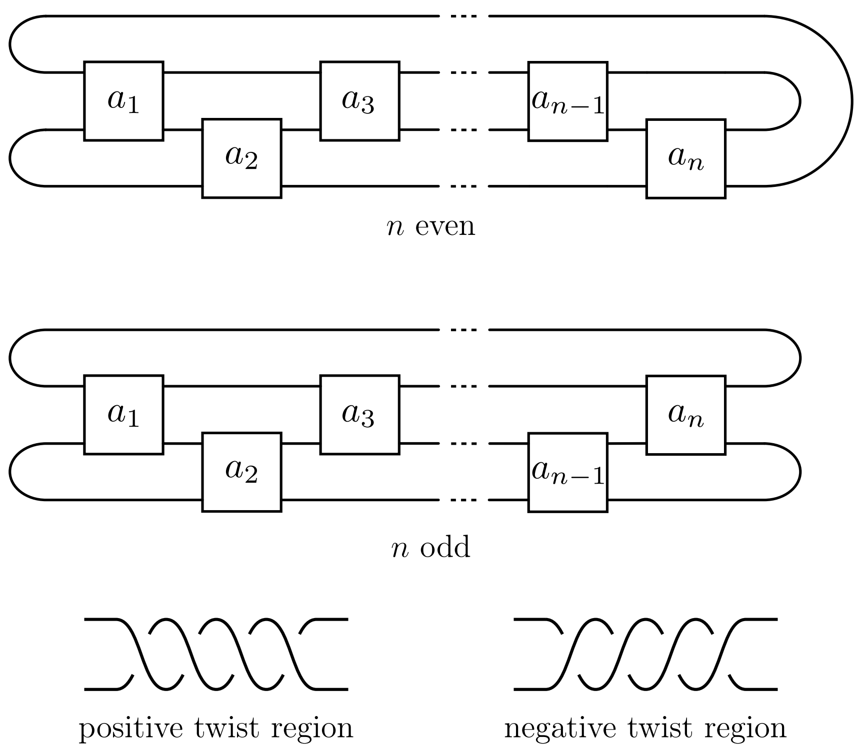


Figure 2.1: Schematic of two-bridge links $[a_1, \ldots, a_n]$

Each box in the figure corresponds to a twist region with a_i half-twists. Every two-bridge link can be expressed as a ratio of coprime integers p and q

where as a finite continued fraction ([39, Section 10.1], [44]):

$$\frac{p}{q} = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{\cdots + \frac{1}{a_n}}}}$$

This identification of p/q to such a continued fraction is a bijection ([3, Section 3.2]). Note that in some notation, it is allowed for some a_i to be negative integers (e.g. [49]), but it is always possible to convert this to a tuple where each block is positive. We will use $\mathbf{b}(p, q)$ to denote the two-bridge knot or link corresponding to the ratio p/q; furthermore, we denote the link complement $\mathbf{B}(p, q) := S^3 \setminus \mathbf{b}(p, q)$.

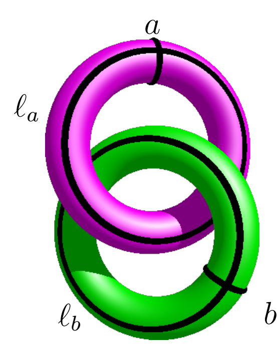


Figure 2.2: Two-bridge Hopf link with meridians and longitudes

A presentation of the fundamental group of $\mathbf{B}(p, q)$ follows from the canonical Schubert normal form in [44] of the two-bridge diagram ([42, Proposition 1], [32, (2.1)], [28, Proposition 1]). Each link compoment in Figure 2.1 admits a **meridian**, which is a loop corresponding topologically to the meridian of a torus neighborhood. For example, a and b in Figure 2.2 represent the meridians of their respective components. Let

$$iq = k_i p + r_i, \qquad 0 < r_i < p \quad \text{and} \quad e_i = (-1)^{k_i}$$

The fundamental group of a link complement $\mathbf{B}(p, q)$ admits a presentation given in [25, Section 4.5]:

$$\pi_1(\mathbf{B}(p, q)) = \langle a, b \mid aw = wa \rangle \tag{2.6}$$

where

$$w = b^{e_1} a^{e_2} \cdots b^{e_{p-1}} \tag{2.7}$$

Just as each link component admits a meridian associated with the component's torus neighborhood, there is a corresponding longitude. In Figure 2.2, these are labeled ℓ_a and ℓ_b. These will serve an important role in computing the fundamental groups of the Dehn surgeries on the Whitehead link complement in Section 4.1.2.

Any representation $\rho \in R(\mathbf{B}(p, q))$ can be conjugated so that

$$a \mapsto A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda^{-1} \end{pmatrix} \qquad\qquad b \mapsto B = \begin{pmatrix} \kappa & 0 \\ \zeta & \kappa^{-1} \end{pmatrix}$$

By consequence, the entries of $\rho(w) = W$ come from computing the representation of the word (2.7), which is dependent on both p and q. To satisfy the group relation $aw = wa$ of (2.6), we require the following equality of matrices.

$$\begin{pmatrix} \lambda W_1 + W_3 & \lambda W_2 + W_4 \\ \lambda^{-1} W_3 & \lambda^{-1} W_4 \end{pmatrix} = \begin{pmatrix} \lambda W_1 & W_1 + \lambda^{-1} W_2 \\ \lambda W_3 & W_3 + \lambda^{-1} W_4 \end{pmatrix} \tag{2.8}$$

We get a system of equations for matrix equality, noting redundancy:

$$\begin{cases} W_3 = 0 \\ W_1 - W_4 = (\lambda - \lambda^{-1}) W_2 \end{cases}$$

We thus define (dropping p and q from the notation for visual convenience but keeping in mind their continued relevance):

$$\begin{cases} \Phi_1 = W_3 \\ \Phi_2 = W_1 - W_4 - (\lambda - \lambda^{-1}) W_2 \end{cases} \tag{2.9}$$

These equations are in terms of λ, λ^{-1}, κ, κ^{-1}, and ζ. Hence, the representation variety $R(\Gamma)$ is cut out by the ideal $(\Phi_1, \Phi_2) \subset \mathbb{C}[\lambda, \lambda^{-1}, \kappa, \kappa^{-1}, \zeta]$. We will see that in cases such as $\mathbf{B}(2p, 3)$, Φ_1 and Φ_2 are both reducible. In fact, they have a nontrivial greatest common denominator, and the remaining factors will be associated with the abelian representations. We will find the character variety $X(\Gamma)$ by finding a birational map between the vanishing set of (Φ_1, Φ_2) in $\mathbb{C}^5$ (considered with coordinates corresponding to $\{\lambda, \lambda^{-1}, \kappa, \kappa^{-1}, \zeta\}$) and a subspace of $\mathbb{A}^3_{\mathbb{C}}$.

The parity of p distinguishes between two-bridge knots and links ([44]). Namely, when p is odd, $\mathbf{B}(p, q)$ is a knot complement; when p is even, $\mathbf{B}(p, q)$

is a two-component link complement. Two-bridge knots have been extensively explored, including their character varieties (e.g. [24], [33]). We are thus motivated to study links and their character varieties within this dissertation.

2.3.2 $\mathbf{b}(2p, 3)$ two-bridge links

Within this section, we find the character varieties of two-bridge links of the form $\mathbf{b}(2p, 3)$ with $p > 3$. The character varieties of two-bridge links has been studied with differing techniques, such as Chebyshev polynomials in [49] and palindromic symmetry in [7]. We will reproduce those results in this section with for the purpose of completeness to account for our use of Vieta polynomials.

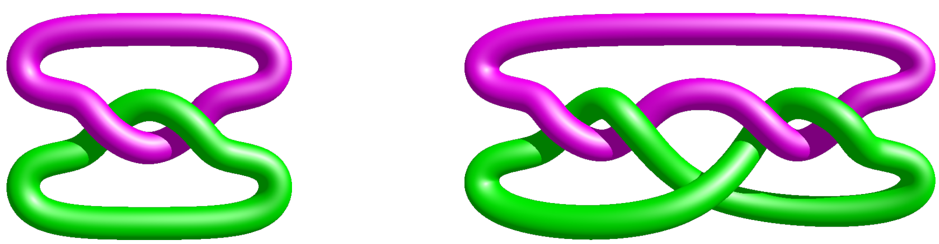


Figure 2.3: Hopf and Whitehead links in two-bridge form

The family of links $\mathbf{b}(2p, 3)$ contains two famous links: the Hopf link and the Whitehead link (see Figure 2.3). Since $\gcd(3n, 3) \neq 1$, we will consider $p = 3n + 1$ and $p = 3n + 2$. While we will make statements about both $p = 3n + 1$ and $p = 3n + 2$, only proofs for the case of $p = 3n + 1$ will be provided for the sake of space. The procedure for the case of $p = 3n+2$ is the same.

The link group of $\mathbf{b}(2p, 3)$ is

$$\pi_1(S^3 \setminus \mathbf{b}(2p,3)) = \langle a, b \mid aw_p = w_p a \rangle \tag{2.10}$$

where

$$w_p = \begin{cases} (ba)^n b^{-1}(a^{-1}b^{-1})^n a(ba)^n a^{-1} & p = 3n+1 \\ \\ (ba)^n b(a^{-1}b^{-1})^n a^{-1}(ba)^n b & p = 3n+2 \end{cases} \tag{2.11}$$

as given in [25]. The word w_p for $p = 3n+1$ reduces to $(ba)^n b^{-1}(a^{-1}b^{-1})^n(ab)^n$. We have intentionally written the form of w_p in (2.11) to evoke the parallels between the cases of $p = 3n+1$ and $p = 3n+2$.

Trace is invariant under conjugacy and inversion, so $\operatorname{tr}(AB) = \operatorname{tr}(BA) = \operatorname{tr}(A^{-1}B^{-1}) = z$. In particular, the powers of AB, BA, and $A^{-1}B^{-1}$ as matrices can be expressed with Vieta–Fibonacci polynomials where the polynomials are evaluated with the same argument:

$$(AB)^n = V_n(z)AB - V_{n-1}(z)$$

$$(BA)^n = V_n(z)BA - V_{n-1}(z)$$

$$(A^{-1}B^{-1})^n = V_n(z)A^{-1}B^{-1} - V_{n-1}(z)$$

The entries of matrix W_p can thus be neatly written in terms of λ, λ^{-1}, κ, κ^{-1}, ζ, and z, where

$$z = \zeta + \lambda\kappa + \lambda^{-1}\kappa^{-1}$$

is the argument of a Vieta–Fibonacci polynomial. For example, for $p = 3n+1$:

$$\begin{aligned} W_p = &(V_n(z)BA - V_{n-1}(z)) \cdot B^{-1} \\ &\cdot \left(V_n(z)A^{-1}B^{-1} - V_{n-1}(z)\right) \cdot (V_n(z)AB - V_{n-1}(z)) \end{aligned} \tag{2.12}$$

We use the entries of the above (after expansive multiplication) to compute the expressions of interest from (2.9): Φ_1 and Φ_2. We gather terms by the occurrence of the Vieta–Fibonacci polynomials with argument z, using identities to simplify using higher powers of Vieta–Fibonacci polynomials. Note that for convenience of reading, we drop p from our notation here; however, the exact expressions for Φ_1 and Φ_2 rely on p. For $p = 3n+1$:

$$\begin{aligned} \Phi_1 = &- \zeta V_{n+1}(z)^3 - \zeta(\zeta - \lambda\kappa^{-1} - \lambda^{-1}\kappa)V_n(z)^3 \\ &+ \zeta(\lambda + \lambda^{-1})(\kappa + \kappa^{-1})V_{n+1}(z)^2 V_n(z) \\ &- \zeta(\lambda^2 + \lambda^{-2} + \kappa^2 + \kappa^{-2} + 1)V_{n+1}(z)V_n(z)^2 \\ \Phi_2 = &- (\kappa - \kappa^{-1})V_{n+1}(z)^3 - (\kappa - \kappa^{-1})(\zeta - \lambda\kappa^{-1} - \lambda^{-1}\kappa)V_n(z)^3 \\ &+ (\kappa - \kappa^{-1})(\lambda + \lambda^{-1})(\kappa + \kappa^{-1})V_{n+1}(z)^2 V_n(z) \\ &- (\kappa - \kappa^{-1})(\lambda^2 + \lambda^{-2} + \kappa^2 + \kappa^{-2} + 1)V_{n+1}(z)V_n(z)^2 \end{aligned}$$

These each nicely convert to a polynomial of two factors. The first factor is purely in terms of κ, κ^{-1}, and ζ. The second polynomial has vanishing set birational to the vanishing set of a polynomial in terms of x, y, and z where $x = \lambda + \lambda^{-1}$, $y = \kappa + \kappa^{-1}$, and $z = \zeta + \lambda\kappa + \lambda^{-1}\kappa^{-1}$. For $p = 3n + 1$:

$$\begin{aligned}
\Phi_1 &= -\zeta \left(V_{n+1}(z)^3 - xy\, V_n(z) V_{n+1}(z)^2 \right. \\
&\qquad \left. + (x^2 + y^2 - 3)\, V_n(z)^2 V_{n+1}(z) - (xy - z)\, V_n(z)^3 \right) \\
\Phi_2 &= -(\kappa - \kappa^{-1}) \left(V_{n+1}(z)^3 - xy\, V_n(z) V_{n+1}(z)^2 \right. \\
&\qquad \left. + (x^2 + y^2 - 3)\, V_n(z)^2 V_{n+1}(z) - (xy - z)\, V_n(z)^3 \right)
\end{aligned}$$

Notably, Φ_1 and Φ_2 have a nonzero common denominator. Regardless of which case of p we consider, we may set:

$$\begin{aligned}
&\Psi_p(x, y, z) \\
&= \begin{cases}
V_{n+1}(z)^3 - xy\, V_n(z) V_{n+1}(z)^2 & \\
\qquad + (x^2 + y^2 - 3)\, V_n(z)^2 V_{n+1}(z) - (xy - z)\, V_n(z)^3, & p = 3n + 1 \\
V_n(z)^3 - xy\, V_{n+1}(z) V_n(z)^2 & \\
\qquad + (x^2 + y^2 - 3)\, V_{n+1}(z)^2 V_n(z) - (xy - z)\, V_{n+1}(z)^3, & p = 3n + 2
\end{cases}
\end{aligned} \tag{2.13}$$

Hence we find that $\Phi_1 = -\zeta \cdot \Psi_p(x, y, z)$ and $\Phi_2 = -(\kappa - \kappa^{-1}) \cdot \Psi_p(x, y, z)$.

Lemma 2.3.1. *The polynomial $\Psi_p(x, y, z)$ is nonconstant and irreducible in $\mathbb{C}[x, y, z]$ for $p > 3$.*

Proof. If $p > 3$, then $n > 1$, so neither $V_n(z)$ nor $V_{n+1}(z)$ are equal to 0. Since their product is the coefficient of x^2, we have that Ψ_p is nonconstant. The polynomial is quadratic in both x and y; that is, there are terms with x^2, y^2, xy, and neither x nor y. If Ψ_p is reducible, then it must be able to be factored into the product of three irreducible factors of form:

$$\Psi_p(x,y,z) = h_1(z)\cdot\big(f_1(z)+xf_2(z)+yf_3(z)\big)\cdot\big(g_1(z)+xg_2(z)+yg_3(z)\big)$$

for some $f_1, f_2, f_3, g_1, g_2, g_3, h_1 \in \mathbb{C}[z]$ where $\{f_1, f_2, f_3\}$ (resp. $\{g_1, g_2, g_3\}$) are pairwise coprime. The occurrences of x and y are symmetric in (2.13), so another valid factorization into three irreducible factors is:

$$\Psi_p(x,y,z) = h_1(z)\cdot\big(f_1(z)+yf_2(z)+xf_3(z)\big)\cdot\big(g_1(z)+yg_2(z)+xg_3(z)\big)$$

Because each factor is irreducible, there are two cases. For $p = 3n+1$, these cases are as follows.

Case 1: Let $f_1(z)+xf_2(z)+yf_3(z) = f_1(z)+yf_2(z)+xf_3(z)$. Then $f_2 = f_3$ and $g_2 = g_1$:

$$\begin{aligned}\Psi_p(x,y,z) &= h_1(z)\cdot\big(f_1(z)+f_2(z)(x+y)\big)\cdot\big(g_1(z)+g_2(z)(x+y)\big)\\ &= h_1(z)f_1(z)g_1(z)+(x+y)\cdot h_1(z)\big(f_1(z)g_2(z)+f_2(z)g_1(z)\big)\\ &\quad + (x+y)^2\cdot h_1(z)f_2(z)g_2(z)\end{aligned}$$

This implies that the coefficient of xy in Ψ_p is precisely twice the coefficient of x^2 (and also y^2). But by (2.13), this would mean that

$$\begin{aligned} -V_n(z)V_{n+1}(z)^2 - V_n(z)^3 &= 2V_n(z)^2V_{n+1}(z) \\ V_{n+1}(z)^2 + V_n(z)^2 &= -2V_n(z)V_{n+1}(z) \\ \big(V_{n+1}(z) + V_n(z)\big)^2 &= 0 \end{aligned}$$

which never occurs. Thus we have a contradiction.

Case 2: Let $f_1(z) + xf_2(z) + yf_3(z) = g_1(z) + yg_2(z) + xg_3(z)$. Then $f_2 = g_3$ and $f_3 = g_2$:

$$\Psi_p(x, y, z) = h_1(z) \cdot \big(f_1(z) + xf_2(z) + yf_3(z)\big) \cdot \big(f_1(z) + xf_3(z) + yf_2(z)\big)$$

We again look at the coefficients of x^2, y^2, and xy in comparison to (2.13) to deduce:

$$\begin{aligned} f_2(z)f_3(z) &= V_n(z)^2V_{n+1}(z) \\ f_2(z)^2 + f_3(z)^2 &= -V_n(z)V_{n+1}(z)^2 - V_n(z)^3 \end{aligned}$$

Thus,

$$\big(f_2(z)^2 + f_3(z)\big)^2 = -V_n(z)(V_{n+1}(z) - V_n(z))^2$$

If this were true, then $-V_n(z)$ would be a square in $\mathbb{C}[x, y, z]$. However we know that there are no repeated roots of $-V_n(z)$ by Lemma 2.2.8. Thus we have our contradiction.

The exact same procedure holds for $p = 3n + 2$. $\square$

With this irreducible polynomial in hand, we now find the character variety of this family of two-bridge links. The following theorem can also be found in [49] in the language of Chebyshev polynomials of the second kind with a different choice of final expression. We will be introducing a new substitution into this polynomial that turns out to be very useful when discussing quaternion algebras:

$$\hat{\beta} := \mathrm{tr}[a, b] - 2 = x^2 + y^2 + z^2 - xyz - 4 \tag{2.14}$$

Theorem 2.3.2. *The canonical component of $X(\pi_1(\mathbf{B}(p, q)))$ is precisely the vanishing set of the polynomial $\Psi_p(x, y, z)$; that is, when $p = 3n + 1$,*

$$\hat{\beta} \cdot V_n(z)^2 V_{n+1}(z) + V_{n+1}(z) - (xy - z)V_n(z) = 0, \tag{2.15}$$

and when $p = 3n + 2$,

$$\hat{\beta} \cdot V_n(z)V_{n+1}(z)^2 + V_n(z) - (xy - z)V_{n+1}(z) = 0. \tag{2.16}$$

The abelian representations (for all p) are the union of the two lines in $\mathbb{C}[x, y, z]$

$$(x, 2, x) \quad \textit{and} \quad (x, -2, -x).$$

Proof. The ideal (Φ_1, Φ_2) cutting out the representation variety can be decomposed into the union of two ideals: $(\zeta, \kappa - \kappa^{-1}) \cup (\Psi_p)$. The ideal $(\zeta, \kappa - \kappa^{-1})$ defines the affine variety $R(\Gamma)_{\mathrm{ab}} = \{\lambda, \lambda^{-1}, \pm 1, \pm 1, 0\} \subset \mathbb{A}^5_{\mathbb{C}}$, which is two copies of $\mathbb{A}^1$. These are precisely all the abelian representations of $R(\Gamma)$ because $\kappa - \kappa^{-1} = \zeta = 0$ precisely means that $b \mapsto \mathrm{id}_{2\times 2}$.

All the discrete and faithful representations thus lie into the subvariety of $R(\Gamma)$ defined by the ideal (Ψ_p) where x, y, and z equal $\lambda + \lambda^{-1}$, $\kappa + \kappa^{-1}$, and $\zeta + \lambda\kappa + \lambda^{-1}\kappa^{-1}$, respectively. This subvariety has precisely one component because Ψ_p is irreducible by Lemma 2.3.1. Under this map, Ψ_p is a natural polynomial to define the canonical component and the affine lines defining the abelian representations are $(x, \pm 2, \pm x)$.

Lastly, we verify that (2.15) and (2.16) are, in fact, equivalent to Ψ_p under this change of variables. For $p = 3n + 1$, consider $\Psi_p - \hat{\beta} \cdot V_n(z)^2 V_{n+1}(z)$. We recall Properties 1 and 5 from Lemma 2.2.5.

$$\begin{aligned}
&\Psi_p - \hat{\beta} \cdot V_n(z)^2 V_{n+1}(z) \\
&\quad = V_{n+1}(z)^3 - xy\, V_n(z) V_{n+1}(z)^2 - \left(z^2 - xyz - 1\right) V_n(z)^2 V_{n+1}(z) \\
&\qquad - (xy - z)\, V_n(z)^3 \\
&\quad = xy \left(-V_n(z) V_{n+1}(z)^2 + z V_n(z)^2 V_{n+1}(z) - V_n(z)^3\right) \\
&\qquad + \left(V_{n+1}(z)^3 - z^2 V_n(z)^2 V_{n+1}(z) + V_n(z)^2 V_{n+1}(z) + z V_n(z)^3\right) \\
&\quad = -xy\, V_n(z) \left(V_{n+1}(z)^2 - z V_n(z) V_{n+1}(z) + V_n(z)^2\right) \\
&\qquad - z\, V_n(z)^2 \left(z V_{n+1}(z) - V_n(z)\right) + V_{n+1}(z) \left(V_{n+1}(z)^2 + V_n(z)^2\right) \\
&\quad = -xy\, V_n(z) - z V_n(z)^2 V_{n+2}(z) + V_{n+1}(z) \left(z\, V_n(z) V_{n+1}(z) + 1\right) \\
&\quad = -xy\, V_n(z) + V_{n+1}(z) + z V_n(z) \left(V_{n+1}(z)^2 - V_n(z) V_{n+2}(z)\right)
\end{aligned}$$

$$= -xy\,V_n(z) + V_{n+1}(z) + zV_n(z)$$

$$= V_{n+1}(z) - (xy - z)\,V_n(z)$$

The same procedure can be performed for $p = 3n + 2$. This completes our proof. □

It is interesting to note that Ψ_p is almost identical for $p = 3n + 1$ and $p = 3n + 2$, save for the exchanging of $V_n(z)$ and $V_{n+1}(z)$. This parallel will appear repeatedly throughout our computations.

CHAPTER 3

QUATERNION ALGEBRAS

Hamilton committed the most famous act of mathematical vandalism in history. . .

Matroids: A Geometric Introduction [19]

3.1 Quaternion algebra as an algebraic object

The complex numbers $\mathbb{C}$ are a two-dimensional real algebra in the sense that the addition and multiplication of complex numbers happen within the $\mathbb{R}$-span of the basis $\{1, i\}$. A next sensible step is to attempy to model a real three-dimensional space with a similar structure. In 1843, Hamilton had a spark of inspiration and in his excitement carved the following equation into the stone of Brougham Bridge in Dublin:

$$i^2 = j^2 = k^2 = ijk = -1$$

This became the defining equation for the so-called Hamilton's quaternions which can be expressed as the following set:

$$\mathcal{H} := \left\{ r_0 + r_1 i + r_2 j + r_3 k \;\middle|\; i^2 = j^2 = k^2 = ijk = -1,\;\; r_0, r_1, r_2, r_3 \in \mathbb{R} \right\} \tag{3.1}$$

This has the structure of an $\mathbb{R}$-algebra. The core structure of this algebra are can be expressed with only a couple of features: the underlying field and the two standard generators are i and j which skew-commute ($ij = ji$) and individually square to elements of that field. The other two basis elements are the identity of the field and the product ij.

The generalized form was written down by Dickson in [16]. Since we already used k in Chapter 2 to denote a field, we now reaffirm our notation: "k" will always be a field and not a basis element as given in Hamilton's original demonstration. Also, to avoid confusion with indices and the traditional use of $i \in \mathbb{C}$ as $\sqrt{-1}$, we will use I and J as the other two nontrivial generators.

Definition 3.1.1. Let k be a field of characteristic not equal to 2. A **quaternion algebra** $\mathcal{A}$ over k is a k-algebra of the form

$$\begin{aligned} \{r_0 1 + r_1 I + r_2 J + r_3 IJ |\; & I^2 = \alpha, J^2 = \beta, IJ = -JI, \\ & r_0, r_1, r_2, r_3 \in k,\; \alpha, \beta \in k^*\} \end{aligned} \tag{3.2}$$

where k^* denotes the multiplicative units of k.

In words, $\mathcal{A}$ will be a k-span of the basis $\{1, I, J, IJ\}$, where we call I and J the **standard generators** for $\mathcal{A}$. There are three essential components to a quaternion algebra as seen in (3.2): the underlying field k and the squares α and β of the two standard generators. This triple encodes the quaternion algebra. To that end, we have a classical expression:

Notation. A **Hilbert symbol** for the quaternion algebra given in (3.2) is

$$\left(\frac{\alpha, \beta}{k}\right). \tag{3.3}$$

Hilbert symbols are far from unique. We will take advantage of the many congruences, such as:

Lemma 3.1.2 ([25, Lemma 2.1.2]). *Let* $\mathcal{A} \cong \left(\frac{\alpha,\beta}{k}\right)$. *Then*

$$\mathcal{A} \cong \left(\frac{\beta, \alpha}{k}\right) \cong \left(\frac{\alpha, -\alpha\beta}{k}\right) \cong \left(\frac{\lambda_1^2\alpha, \lambda_2^2\beta}{k}\right) \tag{3.4}$$

where $\lambda_1, \lambda_2 \in k^*$.

Proof. These isomorphisms arise from a change of basis of (3.2): $\{1, J, I, JI\}$; $\{1, I, IJ, -\alpha J\}$; and $\{1, \lambda_1 I, \lambda_2 J, \lambda_1\lambda_2 IJ\}$. □

Let $F \supseteq k$ be a finite degree field extension of k. There is a natural extension of a k-algebra to an F-algebra by tensoring. In the case of Hilbert symbol expressions of quaternion algebra, there is an isomorphism

$$\left(\frac{\alpha, \beta}{k}\right) \otimes_k F = \left(\frac{\alpha, \beta}{F}\right) \tag{3.5}$$

extending the scalar field while maintaining the basis.

3.1.1 Splitting quaternion algebras

By construction, quaternion algebras are 4-dimensional algebras. More specifically, they are **central simple algebras**, which is to say that the center of $\mathcal{A}$ over k is precisely k and $\mathcal{A}$ contains no nontrivial proper (two-sided) ideals. This structure lends itself to a traditional algebra: 2-by-2 matrices over a field.

Proposition 3.1.3. *[52, Proposition 2.2.8] Let* $\mathcal{A} \cong \left(\frac{\alpha,\beta}{k}\right)$ *be a quaternion algebra over* k*, and let* $k(\sqrt{\alpha})$ *be a splitting field over* k *for the polynomial* $x^2 - \alpha$*. Then the map*

$$\mathcal{A} \to \mathrm{Mat}_2(k(\sqrt{\alpha}))$$
$$I,\ J \mapsto \begin{pmatrix} \sqrt{\alpha} & 0 \\ 0 & -\sqrt{\alpha} \end{pmatrix},\quad \begin{pmatrix} 0 & \beta \\ 1 & 0 \end{pmatrix}$$

is an injective k*-algebra homomorphism and an isomorphism onto its image (noting that* $r \in k$ *maps to* rI_2*).*

Every quaternion algebra is thus a subalgebra of a matrix algebra, so it behooves us to see when a quaternion algebra is isomorphic to a matrix algebra.

Corollary 3.1.4. *[52, Corollary 2.2.12] There is an isomorphism*

$$\mathcal{A} \cong \left(\frac{1, \beta}{k}\right) \to \mathrm{Mat}_2(k)$$
$$I,\ J \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & \beta \\ 1 & 0 \end{pmatrix}$$

The application of Lemma 3.1.2 to Corollary 3.1.4 returns

$$\mathrm{Mat}_2(k) \cong \left(\frac{1, \beta}{k}\right) \cong \left(\frac{\beta, -\beta}{k}\right)$$

For a general k-algebra $\mathcal{B}$, we say that $F \supseteq k$ is a **splitting field** for $\mathcal{B}$ if $\mathcal{B} \otimes_k F \cong \mathrm{Mat}_2(F)$. This language is naturally adopted for quaternion algebras.

Definition 3.1.5. A quaternion algebra $\mathcal{A}$ is called **split** if $\mathcal{A}$ is a matrix algebra. If $\mathcal{A} \otimes_k F$ is a matrix algebra, we say that $\mathcal{A}$ **splits over** F. We will (when notationally or linguistically convenient) say a k-algebra $\mathcal{A}$ **splits over** F' if $\mathcal{A} \otimes_k (k \otimes F')$ is a matrix algebra.

The failure of a quaternion algebra to be split is a simple dichotomy in comparison to the following definition (keeping in mind that we suppose $\operatorname{char} k \neq 2$).

Definition 3.1.6. $\mathcal{A}$ is called a **division algebra** if it is an algebra in which every nonzero element has an inverse.

Theorem 3.1.7 ([52, 8.3.4])**.** *If $\mathcal{A}$ is a quaternion algebra over a field k with $\operatorname{char} k \neq 2$, then $\mathcal{A}$ either is isomorphic to $\mathrm{Mat}_2(k)$ or is a division algebra.*

Remark 3.1.8 ([25, Theorem 2.5.1])**.** In particular, for $k = \mathbb{R}$, the only two possibilities are $\mathcal{A} \cong \left(\frac{-1,-1}{\mathbb{R}}\right) =: \mathcal{H}$ or $\mathcal{A} \cong \mathrm{Mat}_2(\mathbb{R})$, according to whether both α and β are negative or not, respectively. We note that $\mathcal{H}$ is precisely Hamilton's quaternions — the quaternion algebra which satisfies (3.1). Moreover, if $k' \subseteq \mathbb{R}$, then $\left(\frac{\alpha,\beta}{k'}\right)$ fails to split over a real extension if and only if $\alpha, \beta < 0$.

There are several ways to verify if $\mathcal{A}$ is split or splits over a given field F. We begin with an approach using the reduced norm. A quaternion algebra being a matrix algebra is equivalent to containing a nonzero zero-divisor ([25, Theorem 2.3.1]). Let $\{1, I, J, IJ\}$ be the generators of $\mathcal{A}$ such that $I^2 = \alpha$ and $J^2 = \beta$. The **conjugate** of an element $r = r_0 + r_I I + r_J J + r_K IJ \in \mathcal{A}$ is defined as $\overline{r} = r_0 - r_I I - r_J J - r_K IJ$.

Definition 3.1.9. Let $r \in \mathcal{A} \cong \left(\frac{\alpha,\beta}{k}\right)$. The **(reduced) trace** is $\mathrm{trd} : \mathcal{A} \to k$ where $\mathrm{trd}(r) = r + \overline{r}$, and the **(reduced) norm** is $\mathrm{nrd} : \mathcal{A} \to k$ where $\mathrm{nrd}(r) = r\overline{r}$.

A nonzero $r \in \mathcal{A}$ is a zero-divisor if and only if $\mathrm{nrd}(r) = 0$ by [52, Lemma 3.3.5]. Furthermore:

Lemma 3.1.10 (within the proof of [25, Theorem 2.3.1])**.** *$\mathcal{A}$ is split if and only if there exists an element $r = r_0 + r_I I + r_J J \in \mathcal{A}$ such that* $\mathrm{nrd}(r) = 0$.

Proof. If such an element exists, then our previous discussion concludes that $\mathcal{A}$ is split. Conversely, let $s = s_0 + s_I I + s_J J + s_K IJ \in \left(\frac{\alpha,\beta}{k}\right)$ such that $\mathrm{nrd}(s) = 0$; that is,

$$s_0{}^2 - s_I{}^2\alpha - s_J{}^2\beta + s_K{}^2\alpha\beta = 0$$
$$s_0{}^2 - s_J{}^2\beta = \alpha(s_I{}^2 - s_K{}^2\beta)$$

If any of $s_0, s_I, s_J, s_K \in k$ are zero, then we are done by multiplying a basis element $\{1, I, J, IJ\}$ to achieve the form given in the statement. We assume that $s_0, s_I, s_J, s_K \neq 0$. Consider the element $r = r_0 + r_I I + r_J J$ where

$$r_0 = s_0 s_I + s_J s_K \beta$$
$$r_I = s_I{}^2 - s_K{}^2\beta$$
$$r_J = s_0 s_K + s_I s_J$$

Then

$$\begin{aligned}
\mathrm{nrd}(r) &= (s_0 s_I + s_J s_K\beta)^2 - \alpha(s_I{}^2 - s_K{}^2\beta)^2 - \beta(s_0 s_K + s_I s_J)^2 \\
&= (s_0 s_I + s_J s_K\beta)^2 - (s_0{}^2 - s_J{}^2\beta)(s_I{}^2 - s_K{}^2\beta) - \beta(s_0 s_K + s_I s_J)^2 \\
&= s_0{}^2 s_I{}^2 + s_J{}^2 s_K{}^2\beta^2 + 2 s_0 s_I s_J s_K\beta - (s_0{}^2 - s_J{}^2\beta)(s_I{}^2 - s_K{}^2\beta) \\
&\qquad - s_0{}^2 s_K{}^2\beta - s_I{}^2 s_J{}^2\beta - 2 s_0 s_I s_J s_K\beta \\
&= \left(s_0{}^2 s_I{}^2 - s_0{}^2 s_K{}^2\beta - s_I{}^2 s_J{}^2\beta + s_J{}^2 s_K{}^2\beta^2\right) - (s_0{}^2 - s_J{}^2\beta)(s_I{}^2 - s_K{}^2\beta) \\
&= 0
\end{aligned}$$

Thus r has the desired form and reduced norm; this existence is necessary and sufficient. $\square$

Let $F \supseteq k$ be a finite field extension, and let $\mathcal{A}' := \mathcal{A} \otimes_k F$. If $r \in \mathcal{A}'$ such that $\mathrm{nrd}_{\mathcal{A}'}(r) = 0$, then $\mathcal{A}'$ is split, and thus $\mathcal{A}$ splits over F. An example of this verification is as below:

Lemma 3.1.11. *Let $k \supseteq \mathbb{Q}$, and let $\mathcal{A} \cong \left(\frac{\alpha, \alpha+\eta^2}{k}\right)$ for $\alpha \in k^*$ and $\eta \in k$ such that $\eta^2 \neq -\alpha$. Then $\mathcal{A}$ splits over $k(i)$ where $i \in \mathbb{C}$ is the usual square root of -1.*

Proof. A straightforward calculation shows that $\eta + iI + J \in \mathcal{A} \otimes_k k(i)$ has reduced norm 0:

$$\mathrm{nrd}(\eta + iI + J) = \eta^2 - i^2 I^2 - J^2 = \eta^2 + \alpha - (\alpha + \eta^2) = 0$$ $\square$

Remark 3.1.12. The caveat $\eta^2 \neq \alpha$ is required or else the second entry of the Hilbert symbol is 0, which would force $\mathcal{A}$ to be strictly less than 4 dimensional and thus not a quaternion algebra.

The concept of splitting a quaternion algebra can also be expressed through an abstraction that considers the local fields. Let $\sigma : k \to F$ be a field embedding. Then, with respect to that embedding, we obtain an isomorphism generalized from (3.5)

$$\left(\frac{\alpha, \beta}{k}\right) \otimes_\sigma F \cong \left(\frac{\sigma(\alpha), \sigma(\beta)}{F}\right)$$

induced by

$$(r_0+r_I I_1+r_J J_1+r_{IJ} I_1 J_1)\otimes_\sigma c \to c\cdot(\sigma(r_0)+\sigma(r_I)I_2+\sigma(r_J)J_2+\sigma(r_{IJ})I_2J_2)$$

where $\{1, I_1, J_1, I_1J_1\}$ and $\{1, I_2, J_2, I_2J_2\}$ are the standard bases of $\left(\frac{\alpha,\beta}{k}\right)$ and $\left(\frac{\sigma(\alpha),\sigma(\beta)}{F}\right)$, respectively.

As seen in Chapter 1 and Chapter 2, we are particularly interested in subfields of $\mathbb{C}$, so we specialize the above to that context. Let $k \subseteq \mathbb{C}$ be a number field; that is, k is a finite degree field extension of $\mathbb{Q}$. For any complex embedding σ,

$$\left(\frac{\alpha,\beta}{k}\right)\otimes_\sigma \mathbb{C} \cong \left(\frac{\sigma(\alpha),\sigma(\beta)}{\mathbb{C}}\right) \cong \mathrm{Mat}_2(\mathbb{C})$$

We now must tend to real embeddings:

Definition 3.1.13. If $\sigma : k \to \mathbb{R}$ is a real embedding of a number field k, then $\left(\frac{\alpha,\beta}{k}\right)$ is said to be **ramified** at σ if $\left(\frac{\sigma(\alpha),\sigma(\beta)}{\mathbb{R}}\right) \cong \mathcal{H}$.

3.1.2 Ramification of quaternion algebras

The ramification defined at the end of the previous section focuses on number fields and complex numbers. This concept can be introduced over any field. To do so requires a discussion about valuations and places.

Definition 3.1.14 ([25, Definitions 0.6.1, 0.6.2])**.** Let k be a field. A **valuation** v on k is a function $v : k \to \mathbb{R}^+$ such that

(i) $v(\alpha) \geq 0$ for all $\alpha \in k$, and $v(\alpha) = 0$ if and only if $\alpha = 0$;

(ii) $v(\alpha\beta) = v(\alpha)\, v(\beta)$ for all $\alpha, \beta \in k$;

(iii) $v(\alpha + \beta) \leq v(\alpha) + v(\beta)$ for all $\alpha, \beta \in k$.

The valuation v is called **non-Archimedean** if $v(\alpha + \beta) \leq \max\{v(\alpha), v(\beta)\}$ for all $\alpha, \beta \in k$. Otherwise, v is called **Archimedean**.

Two valuations v, v' on k are **equivalent** if there exists $n \in \mathbb{R}^+$ such that $v'(\alpha) = [v(\alpha)]^n$ for all $\alpha \in k$. An equivalence class of valuations is called a **place**. Archimedean places are referred to as **infinite places**; non-Archimedean places are referred to as **finite places**.

A familiar example of a non-Archimedean valuation is the p-adic valuation. Consider the valuation $|\cdot|_p : \mathbb{Q} \to \mathbb{R}^+$ defined by mapping $|\alpha/\beta|_p = p^{\mathrm{ord}_p(\alpha) - \mathrm{ord}_p(\beta)}$ where ord_p denotes the highest power of p dividing an integer ([26, Section 4.3.1]). The localization with respect to this valuation is the p-adics $\mathbb{Q}_p$.

Localization leads to a general definition of ramification and splitting that does not explicitly reference matrices or reduced norm.

Definition 3.1.15 ([25, Definition 2.7.1])**.** Let k_σ denote the localization of the field over the place σ. Denote $\mathcal{A}_\sigma := \mathcal{A} \otimes_k k_\sigma$. Then $\mathcal{A}$ is said to be **ramified** at σ if $\mathcal{A}_\sigma$ is a division algebra over k_σ. Otherwise, $\mathcal{A}$ **splits** at σ. The (finite) set of places at which $\mathcal{A}$ is ramified is finite and is denoted $\mathrm{Ram}\,\mathcal{A}$.

The above definitions of ramification rely on places and the associated local fields. There is a local-global principle for splitting behavior that allows for easy and explicit methods to determine over which fields a quaternion algebra can split. The following theorem can be extended to higher-degree extensions such as in [25, Theorem 2.7.2] and [52, Proposition 14.6.7], but in the course of this dissertation, we are primarily concerned with quadratic extensions of $\mathbb{Q}$.

Theorem 3.1.16 ([25, Theorem 7.3.3])**.** *Let $\mathcal{A}$ be a quaternion algebra over a number field k and $F \supset k$ be a quadratic field extension. Then the following are equivalent:*

1. *F embeds in $\mathcal{A}$;*
2. *F splits $\mathcal{A}$;*
3. *$F \otimes_k k_v$ is a field for each $v \in \mathrm{Ram}(\mathcal{A})$.*

3.2 Associated quaternion algebras of hyperbolic manifolds

To set the stage for the canonical quaternion algebras upcoming in Section 3.3, we begin with a review of an analogous study of the geometric representation of a Kleinian group.

3.2.1 Trace fields

The study of character varieties in Chapter 2 centers around traces of matrices. In this subsection, we study properties of traces and, in particular, the traces of the geometric representation of a hyperbolic 3-manifold fundamental group.

Definition 3.2.1. Let Γ be a non-elementary subgroup of $\mathrm{PSL}_2(\mathbb{C})$. Let $\hat{\Gamma} = P^{-1}(\Gamma)$ where $P : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{PSL}_2(\mathbb{C})$ is the usual projection map. Then the **trace field** of Γ is the field

$$\mathbb{Q}(\mathrm{tr}(\Gamma)) := \mathbb{Q}(\mathrm{tr}\,\hat{g} \mid \hat{g} \in \hat{\Gamma}).$$

Theorem 3.2.2 ([25, Theorem 3.1.2])**.** *Let Γ be a Kleinian group of finite covolume. Then the field $\mathbb{Q}(\mathrm{tr}(\Gamma))$ is a finite extension of $\mathbb{Q}$.*

By Mostow rigidity from [30, 27], hyperbolic structure is a topological invariant of a finite volume hyperbolic 3-manifold, so we have the following:

Corollary 3.2.3 ([25, Corollary 3.1.6])**.** *Let $M = \mathbb{H}^3/\Gamma$ be a hyperbolic 3-manifold of finite volume. Then $\mathbb{Q}(\mathrm{tr}(\Gamma))$ is a topological invariant of M.*

The geometric importance of the trace field is the connection to lengths of geodesics in the base manifold. Let M be a hyperbolic 3-manifold with the associated Kleinian group Γ. Every closed geodesic arises as the axis of a loxodromic element $\gamma \in \Gamma$, and the length of that geodesic is related to the translation length $\ell_0(\gamma)$ of γ. The specific conversion between length and trace is $\cosh(\ell(\gamma)/2) = \pm\,\mathrm{tr}(\gamma)/2$ by [25, Lemma 12.1.2] where $\ell(\gamma)$ is the complex length. Thus the trace field determines the set of lengths of geodesics up to rational multiplicities.

In the spirit of Lemma 2.2.10, the computation of the trace set $\{\mathrm{tr}(\hat{g}) \mid \hat{g} \in \Gamma\}$ is given through integer polynomials in terms of a finite set. Firstly, directly from Lemma 2.2.10,

$$\mathrm{tr}(C^{-1}) = \mathrm{tr}(C) \text{ and } \mathrm{tr}(C^2) = \mathrm{tr}(C)^2 - 2$$

Similarly,

$$\mathrm{tr}(C_1C_2) = \mathrm{tr}(C_1)\,\mathrm{tr}(C_2) - \mathrm{tr}(C_1C_2^{-1}).$$

There is a significant identity that will appear in both associated and canonical quaternion algebras regarding commutators. Let $[C_1, C_2]$ denote the commutator $C_1C_2C_1^{-1}C_2^{-1}$. Then

$$\mathrm{tr}([C_1, C_2]) = \mathrm{tr}(C_1)^2 + \mathrm{tr}(C_2)^2 + \mathrm{tr}(C_1C_2)^2 - \mathrm{tr}(C_1)\,\mathrm{tr}(C_2)\,\mathrm{tr}(C_1C_2) - 2.$$

3.2.2 Associated quaternion algebras

Let Γ be the fundamental group of a hyperbolic 3-manifold.

Theorem 3.2.4 ([25, Theorem 3.2.1]). *For $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$ non-elementary, let*

$$\mathcal{A}(\Gamma) = \left\{ \sum a_i\gamma_i \;\middle|\; a_i \in \mathbb{Q}(\mathrm{tr}(\Gamma)),\ \gamma_i \in \Gamma \right\}$$

where only finitely many a_i are nonzero. Then $\mathcal{A}(\Gamma)$ is a quaternion algebra over $\mathbb{Q}(\mathrm{tr}(\Gamma))$.

This presentation of $\mathcal{A}(\Gamma)$ as a(n infinite) set is often intractable. Fortunately, for nice groups, there is a more accessible expression for this algebra encoded in the Hilbert symbol.

Theorem 3.2.5 ([25, Theorem 3.6.2])**.** *If g and h are elements of the nonelementary group Γ such that $\langle g, h\rangle$ is irreducible, g and h do not have order 2 in $\mathrm{PSL}_2(\mathbb{C})$, and g is not parabolic, then*

$$\mathcal{A}(\Gamma) \cong \left(\frac{\mathrm{tr}(g)^2\big(\mathrm{tr}(g)^2 - 4\big), \mathrm{tr}(g)^2\,\mathrm{tr}(h)^2\big(\mathrm{tr}[g,h] - 2\big)}{k(\Gamma)}\right).$$

We know from the end of Section 3.2.1 that the trace field determines the set of all possible lengths of geodesics. We now take a step further to see a geometric implication of the splitting or ramification of a quaternion algebra. The trace $\mathrm{tr}(\gamma)$ decomposes as the sum of the two eigenvalues λ_γ and λ_γ^{-1} of γ. These eigenvalues appear usefully in the following lemma.

Lemma 3.2.6 ([25, Lemma 12.2.1])**.** *Let Γ be a non-elementary group and assume that $k_\Gamma = \mathbb{Q}(\mathrm{tr}\,\Gamma)$ is a number field. For all non-trivial $\gamma \in \Gamma$, $k_\Gamma(\lambda_\gamma)$ embeds isomorphically as a subfield of $\mathcal{A}_\Gamma$.*

Recall from Theorem 3.1.16 that embedding into a quaternion algebra is equivalent to splitting that quaternion algebra. We are therefore motivated to explore a general form of an associated quaternion, its splitting behavior, and the geometric consequences thereof.

3.3 Canonical quaternion algebras

The associated quaternion algebras in the previous subsection are related to the discrete, faithful representation with the associated trace field of a Kleinian group. We now zoom out to view all representations via the character variety (and its canonical component) and we ask: what would an analogous *canonical* quaternion algebra look like?

3.3.1 Brauer groups and splitting with local rings

Let C be a curve. We look at the Brauer group of the function field $k(C)$ corresponding to that curve versus the Brauer group of the curve itself. For the former case, let $\mathcal{A}, \mathcal{A}'$ be central simple F-algebras (not restricted by dimension). We say that $\mathcal{A} \sim \mathcal{A}'$ are **Brauer equivalent** if there exists $n, n' \in \mathbb{N}$ such that $\mathrm{Mat}_n(\mathcal{A}) \cong \mathrm{Mat}_{n'}(\mathcal{A}')$.

Definition 3.3.1 ([52, Definition 8.3.3])**.** The **Brauer group** of F is the set $\mathrm{Br}(F)$ of equivalence classes of central simple F-algebras under Brauer equivalence.

A curve C is, of course, not a field, so Brauer equivalence and the objects thereof must be adapted. The first necessary definition is the generalization when the algebra is not itself central simple.

Definition 3.3.2. A **local ring** R is one with a unique maximal ideal $\mathfrak{m}$ (without loss of generality, suppose $\mathfrak{m}$ is a left ideal). The **residue field** of R is $k := R/\mathfrak{m}$.

Definition 3.3.3 ([29, Section IV.1])**.** An algebra $\mathcal{A}$ over a commutative local ring R with residue field k is called an **Azumaya algebra** if $\mathcal{A}$ is free of finite rank as an R-module and $\mathcal{A} \otimes_R k$ is a central simple algebra over k.

Let $\mathcal{A} \otimes_R k$ be a quaternion algebra, in particular. Then we can talk about the splitting behavior of $\mathcal{A} \otimes_R k$. While a ring R lacks the useful multiplicative inverses of fields, we can address that issue by extending to the **fraction field**:

$$\mathrm{Frac}(R) := \left\{ \frac{r}{s} \;\middle|\; r \in R,\, s \in R^* \right\}$$

that carries the behavior of fractions in a natural way:

- $\dfrac{r_1}{s_1} + \dfrac{r_2}{s_2} = \dfrac{r_1 s_1 + r_2 s_1}{s_1 s_2}$;
- $\dfrac{r_1}{s_1} \cdot \dfrac{r_2}{s_2} = \dfrac{r_1 r_2}{s_1 s_2}$;
- $\dfrac{1_R}{s_1} \cdot \dfrac{s_1}{1_R} = \dfrac{1_R}{1_R} = 1_{\mathrm{Frac}(R)}$;

where 1_R is the multiplicative unit of R. With this in mind, we have the following lemma:

Lemma 3.3.4. *If $\mathcal{A} \otimes_R k$ is a division algebra, then so is $\mathcal{A} \otimes_R \mathrm{Frac}(R)$.*

Proof. We prove this lemma by contrapositive. Suppose $\mathcal{A} \otimes_R \mathrm{Frac}(R)$ is the matrix algebra $\mathrm{Mat}_2(\mathrm{Frac}(R))$.The matrix algebra $\mathcal{A}' = \mathrm{Mat}_2(R)$ is the unique maximal order of $\mathrm{Mat}_2(\mathrm{Frac}(R))$ ([41, Theorem 12.8]), so the R-order $\mathcal{A}$ of $\mathcal{A} \otimes_R \mathrm{Frac}(R)$ is conjugate into $\mathcal{A}'$. Thus $\mathcal{A} \otimes_R k \subseteq \mathcal{A}' \otimes_R k$ are central simple algebras over k of the same dimension with a containment relation, which means that they are equal. Since $\mathcal{A}' \otimes_R k \cong \mathrm{Mat}_2(R \otimes_R k) \cong \mathrm{Mat}_2(k)$, we conclude that $\mathcal{A} \otimes_R k = \mathrm{Mat}_2(k)$, which has nilpotents and thus is not a division algebra. □

The objects of Definition 3.3.3 can be applied to the scheme C with its structure sheaf $\mathcal{O}_C$, with the aim of being able to use Lemma 3.3.4. Let $\chi \in C$ be a point. The stalk $\mathcal{O}_{C,\chi}$ of $\mathcal{O}_C$ is the local ring of C at χ. The residue class field of that point is denoted $k(\chi)$. Thus, over the entirety of the variety:

Definition 3.3.5 ([29, Section IV.2])**.** An **Azumaya algebra** $\mathcal{A}$ on C is a locally free sheaf of $\mathcal{O}_C$-algebras such that $\mathcal{A}_\chi$ is an Azumaya algebra over the local ring $\mathcal{O}_{C,\chi}$ for every $\chi \in C$.

It is helpful to note that the description of Azumaya algebras in Definition 3.3.3 is simply a reduction of Definition 3.3.5 where the variety C is a single point. The Brauer equivalence in this context is

$$\mathcal{A} \otimes_{\mathcal{O}_C} \mathrm{End}_{\mathcal{O}_C}(\mathcal{E}) \cong \mathcal{A}' \otimes_{\mathcal{O}_C} \mathrm{End}_{\mathcal{O}_C}(\mathcal{F})$$

where $\mathcal{E}$ and $\mathcal{F}$ are $\mathcal{O}_C$-modules.

Definition 3.3.6 ([29, Section IV.2])**.** The **Brauer group** of a variety C is the set $\mathrm{Br}(C)$ of equivalence classes of Azumaya algebras under Brauer equivalence.

The desired object is a quaternion algebra. This requires a shift from the Brauer group $\mathrm{Br}(C)$ to the Brauer group $\mathrm{Br}(k(C))$ so that we can use properties discussed in Section 3.1.

Theorem 3.3.7 ([29, Corollary IV.2.6])**.** *The natural homomorphism* $\mathrm{Br}(C) \to \mathrm{Br}(k(C))$ *is injective. An Azumaya algebra* $\mathcal{A}_{k(C)}$ *over* $k(C)$ *is defined up to isomorphism by its image in* $\mathrm{Br}(k(C))$*.*

The formal construction of this homomorphism and proof of this corollary relies on cohomologies as explored in [29, Example III.2.22], but this is beyond the scope of this dissertation.

3.3.2 The Hilbert symbol of a canonical quaternion algebra

We proceed with the construction of the canonical quaternion algebra from [10]. Let C be a canonical component for some character variety. Define $\mathcal{A}(k(C))$ to be the $k(C)$ subalgebra over $\mathrm{Mat}_2(F)$ generated by the elements of $P_C(\Gamma)$ from Section 2.1; that is,

$$\mathcal{A}(k(C)) = \left\{ \sum a_i P_C(\gamma_i) \;\middle|\; a_i \in k(C),\, \gamma_i \in \Gamma \right\} \tag{3.6}$$

where only finitely many a_i are nonzero. The representation P_C is irreducible if C contains the character of an irreducible $\mathrm{SL}_2(\mathbb{C})$ representation by [13, Lemma 1.3.1]. Thus this $\mathcal{A}(k(C))$ is a quaternion algebra over $k(C)$ called the **canonical quaternion algebra**, analogous to $\mathcal{A}(\Gamma)$ in Section 3.2:

Theorem 3.3.8 ([10, Theorem 1.1]). *Suppose that Γ is a finite generated group with $\mathrm{SL}_2(\mathbb{C})$ character variety $X(\Gamma) = X(\Gamma)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$. Let $k \subset \mathbb{C}$ be a number field, and suppose that C is a geometrically integral curve on $X(\Gamma)_k = X(\Gamma)_{\mathbb{Q}} \otimes_{\mathbb{Q}} k$ such that $C \otimes_k \mathbb{C} \subset X(\Gamma)$ has field of definition k. Further assume that C contains an irreducible character. Then taking the $k(C)$-span of $P_C(\Gamma)$ (as in* (3.6)*) defines a $k(C)$-quaternion algebra $\mathcal{A}_{k(C)} \subset \mathrm{Mat}_2(F)$ for some finite extension F of $k(C)$.*

Lemma 3.3.9 ([10, Lemma 2.8]). *There exists a pair of elements $g, h \in \Gamma$ so that the regular functions ${I_g}^2 - 4$ and $I_{[g,h]} - 2$ are not identically zero on C. More specifically, given any $g \in \Gamma$ so that I_g is not constant with value ± 2 on C there is an element $h \in \Gamma$ so that $I_{[g,h]} - 2$ is not identically zero on $\widetilde{C}$.*

Observe that the set presentation in (3.6) resembles the set presentation of an associated quaternion algebra in Theorem 3.2.4 by subsituting rational functions for traces. The substitution similarly appears in the expressibility of a canonical quaternion algebra by a Hilbert symbol:

Corollary 3.3.10 ([10, Corollary 2.9]). *Let Γ be a finitely generated group and C be a geometrically integral curve over k that is a closed subscheme of $X(\Gamma)$. Assume that C contains the character of an irreducible representation, and let $g, h \in \Gamma$ be two elements such that there exists a representation $\rho \in R(\Gamma)$ with character $\chi\rho \in C$ for which the restriction of ρ to $\langle g, h\rangle$ is irreducible. Then the canonical quaternion algebra $\mathcal{A}(k(C))$ is described by the Hilbert symbol*

$$\left(\frac{I_g^2 - 4, I_{[g,h]} - 2}{k(C)}\right).$$

We now discuss how behavior of the canonical quaternion algebra can be determined by its behavior at a point. The algebra $\mathcal{A}(C)$ is called **split** when its image under the injection in Theorem 3.3.7 is the matrix algebra. The local ring $\mathcal{O}_{C,\chi}$ acts as the ring R from Lemma 3.3.4. Since $k(C) \hookrightarrow \mathrm{Frac}(\mathcal{O}_{C,\chi})$, the quaternion algebra $\left(\frac{\alpha,\beta}{k(C)}\right)$ injects into $\left(\frac{\alpha,\beta}{\mathrm{Frac}(\mathcal{O}_{C,\chi})}\right)$. Thus if $\left(\frac{\alpha,\beta}{k(C)}\right)$ is a matrix algebra, then so is $\left(\frac{\alpha,\beta}{\mathrm{Frac}(\mathcal{O}_{C,\chi})}\right)$. Furthermore, if we consider the residue field $k(\chi)$ at χ, Lemma 3.3.4 tells us that $\mathcal{A} \otimes_{\mathcal{O}_{C,\chi}} k(\chi)$ is also a matrix algebra. All in all:

Lemma 3.3.11. *Let $\mathcal{A} = \left(\frac{\alpha,\beta}{k(C)}\right)$ be the canonical quaternion algebra of C over its function field. If there exists a number field F and a point χ in C_F such that $\mathcal{A} \otimes_{\mathcal{O}_{C,\chi}} (k(\chi) \otimes_{\mathbb{Q}} F)$ is a division algebra, then $\mathcal{A}$ is also a divison algebra.*

3.4 Canonical quaternion algebra of two-bridge link complements

Let Γ be the fundamental group of $\mathbf{B}(2p,3)$, as given in (2.10). Let $\mathcal{V}_p = (\Psi_p)$ be the ideal in $\mathbb{C}[x,y,z]$ generated by $\Psi_p(x,y,z)$ from (2.13). The ideal $\mathcal{V}_p$ is prime because Ψ_p is irreducible and $\mathbb{C}[x,y,z]$ is a UFD. Thus $\mathbb{C}[x,y,z]/\mathcal{V}_p$ is a field which we will denote $k(\mathbf{B}(2p,3))$.

Theorem 3.4.1. *The canonical quaternion algebra of* $\mathbf{B}(2p,3)$ *with* $p = 3n+1$ *or* $p = 3n+2$ *is*

$$\mathcal{A}_k(\mathbf{B}(2p,3)) \cong \left(\frac{\alpha_p, \beta_p}{\mathcal{V}_p}\right)$$

where $\alpha_p \in \{x^2-4, y^2-4, z^2-4\}$ *(whichever is nonzero) and*

$$\beta_p = \begin{cases} (xy-z)\, V_n(z)V_{n+1}(z) - V_{n+1}(z)^2, & p = 3n+1 \\ (xy-z)\, V_n(z)V_{n+1}(z) - V_n(z)^2, & p = 3n+2 \end{cases} \tag{3.7}$$

Proof. Recall the expression of the canonical quaternion algebra from Corollary 3.3.10:

$$\left(\frac{I_g^2-4, I_{[g,h]}-2}{k(C)}\right)$$

Here we have a choice of two elements of Γ to satisfy the conditions of Corollary 3.3.10. We may choose any of the following pairs: $\langle a,b\rangle$; $\langle b,a\rangle$; or $\langle ab,a\rangle$. Then the first entry α_p of our Hilbert symbol may be any of the following:

$$I_a^2-4 = x^2-4; \qquad I_b^2-4 = y^2-4; \qquad I_{ab}^2-4 = z^2-4$$

In all three pairs, however,

$$I_{[a,b]} - 2 = I_{[b,a]} - 2 = I_{[ab,a]} - 2 = x^2 + y^2 + z^2 - xyz - 4 = \hat{\beta}$$

where we're recalling $\hat{\beta} = x^2 + y^2 + z^2 - xyz - 4$ from (2.14) for notational shorthand. Thus we have an initial form

$$\mathcal{A}_k(\mathbf{B}(2p,3)) \cong \left(\frac{\alpha_p, x^2 + y^2 + z^2 - xyz - 4}{\mathcal{V}_p}\right)$$

where $\alpha_p \in \{x^2 - 4, y^2 - 4, z^2 - 4\}$ (whichever is nonzero).

The second entry β_p requires more work. We will use Lemma 3.1.2 and multiply $\hat{\beta}$ by $V_n(z)^2 V_{n+1}(z)^2$ (which is not identically 0 except when $n = 0, -1$).

$$\left(\frac{\alpha, x^2 + y^2 + z^2 - xyz - 4}{\mathcal{V}_p}\right) \cong \left(\frac{\alpha, V_n(z)^2 V_{n+1}(z)^2 \left(x^2 + y^2 + z^2 - xyz - 4\right)}{\mathcal{V}_p}\right)$$

Again, we will only prove the situation for $p = 3n + 1$. From Theorem 2.3.2, we know that

$$\begin{aligned}
\hat{\beta} \cdot V_n(z)^2 V_{n+1}(z) + V_{n+1}(z) - (xy - z)V_n(z) = 0 \\
\hat{\beta} \cdot V_n(z)^2 V_{n+1}(z) &= (xy - z)V_n(z) - V_{n+1}(z) \\
\hat{\beta} \cdot V_n(z)^2 V_{n+1}(z)^2 &= V_{n+1}(z)\left((xy - z)V_n(z) - V_{n+1}(z)\right)
\end{aligned}$$

This rightside expression is precisely β_p in (3.7). $\square$

With this Hilbert symbol in hand, we tackle the specific example of the Whitehead link complement and its Dehn surgeries.

CHAPTER 4

THE WHITEHEAD LINK COMPLEMENT

If [J. H. C. Whitehead] expected concentrated attention, he gave it in full measure himself.

M. H. A. Newman [36]
"John Henry Constantine Whitehead.
1904-1960"

4.1 The Whitehead link complement

4.1.1 The canonical quaternion algebra of the Whitehead link complement

The Whitehead link complement is one of the most iconic hyperbolic two-bridge two-component link complements. In the language of this dissertation, we view the Whitehead link as the two-bridge link $\mathbf{b}(8,3)$, which is the form $\mathbf{b}(2(3 \cdot 1 + 1), 3)$. We harness the work from earlier Section 2.3 and Section 3.4.

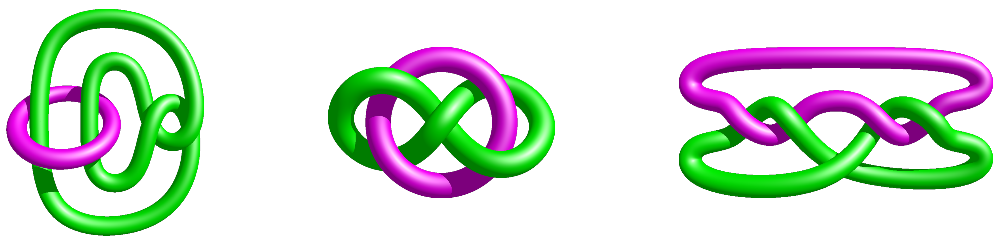


Figure 4.1: Righthanded Whitehead link: twist, lemniscate, and two-bridge

The fundamental group of W is

$$\Gamma_W = \langle a, b \mid aw = wa \rangle$$

where $w = bab^{-1}a^{-1}b^{-1}ab$. The equation defining the Whitehead link complement $\mathrm{SL}_2(\mathbb{C})$ character variety was written down by Landes in [21] and repeatedly verified in work such as [49]. It can also be directly computed from Theorem 2.3.2 to present this dissertation as self-contained in this computation.

Proposition 4.1.1 ([21, Proposition 4])**.** *The canonical component C_W of the $\mathrm{SL}_2(\mathbb{C})$ character variety of the Whitehead link complement is cut out by the polynomial*

$$\Psi_W := z(x^2 + y^2 + z^2 - xyz - 4) - (xy - 2z). \tag{4.1}$$

The set of all the abelian representations in $X(\Gamma_W)$ is the union of the two lines

$$(x, 2, x) \quad \text{and} \quad (x, -2, -x).$$

Remark 4.1.2. Of the abelian representations, the affine lines $(x, 2, x)$ and $(x, -2, -x)$ contain the characters where the image of x is nontrivial.

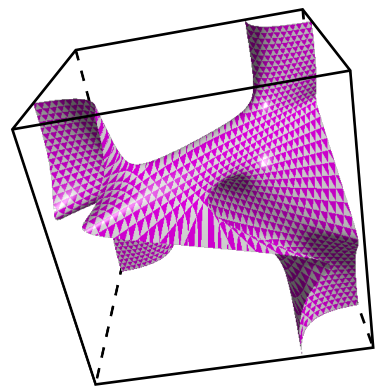


Figure 4.2: Canonical component of the Whitehead link complement in $\mathbb{A}^3_{\mathbb{R}}$

Having the canonical component of the character variety in hand, we apply Theorem 3.4.1 to find a Hilbert symbol expression for the canonical quaternion algebra.

Lemma 4.1.3. *The Hilbert symbol of the quaternion algebra on the canonical component C_W of the Whitehead link complement character variety can be written as*

$$\mathcal{A}_k(C_W) = \left(\frac{\alpha_W, x^2 + y^2 + z^2 - xyz - 4}{k(C_W)}\right) \tag{4.2}$$

for $\alpha_W \in \{x^2-4, y^2-4, z^2-4\}$ nonzero. Furthermore, away from the subvariety comprised of the intersection with the affine lines $(x, 0, 0)$ and $(0, y, 0)$,

$$\mathcal{A}_k(C_W) = \left(\frac{\alpha_W, z(xy - 2z)}{k(C_W)}\right) \tag{4.3}$$

On the intersection with the affine lines $(x,0,0)$ *and* $(0,y,0)$*, the canonical quaternion algebra has the form, respectively,*

$$\mathcal{A}_k(C_W) = \left(\frac{x^2-4,-1}{k(C_W)}\right) \qquad \textit{and} \qquad \mathcal{A}_k(C_W) = \left(\frac{y^2-4,-1}{k(C_W)}\right). \tag{4.4}$$

Proof. The Whitehead link complement is of the form $\mathbf{b}(2p,3)$, so Theorem 3.4.1 immediately gives (4.2). The same theorem also gives (4.3) away from $z \in \{2\cos\frac{\pi}{2}\} = \{0\}$. At this value of z, we know that $xy = z$; moreover, since $z = 0$, either $x = 0$ or $y = 0$. This gives us our exceptions along the affine lines $(x,0,0)$ and $(0,y,0)$. Substituting these affine lines directly into (4.2) and using the congruences from Lemma 3.1.2 returns (4.4). □

4.1.2 Dehn surgery on the Whitehead link complement

A **Dehn filling** of a 3-manifold with a torus boundary component is the procedure of gluing a solid torus to the 3-manifold along that boundary component. A **Dehn surgery** with surgery coefficients (d,m) along a link complement L will mean the removal of a regular neighborhood of the link and then performing a Dehn filling along the resulting torus boundary, such that the meridian of the solid torus T is glued to a simple closed curve in the homotopy class $[d\mu + m\ell]$ where μ is a meridian and ℓ is its corresponding longitude as discussed in Section 2.3.1. Equivalently, one attaches a torus T to the manifold such that the curve of slope d/m bounds a disc in T. Dehn surgery is a method through which one can obtain new manifolds (and orbifolds) whose geometry can often be determined. For example, a powerful result is Thurston's hyperbolic Dehn surgery theorem. We state it here for the relevant case of link complements.

Theorem 4.1.4 ([47, Theorem 5.8.2])**.** *Let M be a hyperbolic link complement. If a finite number of integral pairs (d, m) is excluded from each boundary component, then all remaining manifolds (or orbifolds) obtained by Dehn surgery on M are also hyperbolic. Furthermore, all but finitely many Dehn surgeries on a hyperbolic knot complement produce hyperbolic manifolds (or orbifolds).*

Dehn surgeries produce a manifold when $\gcd(d, m) = 1$ and an orbifold otherwise. We will examine Dehn surgery on $\partial_a W$ with the understanding that surgery can be analogously defined on the $\partial_b W$ component. First recall the homological longitude described in Section 2.3.1. Just as w can be found by using the Wirtinger presentation, we may use the same method to find the longitude by tracking the undercrossings: $\ell_a = a^{-1}w$. Performing (d, m)-surgery here is equivalent to trivializing the element $a^d {\ell_a}^m$ in the fundamental group Γ_W. More details can also be found in [22].

Remark 4.1.5. We note that (d, m)-surgery and $(-d, -m)$-surgery produce the same manifold (or orbifold).

Remark 4.1.6. We take a moment to highlight that we are using the *right-handed* Whitehead link, whose Dehn $(1,-1)$-surgery returns the figure-eight knot complement. Many sources, such as SnapPy [11], use the *left-handed* Whitehead link, where the figure-eight knot complement arises from Dehn $(1,1)$-surgery. We choose to use the right-handed to conform to the standard construction of the Weeks manifold achieved by simultaneously performing $(5,1)$- and $(5,2)$-surgery on the boundary components of the Whitehead link complement, as introduced in [53, Chapter 5].

We now manipulate the trivialized word $a^d {\ell_a}^m = \mathrm{id}$ to be expressed as

$$a^{d-3m} = (a^{-2}w^{-1})^m.$$

We use this form in particular to compensate for the λ and λ^{-1} factors in the diagonal entries of $W := \rho(w)$, so let $\hat{w} := a^{-2}w^{-1}$. The matrix $\hat{W} := \rho(\hat{w})$ can be found by multiplying $A^{-2}W^{-1}$ where W is from (2.12):

$$\hat{W} = \begin{pmatrix} \hat{W}_1 & \hat{W}_2 \\ 0 & \hat{W}_4 \end{pmatrix}$$

where

$$\begin{aligned} \hat{W}_2 &= -(z^2-1) \\ \hat{t} &:= \mathrm{tr}(\hat{W}) \\ &= y(x^2-2)z + x(z^2 - xyz + 1) \\ &= x(z^2-1) + 2(x - yz) \end{aligned} \tag{4.5}$$

Definition 4.1.7. Consider the polynomials

$$\Psi_W := z(x^2 + y^2 + z^2 - xyz - 4) - (xy - 2z)$$

$$\Phi_d^m := V_{d-3m}(x) + (z^2 - 1) \cdot V_m(\hat{t})$$

$$\Theta_d^m := V_{d-3m+1}(x) - \big((x - yz) \cdot V_m(\hat{t}) - V_{m-1}(\hat{t})\big)$$

where $\hat{t}$ is as in (4.5). Let $\mathcal{V}_d^m := (\Psi_W, \Phi_d^m, \Theta_d^m) \subseteq \mathbb{C}[x, y, z]$. We define the algebraic set C_d^m as the vanishing set of $\mathcal{V}_d^m$; that is, the coordinate ring of C_d^m is $\mathbb{C}[x, y, z]/\mathcal{V}_d^m$.

Proposition 4.1.8. *The* $\mathrm{SL}_2(\mathbb{C})$ *character variety of* W_m^d *equals the variety* C_d^m *from Definition 4.1.7, with some additional some abelian characters. Furthermore, when* $m \neq 0$*, the canonical quaternion algebra is*

$$\mathcal{A}_k(C_d^m) \cong \left(\frac{\alpha,\ V_m(\hat{t})\left((x^2 - 2)V_m(\hat{t}) - xV_{m-1}(\hat{t}) - v_{d-3m+1}(x)\right)}{k(C_d^m)} \right) \tag{4.6}$$

where $\alpha \in \{x^2 - 4, y^2 - 4, z^2 - 4\}$.

Proof. Recall from (2.2.10) that $C^n = V_n(\mathrm{tr}(C)) \cdot C - V_{n-1}(\mathrm{tr}(C)) \cdot I_2$. Thus:

$$A^{d-3m} = V_{d-3m}(x) \cdot A - V_{d-3m-1}(x) \cdot I_2 \tag{4.7}$$

$$\hat{W}^m = V_m(\hat{t}) \cdot \hat{W} - V_{m-1}(\hat{t}) \cdot I_2$$

The equality $A^{d-3m} = \hat{W}^m$ relies on the equality of three matrix entries:

$$V_{d-3m}(x) \cdot A_1 - V_{d-3m-1}(x) = V_m(\hat{t}) \cdot \hat{W}_1 - V_{m-1}(\hat{t}) \tag{4.8a}$$

$$V_{d-3m}(x) \cdot A_4 - V_{d-3m-1}(x) = V_m(\hat{t}) \cdot \hat{W}_4 - V_{m-1}(\hat{t}) \tag{4.8b}$$

$$V_{d-3m}(x) \cdot A_2 = V_m(\hat{t}) \cdot \hat{W}_2 \tag{4.8c}$$

We must be the solution set of the triple $\{(4.8a), (4.8b), (4.8c)\}$. This is equivalent to being the solution set of the triple $\{(4.8a) + (4.8b), (4.8a) - (4.8b), (4.8c)\}$, where we use "+" (resp. "−") of equalities here to denote equating the sum (resp. difference) of the lefthand sides with the sum (resp. difference) of the righthand sides. The equations of this new tuple are, respectively,

$$x\, V_{d-3m}(x) - 2V_{d-3m-1}(x) = \hat{t}\, V_m(\hat{t}) - 2V_{m-1}(\hat{t}) \tag{4.9a}$$

$$(\lambda - \lambda^{-1})V_{d-3m}(x) = (\hat{W}_1 - \hat{W}_4)V_m(\hat{t}) \tag{4.9b}$$

$$V_{d-3m}(x) = V_m(\hat{t}) \cdot \hat{W}_2 \tag{4.8c}$$

Since A and $\hat{W}$ commute, we can adapt (2.8) to see that any solution of (4.8c) is also a solution to (4.9b). We can next apply Lemma 2.2.7 to the lefthand side of (4.9a) to get a pair of equalities $\{(4.10a), (4.8c)\}$ with the same solution set as $\{(4.8a), (4.8b), (4.8c)\}$:

$$2V_{d-3m+1}(x) - xV_{d-3m}(x) = \hat{t}\, V_m(\hat{t}) - 2V_{m-1}(\hat{t}) \tag{4.10a}$$

$$V_{d-3m}(x) = V_m(\hat{t}) \cdot \hat{W}_2 \tag{4.8c}$$

The usefulness of (4.10a) is that we can use (4.8c) to convert the $xV_{d-2m}(x)$ term to get a pair of equations with the same solution set:

$$2V_{d-3m+1}(x) = (\hat{t} + x\hat{W}_2)\, V_m(\hat{t}) - 2V_{m-1}(\hat{t}) \tag{4.11a}$$

$$V_{d-3m}(x) = V_m(\hat{t}) \cdot \hat{W}_2 \tag{4.8c}$$

The next step is to use (4.5) to see that $\hat{t} + x\hat{W}_2 = 2(x - yz)$ Thus, we divide both sides of (4.11a) by 2 and get the pair of polynomials

$$V_{d-3m+1}(x) = (x - yz)\; V_m(\hat{t}) - V_{m-1}(\hat{t}) \tag{4.12a}$$

$$V_{d-3m}(x) = -(z^2 - 1)\, V_m(\hat{t}) \tag{4.8c}$$

The solution set of $\{(4.12a), (4.8c)\}$ is precisely the zero set of $\{\Theta_d^m, \Phi_d^m\}$ as given in Definition 4.1.7. Hence B_d^m is the $\mathrm{SL}_2(\mathbb{C})$ character variety, excluding some abelian characters. These abelian characters would lie on the pair of affine lines $(x, 2, x)$ and $(x, -2, -x)$ instead of the variety cut out by Ψ_p, as demonstrated in Theorem 2.3.2.

To write the canonical quaternion algebra, we use the form in Lemma 4.1.3 to get that

$$\mathcal{A}_k(C_d^m) = \left(\frac{\alpha, \beta'}{k(C_d^m)} \right)$$

where $\alpha \in \{x^2 - 4, y^2 - 4, z^2 - 4\}$ and (where β is the second entry in the Hilbert symbol from Lemma 4.1.3):

$$
\begin{aligned}
\beta' &= V_m(\hat{t})^2\beta \\
&= V_m(\hat{t})^2(xyz - 2z^2) \\
&= V_m(\hat{t})^2(-x(x - yz) + x^2 - 2 - 2(z^2 - 1)) \\
&= V_m(\hat{t})\left(-x(x - yz)V_m(\hat{t}) + (x^2 - 2)V_m(\hat{t}) - 2(z^2 - 1)V_m(\hat{t})\right) \\
&= V_m(\hat{t})\left(-x(V_{d-3m+1}(x) + V_{m-1}(\hat{t})) + (x^2 - 2)V_m(\hat{t}) + 2(V_{d-3m}(x))\right) \\
&= V_m(\hat{t})\left((x^2 - 2)V_m(\hat{t}) - xV_{m-1}(\hat{t}) - v_{d-3m+1}(x)\right)
\end{aligned}
$$

which is precisely the second entry of the Hilbert symbol of (4.6). □

Remark 4.1.9. The exclusion of $m = 0$ is rooted in the requirement that $V_m(\hat{t}) \neq 0$, which will be addressed in Section 4.3.

In [34], the set of hyperbolic Dehn filling slopes of one component of the Whitehead link complement is computed and is shown to be all but six slopes (excluding the ill-defined $(0, 0)$): $(0, 1)$, $(1, 0)$, $(1, 1)$, $(2, 1)$, $(3, 1)$, and $(4, 1)$. We will thus look at the contextual behavior in the families of $(d, 1)$ and $(d, 0)$. The surgeries $(d, 1)$ will produce manifolds; $(d, 0)$-surgeries, however, will produce orbifolds.

4.2 Once-punctured torus bundles of tunnel number 1

The construction of character varieties arising from (d, m)-surgery in the previous section can be applied to known families of manifolds. We'll start with the once-punctured torus bundles with tunnel number one, where tunnel number one means that the once-punctured torus bundle admits a genus-two Heegaard splitting ([4]). This is a one parameter family that can be found through Dehn $(d, 1)$-surgery on the Whitehead link complement ([4, Theorems 1.2, 1.3]). For small d, there are some familiar manifolds ([5]):

- W_1^1 is the positive trefoil knot complement;
- W_{-1}^1 is the figure-eight knot complement, which we discuss in more detail in Section 4.2.2;
- W_5^1 is the figure-eight sibling manifold.

The family of manifolds W_d^1 are all hyperbolic except for $d = 0, 1, 2, 3, 4$. Excluding those four cases, the fundamental group $\pi_1(W_d^1)$ has a discrete, faithful representation into $\mathrm{SL}_2(\mathbb{C})$ and so is of interest through the lens of the $\mathrm{SL}_2(\mathbb{C})$ character variety. Much work has been done on the $\mathrm{SL}_2(\mathbb{C})$ and $\mathrm{PSL}_2(\mathbb{C})$ character varieties, typically using a different presentation of the fundamental group (e.g. [5, 48]). Results here are compatible with those works but are presented and proved in the language of this dissertation for consistency.

4.2.1 Quaternion algebra for $(d, 1)$-surgery

Definition 4.2.1. Recall that

$$\Psi_W := z(\hat{\beta}) - (xy - 2z) \tag{4.1}$$

and the specialization of $(d,1)$ on Definition 4.1.7

$$\Theta_d^1 := V_{d-2}(x) - (x - yz) \tag{4.13}$$

$$\Phi_d^1 := V_{d-3}(x) - (1 - z^2) \tag{4.14}$$

Let $I_1 = (\Psi_W, \Theta_d^1, \Phi_d^1)$, and let E_d be the vanishing set of I_1.

Lemma 4.2.2. *The curve E_d is the $\mathrm{SL}_2(\mathbb{C})$ character variety of $(d,1)$-surgery on the Whitehead link complement. Furthermore, the canonical quaternion algebra admits the Hilbert symbol:*

$$\mathcal{A}_k(E_d) \cong \left(\frac{x^2-4, x^2-2-v_{d-2}(x)}{k(E_d)}\right) \tag{4.15a}$$

$$\cong \left(\frac{(V_{d-1}(x)(x)+2)\,(V_{d-3}(x)(x)-2)+x^2+1, x^2-2-v_{d-2}(x)}{k(E_d)}\right) \tag{4.15b}$$

$$\cong \left(\frac{-V_{d-3}(x)-3, x^2-2-v_{d-2}(x)}{k(E_d)}\right) \tag{4.15c}$$

Proof. The character variety follows directly from Proposition 4.1.8 by setting $m = 1$ with an interesting quirk: the element Ψ_W in the ideal is often redundant.

Case 1 Let $z \neq 0$. Then $z \cdot \Psi_W$ is a nonzero polynomial:

$$\begin{aligned} z \cdot \Psi_W &= (z^2)^2 - x(yz)z^2 + z^2(x^2-2) + (yz)^2 - x(yz) \\ &= (z^2-1)^2 - (x-yz)(-x(z^2-1) - (x-yz)) - 1 \\ &= V_{d-3}(x)^2 - V_{d-2}(x)(xV_{d-3}(x) - V_{d-2}(x)) - 1 \end{aligned}$$

$$= V_{d-3}(x)^2 - V_{d-2}(x)V_{d-4}(x) - 1$$

$$= 0$$

Since $z \neq 0$, we must have $\Psi_W = 0$; that is, $\Psi_W \in (\Phi_d^1, \Theta_d^1)$ as an element of an ideal in $\mathbb{C}[x, y, z]$. This means that (4.13) and (4.14) are sufficient to define E_d. We can find the canonical quaternion algebra through Lemma 4.2.2. We first verify the second entry using the form from (4.15c), which we can do since $z \neq 0$. Recalling that $z^2 = 1 - V_{d-3}(x))$ and $yz = x - V_{d-2}(x)$,

$$\begin{aligned} z(xy - 2z) &= x(yz) - 2(z^2) \\ &= x(x - V_{d-2}(x)) - 2(1 - V_{d-3}(x)) \\ &= x^2 - 2 - (2V_{d-3}(x) - xV_{d-2}(x)) \\ &= x^2 - 2 - v_{d-2}(x). \end{aligned}$$

Thus (4.15a) and (4.15c) immediately follow from Proposition 4.1.8. The equation (4.15b) requires a bit more work. By Proposition 4.1.8 when $m = 1$,

$$\mathcal{A}_k(E_d) \cong \left(\frac{y^2 - 4, x^2 - 2 - v_{d-2}(x)}{k(E_d)}\right) \cong \left(\frac{z^2(y^2 - 4), x^2 - 2 - v_{d-2}(x)}{k(E_d)}\right)$$

We can see that:

$$z^2(y^2 - 4) = (yz)^2 - 4z^2$$

$$
\begin{aligned}
&= (x - V_{d-2}(x))^2 - 4(1 - V_{d-3}(x)) \\
&= x^2 - 2x\, V_{d-2}(x) + V_{d-2}(x)^2 - 4 + 4V_{d-3}(x) \\
&= x^2 - 2(V_{d-1}(x) + V_{d-3}(x)) + (V_{d-1}(x)V_{d-3}(x) + 1) \\
&\qquad - 4 + 4V_{d-3}(x) \\
&= (V_{d-1}(x)(x) - 2)\,(V_{d-3}(x)(x) + 2) + x^2 + 1
\end{aligned}
$$

This completes (4.15b).

Case 2 Let $z = 0$. Then $\Psi_W = xy$, $V_{d-2}(x) = x$, and $V_{d-3}(x) = 1$. From this, $V_{d-4}(x) = 0$. The zeroes of this function are $x = 2\cos\frac{j\pi}{d-4}$ by Lemma 2.2.8, and $x \neq 0$ except when $k/(d-4) = 1/2$. If $x \neq 0$, then we are on the affine line $(x, 0, 0)$ and are thus one of the finite characters $\big(2\cos\frac{j\pi}{d-4}, 0, 0\big)$. Let $k(\hat{E}_d)_j := \mathbb{C}[x, y, z]/(x - 2\cos\frac{j\pi}{d-4}, y, z)$ denote the coordinate ring at such a character. Then its specialized quaternion algebra is:

$$
\begin{aligned}
\mathcal{A}_k(E_d) &\cong \left(\frac{(2\cos\frac{j\pi}{d-4})^2 - 4, -1}{k(\hat{E}_d)}\right) \\
&\cong \left(\frac{\cos^2\frac{j\pi}{d-4} - 1, -1}{k(\hat{E}_d)}\right) \\
&\cong \left(\frac{-\sin^2\frac{j\pi}{d-4}, -1}{k(\hat{E}_d)}\right)
\end{aligned}
$$

□

There's an interesting mirror fact that pops up if $d_1 + d_2 = 4$; that is, $d_1 - 2 = 2 - d_2$.

$$V_{d_1-2}(x) = V_{2-d_2}(x) = -V_{d_2-2}(x)$$

$$V_{d_1-1}(x) = V_{(2-d_2)+1}(x) = -V_{d_2-3}(x)$$

$$v_{d_1-2}(x) = v_{2-d_2}(x) = v_{d_2-2}(x)$$

So if $d_1 + d_2 = 4$, working through the quaternion algebras from Lemma 4.2.2 tells us that the entries of the forms (4.15a) and (4.15b) are the same for $\mathcal{A}_k(E_{d_1})$ and $\mathcal{A}_k(E_{d_2})$; the distinguishing feature is the underlying function field:

$$\mathcal{A}_k(E_{d_1}) = \left(\frac{(V_{d_1-1}(x)+2)\,(V_{d_1-3}(x)-2)+x^2+1,\, x^2-2+v_{d_1-2}(x)}{k(E_{d_1})}\right)$$

$$\mathcal{A}_k(E_{d_2}) = \left(\frac{(V_{d_2-1}(x)+2)\,(V_{d_2-3}(x)-2)+x^2+1,\, x^2-2+v_{d_2-2}(x)}{k(E_{d_2})}\right)$$

This applies, for example, to the figure-eight knot complement and its sibling, coming from $(-1, 1)$ and $(5, 1)$-surgery, respectively.

Remark 4.2.3. In [5, Proposition 5.23], Baker and Petersen used the defining polynomials of the form $f_n(y) = y - xz$, etc. This seeming difference between their notation and ours arises because they consider Dehn $-(n + 2)$-surgery on the *left-handed Whitehead link*, and there is an index shift of the recursive polynomial. The conversion of surgery notation is $d = n + 2$. By further recalling the index shift from Section 2.2, we see $f_n(x) = f_{d-2}(x) = V_{d-3}(x)$, which confirms consistency with the result of (4.2.2).

Given the nice form of a character variety and its canonical quaternion algebra, we delve into splitting behavior.

Lemma 4.2.4. *Let $p \in \mathbb{Z}$ such that E_p is hyperbolic (i.e., $p \neq 0, 1, 2, 3, 4$). If $\mathcal{A}_k(E_p)$ splits over $k(E_p) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{d})$, then either $-p$ or $-p(4-p)$ is a square in $k(E_p) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{d})$.*

Proof. The proof will use the reduced norm approach to make claims about splitting. Considering the reduced norm of some element $r_0 + r_I I + r_J J \in \mathcal{A}_k(E_p)$ (which is sufficient by Lemma 3.1.10), we deduce that certain elements must exist inside the candidate field. (Later proofs will rely on their necessary absence.) Let $r_0, r_I, r_J \in k(E_p) \otimes \mathbb{Q}(\sqrt{d})$; then the reduced norm of $r_0 + r_I I + r_J J$ is

$$r_0{}^2 - (-V_{p-3}(x) - 3)\, r_I{}^2 - (x^2 - 2 - v_{p-2}(x))\, r_J{}^2 = 0.$$

Recall that $V_n(2) = n$ and $v_n(2) = 2$, so restricting to $x = 2$ gives $z^2 = 4 - p$. As long as $z \neq 0$ (that is, $p \neq 4$),

$$\begin{aligned} r_0{}^2\big|_{x=2} + p\, r_I{}^2\big|_{x=2} &= 0 \\ \left(\frac{r_0{}^2}{r_I{}^2}\right)\bigg|_{x=2} &= -p \end{aligned}$$

for all $p \neq 4$. However, $r_0 + r_I I + r_J J$ need not be free in x; by (4.13), y is expressible in terms of x and z, but x and z are not linear terms and thus cannot be free. Recall that $z^2 = 1 + V_{p-3}(x)$. Evaluated at $x = 2$, we get $z = \pm\sqrt{1-(p-3)} = \pm\sqrt{4-p}$. Thus $-p$ is a square in $k(E_p) \otimes \mathbb{Q}(\sqrt{d}, \sqrt{4-p})$ for any quadratic splitting extension $\mathbb{Q}(\sqrt{d})$, except possibly when $p = 4$. Because $p \in \mathbb{Z}$, this means that either $-p$ or $-p(4-p)$ is a square in $k(E_p) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{d})$. $\square$

A direct corollary is the beginning of the proof of Theorem A.

Corollary 4.2.5. *Let $d \in \mathbb{Q}$. Then $\mathcal{A}_k(C_W) \otimes k(C_W)(\sqrt{d})$ is a division algebra.*

Proof. If $\mathcal{A}(W) \otimes k(C_W)(\sqrt{d})$ is a matrix algebra, then so is the canonical quaternion algebra of each character subvariety by Lemma 3.3.11. Lemma 4.2.4 says that if $\mathcal{A}(E_p) \otimes k(E_p)(\sqrt{d})$ is a matrix algebra, then either $-p$ or $4-p$ is a square in $k(E_p)(\sqrt{d})$. We proceed by contradiction: if $-p$ is a square in $k(E_p)(\sqrt{d})$ for infinitely many primes p, then $k(E_p)(\sqrt{d})$ must be isomorphic to an infinite field extension of $\mathbb{Q}$, which is a contradiction. If instead only finitely many $-p$ are squares in $k(E_p)(\sqrt{d})$, then there exists some prime p_0 such that $4-p$ is a square in $k(E_p)(\sqrt{d})$ for all prime $p > p_0$; this similarly compels $k(E_p)(\sqrt{d})$ to be isomorphic to an infinite degree field extension of $\mathbb{Q}$. Therefore, $\mathcal{A}(W) \otimes k(C_W)(\sqrt{d})$ is a division algebra. $\square$

The requirement that either $-p$ or $4-p$ is a square in the extension field has thus been addressed across the entirety of C_W. It is interesting to see if there are subvarieties of C_W whose canonical quaternion algebras do, in fact, split over a quadratic number field.

Theorem C. *Let $p \in \mathbb{Z}$ be such that $p = (p_1)^2 p_2$ where $p_1, p_2 \in \mathbb{Z}$ and $|p_2|$ is squarefree with $|p_2| \neq 0, 1$. If*

(i) $p < 0$ or

(ii) $p > 4$ such that $p_2 \equiv 7 \mod 8$

then $\mathcal{A}_k(E_p)$ does not split over any quadratic extension of $\mathbb{Q}$.

Proof. Proving this theorem requires examining the two cases of Lemma 4.2.4: $-p$ or $-p(4-p)$ is a square in $k(E_p) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{d})$; that is,

- $\sqrt{-p} \in k(E_p) \otimes \mathbb{Q}(\sqrt{d})$ or
- $\sqrt{-p(4-p)} \in k(E_p) \otimes \mathbb{Q}(\sqrt{d})$

The next step is to find a character in E_p where neither of the above inclusions can hold for any d. Consider characters where $x = 0$. The satisfying polynomials in (4.13) and (4.14) tell us that $V_{p-2}(0) = -yz$ and $V_{p-3}(0) = z^2 - 1$.

This is dependent on $p \mod 4$:

$$V_n(0) = \begin{cases} 0, & n \equiv 0 \mod 2 \\ 1, & n \equiv 1 \mod 4 \\ -1, & n \equiv 3 \mod 4 \end{cases}$$

This work requires that $z \neq 0$, so we must exclude the case $V_{p-2}(0) = 0$; that is, the following machinery only works for odd p. Fortunately, there are the following naturally arising characters on E_p:

$$\chi_\rho = \begin{cases} (0,1,-1) \Rightarrow k_\rho(E_d) = \mathbb{Q} \text{ and } \mathcal{A}_\rho \cong \left(\frac{-1,-2}{\mathbb{Q}}\right), & p \equiv 1 \mod 4 \\ (0,1,1) \quad \Rightarrow k_\rho(E_d) = \mathbb{Q} \text{ and } \mathcal{A}_\rho \cong \left(\frac{-1,-2}{\mathbb{Q}}\right), & p \equiv 3 \mod 4 \end{cases}$$

The ramification set of $\left(\frac{-1,-2}{\mathbb{Q}}\right)$ is $\mathrm{Ram}(\mathcal{A}_\rho) = \{2, \infty\}$. Theorem 3.1.16 tells us that $\mathcal{A}_\rho$ splits over $\mathbb{Q}(\sqrt{d})$ if and only if every place in $\mathrm{Ram}\,\mathcal{A}_\rho$ does NOT split in $\mathbb{Q}(\sqrt{d})$ (i.e. $\mathbb{Q}(\sqrt{d})_v$ is a field for all $v \in \mathrm{Ram}\,\mathcal{A}_\rho$. We know that the real place returns a field, so $\mathcal{A}_\rho$ splits over $\mathbb{Q}(\sqrt{d})$ if and only if 2 does NOT split in $\mathbb{Q}(\sqrt{d})$. Recall from [46, Section 10] that the prime ideal $(2) \subset \mathbb{Q}(\sqrt{d})$ splits if $d \equiv 1 \mod 8$; remains prime if $d \equiv 5 \mod 8$; and ramifies otherwise.

Let's examine the two cases $-p$ and $-p(4-p)$.

- $\sqrt{-p} \in \mathbb{Q}(\sqrt{d})$: assuming that $-p \notin \mathbb{Q}^{\times 2}$, let $d = -p_2$, and we see that $d < 0$ only for $p_2 > 0$, so we may immediately rule out $p < 0$ with $|p| \notin \mathbb{Q}^{\times 2}$, giving us the necessary failure for (i). If $p > 4$, the necessary failure arises when the prime ideal (2) splits in $\mathbb{Q}(\sqrt{-p_2})$; that is, the necessary failure occurs when $-p_2 \equiv 1 \mod 8$, which gives us (ii).

- $\sqrt{-p(4-p)} \in \mathbb{Q}(\sqrt{d})$: because of the gap between integer squares, we know that $-p(4-p) \notin \mathbb{Q}^{\times 2}$ for all odd $p \in \mathbb{Z}$. Thus, we may set $d = -p(4-p)$. But $d = p^2 - 4p > 0$ for all $p \neq 0, 1, 2, 3, 4$ which means that $\mathcal{A}_\rho$ cannot split over $\mathbb{Q}(\sqrt{d})$ because of Remark 3.1.8. This gives us the necessary conditions of failure for $p < 0$ in (i) and for $p > 4$ in (ii).

To make sure both bullet points are addressed when $p > 4$, we need to ensure that $-p_2 \equiv \mod 1 \mod 8$; that is, $p_2 \equiv 7 \mod 8$. □

Example 4.2.6. The canonical quaternion algebra of the manifold m023 in the SnapPy census [11] does not split over any quadratic extension because it arises from $(-3, 1)$-surgery on one component of the Whitehead link complement.

Example 4.2.7. The canonical quaternion algebra of the manifold m022 in the SnapPy census [11] does not split over any quadratic extension. This manifold is nonarithmetic ([25, Appendix 13.6]) and comes from $(7, 1)$-surgery on one component of the Whitehead link complement.

Remark 4.2.8. Note that p itself need not be square-free. For example, the canonical quaternion algebra of the manifold produced by $(63, 1)$-surgery (which is hyperbolic!) does not split over any quadratic extension because $63 = (3)^2 \cdot 7$.

4.2.2 Figure-8 knot complement

The figure-eight knot complement is an example throughout the theory of hyperbolic manifolds with abundant compelling properties. This manifold arises from $(1,-1)$-surgery on one component of the Whitehead link complement. Chinburg–Reid–Stover proved that $\mathcal{A}_k(E_{-1})$ splits over $\mathbb{Q}(i)$ in [10, Lemma 7.1]. We extend their statement by proving that $\mathbb{Q}(i)$ is the unique quadratic extension.

Theorem D. *The canonical quaternion algebra (over the function field) of the figure-eight knot complement splits over* $\mathbb{Q}(i)$ *and no other quadratic extension of* $\mathbb{Q}$.

Proof. Lemma 4.2.2 requires that $V_{-3}(x) = x - yz$ and $V_{-4}(x) = -(z^2-1)$. The former equation tells us that y is entirely dependent on x and z. The latter equation tells us that $z^2 = x^3 - 2x + 1$. Thus we can neatly transition our curve from $\mathbb{A}^3_{\mathbb{C}}$ to $\mathbb{A}^2_{\mathbb{C}}$ to obtain the curve $y^2 = x^3 - 2x + 1$ in $\mathbb{A}^2_{\mathbb{C}}$. This is the same curve that Chinburg–Reid–Stover consider in [10], with a change of notation that we will follow from the remainder of this proof: their y is our z. They show that this coincides with the canonical component of the figure-eight knot $\mathrm{SL}_2(\mathbb{C})$ character variety. Specifically in [10, Lemma 7.1], they prove that

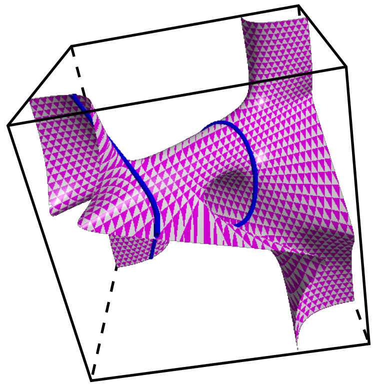
$$\mathcal{A}_k(E_{-1}) \cong \left(\frac{x^3 - 4x^2 + 6x - 3, x - 2}{k(E_{-1})}\right)$$

and that this quaternion algebra splits over $k(E_{-1})(i)$ via the reduced norm of the word $r = r_0 + r_I I + r_J J$ where $r_0 = -2 - i + (1+i)x$, $r_I = i$, and $r_J = 1 - i - x$. We now go further. Now let's assume that there exists some other word $r_0 + r_I I + r_J J$ with reduced norm 0. Then we can evaluate the polynomial equality at $x = 1$ (and so $y = 0$ as in Chinburg–Reid–Stover's notation):

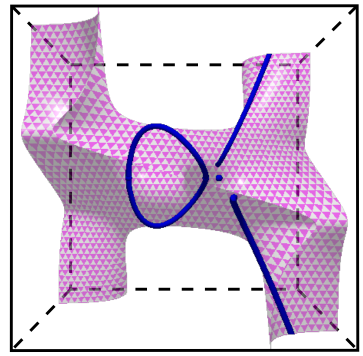
$$r_0{}^2 - r_I{}^2(x^3 - 4x^2 + 6x - 3) - r_J{}^2(x - 2) = 0$$
$$r_0{}^2\Big|_{x=1} - r_J{}^2\Big|(-1) = 0$$
$$\left(\frac{r_0}{r_J}\right)^2 = -1$$

Thus if $\mathcal{A}_k(E_{-1})$ splits over $\mathbb{Q}(\sqrt{d})$, then -1 must be a square in any $k(E_{-1})(\sqrt{d})$. But there exist characters $\rho \in E_{-1}$ (such as constructed in the proof of Theorem C) such that $i \notin k_\rho$, so $d = i$ itself and is the unique quadratic extension. $\square$

One clear distinction between Theorem D and Example 4.2.7 is that the figure-eight knot complement is arithmetic while the manifold m022 is not. However, as we witness through Theorem A, the splitting of the canonical quaternion algebra seems to be unrelated to the arithmeticity of a manifold. An interesting approach would be to find the splitting behavior of the figure-eight sibling manifold (m003 in [11]), which comes from $(5,1)$-surgery on one component of the Whitehead link complement.



(a) E_{-1} as subvariety of C_W



(b) Side view of E_{-1} suggests birational equivalence to projection $z^2 = x^3 - 2x + 1$

Figure 4.3: Canonical component of the figure-eight knot complement in $\mathbb{A}^3_{\mathbb{R}}$

4.3 Dehn filling of the form $(d, 0)$

Dehn $(d, 0)$-surgery surgery has peculiar effects on a manifold. All such surgeries produce orbifolds with the exceptions of $(0, 0)$, $(1, 0)$, and $(-1, 0)$. Dehn $(0, 0)$-surgery fails to produce a manifold of any kind, whereas $(1, 0)$-surgery is the trivial filling. In the case of the Whitehead link complement, $(1, 0)$-surgery on one component produces the unknot complement. The Dehn $(d, 0)$- and $(-d, 0)$-surgeries produce the same orbifold by Remark 4.1.5, so we will assume $d > 0$ for the remainder of this section.

The behavior of $(d, 0)$-surgery has been studied on various knot complements. Of particular relevance to this dissertation is Rouse's work on $(d, 0)$-surgery on the 7_4 knot complement in [43].

Theorem 4.3.1 ([43, Theorem 1.2]). *Let M be the 7_4 knot complement. Let T be the set of rational primes p such that there exists a place $\mathfrak{p}$ lying above p of the trace field of some hyperbolic Dehn $(d, 0)$-surgery on M at which the canonical quaternion algebra associated to that surgery is ramified. Then T is infinite.*

The success of analyzing this orbifold surgery behavior on 7_4 motivates us to pursue the same results for the Whitehead link complement. Let us consider the ideal $(x - y) \subset \mathbb{C}[x, y, z]$ and the corresponding vanishing curve $\mathcal{D} \subset \mathbb{A}^3_{\mathbb{C}}$. The intersection $C_{\text{diag}} := \mathcal{D} \cap C_W$ is defined as the vanishing set of the ideal $(\Psi_W, x - y) \subset \mathbb{C}[x, y, z]$, with the subscript to encode the sense that $(x - y)$ produces a diagonal affine subspace of $\mathbb{A}^3_{\mathbb{C}}$. By substitution,

$$(\Psi_W,\ x - y) = (x - y,\ x^2(z - 1)^2 - z(z^2 - 2))$$

as ideals in $\mathbb{C}[x, y, z]$, so the canonical quaternion algebra over the curve C_{diag} is:

$$\mathcal{A}_k(C_{\text{diag}}) \cong \left(\frac{x^2 - 4, -1}{k(C_{\text{diag}})}\right) \cong \left(\frac{y^2 - 4, -1}{k(C_{\text{diag}})}\right) \cong \left(\frac{z^2 - 4, -1}{k(C_{\text{diag}})}\right)$$

We will take advantage of the above canonical quaternion algebra by pursuing subvariety of $X(W)$ which intersects C_{diag}.

Definition 4.3.2. Let D_d be the set in $\mathbb{A}^3_{\mathbb{C}}$ which is the solution set defined by the ideal $(\Psi_W, x - y, V_d(x))$.

Lemma 4.3.3. *D_d is the finite collection of points which form the $\mathrm{SL}_2(\mathbb{C})$ character variety of the orbifold obtained by performing $(d, 0)$-surgery on both components of the Whitehead link complement. Furthermore, D_d contains real points; that is, there is a character $\chi_\rho = (x, y, z) \in D_d$ such that $x, y, z \in \mathbb{R}$.*

Proof. Let $m = 0$ in Proposition 4.1.8. Then $V_d(x) = 0$, which is satisfied precisely when $x \in \{2\cos\frac{j_a\pi}{d}\}$ where $0 < j_a < d$. The second polynomial in the triple of Proposition 4.1.8 $V_{d+1}(x) = 1$ is also satisfied by this same set of x. If Proposition 4.1.8 were repeated for surgery on the ∂_b component, we would see that there is no direct dependence on the longitude of ∂_b when $m = 0$. Instead, y must satisfy precisely the analogous $V_d(y) = 0$ and $V_{d+1}(y) = 1$ equations. There are finitely many pairs coming from $x \in \{2\cos\frac{j_a\pi}{d}\}$ and $y \in \{2\cos\frac{j_b\pi}{d}\}$ where $0 < j_b < d$. When a pair of x and y are known, Ψ becomes a cubic polynomial of parameter z, which means that there are at most 3 roots for each pair. This is a finite collection of points. Finally, because Ψ is cubic in z with real coefficients, at least one of those roots must be real. □

We now find a character on D_d that holds a property of particular interest: splitting nontrivially over its underlying function field.

Lemma 4.3.4. *Let $p \equiv 3 \mod 4$ be prime. Then there exists a character on D_d whose associated quaternion algebra splits over the quadratic extension $\mathbb{Q}(\sqrt{-p})$.*

Proof. Let $p \equiv 3 \mod 4$ be prime. As curves, $D_p \subset C_{\text{diag}}$ for all $p \neq 0$. This returns that the canonical quaternion algebra for D_d can be written as

$$\mathcal{A}_k(D_d) \cong \left(\frac{x^2-4,-1}{k(D_d)}\right) \cong \left(\frac{y^2-4,-1}{k(D_d)}\right) \cong \left(\frac{z^2-4,-1}{k(D_d)}\right)$$

Consider the real character where $x = y = 2\cos(\frac{\pi}{p})$. We know that there must be a real character because z must be the root of a cubic polynomial with real coefficients. There must be a real character $\chi_\rho \in D_d$ such that

$$A_\rho \cong \left(\frac{(2\cos(\frac{\pi}{p})^2-4,-1}{k(D_d)}\right) \cong \left(\frac{-\sin^2(\frac{\pi}{p}),-1}{k(D_d)}\right)$$

This A_ρ splits over $k(D_d)(i\sin(\frac{\pi}{p}))$. Since $\cos(\frac{\pi}{p}) \in k(D_d)$ already, A_ρ hence splits over $k(D_d)(\omega_p)$ where ω_p is the primitive pth root of unity. In particular, since we already have $\cos(2\pi/p)$ in our field, our quaternion algebra splits over a root ω_p of the p cyclotomic polynomial $\Phi_p(x)$. By [26, Example 4.14],

$$\text{disc}(\mathbb{Q}(\omega_p)) = \text{disc}(\Phi_p(x)) = (-1)^{(p-1)/2}p^{p-2}$$

For $p \equiv 3 \mod 4$,

$$-p = \frac{\text{disc}(\mathbb{Q}(\omega_p))}{p^{p-3}}$$

Both $\text{disc}(\mathbb{Q}(\omega_p))$ and p^{p-3} are squares in $\mathbb{Q}(\omega_p)$, so $-p$ must also be a square in $\mathbb{Q}(\omega_p)$. Because $\mathbb{Q}(\cos(2\pi/p)) \subseteq \mathbb{R}$, we know that

$$\mathbb{Q}(\cos(2\pi/p)) \subsetneq \mathbb{Q}(\cos(2\pi/p), \sqrt{-p}) \subseteq \mathbb{Q}(\omega_p)$$

However, $[\mathbb{Q}(\omega_p) : \mathbb{Q}(\cos(2\pi/p))] = 2p$, so we must have $\mathbb{Q}(\cos(2\pi/p), \sqrt{-p}) = \mathbb{Q}(\omega_p)$. Because $\mathbb{Q}(\cos(2\pi/p), z)$ is real and splits over ω_p, it must also split over $\sqrt{-p}$. □

Corollary 4.3.5. *Let $p \equiv 3 \mod 4$ be prime. No closed manifolds (or possibly orbifolds) arising from performing $(p, 0)$-surgery on both components of the Whitehead link complement admits a canonical quaternion algebra that splits over a real extension of the function field of the canonical component.*

Proof. As shown in the proof of Lemma 4.3.4, there exists a character whose real quaternion algebra has negative values in both entries of its Hilbert symbol. Thus, there is no real extension over which the canonical quaternion algebra splits. □

4.4 Finale: splitting behavior of $\mathcal{A}_k(W)$

We at last return to prove Theorem A:

Theorem A. *Let $d \in \mathbb{Q}$. Then $\mathcal{A}_k(W) \otimes k(C_W)(\sqrt{d})$ is a division algebra. In contrast, for all primes $p \equiv 3 \mod 4$, there exists a character $\chi_\rho \in C_W$ such that the associated quaternion algebra A_ρ splits over $\mathbb{Q}(\sqrt{-p})$.*

Proof. If there existed $d \in \mathbb{Q}$ such that $\mathcal{A}_k(W) \otimes k(C_W)(\sqrt{d})$ where a matrix algebra, then by Lemma 3.3.4, the specialization of the canonical quaternion algebra at each point would also have to split over $\mathbb{Q}(\sqrt{d})$. However from the handling of once-punctured torus bundles from Section 4.2, we know that this is impossible from Theorem C.

Conversely, by Lemma 4.3.4, we have that for $p \equiv 3 \mod 4$ prime, there always exists a representation of Whitehead link complement whose associated quaternion algebra does split over $\mathbb{Q}(\sqrt{-p})$. $\square$

REFERENCES

[1] Persistence of Vision Raytracer, 2004.

[2] Milton Abramowitz and Irene A. Stegun. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. U.S. Government Printing Office, 1972.

[3] Colin Adams. *The knot book: an elementary introduction to the mathematical theory of knots*. W.H. Freeman, New York, 1994.

[4] Kenneth Baker, Jesse Johnson, and Elizabeth Klodginski. Tunnel number one, genus one fibered knots. *Communications in Analysis and Geometry*, 17:1–16, 2009.

[5] Kenneth L. Baker and Kathleen L. Petersen. Character varieties of once-punctured torus bundles with tunnel number one. *International Journal of Mathematics*, 24(06), June 2013.

[6] Chun Cao and G. Robert Meyerhoff. The orientable cusped hyperbolic 3-manifolds of minimum volume. *Inventiones mathematicae*, 146(3):451–478, December 2001.

[7] Alberto Cavicchioli and Fulvia Spaggiari. New formulas for the character varieties of two-bridge links. *Journal of Geometry*, 112(3):43, December 2021.

[8] Haimiao Chen. Character varieties of even classical pretzel knots. *Studia Scientiarum Mathematicarum Hungarica*, 56(4):510–522, December 2019. Publisher: Akadémiai Kiadó Section: Studia Scientiarum Mathematicarum Hungarica.

[9] Ted Chinburg and Alan W. Reid. Closed hyperbolic 3-manifolds whose closed geodesics all are simple. *Journal of Differential Geometry*, 38(3), January 1993.

[10] Ted Chinburg, Alan W Reid, and Matthew Stover. Azumaya Algebras and Canonical Components. *International Mathematics Research Notices*, 2022(7):4969–5036, March 2022.

[11] Marc Culler, Nathan M. Dunfield, Matthias Goerner, and Jeffrey Weeks. SnapPy, a computer program for studying the geometry and topology of 3-manifolds, July 2019.

[12] Marc Culler, C. McA. Gordon, J. Luecke, and Peter B. Shalen. Dehn Surgery on Knots. *Annals of Mathematics*, 125(2):237–300, March 1987.

[13] Marc Culler and Peter B. Shalen. Varieties of Group Representations and Splittings of 3-Manifolds. *The Annals of Mathematics*, 117(1):109–146, January 1983.

[14] Charles W. Curtis. *Pioneers of Representation Theory: Frobenius, Burnside, Schur, and Brauer*. History of Mathematics (Volume 15). American Mathematical Society, London Mathematical Society, new edition, 1999.

[15] Leonard Eugene Dickson. The Analytic Representation of Substitutions on a Power of a Prime Number of Letters with a Discussion of the Linear Group. *Annals of Mathematics*, 11(1/6):65–120, 1896. Publisher: Annals of Mathematics.

[16] Leonard Eugene Dickson. Linear Associative Algebras and Abelian Equations. *Transactions of the American Mathematical Society*, 15(1):31–46, 1914. Publisher: American Mathematical Society.

[17] David Gabai, Robert Haraway, Robert Meyerhoff, Nathaniel Thurston, and Andrew Yarmola. Hyperbolic 3-manifolds of low cusp volume.

[18] C. McA. Gordon and J. Luecke. Knots are determined by their complements. *Journal of the American Mathematical Society*, 2(2):371–415, 1989.

[19] Gary Gordon and Jennifer McNulty. *Matroids: A Geometric Introduction*. Cambridge University Press, August 2012.

[20] Thomas Koshy. *Fibonacci and Lucas numbers with applications*. Pure and applied mathematics: a Wiley series of texts, monographs, and tracts. Wiley & Sons, Hoboken, New Jersey, 2nd edition, 2018.

[21] Emily Landes. Identifying the Canonical Component for the Whitehead Link. *Mathematical Research Letters*, 18(4):715–731, 2011.

[22] Christopher J. Leininger. Surgeries on one component of the Whitehead link are virtually fibered. *Topology*, 41(2):307–320, March 2002.

[23] Edouard Lucas. Theorie des Fonctions Numeriques Simplement Periodiques. *American Journal of Mathematics*, 1(3):197, 1878.

[24] Melissa L. MacAsieb, Kathleen L. Petersen, and Ronald M. Van Luijk. On character varieties of two-bridge knot groups. *Proceedings of the London Mathematical Society*, 103(3):473–507, September 2011.

[25] Colin Maclachlan and Alan W. Reid. *The Arithmetic of Hyperbolic 3-Manifolds*. Graduate Texts in Mathematics. Springer-Verlag, New York, 2003.

[26] Yu. I. Manin and A. A. Panchishkin. *Introduction to modern number theory: fundamental problems, ideas and theories*. Number v. 49 in Encyclopaedia of mathematical sciences. Springer, Berlin; New York, 2nd edition, 2007.

[27] Albert Marden. The Geometry of Finitely Generated Kleinian Groups. *The Annals of Mathematics*, 99(3):383, May 1974.

[28] E. J. Mayland. Two-bridge knots have residually finite groups. In M. F. Newman, editor, *Proceedings of the Second International Conference on The Theory of Groups: Australian National University, August 13–24, 1973*, Lecture Notes in Mathematics, pages 488–493. Springer, Berlin, Heidelberg, 1974.

[29] J. S. Milne. *Etale Cohomology (PMS-33)*, volume 33 of *PMS*. Princeton University Press, April 1980.

[30] G. D. Mostow. Quasi-conformal mappings inn-space and the rigidity of hyperbolic space forms. *Publications mathématiques de l'IHÉS*, 34(1):53–104, January 1968.

[31] Gary L. Mullen, Gerhard Turnwald, and Rudolf Lidl. *Dickson polynomials*. Pitman monographs and surveys in pure and applied mathematics, 0269-3666; 65. New York: Copublished in the United States with John Wiley & Sons, 1993.

[32] Kunio Murasugi. Remarks on knots with two bridges. *Proceedings of the Japan Academy*, 37(6):294–297, January 1961.

[33] Fumikazu Nagasato and Yoshikazu Yamaguchi. On the geometry of the slice of trace-free $SL_2(\mathbb{C})$-characters of a knot group. *Mathematische Annalen*, 354(3):967–1002, November 2012.

[34] Walter D. Neumann and Alan W. Reid. Arithmetic of Hyperbolic Manifolds. In Boris Apanasov, Walter D. Neumann, Alan W. Reid, and Laurent Siebenmann, editors, *Topology '90*. De Gruyter, Berlin, Boston, January 1992.

[35] Walter D. Neumann and Don Zagier. Volumes of hyperbolic three-manifolds. *Topology*, 24(3):307–332, 1985.

[36] M. H. A. Newman. John Henry Constantine Whitehead. 1904-1960. *Biographical Memoirs of Fellows of the Royal Society*, 7:349–363, 1961. Publisher: The Royal Society.

[37] Luisa Paoluzzi and Joan Porti. Invariant character varieties of hyperbolic knots with symmetries. *Mathematical Proceedings of the Cambridge Philosophical Society*, 165(2):193–208, September 2018. Publisher: Cambridge University Press.

[38] Józef H. Przytycki and Adam S. Sikora. On skein algebras and $SL_2(\mathbb{C})$-character varieties. *Topology*, 39(1):115–148, January 2000.

[39] Jessica Purcell. *Hyperbolic Knot Theory*, volume 209 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, Rhode Island, October 2020.

[40] Alan W. Reid. Arithmeticity of Knot Complements. *Journal of the London Mathematical Society*, s2-43(1):171–184, 1991.

[41] Irving Reiner. *Maximal orders*. London; New York: Academic Press, 1975.

[42] Robert Riley. Parabolic Representations of Knot Groups, I. *Proceedings of the London Mathematical Society*, s3-24(2):217–242, 1972.

[43] Nicholas Rouse. Arithmetic of the canonical component of the knot 7_4. *New York Journal of Mathematics*, 27:1494–1523, October 2021.

[44] Horst Schubert. Knoten mit zwei Brücken. *Mathematische Zeitschrift*, 65(1):133–170, December 1956.

[45] Issai Schur. Arithmetisches über die Tschebyscheffschen Polynome. In *Gesammelte Abhandlungen*, volume III, pages 422–453. Springer, Berlin, 1973.

[46] H. P. F. Swinnerton-Dyer. *A Brief Guide to Algebraic Number Theory*. London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 2001.

[47] William P Thurston. The Geometry and Topology of Three-Manifolds. 1979.

[48] Stephan Tillmann and Youheng Yao. On the topology of character varieties of once-punctured torus bundles, June 2022. arXiv:2206.14954.

[49] Anh T. Tran. Character varieties of $(-2, 2m+1, 2n)$-pretzel links and twisted Whitehead links. *Journal of Knot Theory and Its Ramifications*, 25(02):1650007, February 2016.

[50] Konstantinos Varvarezos. Representations of the $(-2, 3, 7)$-pretzel knot and orderability of Dehn surgeries. *Topology and its Applications*, 294:107654, May 2021.

[51] François Viète. *Francisci Vietœi opera mathematica in quibus tractatur canon mathematicus, seu ad triangula. Item Canonion triangulorum laterum rationalium: vnà cum vniuersalium inspectionum ad Canonem mathematicum, libro singulari.* Apud Franciscum Bouuier, London, England, 1589.

[52] John Voight. *Quaternion Algebras.* Graduate Texts in Mathematics. Springer International Publishing, 2021.

[53] Jeffrey Weeks. *Hyperbolic Structures on Three-Manifolds.* PhD thesis, Princeton University, October 1985.