REPRESENTATIONS OF POLYTOPES

A Dissertation
Submitted to
the Temple University Graduate Board

in Partial Fulfillment
of the Requirements for the Degree of
DOCTOR OF PHILOSOPHY

by
Michael Gene Dobbins
May, 2011

Examining Committee Members:

Igor Rivin, Advisory Chair, Mathematics
David Futer, Mathematics
Richard Pollack, External Member, New York University, Mathematics
Daniel Szyld, Mathematics
Louis Theran, Mathematics

ABSTRACT

REPRESENTATIONS OF POLYTOPES

Michael Gene Dobbins
DOCTOR OF PHILOSOPHY

Temple University, May, 2011

Igor Rivin, Chair

Here we investigate a variety of ways to represent polytopes and related objects. We define a class of posets, which includes all abstract polytopes, giving a unique representative among posets having a particular labeled flag graph and characterize the labeled flag graphs of abstract polytopes. We show that determining the realizability of an abstract polytope is equivalent to solving a low rank matrix completion problem. For any given polytope, we provide a new construction for the known result that there is a combinatorial polytope with a specified ridge that is always projectively equivalent to the given polytope, and we show how this makes a naturally arising subclass of intractable problems tractable. We give necessary and sufficient conditions for realizing a polytope's interval poset, which is the polytopal analog of a poset's Hasse diagram. We then provide a counter example to the general realizablity of a polytope's interval poset.

ACKNOWLEDGEMENTS

I would like to thank the people who have played valuable parts in making this thesis a reality. I thank my advisor Igor Rivin for insightful discussions of where opportunities for further progress lay and all manner of topics. His support has helped me pursue many different interests of mine and allowed me to go to the Berlin Summer School and the Rigidity Workshop. Louis Theran picked through my thoughts to help me present them in a more understandable manner. Daniel Szyld advised me before I found a research topic, and patiently pushed me to progress when I was unsure how to go forward. David Futer has made the mathematics department feel more like a community. Sinai Robins always gave me confidence and spurred my curiosity. I am grateful to these and many other Professors at Temple in who's classes I have learned so much. I also appreciate Richard Pollack for coming from New York to participate in my defense. Günter Ziegler gave me the chance to present my research and provided helpful comments. Jeana Mastrangeli introduced me to the world of advanced mathematics and continued to provide so much encouragement and support. My friends John Ratanaprasatporn, Peter de Blanc, Nancy Wong, and many others have made my experience in graduate school much more enjoyable. Regina Wilson and Heon Kim have each introduced me to worlds I would never have known existed, and their friendship has meant so much to me. I am grateful to my family for their love and support, and my wife for going so far and for her nearly infinite patience and understanding.

안경남을 위해서

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

1.1 Motivation and Background

We begin with an overview of the context of concepts addressed in this thesis and how they relate to each other, leaving details for later. We will define basic concepts in the next section and others throughout the thesis as they are needed.

The main impetus for this thesis is the observation that a Hasse diagram of some given polytope's face lattice may resemble a perspective drawing of a larger polytope. Consider for example a triangle's face lattice; this can be drawn as the 1-skeleton of a cube balancing on a vertex as seen in Figure 1.1. We would like the larger polytope itself to represent the combinatorics of the original polytope, and for this we use its interval polytope. This is a polytope that has as its face lattice the poset consisting of the intervals of the original's face lattice ordered by containment. Interval polytopes are related to E-constructions for polytopes and bier spheres, which are both posets consisting of a particular subset of intervals of a poset [3, 8, 17]. In general the interval polytope of a simplex is a hypercube one dimension higher. The question then is given a polytope, can we realize its interval polytope.

This question was asked by Lindström in [13]. Later an equivalent question was considered in [4] where Broadie gives sufficient conditions for the realiz-

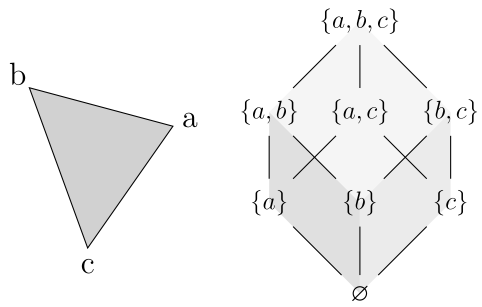


Figure 1.1. A triangle and its face lattice.

ability of the antiprism of a polytope, the dual of the interval polytope, as part of a program to find more efficient linear optimization algorithms, but gives no indication as to whether all polytopes have an antiprism. The conditions require us to find a perfectly centered realization of the original polytope, a term coined latter in [10]. This also appeared more recently as an open problem in [11]. Note that realizability conditions make no distinction between duals since one is realizable if and only if the other is. Here we give necessary and sufficient conditions for the realizability of a polytope's interval polytope. These conditions require the original polytope to be realizable as part of a balanced pair. We will show that if the original polytope is of dimension 3 or lower this can always be realized, but we will give a counter example to this in 4-dimensions.

This puts the question into the more general class of problems of determining if a certain property holds for some realization of every combinatorial type of polytope. Such problems are made difficult by consequences of the universality theorem proved in [20], and recently exposed in [27], that for any primary semialgebraic set, the solutions to a collection of multinomial equations and strict inequalities, we can find a corresponding combinatorial type of 4-polytope that has this set, modulo some trivial transformations, as its realization space. These transformations give a stably equivalent space, which is slightly stronger then homotopy equivalence. Such a space may be disconnected, or have holes, or any manner of unpleasant topological feature we may hope to avoid in some search space.

A natural feature of properties that are general enough to be relevant to polytopes in any dimension is to require an analogous property to hold for the faces of the polytope as polytopes themselves. We say faces inherit such a property. Prefect centeredness has this feature, and balance has an analogous feature as well. For properties that faces inherit, we can use the same ideas in the proof of the universality theorem to make these problems tractable.

If we start with a realization of some face of a combinatorial polytope, it may not always be possible to extend this to a realization of the entire polytope. The universality proof works by constructing a 4-polytope where the combinatorics impose conditions on a ridge this way. The most restrictive such condition possible is to force the ridge to be fixed up to projectivity over all realizations of the combinatorial polytope. We call this a stamp polytope of the projective type of the ridge. Below gave a construction for the stamp polytope of any algebraic polytope in his thesis [1], and we give another construction here. These are polytopes having algebraic coordinates under some projectivity.

For algebraic polytopes, being able to fix the geometry of a polytope as a ridge of a larger polytope reduces the problem of determining whether a property that inherits to faces holds for some realization of every combinatorial type to determining that for every projective type. This is a considerable improvement since under combinatorial equivalence the space of realizations could resemble any semialgebraic set, but under projective equivalence the space, modulo affine transformations, is a convex set. Since the rigid face is of codimension 2 this leaves a gap of 2 dimensions where some other argument must be used to determine whether the property holds.

In contrast to polytopes of dimension $d \geq 4$ we have from Steinitz simple combinatorial rules for determining if a given poset can can be realized as the face lattice of some 3-polytope [23]. In general when we want to determine whether a poset is realizable, we can identify each face as a subset of vertices and use Tarski's theorem on the decidability of quantified algebraic statements to check if vectors can be chosen so that exactly the appropriate subsets satisfy the definition of a face [11]. Here we reformulate realizability of a polytope as a low rank matrix completion problem, which gives simpler conditions. This is related to a result by Díaz in [6] showing that sign conditions on certain minors of the Gram matrix of a polytope's outward normals must hold for every realization.

The realizability conditions we present here use facts about the flag graph of a polytope, which are also used implicitly by Díaz. In [18], Peterin characterizes the labeled flag graphs of graded posets. There may be several graded posets having the same flag graph. Here we present a class of posets, which we call cone connected, having a unique representative of every collection of posets sharing the same labeled flag graph. Additionally we show there is an equivalence of category between these, and we characterize the labeled flag graphs of abstract polytopes. These are posets satisfying some necessary conditions of face lattices, which are stronger than cone connectedness. This is much more than what we would need to show the realizability conditions mentioned earlier, but gives a more complete picture of what is going on.

The content of this thesis is presented in nearly the reverse order of this introduction. We begin with the most abstract and combinatorial results, then proceed to successively more geometric and concrete ones. In Chapter 2, we deal with labeled flag graphs and cone connected polytopes, then consider the restriction to abstract polytopes. In Chapter 3, we give conditions for realizing a polytope with a given lattice as its face lattice. In Chapter 4, we give more detail on polytopes in projective space, and various operations we can preform on polytopes. In Chapter 5, we use these operations to construct a stamp polytope. In Chapter 6, we give conditions for realizing an interval polytope, and we use the stamp polytope to construct a polytope that has no interval polytope.

1.2 Definitions

Here we briefly review definitions used throughout this thesis, and provide sources where the reader may find detailed treatments. We write terms being defined in bold, and we prefer to use the word ‘when’ for the definitional ‘if’ to distinguish it from the logical ‘if’. The reader is advised to skim over the words in bold, and use this section as a quick reference when needed. We assume basic familiarity with predicate logic, quantification, sets, and the natural, rational, and real numbers $\mathbb{N} \subset \mathbb{Q} \subset \mathbb{R}$. As well as the corresponding standard notation.

We denote the disjoint union and product of sets by $\bigsqcup$ and $\times$ respectively. We use superscripts to represent multiple products of the same set $X^n := X \times \cdots \times X$, as well as exponentiation in $\mathbb{R}$. To distinguish between an indexed collection of a product's elements $\{x_i \in X^n\}_{i \in I}$ and the components of a product's element $(x_i)_j \in X$, we write both as a subscript but place the subscript indicating the component outside of parentheses $(\cdot)_j$, writing this as $\{x_i = ((x_i)_1, (x_i)_2, \cdots, (x_i)_n) \in X^n\}_{i \in I}$.

A **relation** R is a predicate on two sets $\mathbf{A}\,\mathbf{B}$, $R : \mathbf{A} \times \mathbf{B} \to \textsc{bool}$. When R is true for a pair of elements we say they are related and write aRb. Otherwise we say a and b are not related and write $a \not{R}\, b$. We call related pairs the **incidences** $\mathbf{I}$ of the relation.

$$\mathbf{I} := \{(a, b) \in \mathbf{A} \times \mathbf{B} : \; aRb\}$$

A **partially ordered set** or **poset** for short is a set $\mathcal{O}$ together with a relation $\leq$ on $\mathcal{O}^2$ satisfying the following axioms.

1. **reflexivity:** $a \leq a$
2. **antisymmetry:** $(a \leq b \,\text{and}\, b \leq a) \Rightarrow a = b$
3. **transitivity:** $(a \leq b \,\text{and}\, b \leq c) \Rightarrow a \leq c$

We call the elements of a poset **clades**, and we say a and b are **comparable** when $a \leq b$ or $b \leq a$, otherwise we say they are **incomparable**. The **dual**

poset $\mathcal{O}^*$ is the poset consisting of the same clades with order reversed. We write $c^\diamond$ instead of c when considering it in the context of $\mathcal{O}^*$, so $b^\diamond \leq a^\diamond$ when $a \leq b$. Frequently when we define notation or terminology for posets we will also define the corresponding term for the dual poset, which we call its **dual**. This will often be the same term with the prefix 'co-'. We denote by $a < b$, $a \leq b$ but $a \neq b$, and we let $\geq$ and $>$ be the respective duals of $\leq$ and $<$. If there is a minimal element it is denoted by $\bot$ and maximal by $\top$, and when these exist we say the poset is **bounded** from below and above respectively. The **interval** $[a, b]$ from a to b is the set of clades between them with the same order.

$$[a, b] := \{c \in \mathcal{O} : \ a \leq c \leq b\}$$

In particular we always have $[\bot, \top] = \mathcal{O}$ for bounded posets. We say a **covers** b when this interval consists of its bounds alone, $[a, b] = \{a, b\}$. Generally the way we visualized a poset is by drawing a **Hasse diagram** of it. This is a diagram with a dot drawn for each clade in the poset and an edge between two dots when one clade covers the other, drawn so the greater clade's dot is higher.

For a subset C of a poset, we denote by $\bigvee C$ those clades that are minimal among the upper bounds of C, and the dual by $\bigwedge C$.

$$\bigvee C := \{a \in \mathcal{O} : \ \forall c \in C(c \leq a), \ \nexists b \leq a \forall c \in C(c \leq b)\}$$

We call a poset $\mathcal{O}$ a **join semi-lattice** when for any pair of clades $a, b \in \mathcal{O}$, there is a single minimal upper bound $|\bigvee\{a, b\}| = 1$, which we call the **join** and denote $a \vee b$. We call the dual a **meet semi-lattice** with **meet** $a \wedge b$ of a and b. A poset is a **lattice** when it is both a meet and join semi-lattice. We call a poset a **complete** lattice when, for any subset C, $\bigvee C$ is a single clade, which is also how we denote that clade. Note that we could also define complete lattices by the dual condition since they are equivalent. More about lattices can be found in [22, 16].

Many times we are interested in posets consisting of a collection $\mathbf{C}$ of subsets of some set X. In particular when this collection is closed under intersection we call it a **closure system** consisting of **closed** sets. From this we get a **closure operator** sending each subset to the smallest closed set containing it $\overline{s} := \bigcap\{c \in \mathbf{C} : c \supset s\}$. We may also define a closure operator as a function satisfying the following axioms.

1. $A \subset \overline{A}$
2. $A \subset B \Rightarrow \overline{A} \subset \overline{B}$
3. $\overline{(\overline{A})} = \overline{A}$

The closure operator of a closure system always satisfies these axioms, and the range of such a function is always a closure system. Also any closure system is a complete lattice with meet and join given by $\bigwedge C := \bigcap C$ and $\bigvee C := \overline{\bigcup C}$, and for any complete lattice the intervals of the form $[\bot, a]$ make up a closure system that is order isomorphic to the lattice. In this way complete lattices, closure systems, and closure operators are three ways of defining the same mathematical object.

We call a closure system a **topology** when the finite union of closed sets is again closed. In this case we call the complement of a closed set **open**. Usually a topology is defined in terms of open sets as a collection where unions and finite intersections of open sets are again open sets. These two definitions are equivalent. We call the largest open subset $A^\circ := \bigcup\{U \subset A : \ U \, \text{open}\}$ of a set A its **interior**. When a topology cannot be expressed as the disjoint union of two nonempty closed sets we say it is **connected**, and we call the maximal connected subsets of a topology its **connected components**.

An **adjacency** is a set of **nodes** N together with a relation $\sim$ satisfying the following axioms.

1. **antireflexivity:** $n \nsim n$
2. **symmetry:** $n \sim m \Rightarrow m \sim n$

We call the incidences E the **edges**, and $G = (N, E)$ a **graph**. For any given node the nodes adjacent to it are its **neighbors** $\text{nei}(n)$. The **induced**

subgraphs of N are graphs consisting of a subset of nodes $M \subset N$ with the edges that are between them $\{\{n,m\} \in E : \; n\, m \in M\}$. These form the closed sets of a topology on $N \sqcup E$, and with this a graph is connected if and only if for any nodes $n\, m \in N$ there is a **path** between them, which is a sequence of nodes $n = n_0, \cdots, n_k = m$ such that consecutive nodes are adjacent $n_i \sim n_{i+1}$. More on graph theory can be found in [7].

When the additional axiom **totality** $a \leq b$ or $b \leq a$ holds for a poset we say it is **totally ordered**. We call a totally ordered subset of a poset a **chain**, and when it is maximal among chains we call it a **flag**. We say a poset is **graded** when all flags are isomorphic. We call the isomorphism class of clades **ranks**, and when the flags are finite we attach a function $\text{rank} : \mathcal{O} \to \mathbb{Z}$ that sends each clade to the number of clades below it in a flag. The **flag graph** of a graded poset is a graph consisting of a node for each flag $\mathcal{F}$, and edges connecting $\mathcal{F}$ to the other flags that contain all but one of $\mathcal{F}$'s clades $\text{nei}(\mathcal{F}) = \{\mathcal{F}' : |\mathcal{F} \setminus \mathcal{F}'| = 1\}$. We say a poset is **flag connected** when its flag graph is connected, and we say it is **strongly connected** when every interval is flag connected.

A **group** is a set Γ with an operation $* : \Gamma^2 \to \Gamma$ satisfying the following axioms.

1. **identity:** $\exists \text{id} \in \Gamma : \text{id} * g = g * \text{id} = g$

2. **inverse:** $\forall g \exists g^{-1} \in \Gamma : g^{-1} * g = g * g^{-1} = \text{id}$

3. **associativity:** $(f * g) * h = f * (g * h)$

When we do not require the inverse axiom, we call this a **monoid**. We will encounter a more general object than monoids in Section 2.2 called a category, but since it is limited to only that section we do not define it here. A **subgroup** is a subset of a group that is itself a group with the same operation. Subgroups form a closure system, and we call the closure of a subset $X \subset \Gamma$ of a group, the subgroup **generated** by X. In many cases we will be interested in groups consisting of functions with composition $\circ$ as the group operation and the identity function as the identity group member. In this case we call function application the **action** of the group.

A **real vector space** is a set of **vectors** V with two operations, **vector addition** $(+) : V^2 \to V$, which must be a group operation, and **scalar multiplication** $\mathbb{R} \times V \to V$, which we denote simply by sv for $s \in \mathbb{R}$ and $v \in V$, satisfying the following axioms.

1. **additive commutativity:** $v + u = u + v$
2. **scalar distribution:** $s(v + u) = sv + su$
3. **vector distribution:** $(s + t)v = sv + tv$

We call the group identity $\mathbb{0}$ the **origin** and a set of vectors $\{v_i\}$ **linearly independent** when the origin cannot be expressed as a nontrivial linear combination of them $\sum_i s_i v_i = \mathbb{0} \Rightarrow \forall i, s_i = 0$, otherwise we say they are **linearly dependent**. When every vector can be expressed as a linear combination of $\{v_i\}$ we say $\{v_i\}$ **spans** the vector space. We call a linearly independent set that spans the vector space a **basis**. Every basis of a vector space has the same size, which is usually called its dimension, but here we will call this size its **rank**. We will restrict ourselves to only considering real vector spaces of finite rank. The standard model for vector spaces we use is the rank r space $\mathbb{R}^r$ with component wise addition and multiplication of reals for vector addition and scalar multiplication respectively, with the standard basis $\{e_i\}_{1 \le i \le r}$ having $(e_i)_i := 1$ and $(e_i)_j := 0$ for $j \ne i$. We call a function between vector spaces preserving the vector space operations a **linear transformation**. For any indexed basis $(v_1, \cdots, v_r)$ of a vector space there is a unique linear bijection to $\mathbb{R}^r$ sending each v_i to e_i. We call the values in the image of a vector under this linear bijection, its **coordinates** with respect to this basis. We then write linear transformations as matrices with columns being the coordinates of the images of the domain's bases vectors. Note that all real vector spaces of the same rank are isomorphic this way. Details on vector spaces and matrices in general, not just real vector spaces, can be found in [12].

We denote by V^* the space of linear functionals $a^* : V \to \mathbb{R}$ and we recall this is a vector space of the same dimension as V. We represent a^* by a vector $a \in \mathbb{R}^r$ where $(a)_i = a^* e_i$, which is enough to uniquely identify every linear functional as $a^* v := \sum_i (a)_i (v)_i$. We denote the **inner product** of two vectors by $\langle a, v \rangle := a^* v$, and may attach an inner product to any vector space V given this way by a linear bijection between V and V^*. We denote by $\|v\| := \sqrt{\langle v, v \rangle}$, and we attach a topology to every real vector space with open sets given by unions of open balls $\bigcup_{i \in \mathcal{I}} \{u : \ \|v_i - u\| < \epsilon_i\}$. Note that any choice of inner product will generate the same topology. With this in mind we may refer to the topology of a finite rank real vector space without specifying an inner product.

We call the monoid generated by linear transformations and **translations**, which are functions $f_t : \ V \to V$ for $t \in V$ of the form $f_t(v) := v + t$, **affine transformations**, and we call a space X of **points** that this monoid acts on an **affine space** when for any point x the monoid action gives a bijection between translations and X by its images $t \in V \leftrightarrow f_t(x) \in X$. We also use $\mathbb{R}^d$ as the standard model for affine spaces. If we then choose a point to be the origin we get back a vector space.

A **linear subspace** of a vector space is a subset that is closed under vector addition and scaler multiplication. That is, it is a vector space with the same operations as the ambient space. The set of linear subspaces ordered by containment forms a closure system with linear span as its closure operator, which we call a real **projective space**. We treat this closure system as a space consisting of the set of rank 1 linear subspaces, which would be lines in the vector space, but we consider to be the **points** of this space, and we say the set of points corresponding to a rank r vector space has dimension $r - 1$. The origin in particular corresponds to the empty set in projective space and has dimension -1. We call the action of the linear transformations of the underlying vector space on the rank 1 subspaces a **projective transformation**. We call a projective transformation a **projectivity** when it is invertible. In both contexts we call the largest subspaces **hyperplanes**. The standard model we use for real projective space is the space of linear subspaces of $\mathbb{R}^{d+1}$, which we denote by $\mathbb{RP}^d$. We may represent a point by the coordinates of a vector spanning the corresponding subspace, which we call **homogeneous coordinates**. An axiomatic treatment of projective spaces in general as lattices satisfying some additional conditions, not just those coming from real vector spaces, is presented in [9].

If we consider only those projective transformations leaving a particular hyperplane fixed, which we call the **horizon**, we get a space of affine transformations acting on the space with the horizon deleted. In particular, in $\mathbb{RP}^d$ we will often delete the hyperplane $x^{d+1} = 0$, and send points in this affine space to $\mathbb{R}^d$ by the map $x^i \to x^i/x^{d+1}$. We will also sometimes represent projective transformations on $\mathbb{R}^d$ by their action under this map. These have the form $x \to \frac{Ax+b}{c^*x+1}$. Since we delete the horizon, this may not be well defined everywhere, in particular where $c^*x = 1$. When $c^* = 0$, this is an affine transformation and when $A = I$ is the identity and $b = 0$ this is a **perspectivity**. Affine space and projective space both inherit span and topology from vector spaces. For projective space the span and the closed subsets come directly from those of the underlying vector space, and for open subsets we ignore the origin. In an affine space the span of a subset is the same as that of the vector space with some point in the subset chosen as the origin, and open sets are given by the open sets of this vector space. The **relative interior** of a subset S of a vector, affine, or projective space is the interior of S in the topology attached to the subspace spanned by S.

The **convex hull** $\operatorname{conv}(S)$ of a set $S \subset \mathbb{R}^d$, or in general any real affine or vector space, is the set of all points that can be expressed as a weighted average of points in S.

$$\operatorname{conv}(S) := \left\{ \sum_i \lambda_i s_i : \; s_i \in S, \, \sum_i \lambda_i = 1 \right\}$$

We note that convex hull is a closure operator, and we denote the convex join by $\uplus$ to distinguish it from the join of other closure systems in the same space. We also define the convex hull $\operatorname{conv}_h(S)$ of a subset of a real projective with respect to a horizon h as the convex hull of that set in the corresponding affine space. Equivalently this is the positive and negative linear combinations of vectors in some half space H with boundary h.

$$\operatorname{conv}_h(S) := \left\{ \pm \sum_i \lambda_i s_i : \ s_i \in S \cap H,\, h = \partial H,\, \lambda_i \geq 0 \right\}$$

We note that in projective space convexity looses the property being a closure system, and we give more details in Section 4.3.

A **polytope** $P = \operatorname{conv}(\{v_i\})$ is the convex hull of finitely many points v_i. In projective space a set is a polytope when it is a polytope in the affine space with respect some horizon $P = \operatorname{conv}_h(\{v_i\})$. The **faces** F of a polytope P are subsets of P where a linear inequality that is satisfied at every point in P, $P \subset H_{a,b}$ s.t. $H_{a,b} := \langle a, x \rangle \leq b \ \forall x \in P$, is an equality, $F := P \cap H_{a,b} = \{x \in P : \langle a, x \rangle = b\}$. In this case we say the half space $H_{a,b}$ and hyperplane $\partial H_{a,b}$ **support** the face F, and a is an **outward normal** vector of F. The **face lattice** of a polytope is the partially ordered set consisting of its faces ordered by containment. The faces of a polytope form a closure system, so as the name suggests, this is always a lattice. The dimension of a face is $r - 1$ where r is its rank in the lattice. For a polytope of dimension d we call the 0, 1, $d-2$, and $d-1$ dimensional faces **vertices**, **edges**, **ridges**, and **facets** respectively, and the empty set $\varnothing$ has dimension -1. Details on polytopes can be found in [26, 11].

A poset $\mathcal{P}$ is a **combinatorial polytope** when there exists a polytope P with face lattice isomorphic to $\mathcal{P}$. We say P is a **realization** of $\mathcal{P}$. When two polytopes are realizations of the same poset, we say they are the same **combinatorial type**. Our convention is to write geometric objects in a regular font and use the same letters in a script font for the corresponding combinatorial objects. The only combinatorial 1-polytope is an edge, and its face lattice is the only rank 2 poset with 4 clades. When every rank 2 interval of a poset has 4 clades we say the poset satisfies the **diamond** condition. An **abstract polytope** is a poset that satisfies the following conditions.

1. It is bounded.
2. It is graded.
3. It is strongly connected.
4. It satisfies the diamond condition.

Note that these conditions are necessary for a poset to be a combinatorial polytope. Abstract polytopes are presented in [15], which is concerned with regular abstract polytopes and understanding their symmetry. These have additional symmetry requirements. We will not, however, give the details of regular abstract polytopes here.

We call a polytope with the dual face lattice $\mathcal{P}^*$ a combinatorial **dual** and in this context we call the orginal polytope the **primal**. We call a polytope in a real vector **centered** when it contains the origin $\mathbb{0} \in P$. For every centered polytope we have a connonical realization of its dual, its **polar** $P^* := \{a : \forall x \in P : (\langle a, x \rangle \leq 1)\}$, consisting of linear functionals that are bounded by 1 on P.

CHAPTER 2

LABELED FLAG GRAPHS

2.1 Cone Connected Posets

We call the flag graph of a graded poset with each edges labeled by the rank of the clade where its flags differ, the **labeled flag graph** of the poset. Peterin gives gives the following conditions for an edge labeled graph with totally ordered labels to be a connected component of a graded poset's labeled flag graph [18]. We take the disjoint union of connected graphs satisfying the conditions of this theorem to be axioms of an alternate definition of a labeled flag graph, and we denote the space of such objects LFG.

Theorem 2.1. (Peterin) *A connected edge labeled graph is the flag graph of a graded poset if and only if it satisfies*

LFG-*1. For every triangle, all edges have the same label.*

LFG-*2. For every nonadjacent pair of distinct nodes u v, there are two labels that appear on every path between u and v.*

LFG-*3. For every nonadjacent pair of distinct nodes u v with a unique common neighbor z, the labels on uz and vz are consecutive.*

The definition of a labeled flag graph implicitly describes a map ψ from the space of graded posets to the space of edge labeled graphs. We may like to provide the inverse of ψ, but this function is not invertable. Instead we provide a function χ constructing a poset that is in a sense universal among the posets having isomorphic labeled flag graphs, and we will show that χ and ψ restricted to these posets are inverses up to isomorphism. In the next section we make the universality of these posets explicit by giving a round about way of constructing these maps showing that ψ is a left-adjoint functor and χ is its corresponding right-adjoint.

We start by seeing how to construct the appropriate poset from a labeled flag graph. From a graph G with edges labeled by some totally ordered set $\mathcal{T}$ where we represent the ranks of a poset $\mathcal{O}$ with $\mathrm{rank} : \mathcal{O} \to \mathcal{T}$, we would like to construct a poset $\chi(G)$ defined in terms of the graph in such a way that $\chi \circ \psi(\mathcal{O}) \cong \mathcal{O}$ and $\psi \circ \chi(G) \cong G$. We will, however, have to impose some conditions on $\mathcal{O}$ and G. For this we define ψ and χ in the following way.

$$\psi(\mathcal{O}) \coloneqq (\mathrm{flag}(\mathcal{O}), \{(\{f, f'\}, \mathrm{rank}(a)) : \{a\} = f \vartriangle f'\})$$
$$\chi(G) \coloneqq (\{(g, r) :\ g \in \mathrm{comp}(G \smallsetminus r), r \in \mathcal{T}\},\ \leq_\cap)$$

Here $\chi(G)$ consists of the connected components of G with edges labeled r removed, denoted $G \smallsetminus r$, for each $r \in \mathcal{T}$, and the relation on $\chi(G)$ is defined by $(g, r) \leq_\cap (h, s)$ when $g \cap h \neq \varnothing$ are intersecting subgraphs of G and $r \leq s$. This relation does not, however, always give a poset as the notation may seem to suggest. Specifically transitivity may fail in general. To avoid this we restrict ourselves to graphs satisfying the following condition. We call an edge labeled graph **gap commuting** when the labels are totally ordered and every intersecting pair of edges with non consecutive labels belongs to an induced square with alternating labels. We can think of this as allowing us to transform a path so that edges identified by non consecutive labels effectively commute. This is in fact an alternative formulation for one of the axioms of a labeled flag graph.

Lemma 2.2. *Under Conditions* LFG-*1 and* LFG-*2 of Theorem 2.1 Condition* LFG-*3 is equivalent to the gap commuting property.*

Proof: We first show that a labeled flag graph is always gap commuting. Let G be a graph where the conditions of Theorem 2.1 hold and let n_0n_1 and n_1n_2 be edges with non consecutive labels. If n_0 and n_2 are adjacent, then $n_0n_1n_2$ is a triangle where more than one label appears, which violates LFG-1, so they cannot be adjacent. Now, assuming that n_1 is the unique common neighbor between them violates LFG-3, so there must be another node n' that is also a common neighbor of n_0 and n_2, which gives us a square containing these edges. If n_1 and n' are adjacent, then this edge either does not have the same label as n_0n_1 or that of n_1n_2, and again LFG-1 is violated, so the square is induced. Finally, by LFG-2 the edge labels are alternating, so the graph is gap commuting.

For the reverse direction suppose G is gap commuting, and let n_0 and n_2 be non adjacent nodes of G with a unique common neighbor n_1. Assuming n_0n_1 and n_1n_2 have non consecutive labels, there are edges $n'n_2$ and n_0n' with these same respective labels, but this violates the uniqueness of n_1, so the labels on n_0n_1 and n_1n_2 must be consecutive and LFG-3 holds. □

We now see that the gap commuting property ensures the relation is that of a poset as desired.

Lemma 2.3. *$\chi(G)$ is poset for any gap commuting graph G, and for any chain $a_{(g_1,r_1)} < \cdots < a_{(g_k,r_k)}$ there is a node $n \in \bigcap_i g_i$.*

Proof: We proceed by induction on the length of chains having such a node. Suppose $a_{(g_1,r_1)} < \cdots < a_{(g_j,r_j)}$ and $a_{(g_j,r_j)} < \cdots < a_{(g_k,r_k)}$ and there are nodes $n_{\leq j} \in \bigcap_{i\leq j} g_i$ and $n_{\geq j} \in \bigcap_{i\geq j} g_i$. Since both $n_{\leq j}, n_{\geq j} \in g_j$ are in the same component of $G \setminus r_j$, there is a path γ between them that does not include any edges labeled r_j. For any adjacent pair of edges of γ with labels r and r' where $r < r_j < r'$, we can find a new path where all edges are the same except where these two edges are replaced by edges that have these two labels appearing in the reverse order. In this way we can construct a new two-part path $\gamma_{>j} \cdot n \cdot \gamma_{<j}$ from $n_{\leq j}$ to $n_{\geq j}$ passing through n where all edge labels of $\gamma_{>j}$ are greater than r_j and all edge labels of $\gamma_{<j}$ are less than r_j. For $i < j$, the path $\gamma_{>j}$ between n

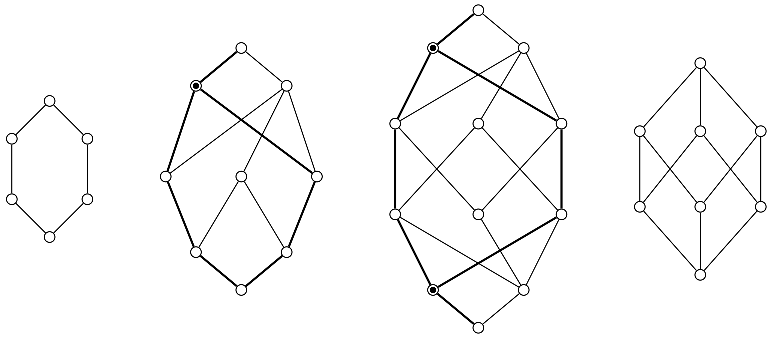


Figure 2.1. Posets from the left are: not flag connect, flag connected but not cone connected, cone connected but not strongly connected, strongly connected.

and $n_{\leq j} \in g_i$ includes no edges labeled r_i, so $n \in g_i$, and for $i > j$, the path $\gamma_{<j}$ between n and $n_{\geq j} \in g_i$ includes no edges labeled r_i, so $n \in g_i$, and none of these edges are labeled r_j, so $n \in g_j$. Thus, $n \in \bigcap_i g_i$. For a chain $a_{(g_1,r_1)} < a_{(g_2,r_2)}$ of length 2 this is trivial since by definition of $\leq_\cap$ the components must intersect giving $n \in g_1 \cap g_2$, and by induction on the length of a chain starting with this case, this holds for any chain. In particular the case $j = 2$, $k = 3$ shows that the relation is transitive. That $\chi(G)$ is reflexive and antisymmetric is an immediate consequence of the definition of $\leq_\cap$, since two components of $G \setminus r$ are exclusively either identical or disjoint, and the ranks are totally ordered. Hence, $\chi(G)$ is a poset. □

In general several graded posets may have the same labeled flag graph, and as such it cannon be the case that every poset arises as the image of some graph under χ. We call a graded poset **cone connected** when for any clade c the poset of all clades comparable to c is flag connected, and we denote the space of these by CCP. As we will see $\chi(G)$ gives the unique cone connected poset that has G as its labeled flag graph. This property is closely related to but slightly weaker than strong connectedness, which is part of the definition of an abstract polytope, but for bounded posets stronger than flag connectedness. Figure 2.1 gives examples illustrating the differences between these.

Theorem 2.4. *$\chi(G)$ is a cone connected poset for any gap commuting graph G.*

Proof: To see this let $c_{(g_r,r)}$ be a clade of $\chi(G)$, and consider two flags with this element

$$f = \{c_{(h_0,0)}, \cdots, c_{(h_r=g_r,r)}, \cdots, c_{(h_t,t)}\}, \; f' = \{c_{(h'_0,0)}, \cdots, c_{(h'_r=g_r,r)}, \cdots, c_{(h'_t,t)}\}$$

For each of these flags, by Lemma 2.3 there is a node $n \in \bigcap_s h_s$, $n' \in \bigcap_s h'_s$ in the respective subgraphs of these flags. Since n and n' are both in the same component of $G \smallsetminus r$, there is some path $\gamma = \{n = n_1 - n_2 - \cdots - n_k = n'\}$ between them that does not include any edge labeled r, and all of the nodes in γ are contained in g_r. Moreover, the components for each rank of each successive node of γ differs from that of the previous node in at most 1 of these components, so the corresponding flags are either adjacent or identical, and we get a path in the labeled flag graph of $\{c \in \chi(G) : c \gtrless c_{(g_r,r)}\}$. Hence, $\chi(G)$ is cone connected. □

Now that we have established the domain and range of ψ and χ we would like to show that ψ and χ are inverses up to isomorphism on their respective ranges. We get this isomorphism by looking at how they act elementwise on the objects of their respective domains. For this we let $\psi_{\mathcal{O}}$ be a function on $\mathcal{O}$ sending each clade c to the subgraph induced by flags containing c and its rank (g, r). We let $\chi_{\mathcal{O}}$ send these components in $\psi(\mathcal{O})$ to the corresponding clades in $\chi \circ \psi(\mathcal{O})$. We let χ_G be a function on G sending each node n to the flag consisting of the connected components of $G \smallsetminus r$ for each $r \in \mathcal{T}$ that contain n. And, we let ψ_G send these flags in $\chi(G)$ to the corresponding nodes in $\psi \circ \chi(G)$.

Theorem 2.5. *ψ and χ give a bijection up to isomorphism between the space of labeled flag graphs* LFG *and cone connected posets* CCP.

Proof: We first show $\chi_{\mathcal{O}} \circ \psi_{\mathcal{O}} \in \mathrm{iso}_{\mathrm{CCP}}$. Since $\chi_{\mathcal{O}}$ is a partial function, we must show that $\chi_{\mathcal{O}} \circ \psi_{\mathcal{O}}$ is a well defined function. Specifically we require that

$$\begin{aligned} \mathrm{dom}(\chi_{\mathcal{O}}) &= \psi_{\mathcal{O}}(\mathcal{O}) \\ \{(g, r) : g \in \mathrm{comp}(\psi(\mathcal{O}) \smallsetminus r)\} &= \{(\{n_f : a \in f \in \mathrm{flag}(\mathcal{O})\}, \mathrm{rank}(a)) : a \in \mathcal{O}\} \end{aligned}$$

For each clade a of the poset there is a flag f that includes this clade, $a \in f$. Furthermore, the induced subgraph $g_{f,r} \in \text{comp}(\psi(\mathcal{O}) \setminus r)$ consisting of all nodes that can be reached by a path that does not include any edges labeled r, where r is the rank of a, must also correspond to flags that include this clade, so $(g_{f,r}, r) \in \text{dom}(\chi_{\mathcal{O}})$ specifies a as the only clade of rank r in the flags corresponding to some node of $g_{f,r}$, and $(g_{f,r}, r) \subset \psi_{\mathcal{O}}(a)$. Under cone connectedness all nodes corresponding to flags containing a can be reached in this manner, by the flag connectedness of the poset of clades comparable to a, so all flags containing a are included in $g_{f,r}$ and $(g_{f,r}, r) = \psi_{\mathcal{O}}(a)$. Furthermore, each (g, r) in $\chi_{\mathcal{O}} \circ \psi_{\mathcal{O}}(\mathcal{O})$ comes from the unique clade of $\mathcal{O}$ that is the rank r clade of a flag in g, and $\chi_{\mathcal{O}} \circ \psi_{\mathcal{O}}$ is a bijection.

To see that order is preserved, consider two clades $a, b \in \mathcal{O}$ of rank $r \leq s$ respectively. If $a \leq b$ are comparable, then there is some flag f that includes both, so $n_f \in \psi_{\mathcal{O}}(a)_1 \cap \psi_{\mathcal{O}}(b)_1$ and $\chi_{\mathcal{O}} \circ \psi_{\mathcal{O}}(a) \leq \chi_{\mathcal{O}} \circ \psi_{\mathcal{O}}(b)$. Otherwise a and b are incomparable, and there is no such flag, so they remain incomparable under $\chi_{\mathcal{O}} \circ \psi_{\mathcal{O}}$. Thus, $\chi_{\mathcal{O}} \circ \psi_{\mathcal{O}}$ is an isomorphism.

We now show $\psi_G \circ \chi_G \in \text{iso}_{\text{LFG}}$. For this we must also show that $\psi_G \circ \chi_G$ is well defined. Specifically we require that

$$\begin{aligned} \text{dom}(\psi_G) &= \chi_G(G) \\ \text{flag}(\chi(G)) &= \{\{a_{(g,r)} : n \in g \in \text{comp}(G \setminus r)\} : n \in G\} \end{aligned}$$

For each node n of the graph, there is a unique sequence of induced subgraphs $\{g_0, \cdots, g_t\}$ with $n \in g_r \in \text{comp}(G \setminus r)$. Since the intersection of any of these at least includes n, we have $a_{(g_r,r)} \leq a_{(g_s,s)}$ for $r \leq s$ and $\{a_{(g_0,0)}, \cdots, a_{(g_t,t)}\} = \chi_G(n)$ is a flag of $\chi(G)$. To see that the map is an injection, we will show that for any two distinct nodes n and m there is some r such that they are in different component of $G \setminus r$. If n and m are not adjacent then by Condition LFG-2 there are two labels that appear on every path between n and m, and they cannot be in the same component of the graph where edges with either label are removed. Alternatively, suppose n and m are adjacent and let r be the label on nm. For them to be in the same component of $G \setminus r$, there must be some path γ between n and m that does not include r. This would mean that $\gamma \cdot nm$ is a cycle with a label appearing only once, but that is impossible. Such a cycle could not be a triangle, since this would violate Condition LFG-1. If the cycle is not a triangle, then the first edge pn of γ with nm forms a path between p and m, which cannot be adjacent since we would otherwise have a triangle, but by Condition LFG-2 the labels on both edges must appear somewhere on the rest of γ, including r. Thus, there is no such path γ and $\psi_G \circ \chi_G$ is injective. By Lemma 2.3 there is some node in every subgraph of each flag of $\chi_G(G)$, so the map is a bijection.

To see that edges and labels are preserved, consider two adjacent nodes n and m with edge nm labeled r. For $s \neq r$, both n and m are in the same component of $G \setminus s$, and since they are distinct nodes and the map is injective they must be in different components of $G \setminus r$. Therefore, $\psi_G \circ \chi_G(n)$ and $\psi_G \circ \chi_G(m)$ are adjacent with edge labeled r. In contrast, by Condition LFG-2 for any two non adjacent nodes will be two labels that appear on every path between them, so their images will not be connected. Thus, $\psi_G \circ \chi_G$ is an isomorphism. □

2.2 Categories

Restricting ourselves to cone connected posets gives us a single representative for the posets having isomorphic labeled flag graphs, but we could have chosen a different class of representatives. In this section we will see how this choice is canonical.

A **category** is a class of objects **O** and morphisms **M** where every object o has an **identity** morphism id_o and every morphism α has a pair of objects, its **domain** $\mathrm{dom}(\alpha)$ and **codomain** $\mathrm{cod}(\alpha)$ and every pair of morphisms $\alpha\,\beta$ with $\mathrm{cod}(\alpha) = \mathrm{dom}(\beta)$ has a **composition** morphism $\beta \circ \alpha$ such that the following conditions hold.

1. $\mathrm{dom}(\beta \circ \alpha) = \mathrm{dom}(\alpha)$ and $\mathrm{cod}(\beta \circ \alpha) = \mathrm{cod}(\beta)$

2. $\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha$

3. $\mathrm{dom}(\mathrm{id}_o) = \mathrm{cod}(\mathrm{id}_o) = o$

4. $\alpha \circ \mathrm{id}_{\mathrm{dom}(\alpha)} = \mathrm{id}_{\mathrm{cod}(\alpha)} \circ \alpha = \alpha$

A **functor** Φ from a category C to a category D is a function sending each object and morphism in C to one in D that preserves identity and composition. That is so that $\Phi(\mathrm{id}_o) = \mathrm{id}_{\Phi(o)}$ and $\Phi(\beta \circ \alpha) = \Phi(\beta) \circ \Phi(\alpha)$. A standard reference text on categories is [14].

An example is the category SET. This consists of all sets with morphism being functions and composition and identity the usual composition and identity. Many categories consist of sets with some structure and with morphisms being functions that preserve this structure, though this is not always the case. The categories we are interested in here are GPOSET or CCP and LFG. So far we have specified the objects, but we still need to specify the morphism.

There is not presently a standard category for posets. Monotonicity is commonly required for a function to be a morphism between preorders, but may not preserve transitivity. Pfaltz suggests an alternative in [19] that does and has other nice properties, but we will not give their details here. We want ψ and χ to carry morphisms between GPOSET and LFG, but they do not send clades to nodes and back, so the morphisms we use cannot be functions on these with special restrictions in both categories. In fact we have a lot of leeway in what we could use and still get the same results. We only need to require the isomorphisms to agree with Theorem 2.5. We provide a natural choice in this setting setting below, which also uses functions between powersets.

As mentioned earlier we will present ψ in a way that acts on chains. For this we denote the chains of the poset as $\mathrm{chain}(\mathcal{O})$, and ψ acting on the chains of a particular poset as $\psi_{\hat{\mathcal{O}}}$. We also include among the chains all of $\mathcal{O}$ with $\mathrm{chain}(\mathcal{O})$ so that it is a closure system.

With this $\text{chain}^*(\mathcal{O})$ is also a closure system from which we will get the graph $\psi(\mathcal{O})$. We would like to think of $\text{chain}^*(\mathcal{O})$ as a closure system over the flags of $\mathcal{O}$, but $\text{chain}^*(\mathcal{O})$ may not be atomistic. We can instead, however, treat $\text{chain}^*(\mathcal{O})$ as a closure system of $\text{flag}(\mathcal{O}) \times \mathcal{T}$. Here the ranks $\mathcal{T}$ serve to tell us the ranks of clades in the chain, which is necessary in cases where a chain can only be extended to include a certain rank in a unique way. In such a case just knowing what flags contain the chain would not provide enough information to identify it.

A chain that is covered by a coatom in $\text{chain}(\mathcal{O})$ is one clade shy of a flag, and corresponds to a chain covering an atom in $\text{chain}^*(\mathcal{O})$. We treat these as subsets of $\text{flag}(\mathcal{O})$ labeled with the missing rank of the chain. Finally we get to $\psi(\mathcal{O})$ as a graph on the atoms of $\text{chain}^*(\mathcal{O})$ by replacing each of these subsets with a clique of edges all labeled with the same rank as this subset.

Notice this gives a graph on flags of $\mathcal{O}$ with edges between flags that differ in one clade labeled by the rank of that clade, and as such is consistent with ψ as defined earlier. For a particular graded poset this gives us $\psi_{\hat{\mathcal{O}}}$.

$$\psi_{\hat{\mathcal{O}}} : \text{chain}(\mathcal{O}) \to \text{comp}^*_{\hookrightarrow} \circ \psi(\mathcal{O})$$
$$\psi_{\hat{\mathcal{O}}}(\mathcal{C}) := (\{\mathcal{F} \in \text{flag}(\mathcal{O}) : \mathcal{C} \subset \mathcal{F}\}, \text{rank}^{\mathfrak{p}}(\mathcal{C}))$$

Here, for a graph G, $\text{comp}_{\hookrightarrow}(G)$ consists of sets of edge labels $\mathcal{S}$ together with connected components of G with edges having labels not in $\mathcal{S}$ removed.

$$\text{comp}_{\hookrightarrow}(G) := \{(g, \mathcal{S}) : g \in \text{comp}(G \setminus \mathcal{S}), \mathcal{S} \subset \mathcal{T}\}$$

For the other direction, χ acts on these subgraphs of G with subsets of edge labels by passing through isomorphic copies of the same closure systems in the reverse order, and we denote the function on $\text{comp}_{\hookrightarrow}(G)$ as $\chi_{\hat{G}}$ to distinguish it from the function acting on the space of gap commuting graphs.

We get the poset $\chi(G)$ in a way similar to the situation above. The clades of the poset are the atoms of $\text{comp}^*_{\hookrightarrow}(G)$. These atoms, the coatoms of $\text{comp}_{\hookrightarrow}(G)$, consist of a label r together with connected components of G without the edges that have this label, $\text{comp}(G \setminus r)$. Each such atom will correspond to a clade of rank r in $\chi(G)$, and two clades are comparable to each other when the corresponding atoms are covered by their join. Since this gives comparability and rank, it completely describes the poset.

Notice we get the same partial ordering on the same subgraphs as $\chi(G)$ as defined earlier. For a particular gap commuting graph this gives us $\chi_{\hat{G}}$.

$$\chi_{\hat{G}} : \text{comp}_{\hookrightarrow}(G) \to \text{chain}^* \circ \chi(G)$$
$$\chi_{\hat{G}}(h, \mathcal{S}) := \{(g, r) : h \subset g \in \text{comp}(G \setminus r), r \in \mathcal{S}\}$$

To view ψ and χ as functors between categories, we need to specify how they act on what morphisms. For this we use the usual lattice morphisms of these closure systems, functions α such that $\alpha(a \vee b) = \alpha(a) \vee \alpha(b)$ and $\alpha(a \wedge b) = \alpha(a) \wedge \alpha(b)$.

Now that we have seen ψ and χ to be inverses of each other up to isomorphism on their respective ranges it is just a matter of letting them carry the appropriate morphisms along with the objects between GPOSET and LFG. For a morphism α in GPOSET, we define $\psi(\alpha)$ to be the unique morphism in LFG so that the following diagram commutes.

$$\begin{array}{ccc} \hat{\mathcal{O}}_0 & \xrightarrow{\alpha} & \hat{\mathcal{O}}_1 \\ \downarrow \psi_{\hat{\mathcal{O}}_0} & & \downarrow \psi_{\hat{\mathcal{O}}_1} \\ \hat{G}_0 & \xrightarrow{\psi(\alpha)} & \hat{G}_1 \end{array}$$

$$\alpha : \text{chain}(\mathcal{O}_0) \to \text{chain}(\mathcal{O}_1)$$
$$\psi(\alpha) : \text{comp}_{\hookrightarrow} \circ \psi(\mathcal{O}_0) \to \text{comp}_{\hookrightarrow} \circ \psi(\mathcal{O}_1)$$
$$\psi(\alpha) \circ \psi_{\hat{\mathcal{O}}_0} := \psi_{\hat{\mathcal{O}}_1} \circ \alpha$$

We also define $\chi(\beta)$ for a morphism β in LFG in this same way as represented below.

$$\begin{array}{ccc} \hat{G}_0 & \xrightarrow{\beta} & \hat{G}_1 \\ \downarrow \chi_{\hat{G}_0} & & \downarrow \chi_{\hat{G}_1} \\ \hat{\mathcal{O}}_0 & \xrightarrow{\chi(\beta)} & \hat{\mathcal{O}}_1 \end{array}$$

$$\beta : \text{comp}_{\hookrightarrow}(G_0) \to \text{comp}_{\hookrightarrow}(G_1)$$
$$\chi(\beta) : \text{chain} \circ \chi(G_0) \to \text{chain} \circ \chi(G_1)$$
$$\chi(\beta) \circ \chi_{\hat{G}_0} := \chi_{\hat{G}_1} \circ \beta$$

Lemma 2.6. *ψ and χ are well defined functors.*

Proof: For ψ to be well defined we need preimages of elements of the domain of $\psi(\alpha)$ under $\psi_{\hat{\mathcal{O}}_0}$ to give the same result on the right hand side. Given any component with ranks $(g, \mathcal{S})$ of $\hat{G}_0$, we can uniquely identify the chain $\mathcal{C}$ of $\hat{\mathcal{O}}_0$ contained in the flags that are nodes of g and have ranks $\mathcal{S}$. For the diagram to commute we let $\psi(\alpha)(g, \mathcal{S}) \coloneqq \psi_{\hat{\mathcal{O}}_1} \circ \alpha(\mathcal{C})$, so the morphism $\psi(\alpha)$ is uniquely defined. For χ this is well defined since $\chi_{\hat{G}_0}$ is invertable so we let $\chi(\beta)(\mathcal{C}) \coloneqq \chi_{\hat{G}_1} \circ \beta \circ \chi^{-1}_{\hat{G}_0}(\mathcal{C})$. □

A **natural transformation** $\eta : \Phi \dot{\to} \Gamma$ from a functor Φ to a functor Γ that are themselves both from C to D, is a function sending each object o in C to a morphism $\eta_o : \Phi(o) \to \Gamma(o)$ in D such that for every morphism α in D we have $\eta_{\mathrm{cod}(\alpha)} \circ \Phi(\alpha) = \Gamma(\alpha) \circ \eta_{\mathrm{dom}(\alpha)}$. An **isomorphism** is a morphism α with an inverse α^{-1} such that $\alpha^{-1} \circ \alpha$ and $\alpha \circ \alpha^{-1}$ are identity morphisms. When η_o is an isomorphism for every object o we call η a **natural isomorphism**. An **equivalence of categories** is a pair of functors Φ and Γ such that $\Gamma \circ \Phi$ and $\Phi \circ \Gamma$ are naturally isomorphic to the identity functors in their respective categories.

Theorem 2.7. *ψ and χ give an equivalence of categories between* CCP *and* LFG.

Proof: For these to be equivalent categories there must be natural isomorphisms

$$\theta : \mathrm{id}_{\mathrm{CCP}} \dot{\to} (\chi \circ \psi) \qquad \varepsilon : \mathrm{id}_{\mathrm{LFG}} \dot{\to} (\psi \circ \chi)$$

$$\begin{array}{ccc} \mathcal{O}_0 & \xrightarrow{\alpha} & \mathcal{O}_1 \\ \downarrow \theta_{\mathcal{O}_0} & & \downarrow \theta_{\mathcal{O}_1} \\ \chi\psi(\mathcal{O}_0) & \xrightarrow{\chi\psi(\alpha)} & \chi\psi(\mathcal{O}_1) \end{array} \qquad \begin{array}{ccc} G_0 & \xrightarrow{\beta} & G_1 \\ \downarrow \varepsilon_{G_0} & & \downarrow \varepsilon_{G_1} \\ \psi\chi(G_0) & \xrightarrow{\psi\chi(\beta)} & \psi\chi(G_1) \end{array}$$

We have these already as $\theta_{\mathcal{O}} \coloneqq \chi_{\hat{\mathcal{O}}} \circ \psi_{\hat{\mathcal{O}}}$ and $\varepsilon_G \coloneqq \psi_{\hat{G}} \circ \chi_{\hat{G}}$. Since $\chi_{\mathcal{O}} \circ \psi_{\mathcal{O}}$ is an order isomorphism, and chain$(\mathcal{O})$ and its lattice structure are defined only in terms of $\mathcal{O}$, we have that $\theta_{\mathcal{O}}$ is an isomorphism. We have only to see that these are natural transformations, which comes directly from the way we defined the morphism, we just unpack the notation.

$$\begin{aligned}
\chi\psi(\alpha) \circ \theta_{\mathcal{O}_0} &= \chi\psi(\alpha) \circ \chi_{\hat{\mathcal{O}}_0} \circ \psi_{\hat{\mathcal{O}}_0} \\
&= \chi\psi(\alpha) \circ \chi_{\hat{\psi}(\mathcal{O}_0)} \circ \psi_{\hat{\mathcal{O}}_0} \\
&= \chi_{\hat{\psi}(\mathcal{O}_1)} \circ \psi(\alpha) \circ \psi_{\hat{\mathcal{O}}_0} \\
&= \chi_{\hat{\psi}(\mathcal{O}_1)} \circ \psi_{\hat{\mathcal{O}}_1} \circ \alpha \\
&= \chi_{\hat{\mathcal{O}}_1} \circ \psi_{\hat{\mathcal{O}}_1} \circ \alpha \\
&= \theta_{\mathcal{O}_1} \circ \alpha
\end{aligned}$$

The same calculation for $\psi \circ \chi$ shows that ε is also a natural isomorphism. □

A **left adjoint functor** Φ is a functor from C to D such that for every object o in D there is a **universal arrow** consisting of an object p in C and morphism $\alpha : \Phi(p) \to o$ in D that is universal in the sense that α factors through any other such pair in a unique way. That is for any other object q and morphism $\beta : \Phi(q) \to o$ there is a unique morphism $\gamma : q \to p$ such that $\alpha \circ \Phi(\gamma) = \beta$. When the universal arrow comes from a functor Γ giving an object $\Gamma(o)$ and natural transformation η giving a morphism $\eta_o : \Phi\Gamma(o) \to o$ we say Γ is a **right adjoint** for Φ.

Theorem 2.8. *ψ is a left-adjoint functor from* GPOSET *to* LFG*, with right adjoint χ.*

Proof: For any labeled flag graph G the universal arrow is the cone connected poset $\chi(G)$ and morphism ε_G^{-1}. For any other graded poset $\mathcal{O}$ and morphism $\beta : \psi(\mathcal{O}) \to G$, we get the required morphism as $\chi(\beta) \circ \theta_{\mathcal{O}}$.

To see that this is the right morphism we first look at the relationship between θ and ε.

$$\begin{aligned}
\varepsilon_{\psi(\mathcal{O})} \circ \psi_{\hat{\mathcal{O}}} &= \psi_{\hat{\psi}(\mathcal{O})} \circ \chi_{\hat{\psi}(\mathcal{O})} \circ \psi_{\hat{\mathcal{O}}} \\
&= \psi_{\hat{\chi}\psi(\mathcal{O})} \circ \chi_{\hat{\psi}(\mathcal{O})} \circ \psi_{\hat{\mathcal{O}}} \\
&= \psi_{\hat{\chi}\psi(\mathcal{O})} \circ \chi_{\hat{\mathcal{O}}} \circ \psi_{\hat{\mathcal{O}}} \\
&= \psi_{\hat{\chi}\psi(\mathcal{O})} \circ \theta_{\mathcal{O}}
\end{aligned}$$

This exactly matches the defining equation for how ψ acts on morphism, which we have seen to be unique, so $\psi(\theta_{\mathcal{O}}) = \varepsilon_{\psi(\mathcal{O})}$. Now to check that the conditions for being a left-adjoint are satisfied we see that β factors into this morphism's image under ψ and ε_G^{-1}.

$$\begin{aligned}
\varepsilon_G^{-1} \circ \psi(\chi(\beta) \circ \theta_{\mathcal{O}}) &= \varepsilon_G^{-1} \circ \psi\chi(\beta) \circ \psi(\theta_{\mathcal{O}}) \\
&= \varepsilon_G^{-1} \circ \psi\chi(\beta) \circ \varepsilon_{\psi(\mathcal{O})} \\
&= \varepsilon_G^{-1} \circ \varepsilon_G \circ \beta = \beta
\end{aligned}$$

□

2.3 Abstract Polytopes

The primary concern of this thesis is with polytopes, and in particular their face lattices. In the introduction we provided a combinatorial object with simple properties resembling that of a combinatorial polytope, namely abstract polytopes. In this section, we will see how these properties translate to their labeled flag graphs, and in doing so characterize the labeled flag graphs of abstract polytopes. Alternatively, Eulerian posets, which are presented by Stanley in [22, p. 135], may be used for this purpose. Shellable and Eulerian lattices resemble the face lattices of polytopes much more closely, and both properties are used by Paffenholz this way [17], but we do not consider them here.

We first notice that all abstract polytopes are cone connected, and hence so are all combinatorial polytopes, since cone connectedness is a weaker condition then strong connectedness. This means their labeled flag graphs provide the same information as the poset and the properties we will give could be used as an alternate definition of abstract polytopes. We start by seeing what strong connectedness means for the labeled flag graph.

Theorem 2.9. *for $G = \psi(\mathcal{O})$ where $\mathcal{O}$ is a cone connected bounded poset, the following conditions are equivalent*

1. *$\mathcal{O}$ is strongly connected.*

2. *For any chain of clades $\{c_i\}$ the poset of all clades comparable to every c_i of the chain is flag connected.*

3. *For any pair of nodes of G and* **collection** *of edge labels such that for each label there is a path between the nodes where* **this** *label does not appear, there is a path between the nodes where* **none** *of these labels appear.*

4. *For any pair of nodes of G and* **pair** *of edge labels such that for each label there is a path between the nodes where* **this** *label does not appear, there is a path between the nodes where* **neither** *label appears.*

Proof: To see $1 \Rightarrow 2$, let 1 hold and consider a pair of flags f f' and a collection of ranks $\{\text{rank}(\bot) = r_0, r_1, \cdots, r_k = \text{rank}(\top)\}$ where the clades of these flags are equal $c_i := f_{r_i} = f'_{r_i}$. We now form a path between these flags by concatenating a path for each interval, to show 2. Formally, let $f^i_s := f_s$ for $s \le r_i$ and $f^i_s := f'_s$ for $s \ge r_i$. By the flag connectedness of intervals for each pair f^{i-1} f^i there is a path γ^i between these flags in $\psi([c_{i-1}, c_i]) \hookrightarrow \psi(\bigcap_{c \in f_{\le r_{i-1}} \cup f'_{\ge r_i}} \mathcal{O}_c) \hookrightarrow \psi(\bigcap_j \mathcal{O}_{c_j}) \hookrightarrow G$ where $\mathcal{O}_c := \{a : a \gtreqless c\}$. Taking these paths together gives us a path $\gamma = \gamma^1 \cdots \gamma^k$ between f and f' in $\psi(\bigcap_j \mathcal{O}_{c_j})$, and since this holds for any pair of flags, 2 holds.

To see $2 \Rightarrow 3$, let 2 hold and consider a pair of nodes n n', a collection of labels $\{r_i\}$, and a path between these nodes for each label as in the first part of 3. With this we have $c_i := \chi(n)_{r_i} = \chi(n')_{r_i}$ is the same component of $G \setminus r_i)$ for all i, so by 2 there is a path γ between $\chi(n)$ and $\chi(n')$ in $\psi(\bigcap_i \mathcal{O}_{c_i}) \hookrightarrow G$. This path will have no edges labeled r_i, and since there is always such a change in path, 3 holds.

We have $3 \Rightarrow 4$ immediately, since 4 is just a special case of 3 when the collection of labels avoided consists of only two.

To see $4 \Rightarrow 1$, let 4 hold and consider an interval $[c, c']$ of $\mathcal{O}$ with bounds of respective ranks r r' and a pair of flags f_I f'_I of this interval. These flags can each be extended respectively to flags $f_{\mathcal{O}}$ $f'_{\mathcal{O}}$ of $\mathcal{O}$. These extended flags will have the same rank r clade $(f_{\mathcal{O}})_r = (f'_{\mathcal{O}})_r = c$, so they will be in the same component of $G \setminus r$ and there will be a path γ_r between them with no edges labeled r, and likewise a path $\gamma_{r'}$ for r'. By 4 there will be a path γ between these flags where neither label appears. Since G is gap commuting, from γ we can produce a three part path $\gamma_{<r} \cdot \gamma_{(r,r')} \cdot \gamma_{>r'}$ by moving labels less than r to the first part, those greater then r' to the last part, and leaving the rest for the middle part. $\gamma_{(r,r')}$ now projects to a path between f_I and f'_I in $\psi([c,c'])$ by removing those clades with rank outside $[r,r']$, and since this can be done for any pair of flags in the interval, it is flag connected. Finally, since this can be done for any interval 1 holds. □

We call an edge labeled graph that satisfies the conditions of Theorem 2.9 **simultaneously connected**. To be consistent with posets we may want to call this strongly connected, but a strongly connected graph already has an unrelated meaning that is widely used. We now see what the diamond property, that every rank 2 interval has 4 clades, means for labeled flag graphs.

Lemma 2.10. *For $G = \psi(\mathcal{O})$ where $\mathcal{O}$ is a bounded poset, the diamond property holds for $\mathcal{O}$ if and only if every label gives a* **perfect matching**, *that is each node has exactly one edge with each label.*

Proof: Suppose the diamond property holds for $\mathcal{O}$, then for $f \in \text{flag}(\mathcal{O})$ and $0 < r < d$, the interval $[f_{r-1}, f_{r+1}]$ is a diamond consisting of $f_{r-1} < c, f_r < f_{r+1}$. $f'_s = \{c \; s{=}r; \; f_s \; s{\neq}r$ is a flag of $\mathcal{O}$, but $f(a)_s = \{a \; s{=}r; \; f_s \; s{\neq}r$ is not a flag for any clade other then c or f_r. Thus, $\psi_G(f)$ has exactly exactly one edge labeled r. Moreover this holds for any choice of f and r, and every label gives a perfect matching.

For the reverse direction suppose every label gives a perfect matching in G, and consider a rank 1 interval $[c,c']$ of $\mathcal{O}$. The bounds of this interval can be extended to a flag $f \ni c, c'$. Now $\psi_G(f)$ has exactly one neighbor in G_r for each clade in (c,c') other than the rank r clade of f, and since this is a perfect matching, there is only one other such clade. Thus, the interval is a diamond, and this holds for all intervals of $\mathcal{O}$. □

We now have all the conditions we require. We could simply add these to the list of conditions for an edge labeled graph to be a labeled flag graph, and we would be done, but with the added conditions some of the conditions of being a labeled flag graph would be redundant. We see first how simultaneous connectedness relates to Condition LFG-2 of Theorem 2.1, then list the conditions without the redundancies.

Lemma 2.11. *A simultaneously connected graph cannot have a pair of paths with distinct common end nodes that do not share any labels.*

Proof: Let G be a simultaneously connected graph and assume there are distinct nodes n and n' with paths γ and γ' between them such that no label appears on both paths. With this, there must be a path between n and n' with only labels that appear on both γ and γ'. Such a path would have no labels at all, which is impossible, so any pair of paths between n and n' must share a label. □

Theorem 2.12. *An edge labeled graph G is the labeled flag graph of a an abstract polytope $\mathcal{P}$ if and only if it satisfies the following conditions*

AP-*1. Every label gives a perfect matching*

AP-*2. G is simultaneously connected*

AP-*3. G is gap commuting*

Moreover, $\mathcal{P} \cong \chi(G)$

Proof: Suppose $G = \psi(\mathcal{P})$ is the labeled flag graph of an abstract polytope. We have that G is gap commuting by Theorem 2.1 and Lemma 2.2, since it is the labeled flag graph of a poset; G is simultaneously connected by Theorem 2.9, since $\mathcal{P}$ is strongly connected; and every label gives a perfect matching by Lemma 2.10, since the diamond property holds for $\mathcal{P}$.

For the reverse direction, suppose G satisfies these conditions. We will show that G also satisfies the conditions of Theorem 2.1 and as such is the flag graph of a graded poset. To see that Condition LFG-1 holds assume there is a triangle $n_0n_1n_2$ where more then one label appears. One of these labels r must appear only once in the triangle, so let n_0n_1 be the edge with that label. Now we have two paths between n_0 and n_1, one where only r appears, and one where r does not appear, but the contradicts Lemma 2.11, so there can be no such triangle, and Condition LFG-1 of holds. To see that Condition LFG-2 holds, consider a pair of distinct nodes n, n' of G and the paths between them. By Lemma 2.11 the paths must share at least one common label. Now, If there is only one common label appearing on all paths, in which case there is a path between them where only this label appears, the nodes must be adjacent by AP-1, since such a path consists of a single edge. Thus, Condition LFG-2 holds. With this, Condition LFG-3 is equivalent to Condition AP-3 by Lemma 2.2, so G is the labeled flag graph of a graded poset $\mathcal{P}$. We will now show that G satisfies the conditions of an abstract polytope. For $\mathcal{P}$ to be bounded we choose the ranks of $\mathcal{P}$ to include one rank greater than and one less than those that appear in the graph. Condition AP-1 gives us the diamond property by Lemma 2.10, and Condition AP-2 gives us strong connectedness by Theorem 2.9, so $\mathcal{P}$ is an abstract polytope. Finally, we have that this correspondence is an isomorphism by Theorem 2.5 $\square$

CHAPTER 3

REALIZABILITY

3.1 Filled Incidence Matricies

In this chapter we show that the problem of realizing a polytope with specified combinatorics is equivalent to a low rank matrix completion problem. For an abstract polytope lattice $\mathcal{P}$, we call a $|\text{facet}(\mathcal{P})| \times |\text{vert}(\mathcal{P})|$ matrix N with entries $(N)_{i,j} = 0$ when vertex j is contained in facet i, $v_j \in F_i$, and $(N)_{i,j} < 0$ for all other entries, a **filled 0-incidence matrix** of $\mathcal{P}$, and when we add 1 to all entries we call this a **filled 1-incidence matrix**.

We will see the rank d filled 1-incidence matrices of a combinatorial polytope are the matrices of inner products of some realization's vertices and covertices. Robertson used this fact to count the dimension of a polytopes realization space by showing that an infinitesimal perturbation of a polytope's vertices and covertices are that of some realization of its combinatorial type if and only if the combinatorics of its filled 1-incidence matrix is preserved [21, p. 14]. Similarly Díaz showed that the rank d+1 filled 0-incidence matrices are that of a polytopal cone [6]. Both results, however, require the combinatorics to be that of a polytope.

Díaz provides alternative conditions for finding realizations of polytopes in non-Euclidean spaces. To state her theorem we need some additional definitions. A **bilinear form** $f : V^2 \to \mathbb{R}$ is given by $f(x, y) = x^* F y$ for some matrix

F on a real vector space V, which we say has a particular property of matrices, such as being symmetric or definite, when the matrix F has that property. We call $f(x) = f(x,x)$ a **quadratic form**. The **signature** of a matrix, and likewise a bilinear form, is the pair (p,n) where p and n are the total ranks of eigenspaces with positive and negative eigenvalues respectively. An inner product is a symmetric positive definite bilinear form. That is, it has signature $(r,0)$. A **polytopal cone** $\hat{P} = \text{con}(\{v_j\})$ generated by vectors $\{v_j\}$ is the set of positive linear combinations $\text{con}(\{v_j\}) := \{\sum_j \lambda_j v_j : \lambda_j \geq 0\}$ when this does not contain a line $v \in \hat{P} \Rightarrow -v \notin \hat{P}$, or equivalently by half spaces H_i^- is the intersection $\hat{P} := \bigcap_i H_i^-$ when the intersection of their supporting hyperplanes $H_i = \partial H_i^-$ is the origin $\bigcap_i H_i = \mathbb{0}$. For a half space H^-, where the restriction of f to its supporting hyperplane H has full rank, there is a unique vector h such that $H^- = \{x : f(h,x) \leq 0\}$ and $|f(h)| = 1$, called the **outward normal**, otherwise we say H is **lightlike**. We call the rays $\mathbb{R}^+ v_j$ and $\mathbb{R}^+ h_i$ the **vertices** and **covertices** of $\hat{P}$ respectively. Díaz considers **polytopes** $P := \hat{P} \cap f^{-1}(c)$ in a level set at $c \neq 0$ of a symmetric definite quadratic form f given by its intersection with a polytopal cone $\hat{P}$ having no lightlike facet supporting hyperplanes, and in particular its **Grammian** matrix G with $(G)_{i,j} = f(h_i, h_j)$ where $\{h_i\}$ are their outward normals.

We call a sequence of facets $F_{i_1}, \cdots, F_{i_s}$ of a polytope a **truncated oriented cycle** when $\bigcap_{k=1}^s F_{i_k}$ is a face of the polytope, and when $s = d$ we call this an **oriented cycle** and when $s = d+1$ a **maximal oriented cycle**. We say two maximal oriented cycles have the **same orientation** when the induced flags $\varnothing = \bigcap_{k=1}^{d+1} F_{i_k}, \bigcap_{k=1}^{d} F_{i_k}, \cdots, F_{i_1}, \mathcal{P}$ are an even distance apart in the flag graph. Here we denote the minor of a matrix G with rows $i_1 \cdots i_n$ and columns $j_1 \cdots j_n$ by $G\left[\begin{smallmatrix} i_1 \cdots i_m \\ j_1 \cdots j_n \end{smallmatrix}\right]$.

Theorem 3.1. (Díaz) *Let $\mathcal{P}$ be a combinatorial d-polytope and G be a $|\text{facet}(\mathcal{P})| \times |\text{facet}(\mathcal{P})|$ symmetric matrix with diagonals ± 1 and the same signature as f. There is a polytopal cone of type $\mathcal{P}$ with Grammian G if and only if G satisfies the following:*

1. *For every vertex of* $\mathcal{P}$ *and all facets* $F_{i_1} \cdots F_{i_n}$ *incident to it, the submatrix* $G\left[\begin{smallmatrix} i_1 \cdots i_n \\ i_1 \cdots i_n \end{smallmatrix}\right]$ *has rank* $\le d$.

2. *For every pair of maximal cycles* $F_{i_1}, \cdots, F_{i_{d+1}}$ *and* $F_{j_1}, \cdots, F_{j_{d+1}}$ *with the same orientation,* $\det(G\left[\begin{smallmatrix} i_1 \cdots i_{d+1} \\ j_1 \cdots j_{d+1} \end{smallmatrix}\right])\det(f) > 0$.

In the case of the usual inner product $f(\cdot,\cdot) = \langle \cdot,\cdot \rangle$ we get the d-sphere $\mathbb{S}^d = \{x \in \mathbb{R}^{d+1} : \|x\| = 1\}$, and we can specify a polytope by its vertices $\{v_j\}$,

$$P := \left\{ x \in \mathbb{S}^d : x = \sum_j \lambda_j v_j, \lambda_j \ge 0 \right\}$$

or we can specify it by its covertices $\{h_i\}$.

$$P := \left\{ x \in \mathbb{S}^d : \langle h_i, x \rangle \le 0 \right\}$$

Note that for any polytope in the sphere $\mathbb{R}P$ is a polytope in $\mathbb{RP}^d$, and every polytope in projective space is the image of some polytope in the sphere under this map. Here we can restate Theorem 3.1 as follows.

Theorem 3.2. (Díaz) *Let* $\mathcal{P}$ *be a combinatorial polytope and* G *be a* $|\text{facet}(\mathcal{P})| \times |\text{facet}(\mathcal{P})|$ *symmetric matrix with diagonals all 1. There is a polytope* $P \subset \mathbb{S}^d$ *of type* $\mathcal{P}$ *with Grammian* G *if and only if* G *satisfies the following:*

1. *For every vertex of* $\mathcal{P}$ *and all facets* $F_{i_1} \cdots F_{i_n}$ *incident to it, the submatrix* $G\left[\begin{smallmatrix} i_1 \cdots i_n \\ i_1 \cdots i_n \end{smallmatrix}\right]$ *has rank* d.

2. *For every face of rank* $d-s$ *with* $2 \le s \le d+1$ *and truncated oriented cycle* $F_{i_1}, \cdots, F_{i_s}$ *incident to it,* $\det(G\left[\begin{smallmatrix} i_1 \cdots i_s \\ i_1 \cdots i_s \end{smallmatrix}\right]) > 0$.

3. *For every pair of maximal cycles* $F_{i_1}, \cdots, F_{i_{d+1}}$ *and* $F_{j_1}, \cdots, F_{j_{d+1}}$ *with the same orientation,* $\det(G\left[\begin{smallmatrix} i_1 \cdots i_{d+1} \\ j_1 \cdots j_{d+1} \end{smallmatrix}\right]) > 0$.

Condition 2 is just an alternative to requiring that G be positive definite. Having the Grammian is nice because it tells us what the dihedral angles of a polytope can be. On the way to proving this she proves a lemma that is more similar to Theorem 3.6. For both of these Díaz also proves analogous results for finite volume hyperbolic polytopes as well as spherical ones.

Lemma 3.3. (Díaz) *Let $\mathcal{P}$ be a combinatorial polytope, $\{v_j\}$ and $\{h_i\}$ be a vector in $\mathbb{R}^d$ for each vertex i and covertex j of $\mathcal{P}$, and f a symmetric definite bilinear form. These are the vertices and covertices of polytopal cone realizing $\mathcal{P}$ if they satisfy the following.*

1. *For every oriented cycle $i_1 \cdots i_d$ the vectors $h_{i_1} \cdots h_{i_d}$ have rank d.*
2. *$(M)_{i,j} = f(h_i, v_j)$ is a filled 0-incidence matrix for $\mathcal{P}$.*

3.2 Realizations

The most immediate way to get an algebraic statement for determining when posets are realizable is to go directly to the definition, as Grünbaum does in [11, p. 90].

Theorem 3.4. *A poset $\mathcal{P}$, given by a collection of subsets of $I := \{1, \cdots, n\}$ ordered by containment that includes $\{i\} \in \mathcal{P}$ for each $i \in I$ but not $I \notin \mathcal{P}$, is realizable if and only if there are vectors $v_i \in \mathbb{R}^d$ such that for any subset $f \subsetneq I$ there is a vector $a \in \mathbb{R}^d$ with $\langle a, v_i \rangle = 1$ for $i \in f$ and $\langle a, v_i \rangle < 1$ for $i \notin f$ if and only if $f \in \mathcal{P}$.*

We first note that the combinatorics of $\mathcal{P}$ are much more relaxed here than what we will require, but this is compensated for by more stringent algebraic conditions. If we put the required vectors v_i in a matrix V and restrict the conditions to just the maximal clades of $\mathcal{P}$, which would be facets, and put the required vectors a for each in a matrix A, then A^*V would give us a rank d filled 1-incidence matrix of $\mathcal{P}$. The algebraic part of Theorem 3.4 requires us to additionally find such vectors a for all faces, and show that no such vectors exist for all other subsets of I.

The problem of realizability as we have defined it is to find a polytope having a particular poset as its face lattice, but the only combinatorial data appearing in a filled incidence matrix are the incidences between vertices and facets. These are in fact equivalent, and we can generate the lattice with these incidences as we shall see. We could just as well define realizability for an adjacency relation to be the realizability of the lattice generated. We get this lattice in a very intuitive way; we just recognize that each face can be identified by the vertices and facets that are incident to it.

Before generating the lattice $\mathcal{P}$ from the the incidences between vertices and facets we need some definitions. A **bipartite graph** is a graph with nodes in two disjoint parts and edges only between nodes in different parts. A **biclique** is a bipartite graph where every pair with one node from each part is connected by an edge. A **maxbiclique** of a bipartite graph is a maximal set of nodes and the edges between them such that this is a biclique. The **maxbiclique lattice** of a bipartite graph with one part specified as the lower is the poset consisting of it's maxbicliques ordered by containment of nodes in the lower part. A **join irreducible** is a clade of a lattice that cannot be expressed as the join of other clades, and a **meet irreducible** is similarly defined with order reversed.

Lemma 3.5. *Every lattice where all flags are finite is isomorphic to the maxbiclique lattice of its meet and join irreducibles. For abstract polytope lattices these are the vertices and covertices respectively.*

We are now ready to state the main theorem.

Theorem 3.6. *For any bipartite graph $\mathcal{B}$ with a maxbiclique lattice $\mathcal{P}$ that is flag connected and satisfies the diamond condition, the following are equivalent.*

1. *$\mathcal{P}$ is realizable*
2. *$\mathcal{B}$ has a rank d filled 1-incidence matrix*
3. *$\mathcal{B}$ has a rank $d+1$ filled 0-incidence matrix*

Moreover, vectors $w_j \in \mathbb{R}^r$ are the vertices and $h_i \in \mathbb{R}^r$ the covertices of a type $\mathcal{P}$ polytope if and only if $[\langle h_i, w_j\rangle]_{i,j}$ is a rank d filled 1-incidence matrix of $\mathcal{B}$, and w_j and h_i are that of a type $\mathcal{P}$ polytopal cone if and only if $[\langle h_i, w_j\rangle]_{i,j}$ is a rank $d+1$ filled 0-incidence matrix of $\mathcal{B}$.

To prove Lemma 3.5 we will make use of the following lemma about lattices. The argument used here appears in [16, p. 19] as part of a more general result.

Lemma 3.7. *In a lattice with finite flags, every clade c can be identified uniquely as the join of all join irreducibles below it J or the meet of all meet irreducibles above it M.*

Proof: We first note that, allowing Zorn's lemma, all lattices with finite flags are complete, since the sequence of partial joins of $\bigvee(C) = c_1 \vee c_2 \vee \cdots$ is a finite flag, so only finitely many clades $\{c_{i_k}\}$ contribute $\bigvee(C) = c_{i_1} \vee \cdots \vee c_{i_n}$.

Suppose the lemma does not hold, then there is some minimal clade c that is not the join of join irreducibles below it. c cannot itself be a join irreducible since $c = \bigvee\{c\}$, so it can be expressed as $c = \bigvee D$ where each clade $d \in D$ is below c and as such can be expressed as the join of join irreducibles $d = \bigvee J_d$, so we have $c = \bigvee \bigcup_{d \in D} J_d$. Introducing more join irreducibles can only increase their join, so $c = \bigvee J$, and therefor, there can be no such c. Likewise the dual statement holds for meet irreducibles by symmetry. □

Proof of Lemma 3.5: The pair (J, M) we get for a clade c from Lemma 3.7 induces a biclique in the incidence graph of the meet and join irreducibles of the lattice, since for any pair $(j, m) \in (J, M)$, $j \leq c \leq m$. From the definition for clades $c_i = \bigvee J_i$, we have that $c_1 \leq c_2$ if and only if $J_1 \subseteq J_2$. Now we only have to see that these bicliques are maximal. Suppose they are not, then by symmetry we can assume there is a join irreducible $j \notin J$ that is not less than c, but is less than all meet irreducibles M of c, $j \nleq c = \bigwedge M$. We see this is impossible, since by the construction of M we have $j \in \bigcap_{m \in M}\{\cdot \leq m\} \leq c$, so c must correspond to a maxbiclique.

For abstract polytope lattices if a d-face z is not a vertex, $d \neq 0$, then there is some $(d-2)$-face w contained in z, and the interval $[w, z]$ contains exactly two other faces $\{x, y\}$ of dimension $d-1$. This gives $x \vee y = z$, so z is not join irreducible. If z is a vertex then there is only one face below it $\bot$, so it must be join irreducible. By symmetry the facets are the meet irreducibles. □

The proof of Theorem 3.6 works by constructing a polytope and showing that there is an order preserving injection from the abstract polytope lattice given to its face lattice. The following lemma shows that this is sufficient for it to be a realization.

Lemma 3.8. *An order monomorphism between bounded flag connected diamond lattices of the same finite rank is an isomorphism.*

Proof: Allowing the lattice to be cone connected, which we recall abstract polytopes always are, from Theorem 2.7 we saw that there is an equivalence of categories between these posets and their labeled flag graphs given by a functor ψ. This functor then sends any monomorphism between abstract polytopes to a monomorphism between their labeled flag graphs, which we will see must be an isomorphism, so the original monomorphism must also be an isomorphism. We do not, however, really need category theory to prove the claim, so we provide the following alternative argument.

Suppose the lemma fails, then there are flag connected diamond lattices $\mathcal{P}$ and $\mathcal{Q}$ of rank $r = d+1$ with a monomorphism sending $\mathcal{P}$ into $\mathcal{Q}$ that misses some clade $c \in \mathcal{Q}$. Without loss of generality we can take $\mathcal{P}$ to be a subset of $\mathcal{Q}$ and the monomorphism to be the identity, otherwise just replace $\mathcal{P}$ with its image. Consider now the flag graphs G of $\mathcal{P}$ and H of $\mathcal{Q}$. Every flag of $\mathcal{P}$ is a flag of $\mathcal{Q}$ and two flags differ by one clade in $\mathcal{P}$ if and only if they do so in $\mathcal{Q}$, so G is an induced subgraph of H. There is some flag in $\mathcal{P}$, and c must belong to some flag of $\mathcal{Q}$, so $\varnothing \neq G \subsetneq H$. We recall that the labeled flag graphs are perfect matchings by Lemma 2.10 and connected. This means G is a d-regular proper induced subgraph of a connected d-regular graph, namely H, which is impossible.

To see this consider a path from a node that is in G to one that is not. Let n be the last node along this path that is in G. This node must have d neighbors in G and a neighbor that is not in G, the next node in the path, so n must have at least $d+1$ neighbors in H contradicting the fact that H is also d-regular. □

We construct the polytope P from the filled 1-incidence matrix M in Theorem 3.6 by first computing its compact singular value decomposition. The entries of M represent the inner products of corresponding vertices and covertices of the polytope. The **compact singular value decomposition** of a $n \times m$ matrix M is three matrices: a $n \times r$ matrix U with orthonormal columns, a $r \times r$ diagonal matrix Σ with only positive diagonal entries, and a $m \times r$ matrix V with orthonormal columns such that $U\Sigma V^* = M$. It is a theorem from linear algebra that every matrix has such a decomposition [12, p. 414].

P's vertices are given by the rows of $V\sqrt{\Sigma}$, and covertices by the rows of $U\sqrt{\Sigma}$. Each covertex corresponds to a facet of P and scaling it by the distance from the origin to the affine span of this facet gives the outward normal. We can construct the grammian of the vertices and covertices together as follows.

$$\tilde{G} := \begin{bmatrix} \sqrt{MM^*} & M \\ M^* & \sqrt{M^*M} \end{bmatrix} = \begin{bmatrix} U\Sigma U^* & U\Sigma V^* \\ V\Sigma U^* & V\Sigma V^* \end{bmatrix} = \begin{bmatrix} U \\ V \end{bmatrix} \Sigma \begin{bmatrix} U^* & V^* \end{bmatrix}$$

If we instead form $\tilde{G}$ from the filled 0-incidence matrix and symmetrically rescale so the diagonal entries are all 1, we get the matrix from Lemma 3.3 on the off-diagonal blocks and the matrix from Theorem 3.1 on the diagonal blocks for spherical polytopes $\varphi(P)$ and $\varphi(P^*)$ where $\varphi(x) = \left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right] / \left\| \left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right] \right\|$. As we will see neither the span of U nor of V can contain $\mathbb{1}$, the vector with all entries equal to 1, since their rows are the vertices of rank(M)-polytopes. If we first augment these with an orthogonal column so $\mathbb{1}$ is in their span and then complete the augmented matricies to a full singular value decomposition, the additional will columns give a gale transform of these polytopes.

Proof of Theorem 3.6: We refer to the conditions in the theorem by their enumeration. We have that $(1 \Rightarrow 2$ and $3)$ immediately since these are just the matrices with entries equal to the inner products of vertices and covertices. To show $(2 \Rightarrow 1)$ we will construct a realization of the polytope from a rank d filled 1-incidence matrix M with appropriate combinatorics.

Let $\mathcal{A} \subset (I \times J)$ be the pairs of indices for entries of M that are 1, and let the lattice $\mathcal{P}$ of maxbicliques $a = (I_a, J_a)$ of $\mathcal{A}$ be flag connected and satisfy the diamond condition. Now, let $M = U\Sigma V^*$ be the compact singular value decomposition, and let $\{h_i\}$ be the rows of $W = U\sqrt{\Sigma}$, and $\{w_j\}$ be the rows of $H = V\sqrt{\Sigma}$, and $P = \mathrm{conv}(\{w_j\})$. We will show that P is a realization of $\mathcal{P}$. There is no reason why, however, we need to use this decomposition, any pair such that $HW^* = M$ would be fine. Since M has rank d so does W, and P has dimension at most d. Moreover, w_j is in the hyperplane $h_i^{=1} := \{\langle h_i, \cdot\rangle = 1\}$ for $(i,j) \in \mathcal{M}$, but is in the open half space $h_i^{<1} := \{\langle h_i, \cdot\rangle < 1\}$ for $(i,j) \notin \mathcal{M}$. By Lemma 3.5 if $\mathcal{M}$ comes from the vertices and covertices of an abstract polytope $\mathcal{P}'$, then $\mathcal{P}'$ is isomorphic to $\mathcal{P}$. We use the terminology of abstract polytopes for $\mathcal{P}$ regardless.

We will now construct a map from $\mathcal{P}$ to the face lattice of P and show that it is an isomorphism. Let $F_a := P \bigcap_{i\in I_a} h_i^{=1}$ be the face of P we get by intersecting it with the hyperplanes corresponding to covertices of a. We know this is a face of P since these are all supporting hyperplanes of P.

First we see that F_a as a function on a preserves order. In this context we require the very strong condition that $F_a \subseteq F_b$ if and only if $a \leq b$. Suppose $a \leq b$, then $I_a \subseteq I_b$ and $\bigcap_{i\in I_a} h_i^{=1} \subseteq \bigcap_{i\in I_b} h_i^{=1}$ so $F_a \subseteq F_b$. For the other direction suppose $a \nleq b$, then there is some $j \in J_a$ but $j \notin J_b$, so $w_i \in F_a$ but $w_j \notin F_b$ and $F_a \nsubseteq F_b$. Thus order is maintained.

We also have that F_a is an injection. To see this consider a pair of faces a, b of $\mathcal{P}$ that map to the same face $F := F_a = F_b$ of P. With this $w_j \in F \subset h_i^{=1}$ for any $i \in I_a \cup I_b$ and $j \in J_a \cup J_b$, so $m_{ij} = 1$ and $v_j \leq \mathcal{F}_i$ where v_j and $\mathcal{F}_i$ are the corresponding vertices and facets of $\mathcal{P}$ respectively. Since (I_a, J_a) and (I_b, J_b) are maxbicliques that are subsets of the same biclique $(I_a \cup I_b, J_a \cup J_b)$ we must have that $a = b$, and F_a is a monomorphism.

This induces an injection from a flag of $\mathcal{P}$ to a totally ordered set of P's faces, which must be of the same size or less. A larger set cannot be injected into a smaller one, so they must be the same size, and P must be of dimension d. Now this is a monomorphism between bounded flag connected diamond lattices of the same dimension, and by Lemma 3.8 is therefore an isomorphism. Thus, P is a realization of $\mathcal{P}$.

Suppose we have some other decomposition $M = H'W'^*$. The columns of both H and H' give a basis of the same space, namely the range of M, so there is a nonsingular linear transformation A between them $H' = HA$. For H as given this is $A = \Sigma^{-\frac{1}{2}}U^*H'$. Observe here $\Sigma^{-\frac{1}{2}}U^*$ is the pseudo-inverse of H. We can see this formula more clearly by considering a completion of U to a full orthogonal matrix U'.

$$\begin{aligned}\begin{bmatrix} H & \mathbb{0} \end{bmatrix}\begin{bmatrix} A \\ \mathbb{0} \end{bmatrix} &= U'\begin{bmatrix} \Sigma^{\frac{1}{2}} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} \end{bmatrix}\begin{bmatrix} \Sigma^{-\frac{1}{2}} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} \end{bmatrix}U'^*H' \\ &= U'\begin{bmatrix} \Sigma^{\frac{1}{2}} & \mathbb{0} \\ \mathbb{0} & I \end{bmatrix}\begin{bmatrix} \Sigma^{-\frac{1}{2}} & \mathbb{0} \\ \mathbb{0} & I \end{bmatrix}U'^*H' = H'\end{aligned}$$

Note we are able to replace the $\mathbb{0}$ blocks with I blocks, since every column of H' is orthogonal to the extra rows of U'^*. Similarly we have $W' = WA^{-1}$, and for any A nonsingular we get such a decomposition $M = (HA)(WA^{-1})^*$.

We could perhaps show $(3 \Rightarrow 1)$ by going through a variation of the same argument for spherical polytopes, but instead we will show that we can use a rank $r = d+1$ filled 0-incidence matrix N with appropriate combinatorics to find a corresponding rank d filled 1-incidence matrix M, $(3 \Rightarrow 2)$. We first note that $N+E$ is a filled 1-incidence matrix, and can differ from N in rank by at most 1. If the rank of $N+E$ were 1 less we would be done, but that might not be the case. Instead, we will find full diagonal matrices D_1 D_2 such that $M := D_1ND_2 + E$ has rank d. We start by letting $N = U\Sigma V^*$ be the compact singular value decomposition and U' and V' be respective completions of U and V to orthogonal matrices.

We would like to find a vector y with positive entries such that Vy has no zero entries. Suppose every vector in the image of the positive orthant under V has some zero entries. Since the range of V is a linear subspace and the set of positive linear combinations of values v including non-zeros that give zero is of measure 0 among positive linear combinations $v \neq \mathbb{0} \Rightarrow \mu(\{y : \langle v, y\rangle = 0\}) = 0$, the assumption can only hold if V has a row of all zeros, and therefor N must have a column of all zeros, which is impossible, since no meet irreducible of a finite non-trivial diamond lattice can be incident to all join irreducibles. Thus we can find a vector y with all positive entries such that Vy has all non-zero entries, and similarly we can find such a vector x for U. Furthermore, we can normalize these so that $\langle x, \Sigma^{-1}y\rangle = 1$.

With this in mind we let $D_1 = \operatorname{diag}(-Ux)^{-1}$ and $D_2 = \operatorname{diag}(Vy)^{-1}$, and we preform a change of bases to make the rank of the resulting matrix more apparent.

$$\begin{aligned}
(U'^* D_1^{-1})M(D_2^{-1}V') &= U'^* N V' + U'^* D_1^{-1} E D_2^{-1} V' \\
&= U'^* U \Sigma V^* V' + U'^* D_1^{-1} \mathbb{1}\mathbb{1}^* D_2^{-1} V' \\
&= U'^* U (\Sigma - xy^*) V^* V' \\
&= \begin{bmatrix} (\Sigma - xy^*) & \mathbb{0} \\ \mathbb{0} & \mathbb{0} \end{bmatrix}
\end{aligned}$$

Under this change of bases we see e_i is sent to $\mathbb{0}$ for $i > r$, so the rank is at most r. We observe that it is actually $r - 1$, by showing the range is orthogonal to $\begin{bmatrix} \Sigma^{-1}y \\ \mathbb{0} \end{bmatrix}$. For this we consider only the first r coordinates.

$$\begin{aligned}
\left\langle (\Sigma - xy^*)e_i, \Sigma^{-1}y \right\rangle &= \left\langle \Sigma e_i - (y)_i x, \Sigma^{-1}y \right\rangle \\
&= (y)_i - (y)_i = 0
\end{aligned}$$

Thus M has rank d and $(3 \Rightarrow 2)$.

For any decomposition $M = HW^*$ we get one for N by $N = HW^* + E = [H\ \mathbb{1}][W\ \mathbb{1}]^*$, and the rows of $[W'\ \mathbb{1}]$ generate a polytopal cone of type $\mathcal{P}$. Multiplying by a diagonal matrix on the left or right of N corresponds to scaling these vectors, and as such does not change the resulting polytopal cone. Consequently, any decomposition of $N = H'W'^*$ can be expressed as one coming in this way from $M = HW^*$ that gives the same polytopal cone by scaling the rows so the last entry of each is 1, and by the same argument we gave for M such a decomposition is unique up to linear transformations. As a final point we note that the diagonal matrices D_i scale the vertices and covertices of a polytopal cone and its dual to be those of the polytopes given by the intersection of each cone with a hyperplane, and the choice of x and y amount to choosing these hyperplanes. □

Theorem 3.6 can be viewed as a stronger version of Lemma 3.3, which Díaz used to prove Theorem 3.1. Consequently we can use this to prove a stronger version of Theorem 3.1. This will not be substantially different from the proof in [6]. We will just extract the part of the proof that uses the lemma and only change it by using Theorem 3.6 instead. For this we must make use of notation used in that proof. The **exterior power** $\bigwedge^m V$ of a vector space V is itself a vector space consisting of that power of V modulo the addition of linearly dependent vectors with scalar multiplication and vector addition acting componentwise and with vectors denoted by $v_1 \wedge \cdots \wedge v_m$, and $\bigwedge^0 V := \mathbb{R}$ by convention for a real vector space. That is $\bigwedge^m V := V^m / \sim$ where $u_1 \wedge \cdots \wedge u_n \;\sim\; v_1 \wedge \cdots \wedge v_n$ when $\{u_1 - v_1, \cdots, u_m - v_m\}$ is linearly dependent. This is equivalent to anti-commutativity $(\cdots \wedge v_i \wedge \cdots \wedge v_j \wedge \cdots) \;\sim\; -(\cdots \wedge v_j \wedge \cdots \wedge v_i \wedge \cdots)$. We also treat $(\wedge) : \bigwedge^n V \times \bigwedge^m V \to \bigwedge^{n+m} V$ as a function acting on exterior powers by concatenation $(v_1 \wedge \cdots \wedge v_n) \wedge (v_{n+1} \wedge \cdots \wedge v_{n+m}) = v_1 \wedge \cdots \wedge v_{n+m}$. The standard basis for $\bigwedge^m \mathbb{R}^r$ is $\{e_{i_1} \wedge \cdots \wedge e_{i_m} : 1 \le i_1 < \cdots < i_m \le m\}$. Given a bilinear form f on V we define a bilinear form on $\bigwedge^m V$ by $(\bigwedge^m f)(u_1 \wedge \cdots \wedge u_n, v_1 \wedge \cdots \wedge v_n) := \det([f(u_i, v_j)]_{i,j})$. Recall that real vector spaces of the same rank are isomorphic, and notice the space $\bigwedge^m V$ has rank $\binom{r}{m}$ and $\bigwedge^{r-m} V$ has rank $\binom{r}{r-m} = \binom{r}{m}$. The **Hodge star operator** $\star : \bigwedge^m V \to \bigwedge^{r-m} V$ defined with respect to f is the canonical isomorphism between these given for $m = r$ by $\star(u_1 \wedge \cdots \wedge u_r) := 1$ for a positively oriented basis $\{u_i\}$ of V that is orthonormal $|f(u_i)| = 1$ with respect to f and otherwise by $\star := \tau^{-1} \circ \psi$, which are as follows.

$$\psi : \bigwedge^m V \to \left(\bigwedge^{r-m} V\right)^*, \quad \psi(x) := v \to \star(x \wedge v)$$
$$\tau : \bigwedge^{r-m} V \to \left(\bigwedge^{r-m} V\right)^*, \quad \tau(x) := v \to \left(\bigwedge^{r-m} f\right)(x, v)$$

We review the following property of $\star$.

$$f(\star(v_1 \wedge \cdots \wedge v_{r-1}), v_r) = \star(v_1 \wedge \cdots \wedge v_{r-1} \wedge v_r)$$

Theorem 3.9. *Let $\mathcal{P}$ be a flag connected diamond lattice and G be a* $|\text{facet}(\mathcal{P})| \times |\text{facet}(\mathcal{P})|$ *symmetric matrix with diagonals* ± 1 *and the same signature as f. There is a polytopal cone of type $\mathcal{P}$ with Grammian G if and only if G satisfies the following:*

2. *For every pair of maximal cycles $F_{i_1}, \cdots, F_{i_{d+1}}$ and $F_{j_1}, \cdots, F_{j_{d+1}}$ with the same orientation,* $\det(G\left[\begin{smallmatrix} i_1 \cdots i_{d+1} \\ j_1 \cdots j_{d+1} \end{smallmatrix}\right])\det(f) > 0$.

Proof: Since we are only strengthening one side of a biconditional statement, we only have to consider the argument showing one direction. Since G and F, the matrix representation of f, are real symmetric matrices with the same signature, there exists a $|\text{facet}(\mathcal{P})| \times (d+1)$ matrix H such that $G = HFH^*$. We denote rows of H by h_j, and we claim $\hat{P} := \bigcap_j H_j^-$ where $H_j^- := \{x : f(h_j, x) \leq 0\}$ is a polytopal cone of type $\mathcal{P}$. Following in the footsteps of [6], we show this by using the Hodge star operator to find vertices of $\hat{P}$, which we collect in a matrix W, and show that $N = HW^*$ is a rank $d+1$ filled 0-incidence matrix of $\mathcal{P}$.

With this in mind, choose an oriented cycle $i_{j,1}, \cdots, i_{j,d}$ incident to each vertex j of $\mathcal{P}$ such that these all have the same orientation, and let $w_j = \star(h_{i_{j,1}} \wedge \cdots \wedge h_{i_{j,d}})$. Since $\mathcal{P}$ is a flag connected diamond lattice we can choose another covertex $i_{j,d+1}$ that is not incident to j. By Condition 2 for all $i\,j$ we have $\det([f(h_{i_{j,k}}, h_{i_{j,l}})]_{k,l}) = \det(G\left[\begin{smallmatrix} i_{j,1} \cdots i_{j,d+1} \\ i_{j,1} \cdots i_{j,d+1} \end{smallmatrix}\right]) \neq 0$, so the vectors $\{h_{i_{j,k}}\}$ are linearly independent and $h_{i_{j,1}} \wedge \cdots \wedge h_{i_{j,d}} \neq \mathbb{0}$. Therefor, $w_j \neq \mathbb{0}$, since $\star$ is an isomorphism. We let $W^* = [w_j]_j$ and $N = HFW^* = H(WF)^*$.

We now show the incidences given by the 0s of N are consistent with $\mathcal{P}$. For $i\,j$ incident we have the following.

$$(N)_{i,j} = f(h_i, w_j) = f(h_i, \star(\cdots \wedge h_i \wedge \cdots)) = \star(h_i \wedge \cdots \wedge h_i \wedge \cdots) = 0$$

Alternatively suppose $i\,j$ are not incident. As we have seen this means $\{h_{i_{j,1}}, \cdots, h_{i_{j,d}}, h_i\}$ are linearly independent, and by counting span $\mathbb{R}^{d+1}$. We have already that $f(h_{i_{j,k}}, w_j) = 0$, and since f is definite this cannot be 0 for every vector of a basis, so we must have $f(h_i, w_j) \neq 0$. Thus $(N)_{i,j} = 0$ if and only if i and j are incident in $\mathcal{P}$.

We would like to show N is nonpositive. We will actually show N is either that or nonnegative, which can be fixed by redefining W to be $-W$, so this is enough. We do this by showing all nonzero entries have the same sign. Let (i, j) and $(\imath, \jmath)$ both be a vertex covertex pair of $\mathcal{P}$ that is not incident. By Condition 2 we have the following, which shows they have the same sign.

$$f(h_i, w_j) f(h_\imath, w_\jmath) = \det\left(G\begin{bmatrix} i_{j,1} \cdots i_{j,d}\ i \\ i_{\jmath,1} \cdots i_{\jmath,d}\ \imath \end{bmatrix} \right) \operatorname{sign}(f) > 0$$

This gives us that $N = H(WF)^*$ is a rank $d+1$ filled 0-incidence matrix of $\mathcal{P}$, adjusting W's sign if needed. Thus, by Theorem 3.6 $\hat{P}$ has type $\mathcal{P}$. $\square$

CHAPTER 4

OPERATIONS ON PROJECTIVE POLYTOPES

4.1 Completion Conditions

When a realization of a face, or in general some portion, of a polytope can be extended to a realization of the entire polytope if and only if it satisfies some conditions, we call these the **completion conditions** of that face. The main theorem of the next chapter is the following.

Theorem 4.1. (Below) *Given an algebraic d-polytope P, the completion condition for the specified face $\mathcal{F}$ of a combinatorial $(d{+}2)$-polytope $\mathcal{S}_P$ is that it be projectively equivalent to P.*

We call $\mathcal{S}_P$ from the theorem a **stamp** of P and the specified face $\mathcal{F}$ the **gum**. Such a polytope was constructed in [1, p. 134], and we give a different construction here. We say a statement about polytopes **inherits** to a face of a polytope when, if the statement holds for the polytope then it holds for the face. Here by a statement about a set's elements we mean a function from that set to the bool values, true or false. We are interested in Theorem 4.1, because for statements about algebraic polytopes that inherit to faces, it reduces the problem of finding some realization of every combinatorial polytope where the statement holds to finding one for every projective type of algebraic polytope. As an example of such a statement, we can ask whether the coordinates of polytope's vertices are all rational. It is straightforward to see that there are polygons with algebraic coordinates that do not have all rational coordinates under any projective transformation. As a consequence of Theorem 4.1 this gives us combinatorial 4-polytopes that cannot be realized with all rational coordinates. Inheritance as we have defined it expresses well the idea of what we want, but what we will actually need to use in the last chapter is the following more technical definition. For statements about an indexed collection of polytopes $\mathbf{P}$ of the same combinatorial type $\mathcal{P}$, we say the statement **inherits projectively** to a face of $\mathcal{P}$ when the statement holds for some projective copies of that face of each of the polytopes in $\mathbf{P}$.

Corollary 4.2. *Let S be a statement about indexed collections of algebraic polytopes of the same combinatorial type that inherits projectively to every ridge. S is true for some indexed realizations of every combinatorial type of polytope if and only if it is true for some indexed realizations of every projective algebraic type. Moreover, there can be a gap of at most 2 dimensions.*

Proof: Since projective equivalence is finer than combinatorial equivalence, we have the 'if' direction trivially. For the other direction, suppose S satisfies the conditions of the corollary and holds for some $\mathcal{I}$ indexed realizations of every combinatorial polytope. Consider an algebraic polytope P. Since the stamp $\mathcal{S}_P$ has an $\mathcal{I}$ indexed collection of realizations where S holds, it must also hold for some $\mathcal{I}$ indexed collection of projective copies of $\mathcal{S}_P$'s gum, which itself is projectively equivalent to P by Theorem 4.1. In this way S holds for some $\mathcal{I}$ indexed projective copies of every algebraic polytope. Also, since $\mathcal{S}_P$ is 2 dimensions higher than P, if S holds for some $\mathcal{I}$ indexed realizations of every combinatorial polytope up to dimension $d+2$, then by this argument, it holds for some $\mathcal{I}$ indexed projective copies of every algebraic polytope up to dimension d. □

4.2 Projective Coordinates

The goal of this chapter is to provide the background needed to construct a stamp. Our approach will be to fix the coordinates of a polytope's vertices in projective space, so we need to determine a point's coordinates in a projectively invariant way. We do this, as in linear space, by choosing a basis. That is determining the minimum elements of the space, so the only automorphism preserving these is the identity. This is enough to uniquely determine a projective transformation to a completion of $\mathbb{R}^d$.

For our basis we select $d+2$ points in general position that will correspond to the origin, $\mathbb{0} = (0, \cdots, 0)$, the standard basis vectors $e_1 = (1, 0, \cdots, 0)$ ... $e_d = (0, \cdots, 0, 1)$, and the vector $\mathbb{1} = (1, \cdots, 1)$. As an alternative to $\mathbb{1}$, we may instead choose a hyperplane Δ to be the horizon, the hyperplane at infinity.

We could define a point's coordinates as that of its image in $\mathbb{R}^d$, but we would rather define the coordinates purely in terms of projective operations. We start by first determining the axis hyperplanes $h_0^i := \{x : (x)_i = 0\}$,

$$h_0^i := \bigvee_{\substack{j=1 \\ j \neq i}}^{d} e_j \vee \mathbb{0}$$

then the horizon, unless we started with Δ instead of $\mathbb{1}$,

$$\Delta := \bigvee_{i=1}^{d} \left((e_i \vee \mathbb{1}) \wedge h_0^i \right)$$

then, the points, ∞e_i on the horizon where the pencil of lines that are parallel to the i^{th} axis meet,

$$\infty e_i := (\mathbb{0} \vee e_i) \wedge \Delta$$

then, the facet supporting hyperplanes of the unit cube away from the origin $h_1^i := \{x : (x)_i = 1\}$.

$$h_1^i := \bigvee_{\substack{j=1 \\ j \neq i}}^{d} \infty e_j \vee e_i$$

We are now ready to determine a coordinate of a point by first projecting the point into the two parallel hyperplanes where that coordinate is 0 and 1, and then taking the cross ratio of these points and the point where that coordinate axis meets the horizon.

$$\pi^i_j(\,\cdot\,) \coloneqq (\,\cdot \vee \infty e_i) \wedge h^i_j$$

$$(\,\cdot\,)_i \coloneqq \big(\,\cdot\,, \pi^i_1(\,\cdot\,) | \pi^i_0(\,\cdot\,), \infty e_i\big)$$

4.3 Visibility

An intuitive way of thinking about polytopes, particularly in projective space, is to consider what portion of a polytope's boundary is visible from a point in the space around the polytope. This perspective was used in [5] to prove that all polytopes are shellable. We will define the portion of a polytope P visible from a point p as the portion of P that is on the boundary of the space between them. An issue we must always deal with when determining what is visible in projective space is the distinction between what is in front of and what is behind the observer. For this we will give p an orientation. The details of oriented projective geometry, as well as convexity in oriented projective space, can be found in [24], though we will present it in a different way.

Recall that the points p of a real projective space are the rank 1 subspaces $p \coloneqq \mathbb{R}v$ of a real vector space $V \ni v \neq \mathbb{0}$, and that a polytope P in projective space is the union of a polytopal cone C in the underlying vector space and C's reflection through the origin $P \coloneqq C \cup -C$. Alternatively, we can pass through the unit sphere to get projective spaces by sending each rank 1 subspace to its intersection with the unit sphere and then identifying the resulting antipodal points. For $p \notin P$ there are two minimal convex sets containing both p and P. To distinguish between these two sets we give p an **orientation** p^+ with respect to P by choosing one of them to be the convex join $p^+ \uplus P$. We denote the opposite orientation by p^-, and both together by $p^{\circlearrowleft} \coloneqq \{p^+, p^-\}$. Fixing a polytopal cone C representing P in the underlying vector space, an orientation amounts to choosing a ray $\mathbb{R}^+v$ to represent p. We then get the convex join of p^+ and P in projective space as positive and negative linear combinations of the cones representing each.

$$p^+ \uplus P := \{\pm(\tau_0 v + \tau_1 C) : \tau_i \geq 0\}$$

In projective space considered as a sphere with antipodal points identified, choosing an orientation on p amounts to choosing one of the corresponding antipital points $\frac{v}{\|v\|}$. We define the line of sight region from p^+ to P to be the closure of their convex join with P removed.

$$\text{los}(p^+, P) := \overline{(p^+ \uplus B) \smallsetminus B}$$

We say a face F of P is **front visible** from p^+ when it is on the boundary of its line of sight region region $F \subset \text{los}(p^+, P)$, and we say a face of P is **back visible** when it would be front visible with the opposite orientation $F \subset \text{los}(p^-, P)$. When a face is not visible we say it is **obscured**. This gives 4 posibilities: a face may be doubly visible, front only visible, back only visible, or doubly obscured. We say orientations on a set are **consistent** when the union of the corresponding rays in the underlying vector space is convex.

The space outside a polytope before assigning orientations is a punctured real projective space. For example the space outside of a polygon in $\mathbb{RP}^2$ is a möbius strip. Including orientations makes this an annulus, or in general a thickened sphere, which is how we would like to think of the positions from where we view a polytope. Choosing orientation on a hyperplane in projective space amounts to inverting the identification of antipital points on the underlying sphere. We call orientations on a hyperplane $h^{\circlearrowleft}$ with respect to a polytope P that is disjoint from this hyperplane $P \cap h = \emptyset$ a **celestial sphere**.

We will use visibility from points on a celestial sphere to representing the normal vectors of a polytope's faces. Recall that the **normal cone** $\text{ncon}(F)$ of a face F of a polytope P in a vector space V is the set consisting of those linear functionals that are maximized over the polytope on that face,

$$\begin{aligned} \text{ncon}(F) &:= \left\{ a \in V^* : \ a(q) = \max_{p \in P}(a(p)) \Leftrightarrow q \in F \right\} \\ &= \text{con}(F^\diamond). \end{aligned}$$

Related to the normal cone is the **solid tangent cone** $\text{tcon}(F)$, which is the intersection of every closed half space containing P with F on its boundary

$$\begin{aligned}\text{tcon}(F) &:= \bigcap\{H = H_{a,b} = \{x : \langle a, x\rangle \le b\} : P \subset H,\ F \subset \partial H\}\\ &= \text{con}(F^{\diamond})^{*} + \text{aff}(F)\end{aligned}$$

The **normal fan** of a polytope in V is the collection of its faces' normal cones. We call an element of a face's normal cone a **normal vector** of the face. We represent a vector in the normal fan as a **sky**, an open half space of a celestial sphere. This is a hyperplane with a $(d-2)$-subspace removed and orientation chosen consistently. We say a sky is visible from a face of a polytope when it is visible from every point in the sky. The **celestial view** from a face of a polytope is the set of skies that are visible from that face. This corresponds to the normal cone of a polytope's face. Notice that since skies are defined to be open, this is a closed set. The **celestial complex** of a polytope P is the collection of celestial views from its faces. This provides the same information as the the normal fan of P, but in projective space. We make this correspondence explicit in the following lemma.

Lemma 4.3. *For polytopes in a vector space embedded in projective space, there is a bijection between the unit sphere and skies sending the unit normal vectors of each face to its celestial view.*

Proof: Let $P \subset \mathbb{R}^d$ be a polytope and $\varphi(P)$ be the usual embedding in projective space $\varphi : \mathbb{R}^d \to \mathbb{RP}^d$, $\varphi(x) = \mathbb{R}\begin{bmatrix}x\\1\end{bmatrix}$. Also let c be the hyperplane $\mathbb{RP}^d \setminus \varphi(\mathbb{R}^d)$ with orientations relative to $\varphi(P)$. We claim ψ^{+} where

$$\psi^{+}(a^{*}) := \left\{\mathbb{R}^{+}\begin{bmatrix}v\\0\end{bmatrix} :\ a^{*}v > 0\right\}$$

oriented relative to $\mathbb{R}^{+}\begin{bmatrix}P\\1\end{bmatrix}$, that is

$$\psi^{+}(a^{*}) = \left\{q^{+} :\ q = \mathbb{R}\begin{bmatrix}v\\0\end{bmatrix},\ q^{+} \uplus \varphi(P) = \pm\left\{\tau_0\begin{bmatrix}v\\0\end{bmatrix} + \tau_1\begin{bmatrix}P\\1\end{bmatrix} : \tau_i \ge 0\right\},\ a^{*}v > 0\right\},$$

gives the desired bijection. We have immediately that this is a bijection. We only need to see that, for a^{*} in the normal cone of F, $\psi^{+}(a^{*})$ is in F's celestial view, and every celestial view of F is the image of some normal vector.

The first part holds if F is visible from $q^+ = \mathbb{R}^+\begin{bmatrix} v \\ 0 \end{bmatrix}$ for all $q^+ \in \psi^+(a^*)$. Since a^* is in the normal cone of F, we have that $\forall p \in P\ \forall f \in F$, $\beta := a^* f \geq a^* p$, so for $\epsilon > 0$ we have $\beta < a^*(f + \epsilon v)$ and $f + \epsilon v \notin P$, but $f + \epsilon v \in q^+ \cup \varphi(P)$. Therefor, letting $\epsilon \to 0$, we see F is on the boundary of $\operatorname{los}(q^+, \varphi(P))$, and as such F is front visible from q^+. Thus, $\psi^+(a^*)$ is in the celestial view of F.

Every celestial view of F is the image of some normal vector if for every $q^+ = \mathbb{R}^+\begin{bmatrix} v \\ 0 \end{bmatrix}$ from where F is visible there is some normal vector a^* such that $a^* v > 0$. Since F is visible from q^+, we have $f + \epsilon v \notin P$ for all $f \in F$ and $\epsilon > 0$, so $v \notin \operatorname{tcon}(F)$. Translating so $\mathbb{0} \in F$, by the Farkas lemma there is a linear functional a^* seperating v from $\operatorname{tcon}(F)$ [26, p. 42]. That is $a^* v > 0$ and a^* is a normal vector of F. Thus, every celestial view of F has the form $\psi^+(a^*)$.

The lemma holds for a polytope in any real vector space embedded in a projective space by an appropriate choice of bases. $\square$

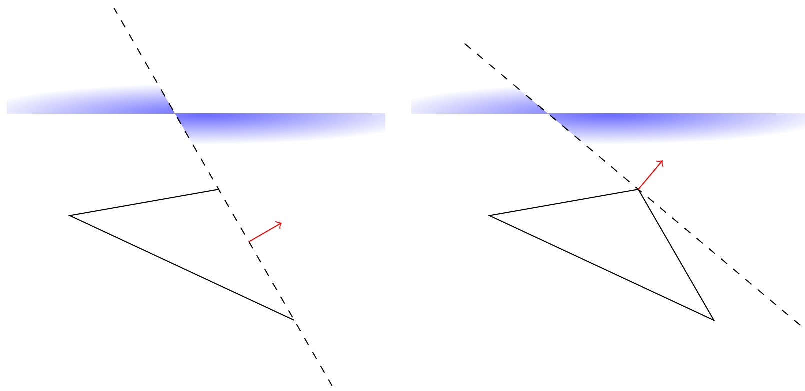


Figure 4.1. Normal vector of an edge and a vertex with the corresponding sky of each. Orientations are indicated by shading.

4.4 Prismoids

A polytope is a **prismoid** when every vertex of the polytope is in one of two nonintersecting faces, the **bases** of the prismoid. We call the remaining proper faces **sides** of the prismoid. When one base is of higher dimension than the other, we will call the smaller one the **apex** and use ‘base’ to refer to only the larger one when the distinction is clear from context.

Any face of a prismoid can be determined by its intersection with each of the bases. With this in mind we define an **abstract prismoid** $\mathcal{P}$ with bases $\mathcal{B}_0$ and $\mathcal{B}_1$, which must be abstract polytopes, to be a sublattice of the product of its bases' faces $\mathcal{P} \subset \mathcal{B}_0 \times \mathcal{B}_1$ ordered by

$$(\mathcal{F}_0, \mathcal{F}_1) \le (\mathcal{F}_0', \mathcal{F}_1') := (\mathcal{F}_0 \le \mathcal{F}_0' \text{ and } \mathcal{F}_1 \le \mathcal{F}_1')$$

satisfying the following. We require that the bases with their subfaces be included in the lattice as $\mathcal{B}_0 \times \bot_1 \subset \mathcal{P}$ and $\bot_0 \times \mathcal{B}_1 \subset \mathcal{P}$ as well as the polytope as a face of itself $\top_{\mathcal{P}} = (\top_0, \top_1) \in \mathcal{P}$. We call all other faces sides and further require that every base face be contained in some side.

A realization of a prismoid is a set of the form

$$P = B_0 \uplus B_1 = \{\tau_0 B_0 + \tau_1 B_1 : \tau_i \ge 0, \tau_0 + \tau_1 = 1\}$$

where $B_0 \bigcap B_1 = \varnothing$ and $\bigvee B_i \cap P = B_i$. We give realizability conditions for prismoids in terms of the following definition. The **Minkowski sum** $X + Y$ of two subsets X, Y of a vector space is the component wise sum of their elements, $X + Y := \{x + y : x \in X, y \in Y\}$.

Lemma 4.4. *An abstract prismoid can be realized if and only if the poset given by its sides* $\mathbf{S}$ *with bounds added can be realized as the Minkowski sum of the bases. Moreover, the completion condition for* $B_i \subset U_i$ *with common celestial sphere* $(U_0 \wedge U_1)^{\circlearrowleft}$ *is that* $\mathbf{S}$ *be the set of pairs of faces having celestial views with intersecting relative interiors,*

$$\mathbf{S} = \{(\mathcal{F}_0, \mathcal{F}_1) \in \mathcal{B}_0 \times \mathcal{B}_1 : \text{view}(F_0)^\circ \cap \text{view}(F_1)^\circ \ne \varnothing\}.$$

The **common refinement** of two collections $\mathbf{X}, \mathbf{Y}$ of subsets is the pair wise intersection of their members $\{X \cap Y : X \in \mathbf{X}, Y \in \mathbf{Y}\}$. Notice that the completion condition in this lemma requires that the pairs of faces in $\mathbf{S}$ form the common refinement of the bases' celestial complexes.

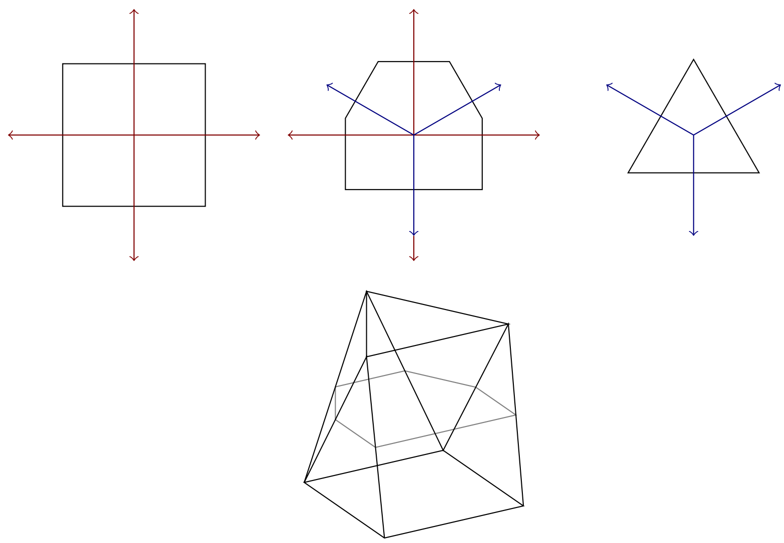


Figure 4.2. A horizontal slice of a prismoid with a square and a triangular base. These slices are weighted Minkowski sums of the bases. The common refinement of the normal fans gives the combinatorics of the sides of the prismoid.

Proof of Lemma 4.4: For the first part, given a realization P of a prismoid, we can assume without loss of generality, that bases B_0 and B_1 are in the hyperplanes $x_d = 0$ and $x_d = 1$ respectively. If this is not the case already we can apply some projective map to make it so. Now we have

$$P = B_0' \times 0 \cup B_1' \times 1 = \{(\tau_0 B_0' + \tau_1 B_1') \times \tau_1 : \tau_i \geq 0, \tau_0 + \tau_1 = 1\}$$

where B_i' is B_i with the d^{th} coordinate removed. We see from this that the sides of the prismoid correspond to faces of the Minkowski sum of the bases $B_0' + B_1'$. For the other direction if we have a realization of a polytope with nontrivial faces order isomorphic to the sides of $\mathcal{P}$ of the form $B_0' + B_1'$, then we get a realization of $\mathcal{P}$ as given above.

For the second part, first recall that the normal fan of the Minkowski sum of two polytopes in affine space is the common refinement of the normal fans of the polytopes [26, p. 198]. We can see this by considering what happens to faces as the solution spaces to linear optimization problems upon summing polytopes. A face is the complete set of optima for the linear functionals in the relative interior of its normal cone, so for any given linear functional a^*, the complete set of optima in the Minkowski sum is the sum of the face of each constituent polytope having this linear functional in its normal cone interior, and these sets are in turn the faces of the Minkowski sum.

$$\begin{aligned} F \subset B_0 + B_1 \text{ s.t. } a^* \in \operatorname{ncon}(F) &= \left\{ f_0 + f_1 : (f_0, f_1) \in \operatorname*{argmax}_{b_i \in B_i}(a^*(b_0 + b_1)) \right\} \\ &= \left\{ f_0 + f_1 : f_i \in \operatorname*{argmax}_{b_i \in B_i}(a^* b_i) \right\} \\ &= \sum_{i \in \{0,1\}} \left\{ f_i \in B_i : f_i \in \operatorname*{argmax}_{b_i \in B_i}(a^* b_i) \right\} \\ &= F_0 + F_1 \text{ s.t. } F_i \subset B_i,\ a^* \in \operatorname{ncon}(F_i) \end{aligned}$$

Returning to projective space, by Lemma 4.3 the solutions to a linear optimization problem form the face from where the corresponding sky is visible. Finally, from the first part of this lemma, we see that $(\mathcal{F}_0, \mathcal{F}_1)$ gives a side of the prismoid if and only if there is a sky in the relative interior of the view from each face. □

Some examples of prismoids are prisms and pyramids. Earlier we visited the antiprism, which is also a prismoid. Another prismoid that will appear extensively is the Lawrence extension, or tent.

4.5 Tents

A **tent** is a prismoid where one of the bases is an edge, the **apex**. An abstract prismoid is an **abstract tent** when a face of the base forms a side with the apex if and only if it forms a side with either both apex vertices or neither apex vertex.

If a combinatorial polytope $\mathcal{B}$ is to be realized as the base of a tent with specified combinatorics it must satisfy some additional conditions beyond what is required to realize it alone. Specifically the combinatorics of the tent determines the visibility of the faces of B from a point $p \notin B$ as described in the following lemma. A very similar result appears in [20, p. 32] and we will use it for the same purposes. We also call a tent the **Lawrence extension** of B over p. This is one operation we perform on the combinatorics of a polytope alter realizability conditions, and in the next section we will present another.

Lemma 4.5. *The completion conditions of the base $\mathcal{B}$, with nontrivial faces $\mathcal{F}_{x,i}$, of a tent $\mathcal{L}$, with apex Λ having vertices $\lambda_\pm$ and sides of the form*

$$\begin{array}{ccc} (\mathcal{F}_{+,i}, \lambda_+) & (\mathcal{F}_{0,i}, \Lambda) & (\mathcal{F}_{-,i}, \lambda_-) \\ (\mathcal{F}_{*,i}, \lambda_+) & (\mathcal{F}_{*,i}, \Lambda) & (\mathcal{F}_{*,i}, \lambda_-), \end{array}$$

are that the faces $\mathcal{F}_{+,i}$, $\mathcal{F}_{0,i}$, $\mathcal{F}_{-,i}$, $\mathcal{F}_{*,i}$ be front only visible, double obscured, back only visible, and doubly visible respectively from a point $p^+ \in (\Lambda \wedge B)^{\circlearrowleft}$.

Proof: This comes from a special case of the second part of Lemma 4.4. The realizations of a line segment modulo projectivity is trivial The lemma tells us that a realization of the base can be completed to a realization of $\mathcal{L}$ if and only if the the sides come from pairs of faces having celestial views with intersecting relative interiors, which we reformulate into statements about visibility from p^+ to get the lemma.

The interior of the celestial view from $\lambda_\pm$ consists of precisely those skies that include $p^\pm$, and as such the celestial view interior of a face $\mathcal{F}$ intersects that of $\lambda_\pm$ if and only if $\mathcal{F}$ is visible from $p^\pm$. These skies form two open hemispheres among skies, and the remaining skies form the great sphere seperating these hemispheres, which is the celestial view from Λ. The relative interior of any base face's celestial view must be contained in either one of the hemispheres, the great sphere between them, or the great sphere and both hemispheres. Thus we have the four cases stated in the lemma. □

4.6 Gluing

In the next chapter we will present a catalog of polytopal pieces that we will combine to construct a combinatorial polytope with specific realizability

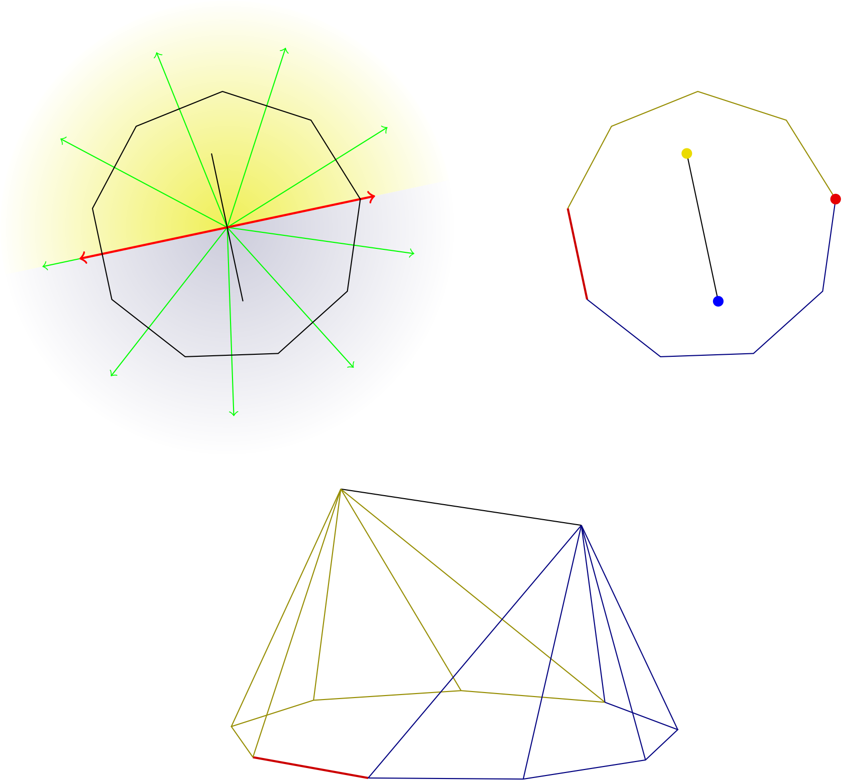


Figure 4.3. The normal fan of a tent's apex and its base. For the apex this is two half spaces separated by a hyperplane. The sides of the tent are determined by how this hyperplane separates the outward normals of its base faces.

conditions, but first we will say a bit about how to combine them. To this end we define a purely combinatorial operation for combining polytopes, gluing them together, and give realizability conditions in terms of the components.

Given combinatorial polytopes $\mathcal{P}_0$ $\mathcal{P}_1$ and an isomorphism between the face lattices of a facet of each $\varphi : [\varnothing, F_0] \to [\varnothing, F_1]$, we **glue** these abstract polytopes along this isomorphism by removing the facet from each one and identifying a face of one with its image in the other under the given isomorphism.

$$\mathcal{P}_0 \#_\varphi \mathcal{P}_1 := (\mathcal{P}_0 \smallsetminus F_0) \bigsqcup (\mathcal{P}_1 \smallsetminus F_1)/\varphi$$

We may also denote this $\mathcal{P}_0 \#_F \mathcal{P}_1$ for F either F_0 or F_1 when φ is understood. We see examples of polytopes glued together in 3 and 4 dimensions in Figure

5.2. We call the dual operation **whittling** $(\mathcal{P}_0^* \#_{v^*} \mathcal{P}_1^*)^*$, and we say a vertex v of a polytope $\mathcal{P}_0$ was whittled when its dual has been glued to another polytope along the facet that is dual to v.

Realizability conditions are much simpler when gluing along facets that are necessarily flat, which all facets we glue along will be. A d-polytope is **necessarily flat** when any realization of its $(d-1)$-skeleton in space of arbitrarily high dimension will be contained in a subspace of dimension d. The following results are shown in [20, p. 29].

Lemma 4.6. *For any pair of polytopes P_1, P_2 where a facet F of one is projectively equivalent to that of the other, there is a projective map ϕ gluing them along this facet, so*

$$P_1 \cap \phi(P_2) = F, \qquad P_1 \cup \phi(P_2) = P, \qquad \mathcal{P} = \mathcal{P}_1 \#_{\mathcal{F}} \mathcal{P}_2.$$

If $\mathcal{F}$ is necessarily flat then all realizations of $\mathcal{P}$ come from realization of $\mathcal{P}_1$, $\mathcal{P}_2$ in this way.

Proof: For the first part we will construct the appropriate transformation. We begin by noting that for each facet of a polytope there is some point with orientation from where only that facet is front visible, and all other facets are back visible. We find such a point in the following way. We can consider a polytope to be the intersection of the half spaces supporting some facet of the polytope. None of these half spaces can be redundant, since they would otherwise not support a facet, so by removing one we get a strictly larger set. Any point in this new portion of this larger sets has the desired property. For the given polytopes and facets denote this new portion and a point in it by $p_i \in \hat{F}_i$.

Since we assume projectively equivalent facets we already have a projectivity ϕ' sending F_2 to F_1. We now need to extend this to a projectivity between the full spaces of the polytopes. For this we let $\phi|_{F_2} = \phi'$ and $\phi(p_2) = p_1$. Under this map we have $\phi(P_2) \subset \phi \circ \operatorname{los}(p_2^-, F_2) = \operatorname{los}(p_1^+, F_1) \subset \hat{F}_1$

For the second part, since F is necessarily flat, we can subdivide P by the hyperplane containing F to get a realization of $\mathcal{P}_1$ and $\mathcal{P}_2$. □

Lemma 4.7. *Pyramids and Prisms over a polytope of dimension at least 2 are necessarily flat.*

Proof: For Pyramids this is because adding a point can at most increase the dimension by 1. For Prisms, consider starting with one base and adding sides. Each side of the prism could potentially introduce a new dimension, but if one side is contained in another then together they are still contained in a space only one dimension higher. Since the base has dimension $d \geq 2$, any side can be reached through a sequence of sides where each side intersects its successor in a side of the prism. Thus, one base and all sides must be contained in a space of dimension $d + 1$. Adding the other base can not increase the dimension, since this is the convex hull of faces we have already added, the Prism must also be necessarily flat. □

CHAPTER 5

PRESCRIBED RIDGES

5.1 Transmitters

Here we define a catalog of polytopes culminating in the stamp. In [20] a language is developed for encoding arithmetic operations into polytopes, which is where the pieces described here come from, with some modifications and additions. Before defining these polytopes we will say a bit about their purpose and give the relevant properties motivating their construction.

The purpose of a transmitter is to impose a relationship between two of its faces, the sockets, in the form of a projective transformation. In a sense the most restrictive example of this, which we will use widely, is the full transmitter. This forces the sockets to be projectively equivalent. Before saying more about full transmitters we define the most general type of transmitter we will use.

A **projective transmitter** T_{B_+,B_-} of two polytopes B_+ and B_-, its **sockets**, is a Lawrence extension of a prismoid, its **trunk** $P = B_+ \uplus B_-$, having the sockets as bases over a point from where B_+ is front only visible and B_- is back only visible. That is, it is a tent over the trunk such that each socket forms a side of the tent with exactly one of each of the vertices of the apex; $B_+ \uplus \lambda_+$ and $B_- \uplus \lambda_-$ are the only transmitter sides containing the sockets. We refer to sides of this tent as transmitter sides and sides of its base as trunk sides to distinguish them from each other.

Analogously, an **abstract projective transmitter** $\mathcal{T}_{\mathcal{B}_+,\mathcal{B}_-}$ is an abstract tent with an abstract prismoid base as above, but additionally satisfying the following conditions, which are consistent with what can be realized. We require that every face $\mathcal{F}_\pm$ of a socket $\mathcal{B}_\pm$ forms a transmitter side with the corresponding apex vertex $((\mathcal{F}_+, \bot_-), \lambda_+), ((\bot_+, \mathcal{F}_-), \lambda_-) \in \mathcal{T}_{\mathcal{B}_+,\mathcal{B}_-}$, and the remaining transmitter sides be determined by the choice of faces of sockets $\mathcal{B}_\pm$ forming a transmitter side with the other apex vertex $\lambda_\mp$ in the following way. A trunk side $(\mathcal{F}_+, \mathcal{F}_-) \in \mathcal{P}$ forms a transmitter side with λ_- if and only if $\mathcal{F}_+$ does as well, and likewise for λ_+ and $\mathcal{F}_-$,

$$((\mathcal{F}_+, \mathcal{F}_-), \lambda_-) \in \mathcal{T}_{\mathcal{B}_+,\mathcal{B}_-} \Leftrightarrow ((\mathcal{F}_+, \bot_-), \lambda_-) \in \mathcal{T}_{\mathcal{B}_+,\mathcal{B}_-}$$
$$((\mathcal{F}_+, \mathcal{F}_-), \lambda_+) \in \mathcal{T}_{\mathcal{B}_+,\mathcal{B}_-} \Leftrightarrow ((\bot_+, \mathcal{F}_-), \lambda_+) \in \mathcal{T}_{\mathcal{B}_+,\mathcal{B}_-}.$$

Also, being an abstract tent, a face of $\mathcal{P}$ forms a transmitter side with Λ if and only if it forms a side with both or neither $\lambda_\pm$.

This allows us, when specifying the combinatorics of a transmitter, to only indicate those socket faces forming a transmitter side with the other apex vertex. We now show that these additional conditions are necessary for realizability.

Lemma 5.1. *The face lattice of a projective transmitter is always an abstract projective transmitter.*

Proof: For this the three conditions given above must hold for any projective transmitter. The first two conditions must hold since all subfaces of a face are visible from the same point with orientation if and only if the face is visible from that point as well. For the first condition this applies to the sockets, and for the second condition to the trunk sides. The third condition we get directly from the characterization of tents given in Lemma 4.5. □

A projective transmitter is a **full transmitter** $T_\mathcal{B}$ of $\mathcal{B}$ when the trunk P is a prism over $\mathcal{B}$ and every face of a socket forms a transmitter side with only that socket's apex vertex. Equivalently we can define a full transmitter of a polytope as a prism over a pyramid over that polytope $\mathcal{T}_{\mathcal{B}} \coloneqq \mathrm{pris} \circ \mathrm{pyr}(\mathcal{B})$. Intuitively we can think of this as a Lawrence extension specifying a point p from where a light sources would exactly form one copy B_+ of $\mathcal{B}$ as the shadow cast by the other copy B_- on the projective span of the first $\bigvee B_+$.

A slightly more general transmitter that we also use widely is the **forgetful transmitter**. This is similar to a full transmitter except one socket $\mathcal{B}_+$ is a copy of the other socket $\mathcal{B}_-$ with some simple vertices whittled. Specifically, faces formed by whittling form a trunk side with only the corresponding whittled vertex, and other socket faces form a trunk side with only the corresponding face of the other socket; and, the whittled vertices form a transmitter side with both apex vertices, but all other socket faces form a transmitter side with only that socket's apex vertex. Like the full transmitter, the forgetful transmitter forces the whittled socket to be projectively equivalent to a whittled copy of the unwhittled socket.

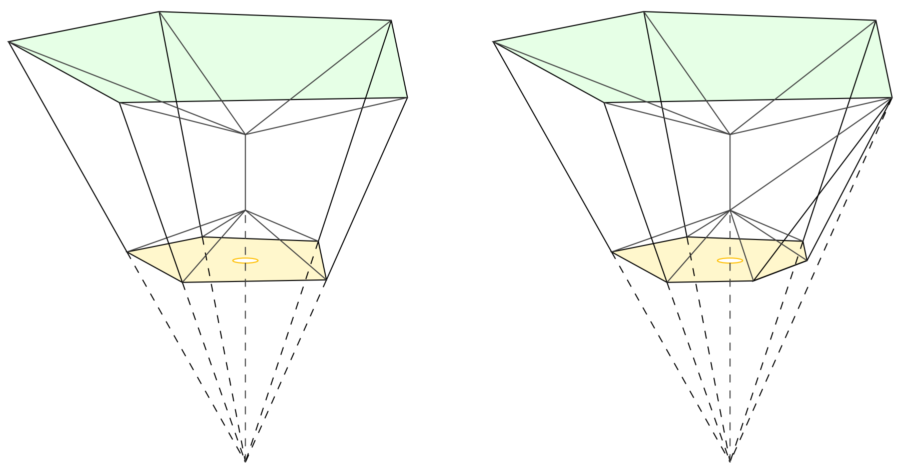


Figure 5.1. Schlegel diagrams for a full transmitter and a forgetful transmitter.

Before proving our claims about full and forgetful transmitters we give a general treatment. We first note that removing two hyperplanes from a real projective space U leaves two components, which we call **half spaces**, and we denote by $\mathrm{half}_h(U)$ the set of half spaces where h is one of the two hyperplanes.

Lemma 5.2. *The completion conditions for the sockets $B_\pm \subset U_\pm$ of an abstract projective transmitter $\mathcal{T}_{\mathcal{B}_-,\mathcal{B}_+}$ with common celestial sphere $c = (U_- \wedge U_+)^{\circlearrowleft}$ are that the common refinement of the celestial complexes be consistent with the trunk sides, and that there be a projectivity $\phi : U_- \to U_+$ fixing c, such that for each face $\mathcal{F}$ of $\mathcal{B}_\pm$, there exists a supporting half space $H \in \mathrm{half}_c(U_\pm)$ of F with the image of the other socket $\phi^\pm(B_\mp)$ in its interior if and only if $\mathcal{F}$ forms a transmitter side with the opposite apex vertex $\lambda_\mp$.*

$$\forall \mathcal{F} \in \mathrm{face}(\mathcal{B}_\pm \hookrightarrow \mathcal{P}) :$$
$$(\exists H \in \mathrm{half}_c(U_\pm) : \ F = B_\pm \setminus H^\circ, \ \phi^\pm(B_\mp) \subset H^\circ) \Leftrightarrow (\mathcal{F}, \lambda_\mp) \in \mathcal{T}_{\mathcal{B}_-,\mathcal{B}_+}$$

Proof: Suppose we have a realization of the projective transmitter, then the pairs of faces appearing in the common refinement of the celestial complexes are the same as those of the trunk sides by Lemma 4.4. Projecting through the point $p = \Lambda \wedge P$, where Λ is the apex and P is the trunk, gives the desired transformation.

$$\phi(\,\cdot\,) := (p \vee \cdot\,) \wedge U_+$$

To see this, suppose there is a half space H supporting a face F of B_+ with $\phi(B_-)$ in its interior, then $p \vee (\partial H)$ does not intersect B_-, so it only intersects P at F. Since $p^- \uplus F \subset p \vee (\partial H)$, we have $p^- \uplus F$ intersects P at F, so $F \subset \mathrm{los}(p^-, P)$ is visible from p^-, and by Lemma 4.5, $F \uplus \lambda_-$ is a transmitter side of T_{B_-,B_+}. Now suppose $F \uplus \lambda_-$ is a transmitter side of T_{B_-,B_+}, then F as a face of P is doubly visible from p and there is some supporting hyperplane ∂H of F containing p. Restricting the half space H containing P bounded by this hyperplane to U_+ gives us an appropriate half space. Similarly these conditions hold for the existence of such a half space supporting a face of B_-.

For the other direction suppose we have realizations of the sockets with a common refinement of their celestial spheres that is consistent with the combinatorics of the trunk sides, and a projective transformation ϕ consistent with the transmitter sides. By Lemma 4.4 we can complete the sockets to a realization of $\mathcal{P}$, since the relative interiors of celestial views intersect in the appropriate way. To be more explicit, we embed B_- in the hyperplane $(x)_d = 0$ in $\mathbb{R}^d$ where B_- had dimension $d-1$, and translate the preimage of B_+ under ϕ to be in the hyperplane $(x)_d = 1$. The convex hull of the sockets with these embeddings gives us a realization of $\mathcal{P}$ and a new projection $\phi'(\cdot \times 0) := (\cdot \times 1)$ through ∞e_d that is also consistent with the sides of the tent. Now by Lemma 4.5, the Lawrence extension over ∞e_d gives us the correct realization. To see this we observe that a face F of $B_\pm$ is doubly visible from ∞e_d if and only if $F \times \mathbb{R}$ intersects P only in F, which happens if and only if it is contained in some half space supporting F. □

We now apply this to full and forgetful projective transmitters. Here we include the full transmitter as a special case of the forgetful transmitter. A variation of these appears in [20, p. 46].

Lemma 5.3. *The completion conditions of a forgetful transmitter's sockets* $\mathcal{B}_0, \mathcal{B}_1$ *are that there be some projective transformation* ϕ *such that* $\bigvee \phi(F) = \bigvee \varphi(F)$ *for all socket facets* F *of* $\mathcal{B}_0$*. In particular, for full transmitters the sockets are projectively equivalent.* $B_0 \overset{\text{proj}}{\simeq} B_1$

Proof: This is just a special case of Lemma 5.2. We have only to see that the common refinement of celestial complexes of a whittled polytope and the original is just that of the whittled. □

5.2 Connectors

Connectors $\mathcal{C}_{n,\mathcal{B}}$ serve the same purpose as full transmitters, but have n rather then just two sockets. A variation of these appears in [20, p. 47]. Here sockets are always the base of a facet of the form $\operatorname{pyr}(\mathcal{B})$. When more then one such facet shares the same base we treat it as a different socket for each such facet. We now begin constructing larger polytopes by gluing smaller polytopes together. Later we will glue along these pyramids, which we simply refer to as gluing along sockets. We refer to the sockets have not been glued along as

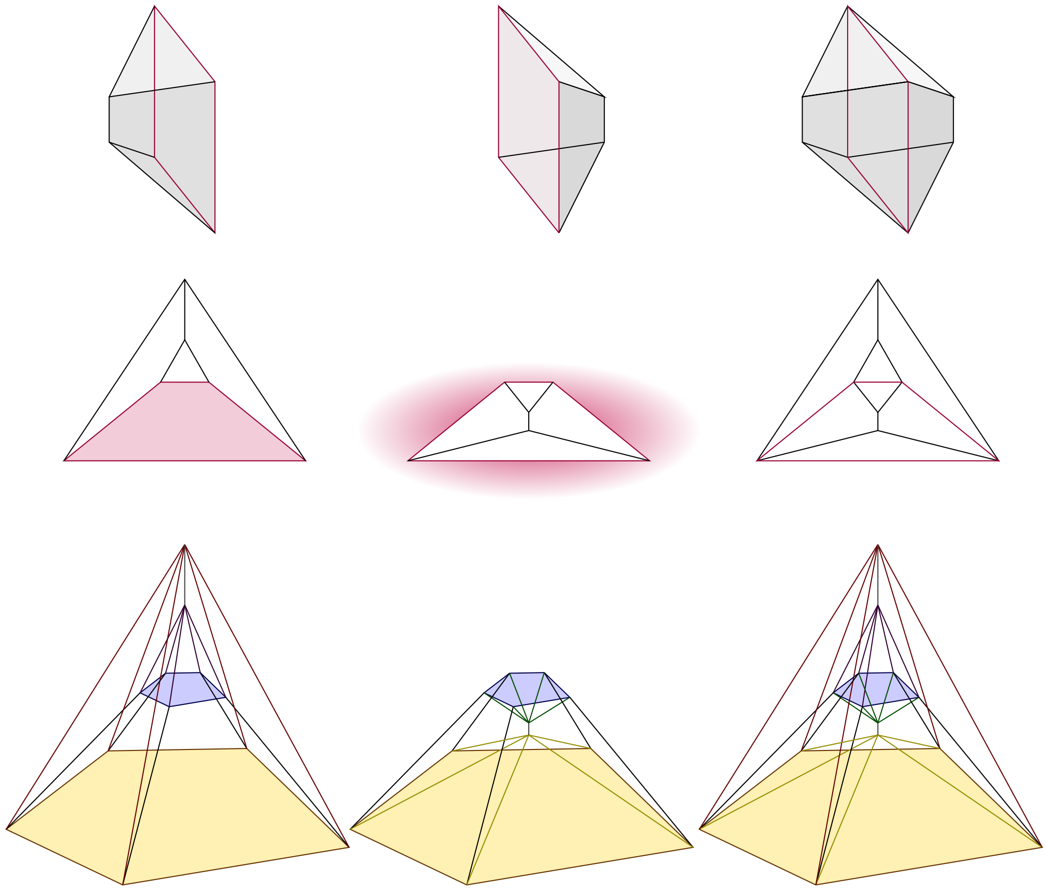


Figure 5.2. A pair of full transmitters glued along their trunks to make a connector with 4 sockets. At the top are two perspective drawings of a transmitter of an edge and a perspective drawing of these glued together. Beneath each of these is a Schlegel diagram of the corrsiponding polytopes. At the bottom are two Schlegel diagrams of a transmitter of a pentagon and a Schlegel diagram of the connector we get by gluing these transmitters together.

open sockets. We will define connectors so that the following lemma about them holds.

Lemma 5.4. *The completion conditions for the n sockets of $\mathcal{C}_{n,\mathcal{B}}$ are that they all be projectively equivalent.*

The following constructions are **connectors**. We begin by letting a full transmitter be a connector with 2 sockets. $\mathcal{C}_{n,\mathcal{B}} := \mathcal{T}_{\mathcal{B}}$ We construct $\mathcal{C}_{4,\mathcal{B}}$ a connector with 4 sockets by gluing two copies of a full transmitter of B together along the trunk. We construct even many socket connectors $\mathcal{C}_{2n,\mathcal{B}}$ for $n > 2$ by gluing together $n-1$ copies of $\mathcal{C}_{4,\mathcal{B}}$ along sockets. The choice of sockets we glue along makes no difference so long as the result is a tree of connectors, so we simply refer to all such polytopes as connectors. Finally, for any n we just ignore extra sockets. We may omit subscripts where doing so is unambiguous.

Proof of Lemma 5.4: That $\mathcal{C}_2$ is a connector is just a restatement of Lemma 5.3 for full transmitters. We see that $\mathcal{C}_4$ is a connector since prisms are necessarily flat, so the two copies of $\mathcal{C}_2$ that make up this polytope must keep their realizability constraints forcing the two bases of the prism to be projectively equivalent, and since each socket is a pyramid over one of these bases, they must also be projectively equivalent. The sockets, as pyramids, are also necessarily flat, so connectors glued along sockets must keep their realizability conditions, with the constraint that the sockets glued along must be projectively equivalent. This forces the remaining sockets of both connectors to be projectively equivalent to each other, so the result for higher n even follows by induction, and for n odd by ignoring one socket of $\mathcal{C}_{n+1}$ □

5.3 Adapters

While gluing connectors and transmitters together, we may in some cases want to glue a socket of lower dimension to an appropriate face of a higher dimensional socket. For this we can repeatedly stellate a facet containing this face, starting with the larger socket's pyramid until we have a facet that is a multipyramid over this face, which is what we glue along. These repeated stellations are equivalent to gluing a single polytope to the larger socket, which we will refer to as an **adapter** [20, p. 149].

Lemma 5.5. *The adapter $\mathcal{A}_{\mathcal{B},\mathcal{F}}$ between a d-polytope $\mathcal{B}$ and its $(d-n)$-face $\mathcal{F}$ has a face of type $\mathcal{F}$ contained in a facet of the form* $\mathrm{pyr}(\mathcal{B})$ *and one of the form* $\mathrm{pyr}^{n+1}(\mathcal{F})$*, and any realization of $\mathcal{B}$ can be completed to a realization of $\mathcal{A}_{\mathcal{B},\mathcal{F}}$.*

When $\mathcal{F}$ is a facet of $\mathcal{B}$ we define the adapter between them to be a double pyramid over the larger socket, $\mathcal{A}_{\mathcal{B},\mathcal{F}} := \mathrm{pyr}^2(\mathcal{B})$. Otherwise we find an unbroken chain $\top, \mathcal{F}_{d-1}, \cdots, \mathcal{F}$ and recursively define the adapter to consist of an adapter between $\mathcal{B}$ and $\mathcal{F}_{d-1}$ glued to a pyramid over an adapter between $\mathcal{F}_{d-1}$ and $\mathcal{F}$.

$$\mathcal{A}_{\mathcal{B},\mathcal{F}} := \mathcal{A}_{\mathcal{B},\mathcal{F}_{d-1}} \#_{\mathrm{pyr}^2(\mathcal{F}_{d-1})} \mathrm{pyr}(\mathcal{A}_{\mathcal{F}_{d-1},\mathcal{F}})$$

Proof of Lemma 5.5: This is immediate from the construction. □

5.4 Hubs

Portions of the polytopes we construct will consist of a collection **H**, which we call a **hub**, of connectors, projective transmitters, and adapters of various sockets **B** glued together along sockets. These give us a collection of projections $\Phi_{\mathbf{H}}$ between the projective spans of their respective sockets. To deal with all of these spaces together we mod out by the equivalence relation generated by these projections, while being careful to treat points that are in more then one of these spaces correctly, to get a single projective space.

$$\mathbb{RP}^k_{\mathbf{H}} := \bigsqcup_{B \in \mathbf{B}} \bigvee B \Big/ \Phi_{\mathbf{H}}$$

We call polytopes glued to a hub's open sockets, its **restricting polytopes**, and we use the hub to combine the completion conditions of the facets along which we glue these polytopes.

Lemma 5.6. *The completion conditions of an open socket of a hub with a collection of restricting polytopes are that it can be completed to a collection satisfying the combined completion conditions of the restricting polytopes.*

Proof: Let us assume at first that our hub consists of only one of the building blocks mentioned. For each of these we have already proved a lemma showing this: for forgetful transmitters Lemma 5.3, for connectors Lemma 5.4, and for adapters Lemma 5.5. We now proceed by induction. Assume the lemma holds for hubs consisting of fewer than n building blocks and consider a hub $\mathcal{H}$ consisting of n. The specified open socket $\mathcal{B}$ is a socket of one of these components $\mathcal{P}$, which has other sockets that are each glued to a portion $\mathcal{H}_i$ of the rest of the hub. This portion of the hub is itself a smaller hub, and by inductive assumption it combines the completion conditions of its restricting polytopes. As such we can treat $\mathcal{H}_i$ with its own restricting polytopes as a restricting polytope of the single component hub $\mathcal{P}$. With this the completion conditions for $\mathcal{B}$ are the combination of those of $\mathcal{P}$'s restricting polytopes, $\mathcal{H}_i$ with their own restricting polytopes, which together are in turn the combination of that of the restricting polytopes of $\mathcal{H}$. $\square$

5.5 Unit Polytopes

Now that we can construct polytopes representing a common space populated by various polytopes sharing supporting hyperplanes, we would like to use this to combine constraints imposed by the tools developed so far. We give a simple example of this now, which is also a stamp $\mathcal{U}^d = \mathcal{S}_{\boxplus}$ of a hypercube $\boxplus$. In the general stamp construction we will use this hypercube as a kind of scaffolding to which we fix points, and to give us a projective coordinate system.

Lemma 5.7. *The completion conditions for the specified facet* $\mathrm{pyr}(\boxplus)$ *of a d-unit polytope is that* $\boxplus$ *be projectively equivalent to the unit hypercube.*

The d**-unit polytope** $\mathcal{U}^d$ with specified facet of combinatorial type $\mathrm{pyr}(\square)$ consists of a connector $\mathcal{C}_{d+1,\square}$ with one of its sockets the specified facet, and for each opposite pair of facets $\mathcal{W}_0^i$ and $\mathcal{W}_1^i$ of $\square$, which we will call walls, a pyramid over a full transmitter $\mathcal{T}_i$ of a $(d-1)$-cube between these walls is glued to a connector socket along the transmitter's trunk.

$$\mathcal{T}_i := \mathcal{T}_{w_0^i, w_1^i}$$

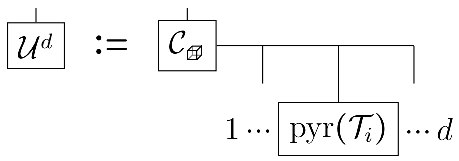


Figure 5.3. Gluing diagram for a unit polytope.

Proof of Lemma 5.7: First we note that the unit cube satisfies all conditions imposed on $\square$ by $\mathcal{U}^d$, namely that as a prism with any choice of an opposite pair of facets to be its bases the projective spans of the sides of the cube all meet at a common point.

Given a realization of $\mathcal{U}^d$ we proceed by selecting a basis of $\mathbb{RP}^d_\square$, and show that with this basis the walls of $\square$ have supporting hyperplanes $h_0^i := \{(x)_i = 0\}$ and $h_1^i := \{(x)_i = 1\}$, and $\square$ must therefor be the unit hypercube. For this we choose one vertex to be $\mathbb{0}$, each of $\mathbb{0}$'s neighbors to be e_i for $i \in \{1, \cdots, d\}$. Each of the transmitters is a Lawrence extension at a point, which we label p_∞^i. To complete the basis we choose $\Delta := \bigvee_{i=1}^d p_\infty^i$ to be the horizon. With this we have that each hyperplane h_0^i supports the facet $\mathcal{W}_0^i$ of $\square$ containing e_j for $j \neq i$, since this hyperplane is defined as the span of these points. For the remaining facets first see that $p_\infty^i \in \mathbb{0} \vee e_i$ since by Lemma 4.5 the edge between $\mathbb{0}$ and e_i is doubly obscured from p_∞^i, so our choice of $p_\infty^i = \infty e_i$ is consistent with the notation in Section 4.2. Also $\mathcal{W}_1^i$ is a trunk side of the transmitter $\mathcal{T}_j$ for $j \neq i$, and as such is doubly obscured from p_∞^j, so the supporting hyperplane of $\mathcal{W}_1^i$

contains p_∞^j as well as e_i, and therefor must be h_1^i. Thus, in every realization of $\mathcal{U}^d$ the projective transformation defined by this basis sends the supporting hyperplanes of $\cube$ to those of the unit cube and as such $\cube$ must be projectively equivalent to the unit cube. $\square$

5.6 Pencil Polytopes

We would like to impose further completion conditions on polytopes. Specifically, we will force collections of lines to meet at a common point. To this end we present the pencil polytope $\mathcal{X}_\mathcal{G}$ of a polygon $\mathcal{G}$ with a specified pair of edges $\mathcal{E}, \mathcal{E}'$ separating a specified pair of vertices v_+, v_- contained in neither edge. This is a varient of a polytope in [20, p. 50], about which we will prove the following.

Lemma 5.8. *The completion conditions for the specified facet* $\operatorname{pyr}(G)$ *of a pencil polytope* $\mathcal{X}_\mathcal{G}$ *with appropriate edges and vertices specified is that the lines* $\bigvee E$, $\bigvee E'$, *and* $v_+ \vee v_-$ *intersect in a single common point. That is, the points* v_+, v_-, *and* $E \wedge E'$ *are collinear.*

A **pencil polytope** X_G is a projective transmitter from the polygon G to an edge Λ_L that sends the specified vertices $v_\pm$ to the vertices $\lambda_{L,\pm}$ of Λ_L. As such the trunk of the transmitter is a tent L, which we require to be the Lawrence extension at the point $E \wedge E'$ where lines supporting the specified edges meet.

A **combinatorial pencil polytope** $\mathcal{X}_\mathcal{G}$ is a combinatorial projective transmitter with a tent $\mathcal{L}$ over $\mathcal{G}$ for its trunk where the faces forming a side of $\mathcal{L}$ with its apex Λ_L are the edges $\mathcal{E}, \mathcal{E}'$. This separates the remaining faces of $\mathcal{G}$ into two components, which each form trunk sides with one of the trunk apex vertices. And, those socket faces forming a transmitter side with both transmitter apex vertices $\lambda_{X,\pm}$ are the faces of $\mathcal{G}$ containing neither $v_\pm$. We say combinatorial rather than abstract because its realizability is not in doubt.

Proof of Lemma 5.8: By Lemma 4.5, G can be completed to a realization of $\mathcal{L}$ if and only if E and E' are doubly obscured from a point $p = \Lambda_L \wedge G$ and the other faces are front or back only visible as appropriate, which is equivalent to the requirement that $\bigvee E$, $\bigvee E'$, and $\bigvee \Lambda_L$ meet at p. Now by Lemma 5.2, L can be completed to a realization of $\mathcal{X}_\mathcal{G}$ if and only if there is a projectivity ϕ sending Λ_L into $\bigvee G$ such that, first, no supporting half space of $\phi(\Lambda_L)$ contains G in its interior, second, $\phi(\Lambda_L)$ is not in the interior of any of the supporting half spaces of the edges containing $v_\pm$, but is for other edges, and third, ϕ preserves p. The first condition is equivalent to requiring that both $\phi(\lambda_\pm)$ be contained in G. For the second condition, notice the space outside the supporting half spaces of edges containing v_+ is the solid tangent cone of v_+ reflected through v_+. Making $\bigvee G$ a vector space, we can express this cone as $2v_+ - \text{tcon}(v_+)$. The second condition then requires that $\phi(\Lambda_L)$ intersect both cones $2v_\pm - \text{tcon}(v_\pm)$. That is, $\phi(\lambda_\pm)$ must be in both cones. Let λ_+ be the one that lands in $2v_+ - \text{tcon}(v_+)$. Since $2v_\pm - \text{tcon}(v_\pm) \cap G = v_\pm$, we must have $\phi(\lambda_\pm) = v_\pm$ and $\phi(p) = p$, in which case the conditions are satisfied. Thus, G can be completed to a realization of $\mathcal{X}_\mathcal{G}$ if and only if $v_+ \vee v_- = \phi(\lambda_+ \vee \lambda_-) \ni p$, or equivalently $\bigvee E$, $\bigvee E'$, and $v_+ \vee v_-$ meet at p. □

5.7 Arithmetic Polytopes

Arithmetic polytopes represent real values and arithmetic operations on them. Values are represented by the cross ratios of 4-tuples among a collection of collinear points determined by edges of a polygon in the following way. A **computational frame** representing n values along with 0, 1, and ∞ is a $2(n+3)$-gon where the meets of pairs of lines supporting opposite edges are all collinear. Three of these point, specified as p_0, p_1, and p_∞, along with their corresponding edges represent the values 0, 1, and ∞ respectively. Each of the remaining points $\{p_{\alpha_i}\}$ along with its pair of edges represents one of the computational frames values as $\alpha_i = (p_{\alpha_i}, p_1 | p_0, p_\infty)$.

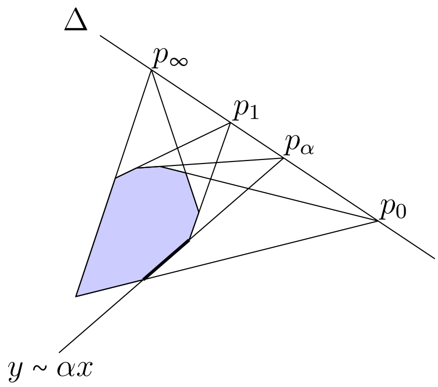


Figure 5.4. A computational frame representing the value α.

As a corollary to a variation of a theorem presented in [20, p. 68], for every algebraic number α, we can construct a 4-polytope $\mathcal{R}_\alpha$ that has, as one of its faces, a computational frame representing α. This number will then appear as a coordinate of a vertex in the stamp construction.

Corollary 5.9. *For any positive algebraic number α, the completion condition for the specified facet* $\mathrm{pyr}(\mathcal{G})$ *of a combinatorial 4-polytope $\mathcal{R}_\alpha$ is that $\mathcal{G}$ be a computational frame representing α.*

To prove this lemma we will use the following theorem.

Theorem 5.10. (Richter-Gebert) *For any pair of triples of indices $\mathbf{A}, \mathbf{M} \subset \{(i,j,k) : 1 \le i \le j < k \le n\}$ the completion condition for the specified facet* $\mathrm{pyr}(\mathcal{G})$ *of an abstract 4-polytope $\mathcal{R}_{\mathbf{A},\mathbf{M}}$ is that $\mathcal{G}$ be a computational frame representing values $1 < x_1 < \cdots < x_n$ satisfying $x_i + x_j = x_k$ $\forall (i,j,k) \in \mathbf{A}$ and $x_i x_j = x_k$ $\forall (i,j,k) \in \mathbf{M}$.*

We will say a bit about what this theorem means, how we will strengthen it, how this strengthening leads to $\mathcal{R}_\alpha$, and why it is true. $\mathcal{R}_{\mathbf{A},\mathbf{M}}$ represents a collection of values with one face a computational frame, like $\mathcal{R}_\alpha$, but with several varying values. We see here that the indices in $\mathbf{A}$ and $\mathbf{M}$ tell us when represented values added or multiplied together respectively must give another of the represented values. We would also like to include 1, for which we use the index 0, as a represented value that can appear as part of these operations. Although this was not stated as part of the theorem in [20], allowing it does not substantially change the construction or the proof. We can actually allow much more.

Theorem 5.11. *Theorem 5.10 holds with positive values*

$$0 < x_{n_-} < \cdots < x_{-1} < x_0 = 1 < x_1 < \cdots < x_{n_+}$$

and indices satisfying the appropriate inequalities.

The appropriate inequalities we refer to here are just what one would first think. For **A** these inequalities are the same as in Theorem 5.10 with bounds adjusted, since the sum of positive numbers is always greater then both arguments, specifically $i \leq j < k$. For **M**, however, we have now three possibilities depending on how the values x_i and x_j compare to $x_0 = 1$.

$$0 < i \leq j < k \quad \text{or} \quad k < i \leq j < 0 \quad \text{or} \quad i < 0 < j \Leftrightarrow i < k < j$$

We do not include 0 as a possible index for i or j in **M** as this would represent multiplication by 1.

With this we can get $\mathcal{R}_\alpha$ by first constructing an appropriate polytope $\mathcal{R}_{\mathbf{A},\mathbf{M}}$ with operations encoded in **A** and **M** forcing a particular represented value x_i to satisfy the minimal polynomial of α as well as rational bounds sufficiently close to distinguish α from the polynomial's other roots, then using a forgetful transmitter to eliminate the extra intermediate variables.

To show Theorem 5.11 we review the proof of Theorem 5.10 with the necessary modification. To this end we use polytopes provided by [20, p. 59] for preforming basic arithmetic operations, doubling $\mathcal{P}^{2x}$ or squaring $\mathcal{P}^{x^2}$ a single, or adding $\mathcal{P}^{x+y}$ or multiplying $\mathcal{P}^{xy}$ two values.

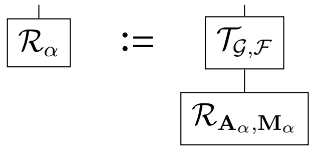


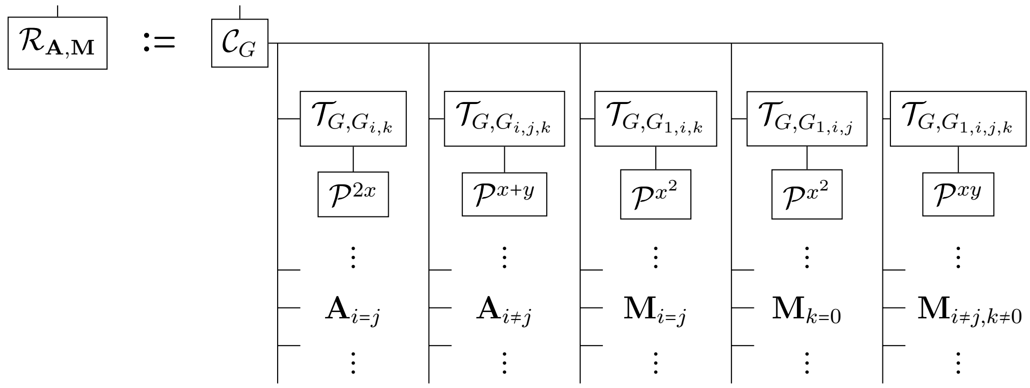
Figure 5.5. Gluing diagram for an arithmetic polytope representing α.

Lemma 5.12. (Richter-Gebert) *Each of the following polytopes has a specified facet* $\text{pyr}(\mathcal{G})$ *with completion condition that* G *be a computational frame representing the following values.*

$$\begin{array}{ll} \mathcal{P}^{2x} & \{0,\ 1,\ 2,\ \qquad\qquad \infty\} \\ \mathcal{P}^{x+y} & \{0,\ 1,\ z,\ 1+z,\ \quad \infty\} \\ \mathcal{P}^{x^2} & \{0,\ 1,\ x,\ x^2,\ \qquad \infty\} \\ \mathcal{P}^{xy} & \{0,\ 1,\ x,\ y,\ xy,\ \infty\} \end{array}$$

We would like the values represented in the completion conditions for $\mathcal{P}^{2x}$ and $\mathcal{P}^{x+1}$ to be $\{0, x, 2x, \infty\}$ and $\{0, x, y, x+y, \infty\}$ respectively, but we have 1 appearing here rather than x. This is because we have defined computational frames to necessarily include 1, but addition is independent of the choice of multiplicative identity, so we not actually need it. To reconcile this we divide by x and let $z = \frac{y}{x}$. Note that scalar multiplication, a subclass of projectivities, does not change the value of a cross ratio, so the completion conditions given in the lemma are what we would like. We include the proof and construction for the basic arithmetic polytopes in Appendix A. We will however now present the construction for $\mathcal{R}_{\mathbf{A},\mathbf{M}}$ using these polytopes.

The **arithmetic polytope** $\mathcal{R}_{\mathbf{A},\mathbf{M}}$ with specified facet of combinatorial type $\text{pyr}(\mathcal{G})$, where $\mathbf{A}$ and $\mathbf{M}$ are triples of indices satisfying the inequalities above and $\mathcal{G}$ is a $2(n_+ - n_- + 3)$-gon, consists of a connector $\mathcal{C}_{1+|\mathbf{A}|+|\mathbf{M}|,\mathcal{G}}$ with one of its sockets the specified facet, and each of the remaining sockets glued to a forgetful transmitter that in turn is glued to some basic arithmetic polytope representing an operation of $\mathbf{A}$ or $\mathbf{M}$. These forgetful transmitters identify those edges of $\mathcal{G}$ representing relevant variables with edges of a basic arithmetic poly-



$\mathbf{A}_{i=j}$	$\mathbf{A}_{i\neq j}$	$\mathbf{M}_{i=j}$	$\mathbf{M}_{k=0}$	$\mathbf{M}_{i\neq j,k\neq 0}$
$1 \quad 2$	$1 \quad z \quad 1+z$	$1 \quad x \quad x^2$	$1 \quad x \quad x^2$	$1 \quad x \quad y \quad xy$
$x_i \quad x_k$	$x_i \quad x_j \quad x_k$	$1 \quad x_i \quad x_k$ $x_k \quad x_i \quad 1$	$x_i \quad 1 \quad x_j$	$1 \quad x_i \quad x_j \quad x_k$ $x_i \quad 1 \quad x_k \quad x_j$ $x_i \quad x_k \quad 1 \quad x_j$ $x_k \quad x_i \quad x_j \quad 1$

Figure 5.6. Gluing diagram for a general arithmetic polytope. Below we list the ways each edge of each basic arithmetic polytope is identified with an edge of G by gluing along computational frames.

tope's computational frame so as to preserve the choice of edges representing 0 and ∞ and the order of the remaining variables. We break up the collections of triples of indices into those satisfying some conditions, which we indicate by subscript. For each triple of indices in $\mathbf{A}_{i=j} := \{(i,j,k) \in \mathbf{A} : i = j\}$ a doubling polytope $\mathcal{P}^{2x}$ is glued by a forgetful transmitter $\mathcal{T}_{\mathcal{G},\mathcal{G}_{i,k}}$ leaving only the edges of $\mathcal{G}$ representing 0, x_i, x_k, and ∞, which are identified with edges of $\mathcal{P}^{2x}$ representing 0, 1, 2, and ∞ respectively, to a socket of $\mathcal{C}_{\mathcal{G}}$. For each triple in $\mathbf{A}_{i\neq j} := \mathbf{A} \setminus \mathbf{A}_{i=j}$ an adding polytope $\mathcal{P}^{x+y}$ is glued by forgetful transmitter $\mathcal{T}_{\mathcal{G},\mathcal{G}_{i,j,k}}$ leaving $\mathcal{G}$'s edges representing 0, x_i, x_j, x_k, ∞. For each triple in $\mathbf{M}_{i=j}$ a squaring polytope $\mathcal{P}^{x^2}$ is glued by forgetful transmitter $\mathcal{T}_{\mathcal{G},\mathcal{G}_{1,i,k}}$ leaving $\mathcal{G}$'s edges representing 0, 1, x_i, x_k, ∞. For each triple in $\mathbf{M}_{k=0}$ a squaring polytope $\mathcal{P}^{x^2}$ is glued by forgetful transmitter $\mathcal{T}_{\mathcal{G},\mathcal{G}_{1,i,j}}$ leaving $\mathcal{G}$'s edges representing 0,

1, x_i, x_j, ∞. Finally, for each triple in $\mathbf{M}_{i\neq j,k\neq 0}$ a multiplying polytope $\mathcal{P}^{xy}$ is glued by forgetful transmitter $\mathcal{T}_{\mathcal{G},\mathcal{G}_{1,i,j,k}}$ leaving $\mathcal{G}$'s edges representing 0, 1, x_i, x_j x_k, ∞.

As mentioned above, when identifying $\mathcal{G}$'s edges with those of a basic arithmetic polytope's computation frame, we preserve the order of the variables, but this may not in general be the order in which we listed the variables in our description. For the triples of $\mathbf{A}$ this order is the same. For $\mathbf{M}_{i=j}$, $\mathcal{P}^{x^2}$'s edges representing 1, x, x^2 may be identified with those of $\mathcal{G}$'s representing either 1, x_i, x_k or x_k, x_i, 1 respectively. for $\mathbf{M}_{k=0}$, $\mathcal{P}^{x^2}$'s edges representing 1, x, x^2 will always be identified with those of $\mathcal{G}$'s representing x_i, 1, x_j respectively. Lastly, for $\mathbf{M}_{i\neq j,k\neq 0}$, $\mathcal{P}^{xy}$'s edges representing 1, x, y, xy may be identified with those of $\mathcal{G}$'s representing either 1, x_i, x_j x_k or x_k, x_i, x_j, 1 or x_i, 1, x_k, x_j or x_i, x_k, 1, x_j respectively. We can think of these computational frames as being rescaled by whatever value has its edges identified with those representing 1.

Proof of Theorem 5.11: By Lemma 5.6 the completion conditions for the open socket of $\mathcal{R}_{\mathbf{A},\mathbf{M}}$ are the combined completion conditions of the restricting polytopes, which by Lemma 5.12 are the conditions of the theorem. □

Proof of Corollary 5.9: This follows immediately from Theorem 5.11, and the fact that every real algebraic number can be uniquely specified as the solution to a polynomial equation in an interval with rational bounds. □

We can explicitly construct $\mathcal{R}_\alpha$ as follows. Note first we can construct $\mathcal{R}_{\mathbf{A}_\alpha,\mathbf{M}_\alpha}$ so that $\mathcal{G}$ represents among its values any finite set of naturals by repeatedly adding 1 in $\mathbf{A}_\alpha$. We also have $\mathcal{G}$ represent some value x that we will eventually force to be α, and we can represent any finite set of powers of x by repeatedly multiplying x in $\mathbf{M}_\alpha$. Since α is algebraic it is a root of some polynomial $p(x) \coloneqq \sum_{k=0}^{n} \pm_k a_k x^k$, which we represent with the polynomial equation $y \coloneqq \sum_{\pm_k=+} a_k x^k = \sum_{\pm_k=-} a_k x^k$ so as to avoid nonpositive values. We impose this on x in $\mathcal{G}$ by multiplying $a_k x^k$ in $\mathbf{M}_\alpha$ to get each monomial term and adding the monomials in $\mathbf{A}_\alpha$ to get y as the result of both sides of the polynomial equation. This polynomial will have finitely many other roots, so there must be some positive minimum distance r between roots. We can now find a pair of rational values $(\frac{m_-}{n_-}, \frac{m_+}{n_+})$ on either side of and within r from α, which we also represent in $\mathcal{G}$ by multiplying $\frac{m_\pm}{n_\pm} n_\pm = m_\pm$ in $\mathbf{M}_\alpha$. Finally by substituting in the value α for x we can determine the correct order for all values represented in $\mathcal{G}$, so that α is the only solution.

5.8 Anchor Polytopes

Now that we have a way to represent any algebraic constant in the completion conditions of a polytope's facet, we would like make this constant the coordinate of a vertex under some projective basis.

Lemma 5.13. *For any algebraic number α, the completion conditions for a specified facet* $\mathrm{pyr}(\pentagon)$ *of a combinatorial 4-polytope Ψ_α are that $\pentagon$ be a projective image of the unit square with* $(1,1)$ *truncated so as to have a vertex p at* $(1,\alpha)$.

The **anchor polytope** Ψ_α with specified facet of combinatorial type $\mathrm{pyr}(\pentagon)$ consists of a hub, with one socket the specified facet, and other sockets glued to the arithmetic polytope $\mathcal{R}_\alpha$ of the constant α and two pencil polytopes $\mathcal{X}_\alpha$ $\mathcal{X}_1$ in the following way. The hub sockets instantiate a common space $\mathbb{RP}^2_{\mathbf{H}}$, which contains an enneagon E and the supporting lines of its edges consecutively labeled l_x, l_y, l_α, l_1, $l_{x'}$, l_h, $l_{y'}$, $l_{\alpha'}$, $l_{1'}$. The hub itself consists of a connector $\mathcal{C}_{4,E}$ glued to one forgetful transmitter leaving l_x, l_y, $l_{x'}$, l_h, $l_{y'}$ supporting the pentagon $\pentagon$, and two forgetful transmitters leaving all but l_h supporting an octagon O. The hub's pentagonal socket is the specified facet $\mathrm{pyr}(\pentagon)$. One of the octagonal sockets is glued to $\mathcal{R}_\alpha$ so that pairs of lines $(l_y, l_{y'})$, $(l_\alpha, l_{\alpha'})$, $(l_1, l_{1'})$, $(l_x, l_{x'})$ are identified with those representing the values 0, α, 1, ∞ respectively. The other octagonal socket is glued to a pencil polytope $\mathcal{X}_1$ with specified edges in l_1, $l_{1'}$ and vertices $l_x \wedge l_y$, $l_{x'} \wedge l_{y'}$. The remaining enneagonal socket is glued to the other pencil polytope $\mathcal{X}_\alpha$ with specified edges in l_α, l_α and vertices $l_x \wedge l_y$, $p \coloneqq l_{x'} \wedge l_h$.

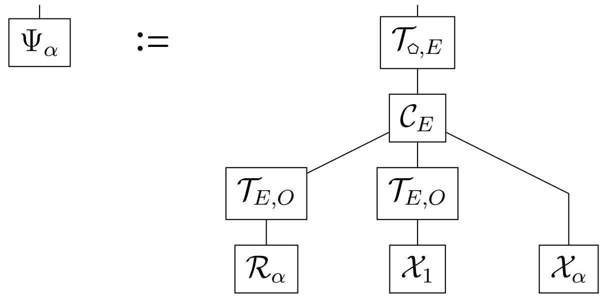


Figure 5.7. Gluing diagram for an anchor polytope representing α.

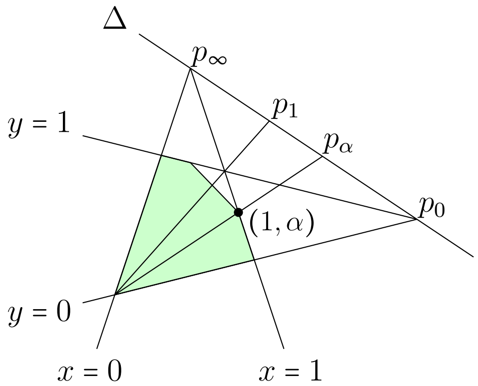


Figure 5.8. Collinearities in the completion conditions of an anchor polytope.

Proof of Lemma 5.13: We start by choosing a coordinate system for $\mathbb{RP}^2_{\mathbf{H}}$ so $\pentagon$ is a truncation of the unit square and $(p)_x = 1$. For this let l_x, l_y, $l_{x'}$, $l_{y'}$ be the line $x = 0$, $y = 0$, $x = 1$, $y = 1$ respectively. The Arithmetic polytope $\mathcal{R}_\alpha$ forces the meets of pairs of lines $p_0 := l_y \wedge l_{y'}$, $p_\alpha := l_\alpha \wedge l_{\alpha'}$, $p_1 := l_1 \wedge l_{1'}$, $p_\infty := l_\infty \wedge l_{\infty'}$ to be on a line Δ and $(p_\alpha, p_1 | p_0, p_\infty) = \alpha$. The pencil polytopes $\mathcal{X}_1$ and $\mathcal{X}_\alpha$ force $(0,0)$, $(1,1)$, p_1 and $(0,0)$, p, p_α to be collinear. The effect of these collinearities is that the projection from $l_{x'}$ to Δ through $(0,0)$ sends $(1,0)$, p, $(1,1)$, p_∞ to p_0, p_α, p_1, p_∞ respectively, thus

$$(p)_y = (p, \pi^y_1(p) | \pi^y_0(p), p_\infty) = (p, (1,1) | (1,0), p_\infty) = (p_\alpha, p_1 | p_0, p_\infty) = \alpha \qquad \square$$

5.9 Stamps

We now give an overview of the construction of the stamp $\mathcal{S}_P$ from Theorem 4.1. We start by choosing a coordinate system for P. We do this so that each wall of the unit cube $\cube$ truncates one of P's vertices, and so the supporting hyperplane $h = \bigvee f$ of each facet f of P does not contain any of $\cube$'s vertices. Let $\mathbf{H}$ be the set of all these hyperplanes h.

We will fix P in projective space by fixing each $h \in \mathbf{H}$ by, in tern, fixing the coordinates of d points spanning h. These points will be on the intersection of h with an edges of $\cube$. This edge determines all of such a point p's coordinates except one. To determine the remaining coordinate we use a hub to identify a pentagon $\pentagon$ of $\cube \cap H$ with the socket of an anchor polytope fixing that coordinate, where H is the half space bounded by h containing P. We see from the following lemma that we can always find d many such pairs $\pentagon$, p. We denote this collection of pairs $\boldsymbol{\pentagon}_h$.

Lemma 5.14. *If H a closed half-space not containing the unit cube $\cube$ and with boundary $h := \partial H$ containing no vertex of $\cube$, but with H intersecting all facets of $\cube$, then there are at least d distinct points in h that are vertices of some pentagonal face $\pentagon$ of $\cube \cap H$.*

Proof: We represent the nonempty faces of $\cube$ as elements of $\{0, 1, *\}^d$ where '$*$' indicates a free coordinate. Here, containment is equivalent to replacing $*$ with 0 or 1, and the vertices are given by their coordinates. We will indicate an appropriate edge and truncated square with 'y' for the free coordinate of the edge containing the point, 'x' for the other free coordinate in the square, and 0 or 1 for the remaining coordinates.

Let $\mathbf{C}_H$ be the set of maximal faces of $\cube$ that do not intersect H. We can assume without loss of generality that H is a lower-half-space and that $\mathbf{C}_H \subset \{1, *\}^d$. If this is not the case, we can relabel vertices of the cube so this assumption holds. To see this, let v be a vertex of $\cube$ such that

$\mathrm{ncon}(h) \subset \mathrm{ncon}(v)$. Now, $\langle \mathrm{ncon}(v), \mathrm{ncon}(h) \rangle \geq 0$ and $v \in \bigcap \mathbf{C}_H$ is in every maximal face away from H, since for any face c of ⧉ that does not intersect H, $c + \mathrm{ncon}(v)$ intersects ⧉ in a face that contains both c and v, but still does not intersect H. Now relabel the faces of ⧉ by switching $0, 1$ in those coordinates so that in the new labeling $v = (1, \cdots, 1)$ and no face in $\mathbf{C}_H$ has a 0 coordinate, since such a face would not contain v.

Now with $\mathbf{C}_H \subset \{1, *\}^d$, let $S := \bigcap_{c \in \mathbf{C}_H} \{i : c_i = *\}$ be the set of indices with coordinate $*$ in all faces of $\mathbf{C}_H$. We claim that for each $c \in \mathbf{C}_H$ and $T \subset S$ and $k \in \{i : c_i = 1\}$, there is an edge $e = e(c, T, k)$ of ⧉ that intersects h in a point p and a $j \in \{i : c_i = 1\}$ with $j \neq k$ and a square $s = s(c, T, k, j)$ of ⧉ with

$$s_i = \begin{cases} x & i = j \\ y & i = k \\ 0 & c_i = *,\ i \notin T \\ 1 & \text{else} \end{cases}$$

that intersects H in a pentagon ⬠, where $e = s|_{x=1}$, and where $s|_{x,y=1,1}$ is the vertex truncated by H.

We can choose any T, since all coordinates of faces in $\mathbf{C}_H$ are $*$ for those with index in S, so the value of these coordinates is not relevant for determining whether a vertex is in H. Among the coordinates with index not in S any vertex with a set of indices with 0 coordinate properly containing the indices of $*$ coordinates of a face in $\mathbf{C}_H$ must be in H, by the condition that the faces of $\mathbf{C}_H$ are maximal and have all fixed coordinates 1. Once we have chosen c, T, k there must be an appropriate j, since otherwise c would be a facet, which it cannot be by assumption. A vertex of s with $x = 0$ or $y = 0$ has a strictly larger set of 0 coordinates then the $*$ coordinates of c among those with index not in S, so these three are in H. The last vertex is also a vertex of c so it is not in H, and the claim holds since s has exactly 3 vertices in H.

We can find d distinct points where h intersects an edge of ⧉ in such a square by selecting $s(\iota) = s(c(\iota), T(\iota), k(\iota), j(\iota))$ for $1 \leq \iota \leq d$ in the following way. For $\iota \in S$ let $c(\iota)$ be any element of $\mathbf{C}_H$ and $k(\iota), j(\iota)$ be any appropriate index and $T = \{\iota\}$. For $\iota \notin S$ let $c(\iota)$ be such that $c(\iota)_\iota = 1$ and $k(\iota) = \iota$ and $j(\iota)$ be any appropriate index and $T = \varnothing$. $\square$

We require that a hyperplane h of $\mathbf{H}$ does not contain any vertex of ⧉ so as to avoid the case where h intersects the 1-skeleton of ⧉ at the points $(*, 1, 1)$, $(1, 0, 1)$, $(1, 1, 0)$, because the point $(*, 1, 1)$ would not be contained in any pentagon.

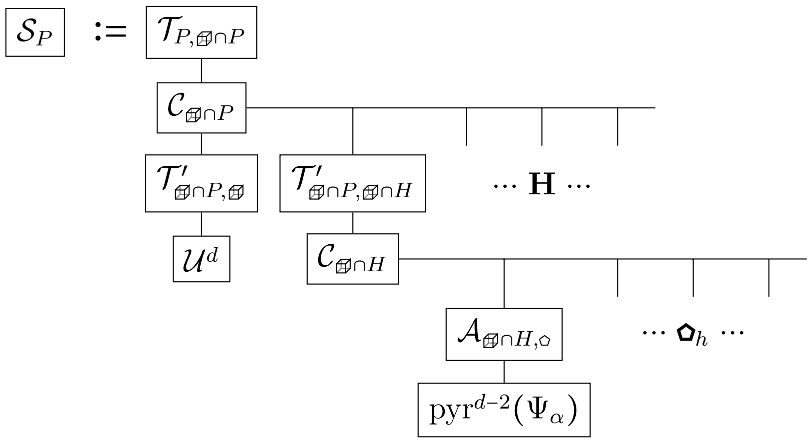


Figure 5.9. Gluing diagram for a stamp of P.

The **stamp** $\mathcal{S}_P$ has a form similar to a hub with restricting polytopes, but rather then forgetful transmitters we allow general projective transmitters. The 'hub' consists of a connector $\mathcal{C}_{d|H|+2,\text{⧉}\cap P}$ glued to a projective transmitter to P and one to ⧉ and one to $\text{⧉}\cap H$ for each half space bounded by a hyperplane $h \in \mathbf{H}$, which are each in turn glued to a connector $\mathcal{C}_{d+1,\text{⧉}\cap H}$ that is glued to an adapter to each pentagon in ⬟_h. The socket P is open and is the specified facet. The socket ⧉ is glued to a unit polytope, and the socket coming from each pentagon and vertex in $\text{⬟}_{\mathbf{H}}$ is glued to an anchor polytope Ψ_α where α is the coordinate of the vertex not determined by the edge of ⧉ containing it when the orientation of the edge is consistent with that of the unit square in Lemma 5.13, and $1 - \alpha$ when they have opposite orientations.

Proof of Theorem 4.1: First we show that $\mathcal{S}_P$ is always realizable. As long as we can realize each piece such that every pair of sockets we glue along is projectively equivalent, by Lemma 4.6 we can actually glue them together to get a realization of $\mathcal{S}_P$. Among the pieces of the 'hub', only the projective transmitters have nontrivial completion conditions for their sockets. However, in defining $\mathcal{S}_P$ we start with realizations of P and ⧉ in the same projective space, and define the combinatorics of the various projective transmitters to be consistent with these, so they can be realized with the appropriate sockets. The unit polytope can be realized with ⧉ as one of its facets, since this is the unit cube, and for each pentagon in ⬠$_{\mathbf{H}}$ we can realize the appropriate anchor polytope Ψ_α, since the algebraic constant α comes from this realization.

For the other direction we show that in every realization the specified facet P' of combinatorial type $\mathcal{P}$ is projectively equivalent to P. We first note that we can treat all sockets of a realization S_P as being in a common projective space $\mathbb{RP}^d_{\mathbf{B}}$ containing P' and the hypercube ⧉$'$. In every realization ⧉$'$ must be projectively equivalent to the unit cube by Lemma 5.7, since it appears as a socket of a unit polytope, so we use ⧉$'$ to determine a coordinate system. For each facet supporting hyperplane $h \in \mathbf{H}$ of P', each of the specified vertices in ⬠$_h$ must have the same coordinates as the corresponding point in that facet supporting hyperplane of P, since it is on an edge of ⧉$'$, which determines $d-1$ coordinates, and appears as the specified vertex of an anchor polytope Ψ_α with appropriate α, which by Lemma 5.13 means the the remaining coordinate must be α. Since this determines d points of h in general position, the hyperplanes must be equivalent with respect to their own coordinate systems, which forces P' and P to be equivalent. Since the coordinate systems determine a projective transformation between P' and P, they must be projectively equivalent. □

CHAPTER 6

ANTIPRISMS AND INTERVAL POLYTOPES

6.1 Balanced Pairs

Recall that a polytope is centered when it contains the origin. A pair of centered polytopes is called a **balanced pair** when they are combinatorial duals of each other, they are centered, and the relative open cones over a face of each intersect in a ray if and only if they are dual faces. We denote that a pair P_0, P_1 is balanced by $P_0 \doteq P_1$. The main result of this chapter is the following.

Theorem 6.1. *In dimensions $d \geq 4$ there exists a combinatorial type of polytope $\mathcal{P}$ that is realizable, but can not be realized as part of a balanced pair, and neither the antiprism nor the interval polytope of $\mathcal{P}$ are realizable.*

When a polytope together with its polar are a balanced pair, $P \doteq P^*$, we say the polytope is **perfectly centered**. Alternatively, this is when the orthogonal projection of the origin into the affine span of each face of the polytope is contained in the relative interior of that face. As mentioned earlier, the following related result appeared in [4].

Theorem 6.2. (Broadie) *If a polytope has a perfectly centered realization then its antiprism is realizable.*

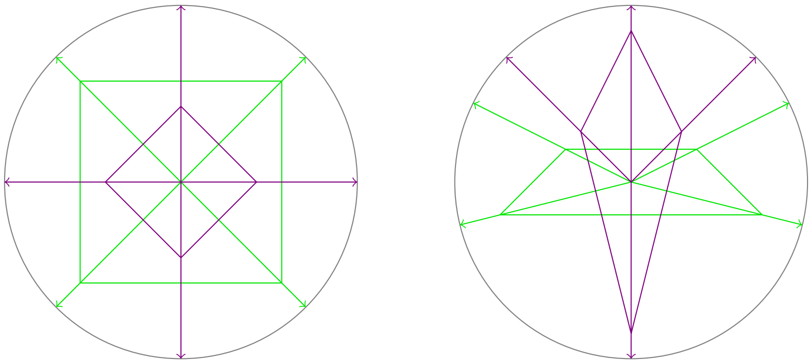


Figure 6.1. A pair of polygons that is balanced and a pair that is not.

As we have stated the situation is simpler for 3-polytopes. From Steinitz we have a complete combinatorial characterization of them [23], and Thurston gave a particularly nice collection of realizations of each 3-polytope called **midscribed**, namely where every edge of the polytope is tangent to the unit sphere [25, p. 13.48]. As a consequence of this we have the following result announced by Björner [2].

Theorem 6.3. *The antiprism of every 3-polytope is realizable.*

Proof: By Theorem 6.2 it is sufficient that any 3-polytope have a perfectly centered realization, which we claim a midscribed polytope is. Since the orthogonal projection of the origin into an affine subspace is the point with minimal norm among points in the subspace, and each edge is tangential to the sphere, the orthogonal projection is exactly the point of tangency. Each facet is a polygon heaving each edge tangent to the unit sphere, so the unit ball intersects the plane of the polygon in a disk, which is contained in the polygon since its edges are also tangent to the disk, and the orthogonal projection of the origin must be in this disk since its norm is minimal. □

On the way to proving Theorem 6.1 we will give several other equivalent ways to define 'balanced pair', including the appropriate definition in projective space. Before this however, we show the obstruction is not combinatorial in nature.

Lemma 6.4. *The interval poset of an abstract polytope is again an abstract polytope, and the property of being a lattice is preserved as well.*

Proof: We will show this by induction on dimension, following the recursive definition of an abstract polytope. We start with a point. The face lattice of a point consists of $\{\bot, \top\}$, so the interval lattice is $\{\varnothing, \{\bot\}, \{\top\}, [\bot, \top]\}$, which is the face lattice for a line, the only 1 dimensional abstract polytope.

By inductive assumption the interval polytopes of the facets and cofacets of $\mathcal{P}$, $[\bot, F]$ and $[v, \top]$ respectively where v and F are vertices and covertices of $\mathcal{P}$, are abstract polytopes. We denote this set of abstract polytopes by $\mathbf{F}$. The facets of these consist of the interval posets of $[\bot, r]$, $[v, F]$, $[e, \top]$ where v e r F are clades of $\mathcal{P}$ of dimension 0, 1, $d-2$, $d-1$ respectively. By the diamond property each interval $[r, \top]$ and $[\bot, e]$ has exactly two elements other than the bounds, so the interval posets of $[\bot, r]$ and $[e, \top]$ are each a facet of exactly two abstract polytopes of $\mathbf{F}$, which come from intervals of the form $[\bot, F] \supset [\bot, r]$ and $[v, \top] \supset [e, \top]$ respectively. Also each interval poset of $[v, F]$ is a facet of exactly two abstract polytopes of $\mathbf{F}$, which come from $[\bot, F]$ and $[v, \top]$. Thus, we have a perfect matching between facets of abstract polytopes in $\mathbf{F}$ without matching facets of the same abstract polytope.

Now we show that the result is connected. This is because $\mathcal{P}$ is flag connected. Consider the intervals from where two elements of $\mathbf{F}$ come, and a path between a flag for each that includes that interval's bounds among its clades. We construct a path between these elements of $\mathbf{F}$ as follows. For each edge of this path where the difference between flags consists of vertices v_0 and v_1, we get $[v_0, \top]$ and $[v_1, \top]$ connected along $[e, \top]$ where e is the rank 1 comparable of both flags. Likewise where the difference consists of covertices, we get the opposite connection. All other edges we ignore, since both flags remain in the same interval. If we get an interval of the form $[\bot, F]$ on one end and $[v, \top]$ on the other of a line of such flags, possibly a single flag, we connect these along $[v, F]$. This gives us the desired path through $\mathbf{F}$.

Finally we observe that we get a new top element $[\bot, \top]$, so the interval poset of an abstract polytope is again an abstract polytope.

To see that this preserves lattices, suppose we started with a lattice, then we have $[a,b] \wedge [c,d] = [a \vee c, b \wedge d]$ and $[a,b] \vee [c,d] = [a \wedge c, b \vee d]$. □

We now show the three conditions of Theorem 6.1 are equivalent. This implicitly provides another definition of balanced pair, namely that $P \doteq P'^*$ when for faces $\mathcal{F}_i \in \mathcal{P}$ we have the following.

$$\mathrm{con}(F_1)^\circ \cap \mathrm{con}(F_2'^\diamond)^\circ \neq \varnothing \Leftrightarrow \mathcal{F}_1 < \mathcal{F}_2$$

Lemma 6.5. *The antiprism, and hence interval polytope, of a polytope is realizable if and only if that polytope can be realized as part of a balanced pair, and the definitions are equivalent.*

Proof: The existence of an antiprism is equivalent to the existence of an interval polytope, since these are dual to each other. For a pair of combinatorial dual polytopes first notice the existence of a balanced realization, by the original deffinition, is equivalent to the existence of a realization where the open normal cone of each face intersects that of its dual in a ray, since the cones over faces of a polytope containing the origin are the normal cones of the polar of that polytope. Thus we can get from a realization satisfying one set of conditions to that of the other by keeping the fans and replacing the polytopes with their polars. Embedding these polytopes in projective space and treating the horizon as a celestial sphere, we see that this is equivalent to the condition that the open celestial view from each face intersects that of its dual in a single sky. From Lemma 4.4 these are exactly the completion conditions for the bases of a prismoid with a side facet $(f, f^\diamond)$ coming from each face f of one base and the corresponding dual face $f^\diamond$ of the other. This gives us all the facets of an antiprism, which means this must actually be an antiprism, because the incidences between vertices and facets include all those of the antiprism, and since this is a lattice, its face lattice contains an isomorphic copy of that of the antiprism by Lemma 3.5. By Lemma 3.8, the face lattice of the prismoid is an isomorphic copy of that of the antiprism. Thus the existence of a balanced realization is equivalent to that of the antiprism, and the combinatorics of the sides of a prismoid shows us the deffinitions of balanced pair are equivalent. □

6.2 A Polytope without an Antiprism

We now set about constructing a counter example for Theorem 6.1. Specifically, a polytope that cannot be realized as part of a balanced pair. We proceed by first paring down the space of realizations we will need to consider.

We will only deal with the case $d = 4$, since this immediately implies the result for all higher dimensions. Also, by corollary 4.2 constructing a pair of polygons that can not be balanced by applying distinct projective transformations to each is sufficient. Furthermore, since applying the same projective transformation to both of a pair of polytopes does not affect balance, the problem of balancing a pair of polytopes by distinct projective transformations on each is equivalent to applying a projective transformation to only one and keeping the other polytope fixed. Finally, since positive scaling of individual vectors does not change the cone of positive linear combinations of these vectors, we can reduce this further to just affine transformations. For this last part we note that a projective transformation consists of an affine part and a part that preserves direction as seen by the following matrix representation of a projective transformation in homogeneous coordinates.

$$\begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix} \frac{x}{1} = \frac{Ax + b}{1} \qquad\qquad \begin{bmatrix} A & b \\ c^* & 1 \end{bmatrix} \frac{x}{1} = \frac{Ax + b}{c^* x + 1}$$

Projecting the vertices of the polygon and its polar to direction vectors on the unit circle, as we see in Figure 6.1, we get that the pair is balanced iff direction vectors of vertices and covertices alternate around the circle. An affine transformation that balances the dual with the primal may have to change some directions of the dual's vertices to be between the appropriate direction vectors of the primal's vertices. We will get a counter example by requiring this affine transformation to turn the directions of vertices in too many alternating orientations.

To get an idea of how many alternating orientations is too many consider the orthogonal linear component (rotating), spd linear component (stretching), and translational component of an orientation preserving affine transformation separately. The orthogonal linear component rotates all vectors in the same way, where as the spd linear component divides the circle into 4 quadrants with direction vectors alternately turning clockwise and counter-clockwise, and similarly the translation component divides the circle into 2 halves with direction vectors turning alternately. Naively adding this up we get 7 regions where direction vectors turn.

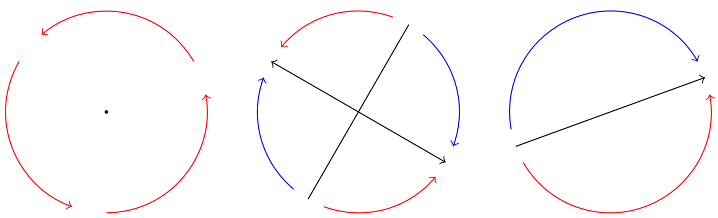


Figure 6.2. From the left, orientations of the directions of vectors turning under: rotation, spd stretching, translation.

With these limitations in mind we now construct the desired polygon. Let G be the polygon with vertices $(8, 5)$, $(7, 7)$ and all permutations and changes of sign. That is the vertices $(8, 5)$, $(7, 7)$, $(5, 8)$, $(-5, 8)$, $(-7, 7)$, $(-8, 5)$, $(-8, -5)$, $(-7, -7)$, $(-5, -8)$, $(5, -8)$, $(7, -7)$, $(8, -5)$. We will see that no affine transformation can accommodate all the alternating orientations in which the direction vectors of vertices must turn to balance G^* with G.

Lemma 6.6. *G and G^* cannot be balanced by projective transformation.*

Proof: The dual of G has vertices $(\frac{1}{8}, 0)$, $(\frac{2}{21}, \frac{1}{21})$ and all permutations and changes of sign. Since scale does not matter, however, we instead use the

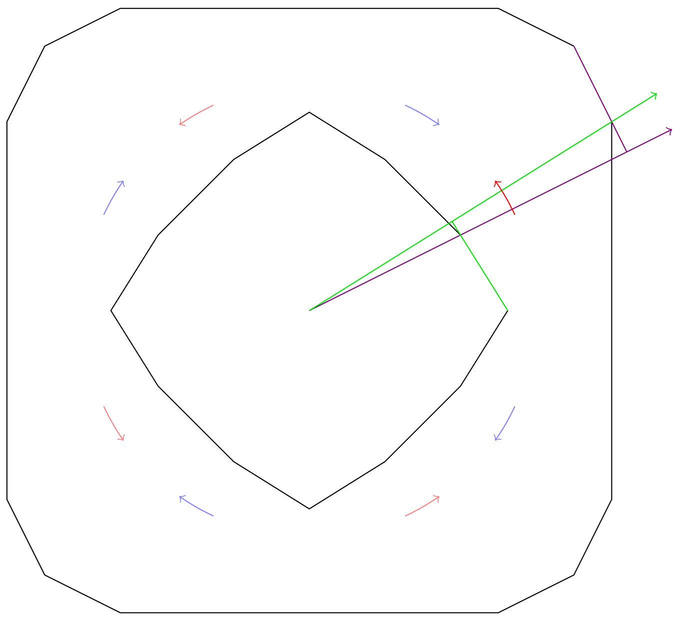


Figure 6.3. A polygon that cannot be balanced with its dual by projective transformations.

vertices $(\frac{21}{8}, 0)$, $(2, 1)$ scaling by 21. We see that G is not perfectly centered since that would require the slope s of the outward normal vector of the edge between $(8, 5)$ and $(7, 7)$ to be between the slopes of the vertices, $\frac{5}{8} < s < 1$, but the slope is $s = \frac{1}{2} < \frac{5}{8}$ as given by the covertex $(2, 1)$. By construction the reflection group of G and G^* is the same as that of a square.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

These transformations give us a total of 8 places where the perfectly centered condition is violated. Moreover, we must turn the directions of the covertices in alternating orientations for the polygons to be a balanced pair.

Consider an affine transformation T.

$$T\begin{bmatrix} \cdot \\ \cdot \end{bmatrix} := \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \cdot \\ \cdot \end{bmatrix} + \begin{bmatrix} s \\ t \end{bmatrix}$$

Here we must have $a, d > 0$. This is because, for a, in order to produce a balanced pair both $T\left(\frac{21}{8}, 0\right)$ must point to the right and $T\left(-\frac{21}{8}, 0\right)$ must point to the left, giving the following.

$$T\left(\tfrac{21}{8}, 0\right)_1 = \tfrac{21}{8}a + s > 0 \qquad T\left(-\tfrac{21}{8}, 0\right)_1 = -\tfrac{21}{8}a + s < 0$$

If $s \leq 0$ then the first inequality implies $a > 0$, and if $s \geq 0$ then the second inequality implies $a > 0$. The same holds for d because of the corresponding inequalities in the 2^{nd} coordinate.

For $T(G^*)$ to be balanced with G, the image of $(2, 1)$ must be a vector with slope greater than $\frac{5}{8}$, and furthermore, this must be the case for images of $(2, 1)$ under the transformations T conjugated by all elements of the reflection group. This gives the following vectors.

$$\begin{bmatrix} a2 + b1 + s \\ c2 + d1 + t \end{bmatrix} \begin{bmatrix} d2 + c1 + t \\ b2 + a1 + s \end{bmatrix} \begin{bmatrix} d2 - c1 - t \\ -b2 + a1 + s \end{bmatrix} \begin{bmatrix} a2 - b1 - s \\ -c2 + d1 + t \end{bmatrix}$$

$$\begin{bmatrix} a2 + b1 - s \\ c2 + d1 - t \end{bmatrix} \begin{bmatrix} d2 + c1 - t \\ b2 + a1 - s \end{bmatrix} \begin{bmatrix} d2 - c1 + t \\ -b2 + a1 - s \end{bmatrix} \begin{bmatrix} a2 - b1 + s \\ -c2 + d1 - t \end{bmatrix}$$

For the first of these vectors the slope requirement is given by the following equivalent inequalities.

$$\frac{T(2,1)_2}{T(2,1)_1} = \frac{c2 + d1 + t}{a2 + b1 + s} > \frac{5}{8}$$

$$-10a - 5b + 16c + 8d - 5s + 8t > 0$$

Putting the inequalities we get from all these slope requirements with the sign requirements of a, d together we get the following matrix inequality.

$$\begin{bmatrix} -10 & -5 & 16 & 8 & -5 & 8 \\ 8 & 16 & -5 & -10 & 8 & -5 \\ 8 & -16 & 5 & -10 & 8 & 5 \\ -10 & 5 & -16 & 8 & 5 & 8 \\ -10 & -5 & 16 & 8 & 5 & -8 \\ 8 & 16 & -5 & -10 & -8 & 5 \\ 8 & -16 & 5 & -10 & -8 & -5 \\ -10 & 5 & -16 & 8 & -5 & -8 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ s \\ t \end{bmatrix} > \mathbb{0}$$

Finding a solution to this inequality amounts to finding a vector in the column space of the matrix that has all positive entries. The columns of this matrix, however, are all perpendicular to $[1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 8\ 8]^*$, which has all positive entries, so the column span of the matrix is outside of the positive orthant, and no values for coefficients satisfy all of these inequalities. Therefor, there is no affine transformation T such that $T(G^*) \doteq G$.

If there were projective transformations π_1, π_2 such that $\pi_1(G) \doteq \pi_2(G^*)$, then we would have $\pi_1^{-1} \circ \pi_2(G^*) \doteq G$, and the affine part T of $\pi_1^{-1} \circ \pi_2$ would balance G with its dual $T(G^*) \doteq G$, which we have just shown to be impossible. Thus G and G^* cannot be balanced by projective transformations □

Lemma 6.7. *The statement that a pair of polytopes is balanced $P \doteq P'^*$, projectively inherits to faces.*

Proof: Let P and P' be a pair of polytopes of the same combinatorial type so that $P \doteq P'^*$, and consider some face F of P and the corresponding face F' of P'. By balance, the cones $\mathbb{R}^{\geq 0}F^\circ$ and $\mathbb{R}^{\geq 0}F'^{\diamond\circ}$ intersect in a ray, and we may assume without loss of generality that they are contained in a common supporting hyperplane. We will show first why the Lemma holds under this assumption, then what we can do when it fails.

Under this assumption, F° and $F'^{\diamond\circ}$ intersect in a point w. To get a combinatorial dual of F in its affine span we dilate the solid tangent cone of $F'^\diamond$ by $r > 1$ and intersect it with this space. We make this a linear space V by choosing the point w to be the origin. We claim this gives us a balanced pair of projective copies of these faces.

$$V := (\mathrm{aff}(F), \oplus, \odot)$$

$$(a + w) \oplus (b + w) := (a + b) + w \qquad s \odot (a + w) := sa + w$$

$$Q := F \hookrightarrow V, \qquad Q'^* := r\,\mathrm{tcon}(F'^\diamond) \cap V \hookrightarrow V$$

We have already that Q is a projective copy of F. From the definition of the solid tangent cone we have $\mathrm{tcon}(F'^\diamond) = \mathrm{con}(F')^* + \mathrm{aff}(F'^\diamond)$. We see now that $C := \mathrm{tcon}(F'^\diamond) \cap \mathrm{lin}(F)$ is the projection of $\mathrm{con}(F')^* + w$ on to $\mathrm{lin}(F)$ along $\mathrm{aff}(F'^\diamond)$, which intersects both $\mathrm{lin}(F')$ and $\mathrm{lin}(F)$ in a point, so C is projectively equivalent to $\mathrm{con}(F')^*$, and since $\mathrm{aff}(F \cup F'^\diamond)$ is a supporting hyperplane of $F'^\diamond \subset P'^*$, $\mathrm{aff}(F)$ is a supporting hyperplane of $w \subset C$ in $\mathrm{lin}(F)$. By construction rC is the cone over Q'^* from rw. Thus, Q'^* is projectively equivalent to F'^*, and therefor Q' is projectively equivalent to F'.

To see that $Q \simeq Q'^*$, we consider what the face cones of Q are. For a point $q \in \partial F$, the projection of $q \vee \mathbb{0}$ into V through rw, the vertex of C, gives $q \vee w$, since $q \in V$ and $\mathbb{0}$ projects to w, which is $\mathrm{lin}(q)$ in V. As such the faces of the cone over F project to the face cones of Q. In contrast, each face of $r\,\mathrm{tcon}(F'^\diamond)$ projects through rw to its intersection with V, the corresponding face of Q'^*. Also, the faces of $\mathrm{tcon}(F'^\diamond)$ consist of the tangent cones $\mathrm{tcon}_{G_1'^\diamond}(F'^\diamond)^\circ$ of $F'^\diamond$ as a face of its superfaces $G_1 \subset F$. We let $D^\circ := \bigcup_{G \subset F} G'^{\diamond\circ}$ denote the union of relative interiors of these faces. By $P \simeq P'^*$ the open faces $\mathrm{tcon}_{G_1'^\diamond}(F'^\diamond)^\circ$ intersect an open face $\mathrm{con}(G_2)^\circ$ of $\mathrm{con}(F)$ if and only if it corresponds to a subface $G_1 \subset G_2$. This is because $\partial\mathrm{tcon}(F'^\diamond) \cap \mathrm{con}(G_2)^\circ$ is connected and does intersect D° but not ∂D since ∂D consists of faces $G_3 \not\subset F$ so we cannot have

$G_3 \subset G_2$, therefor $\partial\mathrm{tcon}(F'^{\diamond}) \cap \mathrm{con}(G_2)^{\circ} \subset D^{\circ}$ and $\mathrm{tcon}_{G_1'^{\diamond}}(F'^{\diamond})^{\circ} \cap \mathrm{con}(G_2)^{\circ} = G_1'^{\diamond} \cap \mathrm{con}(G_2)^{\circ}$. Furthermore, scaling by r does not change this. Consequently the face cones of Q intersect the apropriate faces of Q'^{*}, and we have a balanced pair.

We now have only to justify our assumption. We will see that we can always find balanced projective copies of P and P'^{*} so the assumption holds, and since we are only interested in copies of the faces up to projectivity, this is enough. Specifically we will find projective transformations π and π' so $\pi(F)$ and $\pi'(F'^{\diamond})$ share a common supporting hyperplane, and the face fans of P and P'^{*} remain unchanged.

We still have that the cones over F° and $F'^{\diamond\circ}$ intersect in a ray. Let v and v' be the points where this ray intersects these faces and $v = sv'$ noting $s > 0$. Observe that $\mathrm{lin}(F - v) \cap \mathrm{lin}(F'^{\diamond} - v') = \mathbb{0}$, since $F^{\circ} \cap sF'^{\diamond\circ} = v$, and that $\dim(F) + \dim(F'^{\diamond}) = d - 1$, so the following space l is a line.

$$l := \mathrm{lin}(F - v \cup F'^{\diamond} - v')^{\perp} = \mathrm{lin}(F^{\diamond}) \cap \mathrm{lin}(F')$$

Since $\mathrm{aff}\,(F \cup sF'^{\diamond})$ is a supporting hyperplane of the vertex v of the centered polytope $P \cap sP'^{*}$, we have $v \notin \mathrm{lin}(F - v \cup F'^{\diamond} - v') = l^{\perp}$ and there is some $x \in l$ such that $\langle v, x\rangle = 1$, which means l intersects the supporting hyperplane $h_v := \langle v, \cdot\rangle^{-1}(1)$ of $F^{\diamond}$ at x, and similarly $h_{v'} := \langle v', \cdot\rangle^{-1}(1) \cap l = s^{-1}x$. Since P^{*} is centered, for any $y \in \mathrm{aff}(F^{\diamond})$ and $\varepsilon > 0$ we have $t := y - \varepsilon x \in \mathrm{tcon}(F^{\diamond})^{\circ}$. If $y \in F^{\diamond\circ}$, since the only linear inequalities satisfied on P^{*} that are not strict on y are also satisfied on $\mathrm{tcon}(F^{\diamond})$, for ε sufficiently small $t \in P^{*\circ}$. We have this for P' as well, and taking $u := \varepsilon x$ for the smaller such ε we find points in the following sets.

$$u \in l, \qquad t \in (F^{\diamond\circ} - u) \cap P^{*\circ}, \qquad t' \in (F'^{\circ} - u) \cap P'^{\circ}$$

With this $P^{*} - t$ and $P' - t'$ are centered and the dual faces intersect in a point.

$$(F^{\diamond} - t)^{\circ} \cap (F' - t')^{\circ} = u$$

We claim the following projectivities have the desired properties.

$$\pi := ((\,\cdot\,)^* - t)^*, \qquad \pi' := ((\,\cdot\,)^* - t')^*$$

The images $\pi(P)$ and $\pi'(P'^*)$ respectively have faces $\pi(F)$ and $\pi'(F'^\diamond)$, which share the common supporting hyperplane $h_u := \langle u, \cdot\rangle^{-1}(1)$. Also, since translation in the dual does not change the direction vectors of a polytope's vertices, the face fans are unchanged. Thus we can always find projective transformations such that our assumption holds, and so the statement inherits projectively to faces. □

Proof of Theorem 6.1: By Lemma 6.7, $P \doteq P'^*$ as a statement about P, P' satisfies the conditions of corollary 4.2. As such every combinatorial type of polytope can be realized as part of a balanced pair if and only if every algebraic polytope can be balanced with its polar by projectivities, but G is algebraic and by Lemma 6.6 it cannot. Therefor, there is a combinatorial type of 4-polytope, specifically the stamp $\mathcal{S}_G$, that cannot be realized as part of a balanced pair. Furthermore, by Lemma 6.5 the interval polytope of $\mathcal{S}_G$, and equivalently its antiprism, are not realizable. □

REFERENCES

[1] Alexander Below. *Complexity of Triangulation*. PhD thesis, ETH Zürich, 2002. DISS No. 14672.

[2] Anders Björner. The antiprism fan of a convex polytope. *Abstracts of the AMS*, 18(1):19, 1997. #918-05-688.

[3] Anders Björner, Andreas Paffenholz, Jonas Sjöstrand, and Günter M. Ziegler. Bier spheres and posets. *Discrete and Computational Geometry*, 34, 2005.

[4] Mark N. Broadie. A theorem about antiprisms. *Linear Algebra and its Applications*, 66:99–111, 1985.

[5] Heinz Bruggesser and Peter Mani. Shellable decompositions of cells and spheres. *Math. Scandinavica*, 29:197–205, 1971.

[6] Raquel Díaz. A characterization of gram matrices of polytopes. *Discrete and Computational Geometry*, 21:581–601, 1999.

[7] Reinhard Diestel. *Graph Theory*, volume 173. Springer-Verlag, Berlin Heidelberg, third edition, 2005.

[8] David Eppstein, Greg Kuperberg, and Günter M. Ziegler. Fat 4-polytopes and fatter 3-spheres. *In Discrete Geometry: In honor of W. Kuperberg's 60th birthday, Pure and Applied Math.*, 253:239–265, 2003.

[9] Claude-Alain Faure and Alfred Frölicher. *Modern Projective Geometry*, volume 521 of *Math. and its Applications*. Kluwer Academic Publishers, Dordrecht, 2000.

[10] Komei Fukuda and Christophe Weibel. f-vectors of minkowski additions of convex polytopes. *Discrete and Computational Geometry*, 37:503–516, 2007.

[11] Branko Grünbaum. *Convex Polytopes*, volume 221. Springer-Verlag, New York, second edition, 2003.

[12] Roger A. Horn and Charles R. Johnson. *Matrix Analysis*. Cambridge University Pess, Cambridge, 1985.

[13] Bernt Lindström. Problem p73. *Aequations Math*, 6:113, 1971.

[14] Saunders Mac Lane. *Categories for the Working Mathematician*, volume 5 of *Graduate Texts in Math.* Springer-Verlag, New York, second edition, 1998.

[15] Peter McMullen and Egon Schulte. *Abstract Regular Polytopes*, volume 92 of *Encyclopedia of Math. and its Applications*. Cambridge University Pess, Cambridge, 2002.

[16] James B. Nation. Notes on lattice theory. `http://www.math.hawaii.edu/~jb/books.html`.

[17] Andreas Paffenholz. *Construction for Posets, Lattices, and Polytopes*. PhD thesis, Technischen Universität Berlin, 2005.

[18] Iztok Peterin. Characterizing flag graphs and induced subgraphs of cartesian product graphs. *Order*, 21:283–292, 2004.

[19] John L. Pfaltz. A category of discrete prtially ordered sets, 2004. `http://www.cs.virginia.edu/~jlp/po.category.pdf`.

[20] Jürgen Richter-Gebert. *Realization Spaces of Polytopes*, volume 1643 of *Lecture Notes in Math.* Springer-Verlag, Berlin Heidelberg, 1996.

[21] Stewart A. Robertson. *Polytopes and Symmetry*, volume 90 of *London Math. Society Lecture Note Series.* Cambridge University Pess, Cambridge, 1984.

[22] Richard P. Stanley. *Enumerative Combinatorics Volume I.* Wadsworth & Brooks/Cole Advanced Books & Software, Monterey, 1986.

[23] Ernst Steinitz. Polyeder und raumeinteilungen. *Encyclopädie der math. Wissenschaften*, 3:1–139, 1922.

[24] Jorge Stolfi. *Oriented Projective Geometry: A Framework for Geometric Computations.* Academic Press, New York, 1991.

[25] William P. Thurston. *Geometry and Topology of Three-Manifolds.* Lecture Notes. Princeton University, Princeton, 1978.

[26] Günter M. Ziegler. *Lectures on Polytopes*, volume 152 of *Graduate Texts in Math.* Springer-Verlag, New York, 2007.

[27] Günter M. Ziegler. Non-rational configurations, polytopes, and surfaces. *Math. Intelligencer*, 3(30):36–42, 2008.

APPENDIX A

BASIC ARITHMETIC POLYTOPES

Here we provide constructions for the basic arithmetic polytopes of lemma 5.12. These were presented by Richter-Gebert in [20] as part of the proof of the univsality theorem for 4-polytopes, and are used to preform the basic arithmetic operations, doubling $\mathcal{P}^{2x}$ or squaring $\mathcal{P}^{x^2}$ a single, or adding $\mathcal{P}^{x+y}$ or multiplying $\mathcal{P}^{xy}$ two values represented in a computational frame.

We start with the simplist of these $\mathcal{P}^{2x}$, which we will also use in the construction of the others. For this, however, we will use the polytope to impose relationships among variables other then doubling. In these cases we will denote the polytope $\mathcal{H}$ and call it a harmonic polytope to indicate that it may not be doubling a value. We will construct $\mathcal{H}$ so the following holds, and later see why this is the same.

Lemma A.1. *The completion condition of the specified facet* $\operatorname{pyr}(G)$ *of* $\mathcal{H}$ *is that* G *be a computational frame representing the values* $\{-1,\ 0,\ 1,\ \infty\}$.

For this we construct a polygon F as the socket of a hub with restricting polytopes that impose the collinearities indicated in figure A.4. Such a polygon will have to be a computational frame representing the desired values, but not every computational frame representing the values will satisfy these collinearities. To deal with this we construct the slope transmitter $\mathcal{Z}$ having two specified facets that are octogonal pyramids pyr(F) and pyr(G), and for which the following lemma holds.

Lemma A.2. *The completion conditions, for a realization of $\mathcal{Z}$'s faces F and G where one is a computational frame, are that the other also be a computational frame and represent the same values.*

Throughout this construction we will denote the edge supporting lines of G consecutively by g_{-1}, g_0, g_1, g_∞, g_{-1}, $g_{0'}$, $g_{1'}$, $g_{\infty'}$ and those of F by f_{-1}, f_0, f_1, f_∞, $f_{-1'}$, $f_{0'}$, $f_{1'}$, $f_{\infty'}$. We choose these labels so that subscrips will eventually indicate the values represented by pairs of edges.

$\mathcal{Z}$ consists of two tents $\mathcal{Y}_0$ and $\mathcal{Y}_1$ over an octogonal prism with prism sides between edges having the same index, which are glued along this prism, and two adapters. For $\mathcal{Y}_i$ the octogon edges with index i and i' and the faces containing these do not form a tent side either apex vertex. This leaves the surface of the prism with two components. Each of these components form tent sides with only one of each of the apex vertices. The adapters $\mathcal{A}_F$ and $\mathcal{A}_G$ stelate the tent sides of $\mathcal{Y}_0$ containing the octogons.

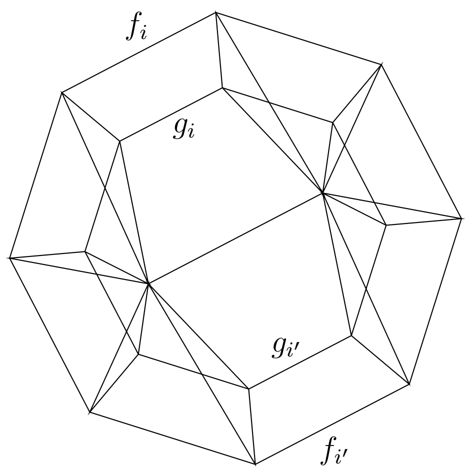


Figure A.1. Schlegel diagram for $\mathcal{Y}_i$.

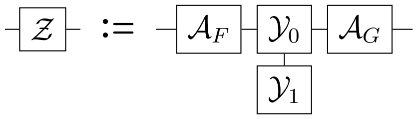
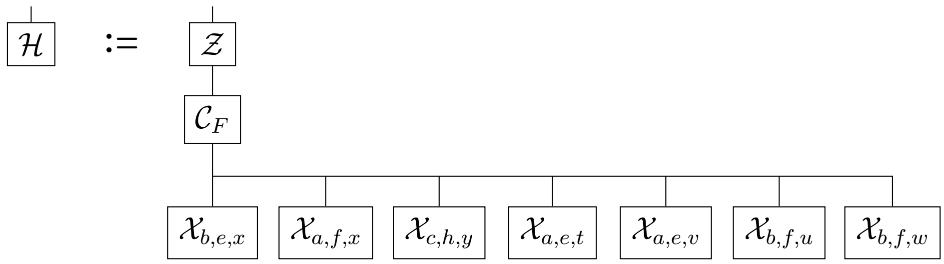


Figure A.2. Gluing diagram for a slope transmitter $\mathcal{Z}$.

Proof of Lemma A.2: We first observe that by lemma 4.5 the completion conditions of the octagonal faces of $\mathcal{Y}_i$ are that edge supporting lines with index i and i' all meet at a common point p_i. Since prisms and pyramids are necessarily flat every realization of $\mathcal{Z}$ must a union of realizations of $\mathcal{Y}_i$ and adapters. That is a pair of octagons F and G can be completed to $\mathcal{Z}$ if and only if they can be completed to an octagonal prism $\mathcal{P}$ that can be completed to $\mathcal{Y}_0$ and $\mathcal{Y}_1$. Since for such a prism P the points p_0 and p_1 are distinct and are both in both $\bigvee F$ and $\bigvee G$ we have $l := F \wedge G = p_0 \vee p_1$. If we further assume F is a computaional frame, then the meet of opposit pairs of F's edge supporting lines are all collinear and in particular are on the line l. By lemma 4.4 F and G can be completed to such a prism $\mathcal{P}$ if and only if the celestial spheres of F and G on l have the same combinatorics as their common refinement, which means they must actually be the same, and the resulting P can be completed to $\mathcal{Z}$ if and only if opposit pairs of G's edge supporting lines meet at the same points as the corresponding pairs of F's. This is equivalent to the requirement that G, considered up to projectivity, be a computational frame representing the same value as F. □

We are now ready to give the construction of $\mathcal{H}$. This consists of an connector with 8 octagonal sockets having a slope transmitter glued to one and 7 pencil polytopes glued to the others. As in lemma 5.6, we identify the connector's sockets by the projectivities between sockets it generates to a single octagon F, and label its vertices a through h so that $a = f_{-1} \cap f_{\infty'}$ and h is the other vertex of f_{-1}. For each pencil polytopes $\mathcal{X}_{i,j,k}$ with this labeling the two specified vertices are i and j and the specified edges are those defining k by their meet as listed in figure A.3 and shown in A.4.



$$x := (c \vee d) \wedge (g \vee h) \qquad t := (c \vee d) \wedge (f \vee g) \qquad u := (a \vee h) \wedge (c \vee d)$$

$$y := (a \vee b) \wedge (e \vee f) \qquad v := (b \vee c) \wedge (g \vee h) \qquad w := (d \vee e) \wedge (g \vee h)$$

Figure A.3. Gluing diagram for $\mathcal{H}$.

Lemma A.3. *Any octagon F with collinearities as indicated in figure A.4 is a computational frame representing the value* -1.

Proof: We embed F in $\mathbb{R}^2 \to \mathbb{RP}^2$ so that $\{a, b, e, f\}$ is symmetric about the x and y axis with $a \vee b$ vertical. That is so that $(a)_1 = (b)_1 = -(e)_1 = -(f)_1$ and $(a)_2 = -(b)_2 = -(e)_2 = (f)_2$. With this we have the slope r of $a \vee e$ is the negative of the slope $-r$ of $b \vee f$. The slope $s > r$ of $b \vee c$ determines the hight of the segment between c and h, since $(c)_2 = ((b \vee c) \wedge (c \vee h))_2$ and $(h)_2 = (v)_2 = ((b \vee c) \wedge (a \vee e))_2$, monotonicly in the following way.

$$(c)_2 - (h)_2 = s((c)_1 - (b)_1) + \tfrac{s}{s-r}((b)_2 - (a)_2)$$

Since $(h)_1 - (a)_1 = (c)_1 - (b)_1$ and the slope of $b \vee f$ is $-r$, we have that the slope of $b \vee c$ and the negative slope of $a \vee h$ determine the hight of the segment monotonicly in the same way, and since this can have only one value the slope of $a \vee h$ must be $-s$. This means that $(c)_2 = -(h)_2$, which gives us that $t = -v$, and since we also have $f = -b$, this gives us that the slope of $f \vee t = f \vee g$ is s. Likewise we have that the slope of $e \vee d$ is $-s$. The result of this is that opposit edges of F are parallel so x, y, $p = (a \vee h) \wedge (e \vee d)$, and $q = (b \vee c) \wedge (f \vee g)$ are all collinear and with these slopes $(p, q \,|\, x, y) = -\frac{s}{s} = -1$ as desired. $\square$

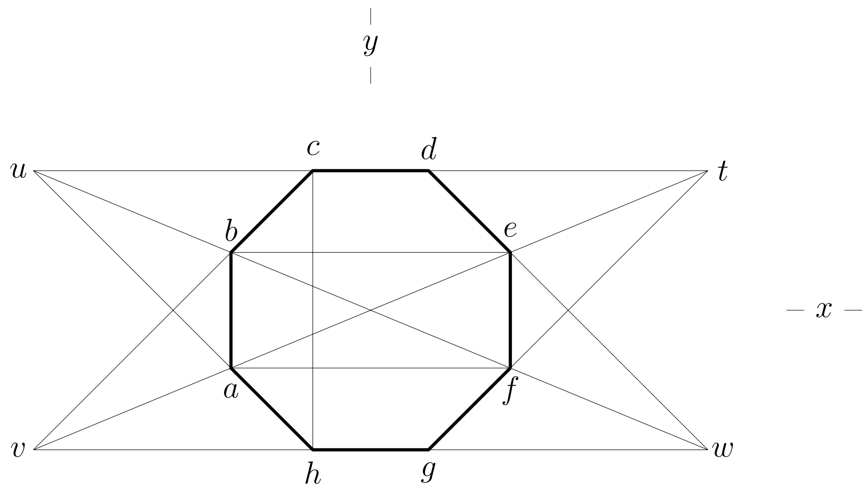


Figure A.4. Collinearities of F. Horizontal and vertical lines meet at the points x and y respectively.

Proof of Lemma A.1: By lemma A.3 the collinearities imposed on F by the pencil polytopes force F to be a computational frame representing -1, so by lemma A.2 the completion conditions for the specified face G of $\mathcal{H}$ are that it be a computational frame representing -1. □

We are now ready to begin giving the constructions for the basic arithmetic polytopes. These will be composed to various polytopes glued together along pyramids over polygons G that in all realizations will be computational frames. We will label opposit pairs of edges by the values i they are ment to represent. In any realization of G we let p_i denote the meet of opposite edge supporting lines labeled i, and $\alpha_i = (p_i, p_1 \,|\, p_0, p_\infty)$ for squaring and multiplying polytopes, and $\alpha_i = (p_i, p_x \,|\, p_0, p_\infty)$ for doubling and adding polytopes, which will amount to letting $\alpha_i = \frac{i}{x}$, since these do not include a multiplicative identity. Not all of these computational frames will reprsent the same set of values, so we indicate the values represented with subscripts. In this way $\mathcal{C}_{c_1,\cdots,c_n}$ denotes a connector with n-gonal sockets and, for any realization as a component of the polytope it belons to, these sockets will be computational frames representing $c_1, \cdots, c_n$. Similarly $\mathcal{H}_{h_1,h_2,h_3,h_4}$ will impose the relation $(h_1, h_3 \,|\, h_2, h_4) = -1$ on the values represented. For forgetful transmitters we inicate the value represented by one socket but not the other with subscripts. We only provide the gluing diagrams for these polytopes since this is enough to fully specify the combinatorics. As we have mentioned, we already have $\mathcal{P}^{2x}$ as $\mathcal{H}$ with the specified octagon's edges labeled apropriately.

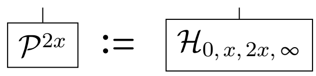


Figure A.5. $\mathcal{P}^{2x}$ as $\mathcal{H}$ with labels.

Theorem A.4. *The completion condition of the specified facet* pyr(G) *of* $\mathcal{P}^{2x}$ *is that* G *be a computational frame representing the values* $\{0,\ 1,\ 2,\ \infty\}$.

Proof: Recall we divide by x so $\alpha_x = 1$. Computational frames representing 2 and computational frames representing -1 are the same up to a relabeling of edges, since adding 1 is a projectivity on $\mathbb{RP}^1$ and sends $\{-1, 0, 1, \infty\}$ to $\{0, 1, 2, \infty\}$ giving us the relabeling, so by lemma A.1 we are done. $\square$

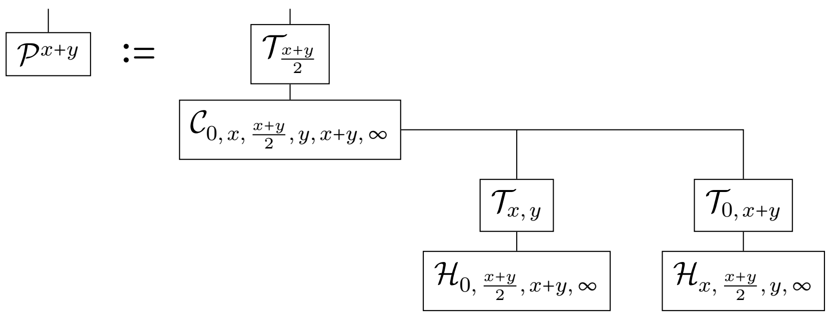


Figure A.6. Gluing diagram of $\mathcal{P}^{x+y}$.

Theorem A.5. *The completion condition of the specified facet* pyr(G) *of* $\mathcal{P}^{x+y}$ *is that* G *be a computational frame representing the values* $\{0,\ 1,\ z,\ 1+z,\ \infty\}$ *for some* $1 < z$.

Proof: We first observe that the completion conditions for $\mathcal{H}_{0,\frac{x+y}{2},x+y,\infty}$ and $\mathcal{H}_{x,\frac{x+y}{2},y,\infty}$ respectively require that p_0, p_{x+y} and p_x, p_y be in the line $p_{\frac{x+y}{2}} \vee p_\infty$, so the requirement that G be a computational frame must be among its completion conditions. Recall we divide x, y, and $x+y$ by x to get 1 represented by α_x in this computational frame, and use $z = \frac{y}{x} = \alpha_y$. With this $\mathcal{H}_{x,\frac{x+y}{2},y,\infty}$ imposes the following relation,

$$\frac{\alpha_x - \alpha_{\frac{x+y}{2}}}{\alpha_y - \alpha_{\frac{x+y}{2}}} = (\alpha_x, \alpha_y \,|\, \alpha_{\frac{x+y}{2}}, \infty) = -1$$

which is equivalent to $\alpha_{\frac{x+y}{2}} = (\alpha_x + \alpha_y)/2$. As we have seen $\mathcal{H}_{0,\frac{x+y}{2},x+y,\infty}$ is a doubling polytope so this imposes the relation $\alpha_{x+y} = 2\alpha_{\frac{x+y}{2}}$. Returning to our choice of values for these variables, this gives us $\alpha_{x+y} = \alpha_x + \alpha_y = 1 + z$. Moreover, this exhausts the completion conditions of the restricting polytopes, so we have the completion conditions for $\mathcal{P}^{x+y}$. □

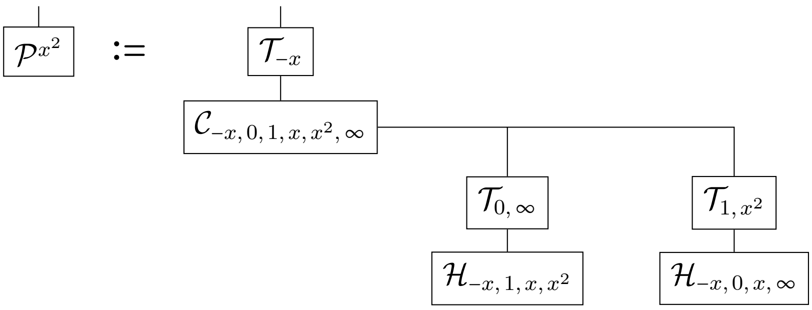


Figure A.7. Gluing diagram of $\mathcal{P}^{x^2}$.

Theorem A.6. *The completion condition of the specified facet* pyr(G) *of* $\mathcal{P}^{x^2}$ *is that G be a computational frame representing the values* $\{0,\ 1,\ x,\ x^2,\ \infty\}$ *for some* $1 < x$.

Proof: As with $\mathcal{P}^{x+y}$, the completion conditions for $\mathcal{H}_{-x,1,x,x^2}$ and $\mathcal{H}_{-x,0,x,\infty}$ respectively require that p_1, p_{x^2} and p_0, p_∞ be in the line $p_{-x} \vee p_x$, so G being a computational frame must be among its completion conditions. As we have seen the completion conditions of $\mathcal{H}_{-x,0,x,\infty}$ require that $\alpha_{-x} = -\alpha_x$. With this $\mathcal{H}_{-x,1,x,x^2}$ imposes the following relation,

$$\frac{(-\alpha_x - 1)(\alpha_x - \alpha_{x^2})}{(-\alpha_x - \alpha_{x^2})(\alpha_x - 1)} = (-x, x \,|\, 1, x^2) = -1$$

which is equivalent to $\alpha_{x^2} = \alpha_x^2$, and since this exhausts the completion conditions of the restricting polytopes, so we have the completion conditions for $\mathcal{P}^{x^2}$. □

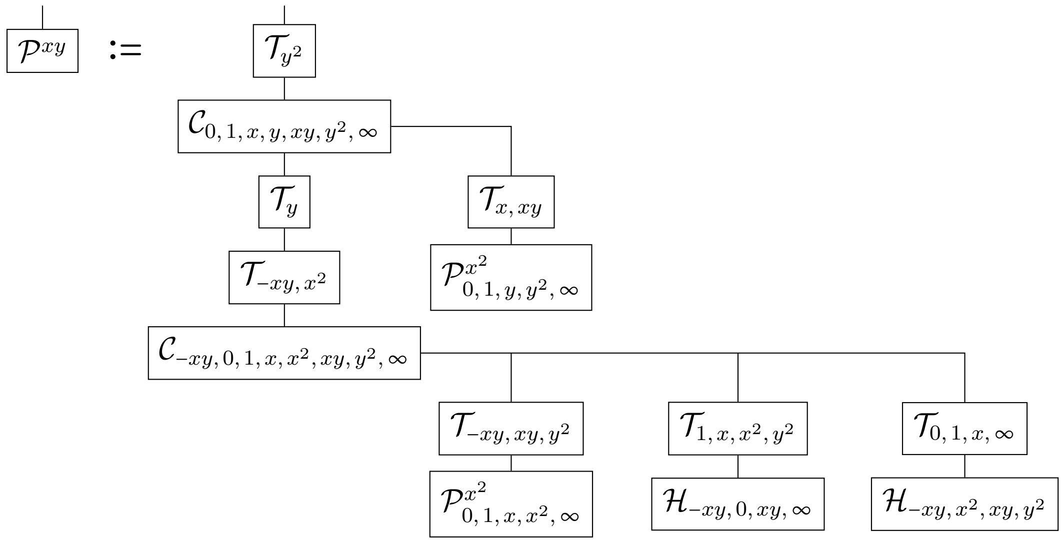


Figure A.8. Gluing diagram of $\mathcal{P}^{xy}$.

Theorem A.7. *The completion condition of the specified facet* $\mathrm{pyr}(G)$ *of* $\mathcal{P}^{xy}$ *is that* G *be a computational frame representing the values* $\{0,\ 1,\ x,\ y,\ xy,\ \infty\}$ *for some* $1 < x < y$.

Proof: The completion conditions for $\mathcal{P}^{x^2}_{0,1,y,y^2,\infty}$ and $\mathcal{P}^{x^2}_{0,1,x,x^2,\infty}$ and $\mathcal{H}_{-xy,0,xy,\infty}$ respectively require that p_1, p_y, p_{y^2} and p_x, p_{x^2} and p_{-xy}, p_{xy} be in the line $p_0 \vee p_\infty$, so G being a computational frame must be among its completion conditions. The completion conditions of each these also respectively require that $\alpha_{y^2} = \alpha_y^2$ and $\alpha_{x^2} = \alpha_x^2$ and $\alpha_{-xy} = -\alpha_{xy}$. With this $\mathcal{H}_{-xy,\,x^2,\,xy,\,y^2}$ imposes the following relation,

$$\frac{(-\alpha_{xy} - \alpha_x^2)(\alpha_{xy} - \alpha_y^2)}{(-\alpha_{xy} - \alpha_y^2)(\alpha_{xy} - \alpha_x^2)} = (\alpha_{-xy}, \alpha_{xy} \,|\, \alpha_{x^2}, \alpha_{y^2}) = -1$$

which is equivalent to $\alpha_{xy}^2 = \alpha_x^2 \alpha_y^2$, and since we have $0 < \alpha_{xy}$ this gives us $\alpha_{xy} = \alpha_x \alpha_y$. This exhausts the completion conditions of the restricting polytopes, so we have the completion conditions for $\mathcal{P}^{xy}$. □