A MEAN VALUE THEOREM FOR DISCRIMINANTS OF ABELIAN EXTENSIONS OF AN ALGEBRAIC NUMBER FIELD

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ABSTRACT

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Let k be an algebraic number field and let $N(k, C_\ell, m)$ denote the number of abelian extensions K of k with $Gal(K/k) \cong C_\ell$, the cyclic group of prime order ℓ, and the relative discriminant $\mathscr{D}(K/k)$ of norm equal to m. In this thesis, we derive an asymptotic formula for $\sum_{m \leq X} N(k, C_\ell, m)$, using the class field theory and a method, developed by Wright [13]. We show that our result is identical to the result of Cohen, Diaz y Diaz and Olivier [1], obtained by the methods of classical algebraic number theory, although our methods allow for a more elegant treatment and reduce a global calculation to a series of local calculations.

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to my parents

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with all my love and gratitude.

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CHAPTER 1

INTRODUCTION AND THE MAIN RESULT

1.1 Motivation and objectives

The principal objective of this thesis is to address the problem of counting the number of finite extensions of an algebraic number field. More specifically, given an algebraic number field k and a finite abelian group G, we would like to count the number $N(k, G; m)$ of Galois extensions K of k with Galois group $Gal(K/k)$ isomorphic to G and the relative discriminant $\mathscr{D}(K/k)$ of absolute norm m. As usual in number theory, we will not be able to pinpoint individual values of $N(k, G; m)$. However, we will be able to determine how large $N(k, G\,; m)$ is on the average. The latter is equivalent to finding an asymptotic formula for $\sum_{m \leq X} N(k, G\,; m)$, the explicit computation of which is given in this thesis.

The study of discriminants of algebraic number fields goes back to Dedekind and Hermite. Hermite was the first to show that the number of extensions K of k with discriminant of a given norm is finite. A breakthrough in the study of the density of discriminants of abelian Galois extensions occurred with the publication of Hasse's Conductor-Discriminant formula [4] (a short proof can be found in [9]). With the help of this formula, one can express the discriminant of an abelian extension in terms of conductors of associated characters. The earliest papers, such as [3] and [11], that gave asymptotics for $k = \mathbb{Q}$ and a cyclic group $G = \mathbb{Z}/\ell\mathbb{Z}$ of prime order ℓ, appeared in the early 1950s. More recently, Mäki [6] gave asymptotics for $k = \mathbb{Q}$ and arbitrary G. The underlying principle invoked in his work is that abelian number fields of absolute conductor f are contained in the field generated by the fth roots of unity, and that the conductor is the smallest such integer. In [10], Taylor produced partial results for an arbitrary number field k by computing the density of conductors of cyclic extensions of a number field. An extensive list of references to related works can be found in [8].

Using class field theory, Wright [13] wrote the definitive paper on this subject. He proved that there exists a positive constant $c(k, G)$ such that

$$\sum_{m \leq X} N(k, G\,; m) \sim \frac{c(k, G)}{(\nu - 1)!} X^{1/\alpha} (\log X)^{\nu - 1} \qquad \text{as} \quad X \to \infty.$$

Here $\alpha = \alpha(G) = |G| \left(1 - \dfrac{1}{Q}\right)$ where Q is the smallest prime divisor of the order of G, and $\nu = \dfrac{\Phi_Q(G)}{d_k}$ where $\Phi_Q(G)$ is the number of elements of G of order Q and $d_k = [k(\zeta_Q) : k]$, where $k(\zeta_Q)$ is the field obtained by adjoining the primitive Qth root of unity ζ_Q to k. (Un)fortunately the work of Wright was so general that he neglected to find the constant $c(k, G)$. Recently, Cohen, Diaz y Diaz and Oliver [1] determined this constant for $G = \mathbb{Z}/\ell\mathbb{Z}$ using classical algebraic number theory. Their methods are entirely global; no class field theory is used in their paper. To state their result, we first introduce the following notations.

Let $k_z = k(\zeta)$ be the field obtained by adjoining the primitive ℓ^{th} root of unity ζ to k. Notice that the extension k_z/k is a cyclic extension whose degree is some divisor d_z of $\ell - 1$. For a detailed study of such cyclotomic extensions, we direct the reader to [2]. For notational simplicity, set $q_z = (\ell - 1)/d_z$. For every divisor d of d_z, let $k_z[d]$ be the unique subextension of k_z/k such that $[k_z : k_z[d]] = d$. If $\mathfrak{p}$ is a prime ideal of k, we denote by $e(\mathfrak{p}_d/\mathfrak{p})$, $f(\mathfrak{p}_d/\mathfrak{p})$, and $g(\mathfrak{p}_d/\mathfrak{p})$ the ramification index, residual index, and the number of prime ideals $\mathfrak{p}_d$ of $k_z[d]$ above $\mathfrak{p}$, respectively. Note that $e(\mathfrak{p}_d/\mathfrak{p})f(\mathfrak{p}_d/\mathfrak{p})g(\mathfrak{p}_d/\mathfrak{p}) = d_z/d$. If, in addition, $\mathfrak{p}|\ell$, we denote by $e(\mathfrak{p}) = e(\mathfrak{p}/\ell)$ the absolute ramification index of $\mathfrak{p}$ over ℓ. Finally, for any integer e, we denote by $r(e)$ the unique integer such that $e \equiv r(e)\, mod\, (\ell - 1)$ with $1 \leq r(e) \leq \ell - 1$.

Theorem 1.1 *Let k be a number field of signature (r_1, r_2). Let $\mathscr{R}$ (resp., $\mathscr{D}$) be the set of prime ideals of k which are ramified (resp., totally split) in k_z/k. Then*

$$\sum_{m \leq X^{\ell-1}} N(k, C_\ell; m) \sim c_1 c_2 c_3 c_4 X \log^{q_z - 1} X$$

with

$$\begin{aligned}
c_1 &= \frac{\left(\prod_{d|d_z} \zeta_{k_z[d]}(d)^{\mu(d)}\right)^{q_z}}{d_z \ell^{r_2 + r_z} q_z!}, \\
c_2 &= \prod_{\mathfrak{p} \in \mathscr{D}} \left(\left(1 + \frac{\ell - 1}{N\mathfrak{p}}\right) \prod_{d|d_z} \left(1 - \frac{1}{N\mathfrak{p}^d}\right)^{(\ell-1)\mu(d)/d} \right), \\
c_3 &= \left(\prod_{\mathfrak{p} \in \mathscr{R}} \prod_{d|d_z} \left(1 - \frac{1}{N\mathfrak{p}^{df(\mathfrak{p}_d/\mathfrak{p})}}\right)^{g(\mathfrak{p}_d/\mathfrak{p})\mu(d)} \right)^{q_z}, \\
c_4 &= \prod_{\substack{\mathfrak{p}|\ell \\ \mathfrak{p} \notin \mathscr{D}}} \left(1 + \frac{\ell - 1}{N\mathfrak{p}} - \frac{\ell - 1 - r(e(\mathfrak{p}))(1 - 1/N\mathfrak{p})}{N\mathfrak{p}^{\lceil e(\mathfrak{p})/(\ell-1) \rceil}}\right).
\end{aligned}$$

Here, $r_z = 0$ if $\zeta \in k$, and $r_z = r_1 - 1$ otherwise. By abuse of notation, for any number field L we write $\zeta_L(1)$ for the residue of the Dedekind zeta function $\zeta_L(s)$ at $s = 1$.

In this thesis, we replicate the result given in [1] using class field theory following the method of Wright's paper [13]. We use the language of places rather than prime ideals to state our theorem. Consequently, let ν be a finite place of k corresponding to the prime ideal $\mathfrak{p}_\nu$ of $\mathcal{O}_k$, and let $q_\nu = N\mathfrak{p}_\nu$. Let $e_\nu = e(\mathfrak{p}_{\nu_1}/\mathfrak{p}_\nu)$, $f_\nu = f(\mathfrak{p}_{\nu_1}/\mathfrak{p}_\nu)$, $g_\nu = g(\mathfrak{p}_{\nu_1}/\mathfrak{p}_\nu)$ be the ramification index, the residual index, and the number of prime ideals dividing $\mathfrak{p}_\nu$ in the cyclotomic extension k_z of k, respectively. We prove the following theorem.

Theorem 1.2 *Let k be a number field of signature (r_1, r_2). For a place ν of k that divides ℓ, let $e(\nu)$ be the ramification index of ν over ℓ, and let $r_0(\nu)$ be the least nonnegative residue of $e(\nu)$ mod $(\ell-1)$. Then*

$$\sum_{m \le X^{\ell-1}} N(k, C_\ell; m) \sim \frac{\zeta_{k_z}(1)^{q_z}}{d_z \ell^{r_2+r_z} q_z!} P_0 X (\log X)^{q_z - 1}$$

with

$$P_0 = \prod_{\substack{\nu|\ell \\ \nu\in\mathscr{D}}} \left(1+(\ell-1)q_\nu^{-1}\right) \prod_{\substack{\nu|\ell \\ \nu\notin\mathscr{D}}} \left(1+(\ell-1)q_\nu^{-1} - \left[\ell-1-r_0(\nu)(1-q_\nu^{-1})\right] q_\nu^{-\lfloor\frac{e(\nu)}{\ell-1}\rfloor - 1}\right)$$
$$\prod_{q_\nu \equiv 1\, mod\, \ell} \left(1+(\ell-1)q_\nu^{-1}\right)\left(1-q_\nu^{-1}\right)^{\ell-1} \prod_{q_\nu \not\equiv 1\, mod\, \ell} \left(1-q_\nu^{-f_\nu}\right)^{q_z g_\nu}$$

where $r_z = 0$ if $\zeta \in k$ and $r_z = r_1 - 1$ otherwise, and $\zeta_{k_z}(1)$ denotes the residue of the Dedekind zeta function $\zeta_{k_z}(s)$ at $s = 1$.

We prove the above theorem by studying the discriminant series

$$D_{C_\ell}(s) = \sum_{Gal(K/k)\cong C_\ell} |\mathscr{D}(K/k)|^{-s}.$$

Using class field theory, we express this series as a finite linear combination of series with Euler products whose Euler factors can be explicitly computed. Following Wright, we study analytic properties of the Euler products by comparing them with appropriate Dedekind zeta functions. As a result, $D_{C_\ell}(s)$ is proved to be analytic in the region $\mathrm{Re}(s) > \dfrac{1}{2(\ell-1)}$ except for a pole at

$s = \dfrac{1}{\ell - 1}$ of multiplicity q_z. Evaluation of the leading coefficient of the Laurent series at this pole leads us to the required constant.

Finally we show that even though our constant appears to be different, it is actually the same as that of Cohen et.al. Thus the methods of Wright and Cohen et. al. produce identical results. The advantage of our method, besides being somewhat simpler than that of Cohen et. al., is that it is very much a method in the spirit of modern algebraic number theory : it reduces a global calculation to a series of local calculations.

1.2 What is class field theory?

The class field theory is one of the crowning achievements of twentieth century number theory. This theory was born in the late nineteenth century in the work of Leopold Kronecker, Henry Weber, and David Hilbert, who together laid out a research program that became the main thrust of algebraic number theory through the 1930s. It relates the arithmetic of an abelian Galois extension of an algebraic number field to the arithmetic of the field itself. Claude Chevalley summarized the main goal:

> The object of class field theory is to show how abelian extensions of an algebraic number field k can be determined by objects taken from our knowledge of k itself; or, if one wishes to present things in dialectical terms, how a field contains within itself the elements of its own surpassing (and this without any internal contradictions).

1.3 Notations

Throughout this thesis, k denotes an algebraic number field and $M(k)$ the set of all places (equivalence classes of absolute values) of k. For $\nu \in M(k)$, let k_ν be the completion of k at ν, $\mathcal{O}_\nu$ the ring of integers in k_ν , $\mathcal{O}_\nu^*$ the group of units in $\mathcal{O}_\nu$, π_ν a fixed uniformizer of the prime ideal of $\mathcal{O}_\nu$, q_ν the number of elements in the residue field $\mathbb{F}_\nu = \mathcal{O}_\nu/\pi_\nu\mathcal{O}_\nu$, and $|\cdot|_\nu$ the absolute value of k_ν normalized so that $|\pi_\nu|_\nu = q_\nu^{-1}$. For any place ν of $M(k)_0$ (the set of finite places of k), let $\mathfrak{p}_\nu$ be the prime ideal of $\mathcal{O}_k$ corresponding to ν. Let $e_\nu = e(\mathfrak{p}_z/\mathfrak{p}_\nu)$, $f_\nu = f(\mathfrak{p}_z/\mathfrak{p}_\nu)$, and $g_\nu = g(\mathfrak{p}_z/\mathfrak{p}_\nu)$ be the ramification index, residual index, and the number of prime ideals $\mathfrak{p}_z$ of k_z above $\mathfrak{p}_\nu$ respectively. Note that $e_\nu f_\nu g_\nu = d_z$.

Let $\mathbf{A}^* = \prod'_{\nu\in M(k)} k_\nu^*$ denote the group of ideles of k. $\prod'$ here means that if $x = (x_\nu) \in \prod'_{\nu\in M(k)} k_\nu^*$ is an element of $\mathbf{A}^*$, then $x_\nu \in \mathcal{O}_\nu^*$ for all but finitely many ν. Endowed with the restricted product topology, $\mathbf{A}^*$ becomes a locally compact topological group. We denote by $|\cdot|_\mathbf{A}$ the idele norm on $\mathbf{A}^*$ given by

$$|x|_\mathbf{A} = \prod_\nu |x_\nu|_\nu$$

for $x = (x_\nu) \in \mathbf{A}^*$. Set $\mathbf{A}^1 = \{x \in \mathbf{A}^* : |x|_\mathbf{A} = 1\}$. k^* can be embedded into $\mathbf{A}^*$ by means of the diagonal embedding: $a \in k^*$ to $(a)_{\nu\in M(k)}$. Then by the product formula $k^* \subset \mathbf{A}^1$. Moreover, $\mathbf{A}^1/k^*$ is compact. We call $\mathbf{A}^*/k^*$ the idele class group of k.

For a ring R, we denote by R^* the group of units of R. Also, for any positive integer n, R^n shall always denote the group of n^{th} powers of units of R.

Let S be the set of infinite places of k. The group of $S-$ideles of k is the following direct product with the usual restricted product topology:

$$\mathbf{A}^*(S) = \prod_{\nu\in S} k_\nu^* \times \prod_{\nu\notin S} \mathcal{O}_\nu^*.$$

Since $\mathcal{O}_\nu^*$ is compact, the $S-$ideles form a locally compact abelian group under multiplication.

For a locally compact abelian group X, a character on X is a continuous homomorphism $\chi : X \mapsto S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Let $\nu \in M(k)_0$ and χ_ν be a character on k_ν^*. Then $\ker\left(\chi_\nu|_{\mathcal{O}_\nu^*}\right)$ is an open subgroup of $\mathcal{O}_\nu^*$. Consequently, there exists a positive integer n such that $\chi_\nu = 1$ on $1 + \pi_\nu^n \mathcal{O}_\nu$. If n_ν is the smallest such integer, we define the conductor $\Phi(\chi_\nu)$ of χ_ν as follows:

$$\Phi(\chi_\nu) = \begin{cases} (\pi_\nu^{n_\nu}) & \text{if } \chi_\nu \neq 1 \quad \text{on} \quad \mathcal{O}_\nu^* \\ (1) & \text{if } \chi_\nu = 1 \quad \text{on} \quad \mathcal{O}_\nu^*. \end{cases}$$

Here we think of $\Phi(\chi_\nu)$ as an ideal of $\mathcal{O}_\nu$.

Now, let χ be a character on $\mathbf{A}^*$. Then $\chi = \prod_{\nu \in M(k)} \chi_\nu$ where χ_ν are characters on k_ν^* such that for all but finitely many ν, $\chi_\nu|_{\mathcal{O}_\nu^*} = 1$. We set the conductor of χ

$$\Phi(\chi) = \prod_{\nu \in M(k)_0} \Phi(\chi_\nu).$$

Alternatively, we can write

$$\Phi(\chi) = \prod_{\nu \in M(k)} \Phi(\chi_\nu)$$

where by convention $\Phi(\chi_\nu) = (1)$ if $\nu \in S$.

Finally, for a character $\chi_S = \prod_{\nu \in M(k)_0} \chi_\nu$ of $\mathbf{A}^*(S)$, define

$$\Phi(\chi_S) = \prod_{\nu \in M(k)_0} \Phi(\chi_\nu).$$

CHAPTER 2

DECOMPOSITION OF THE DISCRIMINANT SERIES

2.1 The Conductor and Discriminant Series

In this section, we introduce the Dirichlet series that counts the discriminants of abelian extensions of a number field with the Galois group isomorphic to a cyclic group of prime order ℓ. We use Hasse's Conductor-Discriminant formula [4] to express this series in terms of series counting conductors of characters. The main result of this chapter is (2.4).

By the class field theory, there is a one-to-one correspondence between finite abelian extensions of k and open subgroups of $\mathbf{A}^*$ containing k^*. Moreover, if K is an abelian extension of k such that

$$K \leftrightarrow \mathcal{U}_K \subset \mathbf{A}^*, \text{ then } Gal(K/k) \cong \mathbf{A}^*/\mathcal{U}_K.$$

Now consider the dual group $\widehat{\mathbf{A}^*/\mathcal{U}_K}$ of $\mathbf{A}^*/\mathcal{U}_K$, that is, the group of all characters χ on $\mathbf{A}^*$ such that $\chi|_{\mathcal{U}_K} = 1$. The Hasse Conductor-Discriminant formula asserts that

$$\mathscr{D}(K/k) = \prod_{\chi \in \widehat{\mathbf{A}^*/\mathcal{U}_K}} \Phi(\chi).$$

Let C_ℓ denote the cyclic group of order ℓ. An abelian extension K of k with $Gal(K/k) \cong C_\ell$ corresponds to an open subgroup $\mathcal{U}_K \subset \mathbf{A}^*$ such that $\mathbf{A}^*/\mathcal{U}_K \cong C_\ell$. Let χ be a character on $\mathbf{A}^*/k^*$. Define $\mathcal{U}_\chi = \ker \chi$. Then $\chi \in \widehat{\mathbf{A}^*/\mathcal{U}_K}$ if and only if $\mathcal{U}_K \subset \mathcal{U}_\chi$. Thus

$$\mathbf{A}^*/\mathcal{U}_\chi = \big(\mathbf{A}^*/\mathcal{U}_K\big)/\big(\mathcal{U}_\chi/\mathcal{U}_K\big)$$

is a factor group of $\mathbf{A}^*/\mathcal{U}_K$. But $\mathbf{A}^*/\mathcal{U}_K$ is a cyclic group of order ℓ. Therefore $\mathcal{U}_\chi = \mathcal{U}_K$ or $\mathcal{U}_\chi = \mathbf{A}^*$. The former case occurs if $\chi \neq 1$ and the latter if $\chi = 1$.

$\widehat{\mathbf{A}^*/\mathcal{U}_K}$ is also a cyclic group of order ℓ. Therefore $\widehat{\mathbf{A}^*/\mathcal{U}_K} =< \chi >$ for some character χ on $\mathbf{A}^*/k^*$ such that $\chi^\ell = 1$. Then

$$\mathscr{D}(K/k) = \prod_{i=0}^{\ell-1} \Phi(\chi^i). \tag{2.1}$$

Note that the choice of generator χ of $\widehat{\mathbf{A}^*/\mathcal{U}_K}$ is not unique. In fact $\widehat{\mathbf{A}^*/\mathcal{U}_K} = < \chi^i >$ for any $1 \leq i \leq \ell - 1$.

Definition 2.1 *For a locally compact group X, let $C_\ell(X)$ denote the group of all continuous characters χ on X such that $\chi^\ell = 1$. Define the discriminant series $D_{C_\ell}(s)$, $s \in \mathbb{C}$, and the conductor series $F_{C_\ell}(s)$ as follows:*

$$D_{C_\ell}(s) = \sum_{Gal(K/k) \cong C_\ell} |\mathscr{D}(K/k)|^{-s} \tag{2.2}$$

and

$$F_{C_\ell}(s) = \sum_{\chi \in C_\ell(\boldsymbol{A}^*/k^*)} |\Phi_{C_\ell}(\chi)|^{-s}, \tag{2.3}$$

where

$$\Phi_{C_\ell}(\chi) = \prod_{i=1}^{\ell-1} \Phi(\chi^i).$$

The previous discussion implies the following identity relating the conductor and the discriminant series:

$$F_{C_\ell}(s) = 1 + (\ell - 1)D_{C_\ell}(s). \tag{2.4}$$

The next sections will be devoted to the decomposition of $F_{C_\ell}(s)$ into a finite linear combination of series that have Euler products.

2.2 Characters on $\mathbf{A}^*(S)$

Let S be the set of all infinite places of k. Recall that the group of S-ideles is $\mathbf{A}^*(\mathrm{S}) = \prod_{\nu \in S} k_\nu^* \times \prod_{\nu \notin S} \mathcal{O}_\nu^*$. If $\chi \in C_\ell(\mathbf{A}^*/k^*)$, then the restriction $\chi|_{\mathbf{A}^*(S)}$ defines a continuous character χ_S on $\mathbf{A}^*(S)$ such that $\chi_S^\ell = 1$. In this section, we will take a closer look at these characters. This in turn will allow us to write $F_{C_\ell}(s)$ as a linear combination of Dirichlet series that have Euler product decompositions.

To begin with, let us investigate the following questions:

- Which characters χ_S on $\mathbf{A}^*(S)$ that satisfy $\chi_S^\ell = 1$ come from the characters of $\mathbf{A}^*/k^*$ with the same property ?

- How many characters on $\mathbf{A}^*/k^*$ of order ℓ induce the same character on $\mathbf{A}^*(\mathrm{S})$?

Proposition 2.1 *A character $\chi_S \in C_\ell(\boldsymbol{A}^*(S))$ is induced by a character $\chi \in C_\ell(\boldsymbol{A}^*/k^*)$ if and only if $\chi_S = 1$ on $\boldsymbol{A}^\ell k^* \cap \boldsymbol{A}^*(S)$.*

Proof: Suppose $\chi|_{\mathbf{A}^*(S)} = \chi_S$ where χ is a character on $\mathbf{A}^*/k^*$ and $\chi^\ell = 1$. Then $\chi = 1$ on $\mathbf{A}^\ell k^*$, and therefore $\chi_S = 1$ on $\mathbf{A}^\ell k^* \cap \mathbf{A}^*(S)$.

Conversely, suppose $\chi_S = 1$ on $\mathbf{A}^\ell k^* \cap \mathbf{A}^*(S)$. Observe that $\ker \chi_S$ is an open subgroup of $\mathbf{A}^*(S)$ (because χ_S is continuous) and therefore an open subgroup of $\mathbf{A}^*$. Now consider the following map:

$$\mathbf{A}^*(S) \hookrightarrow \mathbf{A}^* \mapsto \mathbf{A}^*/\mathbf{A}^\ell k^* \ker \chi_S.$$

Since $\ker \chi_S \subset \mathbf{A}^*(S)$ and $\ker \chi_S$ contains $\mathbf{A}^\ell k^* \cap \mathbf{A}^*(S)$, $\mathbf{A}^\ell k^* \ker \chi_S \cap \mathbf{A}^*(S) = (\mathbf{A}^\ell k^* \cap \mathbf{A}^*(S)) \ker \chi_S = \ker \chi_S$. Then the map

$$\mathbf{A}^*(S) \mapsto \mathbf{A}^*/\mathbf{A}^\ell k^* \ker \chi_S$$

factors through a map

$$\mathbf{A}^*(S)/\ker \chi_S \mapsto \mathbf{A}^*/\mathbf{A}^\ell k^* \ker \chi_S.$$

χ_S can be naturally thought of as a character on the finite group $\mathbf{A}^*(S)/\ker \chi_S$. Any character on $\mathbf{A}^*/\mathbf{A}^\ell k^* \ker \chi_S$ corresponds to a character χ on $\mathbf{A}^*/k^*$ such that $\chi^\ell = 1$. Now let $H = \mathbf{A}^*(S)/ker\chi_S$ and $G = \mathbf{A}^*/\mathbf{A}^\ell k^* ker\chi_S$. Since $\ker \chi_S$ is open in $\mathbf{A}^*(S)$, both G and H are finite and $H \mapsto G$ is an embedding. But for finite groups, every character of a subgroup H of G is induced by a character of G. Hence there exists a character χ on $\mathbf{A}^*/k^*$ such that $\chi^\ell = 1$ and $\chi|_{\mathbf{A}^*(S)} = \chi_S$. This concludes the proof of the proposition.

The next question we want to answer is how many characters on $\mathbf{A}^*/k^*$ of order ℓ induce the same character on $\mathbf{A}^*(S)$. First note that the map

$$\phi : C_\ell(\mathbf{A}^*/k^*) \mapsto C_\ell(\mathbf{A}^*(S)),$$

given by $\phi(\chi) = \chi|_{\mathbf{A}^*(S)}$, is a homomorphism. $\ker \phi$ consists of all characters χ such that $\chi = 1$ on $\mathbf{A}^*(S)$. Since $\chi = 1$ on $\mathbf{A}^\ell k^*$, $\chi = 1$ on $\mathbf{A}^\ell k^* \mathbf{A}^*(S)$, that is, χ can be viewed as a character on $\mathbf{A}^*/\mathbf{A}^\ell k^* \mathbf{A}^*(S)$.

Let $H = \mathbf{A}^*/k^*\mathbf{A}^*(S)$. It is well-known that $H \cong C_k$, the ideal class group of k. Observe that $H^\ell = \mathbf{A}^\ell k^* \mathbf{A}^*(S)/k^*\mathbf{A}^*(S)$. Hence, $\ker\phi$ is the dual of H/H^ℓ. In particular, $|\ker\phi| = |H/H^\ell| = h_{\ell,k}$, where $h_{\ell,k}$ is the number of elements $\bar{\mathcal{M}} \in C_k$ such that $\bar{\mathcal{M}}^\ell = 1$. Therefore, the map $\chi \mapsto \chi_S$ is $h_{\ell,k} - to - 1$.

The map ϕ is not onto. We would like to be able to pinpoint those characters in $C_\ell(\mathbf{A}^*(S))$ that lie in the image of ϕ. Recall that $\chi_S \in C_\ell(A^*(S))$ is $\phi(\chi)$ for some $\chi \in C_\ell(A^*/k^*)$ if and only if $\chi_S = 1$ on $\mathbf{A}^\ell k^* \cap \mathbf{A}^*(S)$.

Define

$$\mathscr{A}_\ell(S) = (\mathbf{A}^\ell k^* \cap \mathbf{A}^*(S))/\mathbf{A}^\ell(S).$$

The next proposition shows that $\mathscr{A}_\ell(S)$ is finite. More precisely, we have

Proposition 2.2 $|\mathscr{A}_\ell(S)| = h_{\ell,k}|\mathcal{O}_k^*/\mathcal{O}_k^\ell|$ *where* $h_{\ell,k} = |\{\bar{\mathcal{M}} \in C_k : \bar{\mathcal{M}}^\ell = 1\}|$.

Proof: First note that the map

$$\varphi : (a_\nu)_\nu \mapsto \prod_{\nu \in M(k)_0} \mathfrak{p}_\nu^{ord_\nu a_\nu}$$

is a homomorphism from $\mathbf{A}^*$ to $I(k)$ where $I(k)$ is the group of fractional ideals of k. This map, composed with the projection

$$I(k) \mapsto I(k)/P(k) = C_k,$$

where $P(k)$ is a group of principal fractional ideals, induces an isomorphism from $\mathbf{A}^*/k^*\mathbf{A}^*(S)$ to the ideal class group C_k of k. Therefore $\mathbf{A}^*$ can be written as disjoint union of the sets $a_i k^* \mathbf{A}^*(S)$, namely,

$$\mathbf{A}^* = \cup_{i=1}^{h_k} a_i k^* \mathbf{A}^*(S),$$

where a_i is a fixed idele corresponding to an ideal class $\bar{\mathcal{M}}_i \in C_k$. Here $\mathcal{M}_i \in I(k)$ and $\varphi(a_i) = \mathcal{M}_i$. Then $\mathbf{A}^\ell k^* = \cup_{i=1}^{h_k} a_i^\ell k^* \mathbf{A}^\ell(S)$.

It is easy to see that $a_i^\ell \alpha u \in \mathbf{A}^*(S)$, where $\alpha \in k^*$ and $u \in \mathbf{A}^\ell(S)$, if and only if $a_i^\ell \alpha \in \mathbf{A}^*(S)$. This in turn holds if and only if the ideal corresponding to $a_i^\ell \alpha$ is (1). Then $\mathcal{M}_i^\ell = (\alpha^{-1})$. In particular, the ideal class $\bar{\mathcal{M}}_i^\ell$ is 1. Hence

$$a_i^\ell \mathbf{A}^\ell(S) k^* \cap \mathbf{A}^*(S) = \begin{cases} \emptyset, & \text{if } \bar{\mathcal{M}}_i^\ell \neq 1 \\ \textit{not empty}, & \text{if } \bar{\mathcal{M}}_i^\ell = 1. \end{cases}$$

Now consider an a_i such that $\bar{\mathcal{M}}_i^\ell = 1$. Then $\mathcal{M}_i^\ell = (\beta)$ for some β in k^*, and $a_i^\ell = \beta u_i$ where $u_i \in \mathbf{A}^*(S)$. Consequently, $a_i^\ell \alpha \in \mathbf{A}^*(S)$ if and only if $\mathcal{M}_i^\ell(\alpha) = (\beta\alpha) = 1$. The last equality holds if and only if $\alpha = \beta^{-1}\gamma$ where $\gamma \in \mathcal{O}_k^*$, which implies that $\alpha \in \beta^{-1}\mathcal{O}_k^*$.

Now we have

$$\begin{aligned} (a_i^\ell \mathbf{A}^\ell(S) k^* \cap \mathbf{A}^*(S))/\mathbf{A}^\ell(S) &= \beta u_i \beta^{-1} \mathcal{O}_k^* \mathbf{A}^\ell(S)/\mathbf{A}^\ell(S) \\ &= u_i \mathcal{O}_k^* \mathbf{A}^\ell(S)/\mathbf{A}^\ell(S) \\ &= u_i \left(\mathcal{O}_k^*/\mathcal{O}_k^* \cap \mathbf{A}^\ell(S)\right) \\ &= u_i \left(\mathcal{O}_k^*/\mathcal{O}_k^\ell\right). \end{aligned}$$

Thus the number of distinct elements in $a_i^\ell \mathbf{A}^\ell(S) k^* \cap \mathbf{A}^*(S)/\mathbf{A}^\ell(S)$ is $|\mathcal{O}_k^*/\mathcal{O}_k^\ell|$. This holds for every a_i such that $\bar{\mathcal{M}}_i^\ell = 1$.

Now suppose $\bar{\mathcal{M}}_i \neq \bar{\mathcal{M}}_j$, and $\bar{\mathcal{M}}_i^\ell = \bar{\mathcal{M}}_j^\ell = 1$. We will show that the sets

$$a_i^\ell \mathbf{A}^\ell(S) k^* \cap \mathbf{A}^*(S)/\mathbf{A}^\ell(S) \qquad \text{and} \qquad a_j^\ell \mathbf{A}^\ell(S) k^* \cap \mathbf{A}^*(S)/\mathbf{A}^\ell(S)$$

are disjoint. This in turn will imply that $|\mathscr{A}_\ell(S)| = h_{\ell,k}|\mathcal{O}_k^*/\mathcal{O}_k^\ell|$.

Assume not. Then for some $\alpha, \beta \in k^*$, $a_i^\ell \alpha = a_j^\ell \beta$ in $\mathscr{A}_\ell(S)$ for some $\alpha, \beta \in k^*$, or alternatively, $a_i^\ell \alpha = a_j^\ell \beta u^\ell$ for some $u \in \mathbf{A}^*(S)$. From the last statement we conclude that $\alpha\beta^{-1} = a_j^\ell a_i^{-\ell} u^\ell$, that is $\alpha\beta^{-1}$ is locally an ℓ^{th}

power for all $\nu \in M(k)_0$. Therefore $\alpha\beta^{-1} = \gamma^\ell$ for some $\gamma \in k^*$. Now, $a_i^\ell \alpha\beta^{-1} = a_j^\ell u^\ell$. Then $a_i^\ell \gamma^\ell = a_j^\ell u^\ell$. Hence for all $\nu \in M(k)_0$

$$(a_{i\nu}\gamma)^\ell = (a_{j\nu}u_\nu)^\ell.$$

Therefore

$$a_{i\nu}\gamma = a_{j\nu}u_\nu\mu_\nu.$$

where μ_ν is an ℓ^{th} root of 1 in k_ν. But $u_\nu\mu_\nu$ is in $\mathcal{O}_\nu^*$. Hence,

$$\begin{aligned} \varphi(a_i\gamma) &= \mathcal{M}_i\gamma \\ &= \varphi(a_j(u_\nu\mu_\nu)) \\ &= \mathcal{M}_j. \end{aligned}$$

This implies that $\bar{\mathcal{M}}_i = \bar{\mathcal{M}}_j$, which is a contradiction. Hence the sets

$$a_i^\ell \mathbf{A}^\ell(S)k^* \cap \mathbf{A}^*(S)/\mathbf{A}^\ell(S) \quad \text{and} \quad a_j^\ell \mathbf{A}^\ell(S)k^* \cap \mathbf{A}^*(S)/\mathbf{A}^\ell(S)$$

are disjoint. This concludes the proof of the proposition.

2.3 Decomposition of The Conductor Series into Summands with Euler Products

In this section, we give decomposition of the conductor series into summands that have Euler products.

Let $a_\ell(S) = |\mathscr{A}_\ell(S)|$, and let $\{\varepsilon_i : 1 \leq i \leq a_\ell(S)\}$ be a set of coset representatives of $\mathbf{A}^\ell(S)$ in $\mathbf{A}^\ell k^* \cap \mathbf{A}^*(S)$. For $\chi_S \in C_\ell(\mathbf{A}^*(S))$, set

$$\delta(\chi_S) = \frac{1}{a_\ell(S)} \sum_{i=1}^{a_\ell(S)} \chi_S(\varepsilon_i).$$

Then by orthogonality of characters

$$\delta(\chi_S) = \begin{cases} 1, & \text{if } \chi_S \in \phi\left(C_\ell(\mathbf{A}^*/k^*)\right) \\ 0, & \text{otherwise.} \end{cases}$$

Note that for a character $\chi \in C_\ell(\mathbf{A}^*/k^*)$

$$\Phi_{C_\ell}(\chi) = \prod_{i=1}^{\ell} \Phi(\chi^i).$$

Similarly, for $\chi_S \in C_\ell\left(\mathbf{A}^*(S)\right)$, define

$$\Phi_{C_\ell}(\chi_S) = \prod_{i=1}^{\ell} \Phi(\chi_S^i).$$

Observe that if $\chi_S = \chi|_{\mathbf{A}^*(S)}$, then $\Phi(\chi_S) = \Phi(\chi)$, and therefore $\Phi_{C_\ell}(\chi) = \Phi_{C_\ell}(\chi_S)$.

Recall the generating series of conductors

$$F_{C_\ell}(s) = \sum_{\chi \in C_\ell(\mathbf{A}^*/k^*)} |\Phi_{C_\ell}(\chi)|^{-s}.$$

For $\varepsilon \in \mathbf{A}^*(S)$, define

$$F_{C_\ell, S}(s, \varepsilon) = \sum_{\chi_S \in C_\ell(\mathbf{A}^*(S))} \chi_S(\varepsilon) |\Phi_{C_\ell}(\chi_S)|^{-s}.$$

Then

$$\begin{aligned} \frac{1}{a_\ell(S)} \sum_{i=1}^{a_\ell(S)} F_{C_\ell, S}(s, \varepsilon_i) &= \sum_{\chi_S \in \phi(C_\ell(\mathbf{A}^*/k^*))} |\Phi_{C_\ell}(\chi_S)|^{-s} \\ &= \frac{1}{h_{\ell,k}} \sum_{\chi \in C_\ell(\mathbf{A}^*/k^*)} |\Phi_{C_\ell}(\chi)|^{-s}. \end{aligned}$$

Hence

$$\begin{aligned} F_{C_\ell}(s) &= \frac{h_{\ell,k}}{a_\ell(S)} \sum_{i=1}^{a_\ell(S)} F_{C_\ell,S}(s,\varepsilon_i) \\ &= \frac{1}{e_\ell(S)} \sum_{i=1}^{a_\ell(S)} F_{C_\ell,S}(s,\varepsilon_i) \end{aligned} \tag{2.5}$$

where $e_\ell(S) = |\mathcal{O}_k^*/\mathcal{O}_k^\ell|$. We will compute $e_\ell(S)$ in Section 4.1.

Note that

$$C_\ell(\mathbf{A}^*(S)) = \prod_{\nu\in S} C_\ell(k_\nu^*) \times {\prod_{\nu\notin S}}' C_\ell(\mathcal{O}_\nu^*)$$

consisting only of those collections $\chi = (\chi_\nu)_{\nu\in M(k)}$ for which $\chi_\nu = 1$ for almost all ν. For a given $\chi \in C_\ell(\mathbf{A}^*(S))$, $\chi(\varepsilon)$ and $\Phi_{C_\ell}(\chi)$ are given as follows:

$$\chi(\varepsilon) = \prod_\nu \chi_\nu(\varepsilon_\nu) \quad \text{and} \quad \Phi_{C_\ell}(\chi) = \prod_\nu \Phi_{C_\ell}(\chi_\nu).$$

This leads to the following Euler product factorization of $F_{C_\ell,S}(s,\varepsilon)$:

$$F_{C_\ell,S}(s,\varepsilon) = \prod_{\nu\in S} \sum_{\chi_\nu\in C_\ell(k_\nu^*)} \chi_\nu(\varepsilon_\nu)\,|\Phi_{C_\ell}(\chi_\nu)|^{-s} \prod_{\nu\notin S} \sum_{\chi_\nu\in C_{C_\ell}(\mathcal{O}_\nu^*)} \chi_\nu(\varepsilon_\nu)\,|\Phi_{C_\ell}(\chi_\nu)|^{-s}.$$

Thus (2.5) is the desired decomposition of the conductor series into a linear combination of series that have Euler products.

CHAPTER 3

ANALYTIC CONTINUATION

3.1 Overview

In Chapter 2, we decomposed the conductor series $F_{C_\ell}(s)$ into a finite linear combination of Euler products $F_{C_\ell,S}(s,\varepsilon)$. The principal goal of this chapter is to study the analytic continuation of $F_{C_\ell,S}(s,\varepsilon)$ where

$$\begin{aligned}
F_{C_\ell,S}(s,\varepsilon) &= \prod_{\nu\in S}\sum_{\chi_\nu\in C_\ell(k_\nu^*)}\chi_\nu(\varepsilon_\nu)|\Phi_{C_\ell}(\chi_\nu)|^{-s}\prod_{\nu\notin S}\sum_{\chi_\nu\in C_\ell(\mathcal{O}_\nu^*)}\chi_\nu(\varepsilon_\nu)|\Phi_{C_\ell}(\chi_\nu)|^{-s}\\
&= \prod_{\nu\in S}\sum_{\chi_\nu\in C_\ell(k_\nu^*)}\chi_\nu(\varepsilon_\nu)|\Phi_{C_\ell}(\chi_\nu)|^{-s}\prod_{\nu|\ell}\sum_{\chi_\nu\in C_\ell(\mathcal{O}_\nu^*)}\chi_\nu(\varepsilon_\nu)|\Phi_{C_\ell}(\chi_\nu)|^{-s}\\
&\quad\prod_{\nu\nmid\ell}\sum_{\chi_\nu\in C_\ell(\mathcal{O}_\nu^*)}\chi_\nu(\varepsilon_\nu)|\Phi_{C_\ell}(\chi_\nu)|^{-s}.
\end{aligned}$$

For $\nu\in S$, $k_\nu=\mathbb{R}$ or $k_\nu=\mathbb{C}$. If ℓ is odd, $k_\nu^\ell=k_\nu^*$. Hence $C_\ell(k_\nu^*)=\{1\}$. On the other hand, if $\ell=2$ and $k_\nu=\mathbb{C}$, then $\mathbb{C}^2=\mathbb{C}^*$ and $C_2(\mathbb{C}^*)=\{1\}$, and if $\ell=2$ and $k_\nu=\mathbb{R}$, then $\mathbb{R}^*/\mathbb{R}^2\cong\{\pm1\}$ and $C_2(\mathbb{R}^*)$ has two characters: $\chi_\nu\equiv1$ and $\chi_\nu(x)=sgn(x)$, $x\in\mathbb{R}^*$. Therefore

$$\sum_{\chi_\nu\in C_2(\mathbb{R}^*)}\chi_\nu(\varepsilon_\nu)|\Phi_{C_\ell}(\chi_\nu)|^{-s}=\sum_{\chi_\nu\in C_2(\mathbb{R}^*)}\chi_\nu(\varepsilon_\nu)=\begin{cases}2 & \text{if } \varepsilon_\nu>0;\\ 0 & \text{if } \varepsilon_\nu<0.\end{cases}$$

In any event,

$$\sum_{\chi_\nu \in C_\ell(k_\nu^*)} \chi_\nu(\varepsilon_\nu)|\Phi_{C_\ell}(\chi_\nu)|^{-s}$$

is a constant for any $\nu \in S$.

For $\nu|\ell$, $\sum_{\chi_\nu \in C_\ell(\mathcal{O}_\nu^*)} \chi_\nu(\varepsilon_\nu)|\Phi_{C_\ell}(\chi_\nu)|^{-s}$ is a polynomial in q_ν^{-s} and is therefore entire. Hence

$$\prod_{\nu|\ell} \sum_{\chi_\nu \in C_\ell(\mathcal{O}_\nu^*)} \chi_\nu(\varepsilon_\nu)|\Phi_{C_\ell}(\chi_\nu)|^{-s}$$

is entire. Consequently, those places ν of $M(k)_0$ dividing ℓ will not affect the analyticity of $F_{C_\ell,S}(s,\varepsilon)$.

It remains to compute

$$\sum_{\chi_\nu \in C_\ell(\mathcal{O}_\nu^*)} \chi_\nu(\varepsilon_\nu)|\Phi_{C_\ell}(\chi)|^{-s}$$

for $\nu \nmid \ell$.

3.2 Finite places not dividing ℓ

In this section, we will study the finite places ν of k that do not divide ℓ.

Since $\nu \nmid \ell$, q_ν and ℓ are relatively prime. We will first show in the following lemma that all nontrivial characters of $C_\ell(\mathcal{O}_\nu^*)$ have the same conductor.

Lemma 3.1 *If ν is a finite place of k such that $\nu \nmid \ell$, then $1+\pi_\nu\mathcal{O}_\nu \subset \mathcal{O}_\nu^\ell$.*

Proof: Let $\alpha \in 1+\pi_\nu\mathcal{O}_\nu$, that is, $\alpha \equiv 1(mod\,\pi_\nu)$, and consider a polynomial $p(x) = x^\ell - \alpha \in \mathcal{O}_\nu[x]$. Observe that $|p(1)|_\nu = |1-\alpha|_\nu \leq \frac{1}{q_\nu} < 1$, and since $p'(x) = \ell x^{\ell-1}$, $|p'(1)|_\nu^2 = |\ell|_\nu^2 = 1$. Consequently $|p(1)|_\nu < |p'(1)|_\nu^2$. Thus, by Hensel's lemma, there exists a $\beta \in \mathcal{O}_\nu^*$ such that $p(\beta) = 0$, that is, $\beta^\ell = \alpha$. Therefore $1+\pi_\nu\mathcal{O}_\nu \subset \mathcal{O}_\nu^\ell$.

Lemma 3.1 implies that if χ_ν is a non-trivial character in $C_\ell(\mathcal{O}_\nu^*)$ then $\chi_\nu = 1$ on $1 + \pi_\nu \mathcal{O}_\nu$. Hence

$$\Phi(\chi_\nu) = \begin{cases} (\pi_\nu) & \text{if } \chi_\nu \neq 1 \\ (1) & \text{if } \chi_\nu = 1 \ . \end{cases} \tag{3.1}$$

If $q_\nu \not\equiv 1 \, mod \, \ell$, $\mathbb{F}_{q_\nu}^* = \mathbb{F}_{q_\nu}^\ell$. Thus $\mathcal{O}_\nu^* = (1 + \pi_\nu \mathcal{O}_\nu)\mathcal{O}_\nu^\ell$. But by Lemma 3.1, $1 + \pi_\nu \mathcal{O}_\nu \subset \mathcal{O}_\nu^\ell$, which implies that $\mathcal{O}_\nu^* = \mathcal{O}_\nu^\ell$. Therefore, any $\chi_\nu \in C_\ell(\mathcal{O}_\nu^*)$ is identically equal to 1 and

$$\sum_{\chi_\nu \in C_\ell(\mathcal{O}_\nu^*)} |\Phi_{C_\ell}(\chi_\nu)|^{-s} = 1.$$

If $q_\nu \equiv 1 \, mod \, \ell$, any character χ_ν on $\mathbb{F}_{q_\nu}^*$ such that $\chi_\nu^\ell = 1$ can be viewed as a character on $\mathbb{F}_{q_\nu}^* / \mathbb{F}_{q_\nu}^\ell$. The latter is a cyclic group of order ℓ. Hence $\widehat{\mathbb{F}_{q_\nu}^* / \mathbb{F}_{q_\nu}^\ell}$ consists of precisely ℓ characters, $(\ell - 1)$ of which are nontrivial. Therefore

$$\sum_{\chi_\nu \in C_\ell(\mathcal{O}_\nu^*)} \chi_\nu(\varepsilon_\nu) |\Phi_{C_\ell}(\chi_\nu)|^{-s} = 1 + \sum_{\substack{\chi_\nu \in \widehat{\mathbb{F}_{q_\nu}^* / \mathbb{F}_{q_\nu}^\ell} \\ \chi_\nu \neq 1}} \chi_\nu(\bar{\varepsilon}_\nu) q_\nu^{-(\ell-1)s}$$

where $\bar{\varepsilon}_\nu$ is the reduction of $\varepsilon_\nu \, mod \, \pi_\nu$.

Finally, note that

$$\sum_{\substack{\chi_\nu \in \widehat{\mathbb{F}_{q_\nu}^* / \mathbb{F}_{q_\nu}^\ell} \\ \chi_\nu \neq 1}} \chi_\nu(\bar{\varepsilon}_\nu) = \begin{cases} -1 & \text{if } \bar{\varepsilon}_\nu \notin \mathbb{F}_{q_\nu}^\ell \\ \ell - 1 & \text{if } \bar{\varepsilon}_\nu \in \mathbb{F}_{q_\nu}^\ell \ . \end{cases}$$

Hence

$$\sum_{\chi_\nu \in C_\ell(\mathcal{O}_\nu^*)} \chi_\nu(\varepsilon_\nu) |\Phi_{C_\ell}(\chi_\nu)|^{-s} = \begin{cases} 1 - q_\nu^{-(\ell-1)s} & \text{if } \bar{\varepsilon}_\nu \notin \mathbb{F}_{q_\nu}^\ell \\ 1 + (\ell - 1) q_\nu^{-(\ell-1)s} & \text{if } \bar{\varepsilon}_\nu \in \mathbb{F}_{q_\nu}^\ell \ . \end{cases}$$

Thus we have proved the following proposition.

Proposition 3.1 *If ν is a finite place of k not dividing ℓ,*

$$\sum_{\chi_\nu \in C_\ell(\mathcal{O}_\nu^*)} \chi_\nu(\varepsilon_\nu)|\Phi_{C_\ell}(\chi_\nu)|^{-s} = \begin{cases} 1 & \text{if } q_\nu \not\equiv 1 \,mod\, \ell \\ 1+(\ell-1)q_\nu^{-(\ell-1)s} & \text{if } q_\nu \equiv 1 \,mod\, \ell \text{ and } \varepsilon_\nu \in \mathcal{O}_\nu^\ell \\ 1-q_\nu^{-(\ell-1)s} & \text{if } q_\nu \equiv 1 \,mod\, \ell \text{ and } \varepsilon_\nu \notin \mathcal{O}_\nu^\ell. \end{cases}$$

The above proposition implies that

$$F_{C_\ell,S}(s,\varepsilon) = \prod_{\nu\in S} \sum_{\chi_\nu \in C_\ell(k_\nu^*)} \chi_\nu(\varepsilon_\nu)|\Phi_{C_\ell}(\chi_\nu)|^{-s} \prod_{\nu|\ell} \sum_{\chi_\nu \in C_\ell(\mathcal{O}_\nu^*)} \chi_\nu(\varepsilon_\nu)|\Phi_{C_\ell}(\chi_\nu)|^{-s}$$
$$\prod_{\substack{q_\nu\equiv 1 \,mod\, \ell \\ \varepsilon_\nu \in \mathcal{O}_\nu^\ell}} \left(1+(\ell-1)q_\nu^{-(\ell-1)s}\right) \prod_{\substack{q_\nu\equiv 1 \,mod\, \ell \\ \varepsilon_\nu \notin \mathcal{O}_\nu^\ell}} \left(1-q_\nu^{-(\ell-1)s}\right). \quad (3.2)$$

Since $\sum_{\nu\in M(k)} q_\nu^{-(\ell-1)s}$ converges absolutely for $\mathrm{Re}(s) > \dfrac{1}{\ell-1}$, the series defining $F_{C_\ell,S}(s,\varepsilon)$ also converges absolutely for $\mathrm{Re}(s) > \dfrac{1}{\ell-1}$.

Before studying the analytic continuation of $F_{C_\ell,S}(s,\varepsilon)$, let us adopt the following notation: for meromorphic functions $F(s)$ and $G(s)$, we write $F(s) \approx G(s)$ if $\dfrac{F(s)}{G(s)}$ is analytic in $\mathrm{Re}(s) > \sigma$ for some $\sigma < \dfrac{1}{\ell-1}$. In view of this notation,

$$F_{C_\ell,S}(s,\varepsilon) \approx \prod_{\substack{q_\nu\equiv 1 \,mod\, \ell \\ \varepsilon_\nu \in \mathcal{O}_\nu^\ell}} \left(1+(\ell-1)q_\nu^{-(\ell-1)s}\right) \prod_{\substack{q_\nu\equiv 1 \,mod\, \ell \\ \varepsilon_\nu \notin \mathcal{O}_\nu^\ell}} \left(1-q_\nu^{-(\ell-1)s}\right).$$

To continue $F_{C_\ell,S}(s,\varepsilon)$ analytically beyond $\mathrm{Re}(s) > \dfrac{1}{\ell-1}$, we will need to use the Dedekind zeta functions $\zeta_{k_z}\big((\ell-1)s\big)$ and $\zeta_{k_\alpha}\big((\ell-1)s\big)$ where for $\alpha \in k^*$, $k_\alpha = k(\zeta, \alpha^{1/\ell}) = k_z(\alpha^{1/\ell})$. Set $d_\alpha = [k_\alpha : k]$ and $d_z = [k_z : k]$.

Let L be an algebraic number field. Observe that

$$\zeta_L(s) = \prod_{\nu \in M(L)_0} (1 - q_\nu^{-s})^{-1}.$$

Here $q_\nu = p_\nu^{f_\nu}$ where $p_\nu \in \mathbb{Z}$ is a rational prime such that $\nu | p_\nu$ and $f_\nu = f(\nu/p_\nu)$.

Let T be any subset of $M(L)_0$ that contains all but finitely many ν with $f_\nu = 1$. Then

$$\zeta_L(s) = \prod_{\nu \in T} (1 - q_\nu^{-s})^{-1} \prod_{\nu \notin T} (1 - q_\nu^{-s})^{-1},$$

and $\prod\limits_{\nu \notin T} \left(1 - q_\nu^{-s}\right)^{-1}$ converges for $\mathrm{Re}(s) > \frac{1}{2}$. Consequently

$$\zeta_L((\ell-1)s) \approx \prod_{\nu \in T} \left(1 - q_\nu^{-(\ell-1)s}\right)^{-1}.$$

Now let L be an extension of k. If $\omega \in M(L)_0$, $\omega|\nu$, $\nu \in M(k)_0$, and $\nu | p_\nu$ where p_ν is a prime in $\mathbb{Z}$, then $f(\omega/p_\nu) = f(\omega/\nu) f(\nu/p_\nu)$. Therefore $f(\omega/p_\nu) > 1$ unless $f(\omega/\nu) = 1$. Finally, assume that L is Galois over k, and let $\mathscr{D}_L$ denote all places of k that split completely in L. Note that $\nu \in M(k)_0$ splits completely in L if and only if $f(\omega/\nu) = e(\omega/\nu) = 1$. Also note that $e(\omega/\nu) > 1$ for only finitely many ν.

Therefore

$$\begin{aligned}
\zeta_L\left((\ell-1)s\right) &\approx \prod_{\substack{\omega|\nu \\ \nu \in \mathscr{D}_L}} (1 - q_\omega^{-(\ell-1)s})^{-1} \\
&= \prod_{\nu \in \mathscr{D}_L} (1 - q_\nu^{-(\ell-1)s})^{-d_L}
\end{aligned}$$

where $d_L = [L : k]$.

Let us apply the preceding discussion to the Galois extension k_z of k. To this end, let $\nu \in M(k)_0$, $\nu \nmid \ell$, and let $\omega \in M(k_z)_0$ be such that $\omega|\nu$. It is well known that $[(k_z)_\omega : k_\nu] = [k_\nu(\zeta) : k_\nu] = e_\nu f_\nu$, where ζ is a primitive ℓ^{th} root of unity. Since $\nu \nmid \ell$, $e_\nu = 1$. Therefore $f_\nu = 1$ if and only if k_ν contains an ℓ^{th} primitive root of unity.

Proposition 3.2 *Let ν be a finite place of k such that $\nu \nmid \ell$. Then k_ν contains an ℓ^{th} primitive root of 1 if and only if $q_\nu \equiv 1 \, mod \, \ell$.*

Proof: First note that if $\nu \nmid \ell$, then $\zeta \neq 1 \, mod \, \pi_\nu$. Otherwise $\pi_\nu | 1 - \zeta$ and the fact $1 - \zeta | \ell$ would imply that $\pi_\nu | \ell$, a contradiction. Now let $\bar{\zeta}$ be the residue of of $\zeta \, mod \, \pi_\nu$. By definition, $\bar{\zeta} \in \mathbb{F}_{q_\nu}$, $\bar{\zeta} \neq 1$, and $\bar{\zeta}^\ell = 1$. This implies that the order of $\bar{\zeta}$ in $\mathbb{F}^*_{q_\nu}$ is ℓ. Since $\mathbb{F}^*_{q_\nu}$ is a group, then ℓ divides the order $q_\nu - 1$ of $\mathbb{F}^*_\nu$. Therefore, $q_\nu \equiv 1 \, mod \, \ell$.

To show the converse, we use Hensel's lemma. Consider the polynomial $p(x) = x^\ell - 1$. Let $\bar{\alpha}$ be the residue of $\alpha \; mod \, \pi_\nu$. Since $q_\nu \equiv 1 \mod \ell$, there exists an $\bar{\alpha} \in \mathbb{F}^*_{q_\nu}$, $\bar{\alpha} \neq 1$, such that $\bar{\alpha}^\ell = 1$. Then $p(\alpha) = \alpha^\ell - 1 \equiv 0 \; mod \; \pi_\nu$ and $p'(\alpha) = \ell \alpha^{\ell - 1}$ is not divisible by π_ν. Hence by Hensel's lemma, there exists $\beta \equiv \alpha \; mod \, \pi_\nu$ such that $\beta^\ell = 1$. But β is nontrivial because $\beta \equiv \alpha \; mod \, \pi_\nu$ and $\alpha \neq 1 \, mod \, \pi_\nu$. Therefore k_ν contains an ℓ^{th} primitive root of 1, which concludes the proof of the proposition.

The above proposition implies that

$$\zeta_{k_z}((\ell - 1)s) \approx \prod_{q_\nu \equiv 1 \, mod \, \ell} \left(1 - q_\nu^{-(\ell-1)s}\right)^{-d_z}.$$

Similarly, since the prime ideal $\mathfrak{p}_\nu$ corresponding to the place $\nu \nmid \ell$ splits completely in k_α if and only if $q_\nu \equiv 1 \, mod \, \ell$ and $\alpha \in k_\nu^\ell$, we obtain

$$\zeta_{k_\alpha}((\ell-1)s) \approx \prod_{\substack{q_\nu \equiv 1 \, mod \, \ell \\ \alpha \in k_\nu^\ell}} \left(1 - q_\nu^{-(\ell-1)s}\right)^{-d_\alpha}.$$

Moreover,

$$\frac{\zeta_{k_z}((\ell-1)s)}{\prod\limits_{q_\nu \equiv 1 \, mod \, \ell} \left(1 - q_\nu^{-(\ell-1)s}\right)^{-d_z}} \quad \text{and} \quad \frac{\zeta_{k_\alpha}((\ell-1)s)}{\prod\limits_{\substack{q_\nu \equiv 1 \, mod \, \ell \\ \alpha \in k_\nu^\ell}} \left(1 - q_\nu^{-(\ell-1)s}\right)^{-d_\alpha}}$$

are analytic in $\mathrm{Re}(s) > \dfrac{1}{2(\ell-1)}$.

Note that since $\varepsilon \in \mathbf{A}^\ell k^* \cap \mathbf{A}^*(S)$, $\varepsilon = a^\ell \alpha$ for some $a \in \mathbf{A}^*$, $\alpha \in k^*$. In addition, $\varepsilon_\nu \in \mathcal{O}_\nu^*$ for any $\nu \in M(k)_0$. Since $\mathcal{O}_\nu^* \cap k_\nu^\ell = \mathcal{O}_\nu^\ell$, $\varepsilon_\nu \in \mathcal{O}_\nu^\ell$ if and only if $\varepsilon_\nu \in k_\nu^\ell$. But $\varepsilon_\nu = a_\nu^\ell \alpha$. Hence $\varepsilon_\nu \in \mathcal{O}_\nu^\ell$ if and only if $\alpha \in k_\nu^\ell$. Therefore

$$F_{C_\ell,S}(s,\varepsilon) \approx \prod_{\substack{q_\nu \equiv 1 \, mod \, \ell \\ \alpha \in k_\nu^\ell}} \left(1 + (\ell-1) q_\nu^{-(\ell-1)s}\right) \prod_{\substack{q_\nu \equiv 1 \, mod \, \ell \\ \alpha \notin k_\nu^\ell}} \left(1 - q_\nu^{-(\ell-1)s}\right).$$

Observe that

$$\prod_{q_\nu \equiv 1 \, mod \, \ell} \left(1 + (\ell-1) q_\nu^{-(\ell-1)s}\right) \approx \prod_{q_\nu \equiv 1 \, mod \, \ell} \left(1 - q_\nu^{-(\ell-1)s}\right)^{-(\ell-1)}$$

since

$$\left(1 + (\ell-1) q_\nu^{-(\ell-1)s}\right)\left(1 - q_\nu^{-(\ell-1)s}\right)^{(\ell-1)} = 1 + O\left(q_\nu^{-2(\ell-1)s}\right),$$

and therefore

$$\prod_{q_\nu \equiv 1 \, mod \, \ell} \left(1 + (\ell-1) q_\nu^{-(\ell-1)s}\right)\left(1 - q_\nu^{-(\ell-1)s}\right)^{(\ell-1)}$$

converges absolutely for $\mathrm{Re}(s) > \dfrac{1}{2(\ell-1)}$.
Hence

$$F_{C_\ell,S}(s,\varepsilon) \approx \prod_{\substack{q_\nu \equiv 1 \, mod \, \ell \\ \alpha \in k_\nu^\ell}} \left(1 - q_\nu^{-(\ell-1)s}\right)^{-(\ell-1)} \prod_{\substack{q_\nu \equiv 1 \, mod \, \ell \\ \alpha \notin k_\nu^\ell}} \left(1 - q_\nu^{-(\ell-1)s}\right).$$

Consequently

$$\begin{aligned} F_{C_\ell,S}(s,\varepsilon) &\approx \prod_{\substack{q_\nu \equiv 1 \, mod \, \ell \\ \alpha \in k_\nu^\ell}} \left(1 - q_\nu^{-(\ell-1)s}\right)^{-(\ell-1)} \prod_{\substack{q_\nu \equiv 1 \, mod \, \ell \\ \alpha \notin k_\nu^\ell}} \left(1 - q_\nu^{-(\ell-1)s}\right) \\ &= \prod_{\substack{q_\nu \equiv 1 \, mod \, \ell \\ \alpha \in k_\nu^\ell}} \left(1 - q_\nu^{-(\ell-1)s}\right)^{-\ell} \prod_{q_\nu \equiv 1 \, mod \, \ell} \left(1 - q_\nu^{-(\ell-1)s}\right) \\ &\approx \frac{\zeta_{k_\alpha}((\ell-1)s)^{\ell/d_\alpha}}{\zeta_{k_z}((\ell-1)s)^{1/d_z}}. \end{aligned}$$

Proposition 3.3 *Let $\alpha \in k$, and ℓ be a prime. Then either $\alpha \in k^\ell$ or $\left[k(\alpha^{1/\ell}) : k\right] = \ell$.*

Proof: Observe that $\alpha^{1/\ell}$ is a root of a polynomial $p(x) = x^\ell - \alpha \in k[x]$ and that in $k_z[x]$ $p(x) = \prod_{i=1}^{\ell}(x - \zeta^i \alpha^{1/\ell})$. If $p(x)$ is irreducible in $k[x]$, then $\left[k(\alpha^{1/\ell}) : k\right] = \ell$. Otherwise, $p(x) = g(x)h(x)$ for some $g(x), h(x) \in k[x]$ with $\deg g$, $\deg h < \ell$. But then $g(x) = \prod_{j=1}^{m}(x - \zeta^{i_j}\alpha^{1/\ell}) = x^m + \ldots + (-1)^m \zeta' \alpha^{m/\ell}$ where $\zeta' = \prod_{j=1}^{m} \zeta^{i_j}$. Hence $\beta = \zeta' \alpha^{m/\ell} \in k$ for some $0 < m < \ell$. Thus $\beta^\ell = \alpha^m$. Since m and ℓ are relatively prime, there exist rational integers r and s such that $rm + s\ell = 1$, which in turn implies that $\alpha = \alpha^{rm+s\ell} = \alpha^{rm}\alpha^{s\ell} = \beta^{r\ell}\alpha^{s\ell} = (\beta^r \alpha^s)^\ell \in k^\ell$. This proves the proposition.

Corollary 3.1 *Let $\alpha \in k$. Then*

$$d_\alpha = [k_\alpha : k_z][k_z : k] = \begin{cases} d_z & \text{if } \alpha \in k^\ell \\ \ell d_z & \text{if } \alpha \notin k^\ell. \end{cases}$$

Proof: If $\alpha \in k^\ell$, then $\alpha^{1/\ell} \in k \subset k_z$. Hence $k_\alpha = k_z(\alpha^{1/\ell}) = k_z$ and $\big[k_\alpha : k\big] = \big[k_z : k\big] = d_z$. On the other hand, if $\alpha \notin k^\ell$, then $\big[k(\alpha^{1/\ell}) : k\big] = \ell$. Since $\big[k_z : k\big] = d_z$ and $d_z \mid \ell - 1$, $\big[k(\alpha^{1/\ell}) : k\big]$ and $\big[k_z : k\big]$ are relatively prime. Now, $k_\alpha = k_z k(\alpha^{1/\ell})$. Since $\big[k_z : k\big]$ and $\big[k(\alpha^{1/\ell}) : k\big]$ are relatively prime,

$$\big[k_\alpha : k\big] = \big[k_z : k\big]\big[k(\alpha^{1/\ell}) : k\big] = \ell d_z,$$

proving the corollary.

Corollary 3.1 implies that

$$F_{C_\ell,S}(s,\varepsilon) \approx \frac{\zeta_{k_\alpha}((\ell-1)s)^{\ell/d_\alpha}}{\zeta_{k_z}((\ell-1)s)^{1/d_z}} = \begin{cases} \zeta_{k_z}((\ell-1)s)^{\frac{\ell-1}{d_z}} & \text{if } \alpha \in k^\ell \\ \\ \dfrac{\zeta_{k_\alpha}((\ell-1)s)^{1/d_z}}{\zeta_{k_z}((\ell-1)s)^{1/d_z}} & \text{if } \alpha \notin k^\ell. \end{cases}$$

If $\alpha \in k^\ell$, then $\alpha = \beta^\ell$ for some $\beta \in k^*$. Therefore $\varepsilon = a^\ell \alpha = (a\beta)^\ell \in \mathbf{A}^\ell \cap \mathbf{A}^*(S) = \mathbf{A}^\ell(S)$. Conversely, if $\varepsilon \in \mathbf{A}^\ell(S)$ then for all $\nu \in M(k)$, $a_\nu^\ell \alpha = b_\nu^\ell$ for some $b_\nu \in k_\nu^*$. Then $\alpha \in k_\nu^\ell$ for any $\nu \in M(k)$. Therefore (by the local-global principle) $\alpha \in k^\ell$. Thus $\alpha \in k^\ell$ if and only if $\varepsilon \in \mathbf{A}^\ell(S)$, and $\varepsilon \in \mathbf{A}^\ell(S)$ if and only if $\bar{\varepsilon} = 1$ where $\bar{\varepsilon}$ is the image of ε in $\mathscr{A}_\ell(S) = \big(\mathbf{A}^\ell k^* \cap \mathbf{A}^*(S)\big)/(\mathbf{A}^\ell(S))$.

Now if $\varepsilon \in \mathbf{A}^\ell(S)$ (that is, $\bar{\varepsilon} = 1$), then

$$F_{C_\ell,S}(s,\varepsilon) \approx \zeta_{k_z}((\ell-1)s)^{q_z} \quad \text{where} \quad q_z = \frac{\ell-1}{d_z}.$$

Therefore $F_{C_\ell,S}(s,\varepsilon)$ has a pole of order q_z at $s = \dfrac{1}{\ell-1}$ and can be continued analytically to the half-plane $\mathrm{Re}(s) > \dfrac{1}{2(\ell-1)}$ with the exception of the pole at $s = \dfrac{1}{\ell-1}$.

On the other hand, if $\varepsilon \notin \mathbf{A}^\ell(S)$, then

$$F_{C_\ell,S}(s,\varepsilon) \approx \left(\frac{\zeta_{k_\alpha}((\ell-1)s)}{\zeta_{k_z}((\ell-1)s)}\right)^{1/d_z}.$$

In this case,

$$\zeta_{k_\alpha}((\ell-1)s) = \prod_{\chi\in\widehat{Gal(k_\alpha/k_z)}} L((\ell-1)s,\chi) = \zeta_{k_z}((\ell-1)s)\prod_{\chi\neq 1} L((\ell-1)s,\chi),$$

and

$$\frac{\zeta_{k_\alpha}((\ell-1)s)}{\zeta_{k_z}((\ell-1)s)} = \prod_{\substack{\chi\in\widehat{Gal(k_\alpha/k_z)}\\ \chi\neq 1}} L((\ell-1)s,\chi).$$

Since for $\chi \neq 1$, $L((\ell-1)s,\chi)$ is entire, $\dfrac{\zeta_{k_\alpha}((\ell-1)s)}{\zeta_{k_z}((\ell-1)s)}$ is also entire. Also, $L(s,\chi) = \displaystyle\prod_{\wp\subset\mathcal{O}_{k_z}}\left(1-\frac{\chi(\wp)}{N(\wp)^s}\right)^{-1}$. Here $\chi(\wp)$ stands for $\chi((\wp,k_\alpha/k_z))$ where $(\wp,k_\alpha/k_z)$ is the Artin symbol.

Recall that $k_\alpha = k_z(\alpha^{1/\ell}) = k(\zeta,\alpha^{1/\ell})$, which in turn gives $k_\alpha = k_z k(\alpha^{1/\ell})$. Using degree considerations, one can show that $k_z \cap k(\alpha^{1/\ell}) = k$. Let $G = Gal(k_\alpha/k)$. Then $G = HK$ where $H = Gal(k_\alpha/k(\alpha^{1/\ell}))$, $K = Gal(k_\alpha/k_z)$ and K is normal in G. Since $H \cong Gal(k_z/k)$, $|H| = d_z$. Let $\sigma \in H$. Then $\sigma(\alpha^{1/\ell}) = \alpha^{1/\ell}$ and $\sigma(\zeta) = \zeta^i$ for some i relatively prime to ℓ. We will denote $\sigma \in H$ such that $\sigma(\zeta) = \zeta^i$ by σ_i. If $\tau \in K$, then $\tau(\zeta) = \zeta$ and $\tau(\alpha^{1/\ell}) = \alpha^{1/\ell}\zeta^m$ for some $0 \leq m \leq \ell-1$. Then $\sigma_i\tau\sigma_i^{-1}(\zeta) = \zeta$ and

$$\begin{aligned}
\sigma_i\tau\sigma_i^{-1}(\alpha^{1/\ell}) &= \sigma_i(\tau(\alpha^{1/\ell})) \\
&= \sigma_i(\alpha^{1/\ell}\zeta^m) \\
&= \sigma_i(\alpha^{1/\ell})\sigma_i(\zeta)^m \\
&= \alpha^{1/\ell}(\zeta^i)^m = \alpha^{1/\ell}\zeta^{im}.
\end{aligned}$$

But $\tau^i(\alpha^{1/\ell}) = \alpha^{1/\ell}\zeta^{mi}$. Hence $\sigma_i\tau\sigma_i^{-1} = \tau^i$.

Let $\sigma_i \in H$. Then σ_i maps k_α to k_α and k_z to k_z setwise. If $\wp \subset \mathcal{O}_{k_z}$ is prime then $\sigma_i(\wp) \subset \mathcal{O}_{k_z}$ is prime and $N(\sigma_i(\wp)) = N(\wp)$. Moreover, by the properties of the Artin symbol, we obtain

$$\begin{aligned}
(\sigma_i(\wp), k_\alpha/k_z) &= (\sigma_i(\wp), \sigma_i(k_\alpha)/\sigma_i(k_z)) \\
&= \sigma_i(\wp, k_\alpha/k_z)\sigma_i^{-1} \\
&= (\wp, k_\alpha/k_z)^i .
\end{aligned}$$

If $\chi \in \widehat{Gal\,(k_\alpha/k_z)}$, and $(\wp, k_\alpha/k_z)$ is the Artin symbol,

$$\begin{aligned}
L(s,\chi) &= \prod_{\wp \subset \mathcal{O}_{k_z}} \left(1 - \frac{\chi\left((\wp, k_\alpha/k_z)\right)}{N(\wp)^s}\right)^{-1} \\
&= \prod_{\wp \subset \mathcal{O}_{k_z}} \left(1 - \frac{\chi\left((\sigma_i(\wp), k_\alpha/k_z)\right)}{N(\sigma_i(\wp))^s}\right)^{-1} \\
&= \prod_{\wp \subset \mathcal{O}_{k_z}} \left(1 - \frac{\chi\left((\wp, k_\alpha/k_z)^i\right)}{N(\wp)^s}\right)^{-1} \\
&= L(s, \chi^i).
\end{aligned}$$

Hence $\displaystyle\prod_{\substack{\chi \in \widehat{Gal(k_\alpha/k_z)} \\ \chi \neq 1}} L((\ell-1)s, \chi)$ decomposes into products of d_z identical factors. Therefore

$$\left(\prod_{\substack{\chi \in \widehat{Gal(k_\alpha/k_z)} \\ \chi \neq 1}} L((\ell-1)s, \chi) \right)^{1/d_z}$$

is a single-valued entire function. Hence for $\bar{\varepsilon} \neq 1$, $F_{C_\ell,S}(s,\varepsilon)$ can be analytically continued to the half-plane $\mathrm{Re}(s) > \dfrac{1}{2(\ell-1)}$.

We summarize the above discussions in the following proposition.

Proposition 3.4 *If* $\bar{\varepsilon} = 1$, *then* $F_{C_\ell,S}(s,\varepsilon)$ *can be continued analytically to the half-plane* $\mathrm{Re}(s) > \dfrac{1}{2(\ell-1)}$ *with an exception of a pole of order* q_z *at* $s = \dfrac{1}{\ell-1}$. *If* $\bar{\varepsilon} \neq 1$, *then* $F_{C_\ell,S}(s,\varepsilon)$ *can be continued analytically to the half-plane* $\mathrm{Re}(s) > \dfrac{1}{2(\ell-1)}$.

CHAPTER 4

THE MAIN RESULT

4.1 Overview

We have previously shown that the discriminant series $D_{C_\ell}(s)$ and the conductor series $F_{C_\ell}(s)$ are respectively given by

$$D_{C_\ell}(s) = \frac{F_{C_\ell}(s)}{\ell - 1} - \frac{1}{\ell - 1}$$

and

$$F_{C_\ell}(s) = \frac{1}{e_\ell(S)} \sum_{i=1}^{a_\ell(S)} F_{C_\ell,S}(s, \varepsilon_i). \tag{4.1}$$

By Proposition 3.4, for all $\bar{\varepsilon} \neq 1$, $F_{C_\ell,S}(s,\varepsilon)$ can be analytically continued to the half plane $\mathrm{Re}(s) > \dfrac{1}{2(\ell-1)}$. We also showed that $\dfrac{F_{C_\ell,S}(s,1)}{\zeta_{k_z}((\ell-1)s)^{q_z}}$ is analytic in $\mathrm{Re}(s) > \dfrac{1}{2(\ell-1)}$, which in turn implies that $F_{C_\ell,S}(s,1)$ has a pole at $s = \dfrac{1}{\ell-1}$ of order q_z. Consequently, the following limit exists:

$$c(k, C_\ell) = \lim_{s \to \frac{1}{\ell-1}} \left(s - \frac{1}{\ell - 1}\right)^{q_z} D_{C_\ell}(s). \tag{4.2}$$

Our principal objective in this chapter is to compute the limit in (4.2). For the sake of brevity, we set

$$P(s) = \frac{F_{C_\ell,S}(s,1)}{\zeta_{k_z}((\ell-1)s)^{q_z}}.$$

Then $P(s)$ is analytic in the half plane $\mathrm{Re}(s) > \dfrac{1}{2(\ell-1)}$ and therefore $\lim_{s\to\frac{1}{\ell-1}} P(s) = P\left(\dfrac{1}{\ell-1}\right)$.

Now,

$$\begin{aligned}
c(k, C_\ell) &= \lim_{s\to\frac{1}{\ell-1}} \left(s - \frac{1}{\ell-1}\right)^{q_z} D_{C_\ell}(s) \\
&= \lim_{s\to\frac{1}{\ell-1}} \left(s - \frac{1}{\ell-1}\right)^{q_z} \frac{F_{C_\ell,S}(s,1)}{(\ell-1)e_\ell(S)} \\
&= \lim_{s\to\frac{1}{\ell-1}} \left(s - \frac{1}{\ell-1}\right)^{q_z} \frac{\zeta_{k_z}((\ell-1)s)^{q_z} P(s)}{(\ell-1)e_\ell(S)} \\
&= \frac{\zeta_{k_z}(1)^{q_z} P\left(\frac{1}{\ell-1}\right)}{e_\ell(S)(\ell-1)^{q_z+1}},
\end{aligned}$$

where $e_\ell(S) = |\mathcal{O}_k^*/\mathcal{O}_k^\ell|$. Since $P(s) = \dfrac{F_{C_\ell,S}(s,1)}{\zeta_{k_z}((\ell-1)s)^{q_z}}$, for $\mathrm{Re}(s) > \dfrac{1}{\ell-1}$

$$\begin{aligned}
P(s) = \prod_{\nu\in S} \sum_{\chi_\nu \in C_\ell(k_\nu^*)} |\Phi_{C_\ell}(\chi_\nu)|^{-s} \prod_{\nu|\ell} \sum_{\chi_\nu\in C_\ell(\mathcal{O}_\nu^*)} |\Phi(\chi_\nu)|^{-(\ell-1)s} \\
\prod_{q_\nu\equiv 1 \bmod \ell} \left(1 + (\ell-1)q_\nu^{-(\ell-1)s}\right) \zeta_{k_z}((\ell-1)s)^{-q_z}.
\end{aligned}$$

Set

$$\delta_\infty(\ell) = \prod_{\nu\in S} \sum_{\chi_\nu\in C_\ell(k_\nu^*)} |\Phi_{C_\ell}(\chi_\nu)|^{-s}.$$

Then

$$\delta_\infty(\ell) = \begin{cases} 1 & \text{if } \ell \neq 2 \\ 2^{r_1} & \text{if } \ell = 2. \end{cases}$$

and

$$P\left(\frac{1}{\ell-1}\right) = \delta_\infty(\ell) P_0,$$

where

$$P_0 = \prod_{\nu|\ell} \sum_{\chi_\nu \in C_\ell(\mathcal{O}_\nu^*)} |\Phi(\chi_\nu)|^{-1} \prod_{q_\nu \equiv 1 \, mod \, \ell} \left(1 + (\ell - 1) q_\nu^{-1}\right) (1 - q_\nu^{-1})^{\ell - 1}$$
$$\prod_{q_\nu \not\equiv 1 \, mod \, \ell} \left(1 - q_\nu^{-f_\nu}\right)^{q_z g_\nu} . \tag{4.3}$$

In the next section, we will compute

$$\prod_{\nu|\ell} \sum_{\chi_\nu \in C_\ell(\mathcal{O}_\nu^*)} |\Phi(\chi_\nu)|^{-1}.$$

We will conclude this section by computing $e_\ell(S)$.

By the Dirichlet Unit Theorem,

$$\mathcal{O}_k^* = \mu(k) \times \mathbb{Z}^r.$$

Here $< \mu >= \mu(k)$ is the finite cyclic group of the roots of unity in k^*, and $r = r_1 + r_2 - 1$. Then

$$\mathcal{O}_k^* / \mathcal{O}_k^\ell \cong < \mu > / < \mu^\ell > \times (\mathbb{Z}/\ell\mathbb{Z})^r ,$$

so that

$$e_\ell(S) = \begin{cases} \ell^r & \text{if } \zeta \notin k \\ \ell^{r+1} & \text{if } \zeta \in k. \end{cases}$$

Observe that if $\ell = 2$, $\zeta = -1 \in k$. Then

$$\frac{e_\ell(S)}{\delta_\infty(\ell)} = \frac{2^{r_1 + r_2}}{2^{r_1}} = 2^{r_2}.$$

If ℓ is odd and a primitive ℓ^{th} root of unity $\zeta \in k$, then $r_1 = 0$, and

$$\frac{e_\ell(S)}{\delta_\infty(\ell)} = \ell^{r_1 + r_2} = \ell^{r_2}.$$

In all other cases,

$$\frac{e_\ell(S)}{\delta_\infty(\ell)} = \ell^{r_1+r_2-1}.$$

Hence, $\dfrac{e_\ell(S)}{\delta_\infty(\ell)} = \ell^{r_2+r_z}$, where $r_z = 0$ if $\zeta \in k$ and $r_z = r_1 - 1$ otherwise.

Therefore

$$c(k, C_\ell) = \frac{\zeta_{k_z}(1)^{q_z} P_0}{\ell^{r_2+r_z}(\ell-1)^{q_z+1}}.$$

4.2 Finite Places Dividing ℓ

Suppose $\nu \in M(k)$ and $\nu|\ell$. Let $\mathfrak{p}_\nu$ denote the prime ideal of $\mathcal{O}_k$ corresponding to ν, and let ω be a place of k_z such that $\omega|\nu$. Then $(k_z)_\omega = (k(\zeta))_\omega = k_\nu(\zeta)$. It is well known that $[k_\nu(\zeta) : k_\nu] = e_\nu f_\nu$.

Now, let π_ν be a uniformizer of $\mathfrak{p}_\nu$ in $\mathcal{O}_\nu$. Then $(\ell) = (\pi_\nu)^{e(\nu)}$for some integer $e(\nu) > 0$. Here $e(\nu) = e(\nu/\ell)$ is the ramification index of ν over ℓ. We will write

$$e(\nu) = d(\nu)(\ell-1) + r_0(\nu) \quad \text{where} \quad 0 \le r_0(\nu) \le \ell - 2.$$

Note that $d(\nu) = \left\lfloor \dfrac{e(\nu)}{\ell-1} \right\rfloor$ and $r_0(\nu)$ is the least nonnegative residue of $e(\nu)\, mod\, (\ell-1)$ (here $\lfloor x \rfloor$ denotes the floor function, that is, the greatest integer less than or equal to x).

Our first task is to find an $n > 0$ such that

$$1 + \pi_\nu^n \mathcal{O}_\nu \subset \mathcal{O}_\nu^\ell.$$

Lemma 4.1 *If* $n = d(\nu)\ell + r_0(\nu) + 1,$ *then* $1 + \pi_\nu^n \mathcal{O}_\nu \subset \mathcal{O}_\nu^\ell.$

Proof: Let $x \in 1 + \pi_\nu^{d(\nu)\ell+r_0(\nu)+1}\mathcal{O}_\nu$. Then $x = 1+\alpha$ where $\alpha \in \pi_\nu^{d(\nu)\ell+r_0(\nu)+1}\mathcal{O}_\nu$, and an ℓ^{th} root of x is given by $x^{\frac{1}{\ell}} = \displaystyle\sum_{n=0}^{\infty} \binom{\frac{1}{\ell}}{n} \alpha^n$ where

$$\binom{\frac{1}{\ell}}{n} = \frac{\frac{1}{\ell}(\frac{1}{\ell}-1)\cdots(\frac{1}{\ell}-n+1)}{n!} = \frac{1(1-\ell)\cdots(1-(n-1)\ell)}{\ell^n n!}.$$

Since $1-\ell, 1-2\ell, \ldots, 1-(n-1)\ell$ are relatively prime to ℓ, their ν-adic norms are 1. Consequently

$$\left|\binom{\frac{1}{\ell}}{n}\right|_\nu = \frac{1}{|\ell|_\nu^n |n!|_\nu} \leq \frac{1}{|\ell|_\nu^n |\ell|_\nu^{n/\ell-1}} = |\ell|_\nu^{-n\ell/\ell-1} = |\pi_\nu|_\nu^{-ne(\nu)\,\ell/\ell-1}.$$

Thus

$$\left|\binom{\frac{1}{\ell}}{n}\alpha^n\right|_\nu \leq |\pi_\nu|_\nu^{n\left(d(\nu)\ell+r_0(\nu)+1-\frac{e(\nu)\ell}{\ell-1}\right)} = |\pi_\nu|_\nu^{n\left(1-\frac{r_0(\nu)}{\ell-1}\right)}.$$

But $1-\dfrac{r_0(\nu)}{\ell-1} > 0$. Therefore $\left|\binom{\frac{1}{\ell}}{n}\alpha^n\right|_\nu \to 0$ as $n \to \infty$, and the series defining $x^{\frac{1}{\ell}}$ converges. This concludes the proof of the lemma.

Lemma 4.1 implies that if ν is a finite place dividing ℓ, and χ_ν is a non-trivial character in $C_\ell(\mathcal{O}_\nu^*)$, then $\Phi(\chi_\nu) = (\pi_\nu^n)$ where $0 < n \leq d(\nu)\ell + r_0(\nu) + 1$.

We will next consider two cases: $\mathfrak{p}_\nu$ splits completely in k_z and $\mathfrak{p}_\nu$ does not split completely in k_z.

Case 1 Suppose $\mathfrak{p}_\nu$ splits completely in k_z. This occurs if and only if the ℓ^{th}-root of unity $\zeta \in k_\nu$.

Consider the tower of field extensions

$$\mathbb{Q}_\ell \subset \mathbb{Q}_\ell(\zeta) \subset k_\nu.$$

It is well-known that $(\ell) = (1-\zeta)^{\ell-1}$. Let $(1-\zeta) = (\pi_\nu^{d(\nu)})$. Then $(\ell) = (\pi_\nu)^{d(\nu)(\ell-1)}$, so that $e(\nu) = d(\nu)(\ell-1)$ and $r_0(\nu) = 0$ in this case.

If $\chi_\nu \in C_\ell(\mathcal{O}_\nu^*)$ has conductor (π_ν^n), then $\chi_\nu|_{1+\pi_\nu^n\mathcal{O}_\nu} = 1$. Also, by Lemma 4.1,

$$\Phi(\chi_\nu) = (\pi_\nu^n), \text{ for some } 0 \le n \le d(\nu)\ell + 1.$$

This leads us to consider

$$M_n = \{\chi_\nu \in C_\ell(\mathcal{O}_\nu^*) : \chi_\nu|_{1+\pi_\nu^n \mathcal{O}_\nu} = 1\}$$

for $0 \le n \le d(\nu)\ell + 1$. Actually, M_n consists of all characters $\chi_\nu \in C_\ell(\mathcal{O}_\nu^*)$ with $\Phi(\chi_\nu) = (\pi_\nu^m)$, for some $m \le n$. Therefore the number of characters $\chi_\nu \in C_\ell(\mathcal{O}_\nu^*)$ with $\Phi(\chi_\nu) = (\pi_\nu^n)$ is precisely $|M_n| - |M_{n-1}|$.

Since $\nu|\ell$, $(\ell, q_\nu - 1) = 1$. Hence $\mathbb{F}_{q_\nu}^\ell = \mathbb{F}_{q_\nu}^*$. Therefore, $\mathcal{O}_\nu^* = (1 + \pi_\nu \mathcal{O}_\nu)\mathcal{O}_\nu^\ell$, and

$$\begin{aligned}
\mathcal{O}_\nu^*/\mathcal{O}_\nu^\ell &= (1 + \pi_\nu \mathcal{O}_\nu)\mathcal{O}_\nu^\ell/\mathcal{O}_\nu^\ell \\
&\cong (1 + \pi_\nu \mathcal{O}_\nu)/(1 + \pi_\nu \mathcal{O}_\nu) \cap \mathcal{O}_\nu^\ell \\
&\cong (1 + \pi_\nu \mathcal{O}_\nu)/(1 + \pi_\nu \mathcal{O}_\nu)^\ell.
\end{aligned}$$

Now if we set $G_n = (1 + \pi_\nu \mathcal{O}_\nu)/(1 + \pi_\nu^n \mathcal{O}_\nu)$, then $\chi_\nu|_{1+\pi_\nu^n \mathcal{O}_\nu} = 1$ and $\chi_\nu^\ell = 1$ if and only if χ_ν is a character on G_n/G_n^ℓ. Thus,

$$|M_n| = |\widehat{G_n/G_n^\ell}| = |G_n/G_n^\ell|.$$

Define a map $\varphi_n : G_n \mapsto G_n$, $\varphi_n(x) = x^\ell$. Since φ_n is a group homomorphism, it is easy to see that

$$G_n/\ker \varphi_n \cong \varphi_n(G_n) = G_n^\ell.$$

Consequently, we have

$$|G_n/G_n^\ell| = |\ker \varphi_n|,$$

where $|\ker \varphi_n| = |\{x \in G_n : x^\ell = 1\}| = |\{x \in 1 + \pi_\nu \mathcal{O}_\nu/1 + \pi_\nu^n \mathcal{O}_\nu : x^\ell \equiv 1 \, mod \, \pi_\nu^n\}|$.

Let $ord_{\pi_\nu}(x-1) = m$. We will next determine $ord_{\pi_\nu}(x^\ell - 1)$.

Consider

$$x^\ell - 1 = (x-1)(x-\zeta)(x-\zeta^2)\dots(x-\zeta^{\ell-1}).$$

Since $ord_{\pi_\nu}(1-\zeta^i) = d(\nu)$ for every $i = 1, 2, \dots \ell-1$ and $\zeta^i - \zeta^j = \zeta^i(1-\zeta^{j-i})$, $ord_{\pi_\nu}(\zeta^i - \zeta^j) = d(\nu)$ for all $i \neq j$. Now consider the following cases:

Case 1: Suppose $m < d(\nu)$.
In this case, $ord_{\pi_\nu}(x-\zeta^i) = ord_{\pi_\nu}(x-1+1-\zeta^i) = m$. Thus

$$ord_{\pi_\nu}(x^\ell - 1) = m\ell < d(\nu)\ell.$$

Case 2: Suppose $m > d(\nu)$.
Here $ord_{\pi_\nu}(x-\zeta^i) = ord_{\pi_\nu}(1-\zeta^i+x-1) = d(\nu)$ for all $i \neq 0$. Thus

$$ord_{\pi_\nu}(x^\ell - 1) = m + (\ell-1)d(\nu) \geq d(\nu)\ell + 1.$$

In particular, this implies that $x \in \ker\varphi_n$ for any $n \leq d(\nu)\ell + 1$.

Case 3: Suppose $m = d(\nu)$.
$ord_{\pi_\nu}(x-\zeta^i) = ord_{\pi_\nu}(x-1+1-\zeta^i) \geq d(\nu)$. Thus $ord_{\pi_\nu}(x^\ell - 1) \geq d(\nu)\ell$.

Now, let us consider the characters $\chi_\nu \in M_n$ for $n \leq d(\nu)\ell$.
Cases 2 and 3 imply that for $m \geq d(\nu)$, $ord_{\pi_\nu}(x^\ell - 1) \geq d(\nu)\ell \geq n$, and thus $x^\ell \in 1 + \pi_\nu^n\mathcal{O}_\nu$.

For $m < d(\nu)$, $ord_{\pi_\nu}(x^\ell - 1) = m\ell$. Hence $x^\ell \in 1 + \pi_\nu^n\mathcal{O}_\nu$ if and only if $m\ell \geq n$. Consequently, for all $n \leq d(\nu)\ell$,

$$\ker\varphi_n = (1 + \pi_\nu^{\lceil \frac{n}{\ell} \rceil}\mathcal{O}_\nu)/(1 + \pi_\nu^n\mathcal{O}_\nu),$$

where $\lceil x \rceil$ denotes the ceiling function (the smallest integer greater or equal to x). Hence $|M_n| = q_\nu^{n-\lceil \frac{n}{\ell} \rceil}$.

Therefore for $n < d(\nu)\ell$

$$|\{\chi_\nu \in C_\ell(\mathcal{O}_\nu^*) : \Phi(\chi_\nu) = (\pi_\nu^n), 1 \le n \le d(\nu)\ell\}| = q_\nu^{n-\lceil \frac{n}{\ell} \rceil} - q_\nu^{n-1-\lceil \frac{n-1}{\ell} \rceil},$$

and

$$\begin{aligned}
\sum_{\Phi(\chi_\nu)=(\pi_\nu^n), n \le d(\nu)\ell} |\Phi(\chi_\nu)|^{-1} &= 1 + \sum_{n=1}^{d(\nu)\ell} \left(q_\nu^{n-\lceil \frac{n}{\ell} \rceil} - q_\nu^{n-1-\lceil \frac{n-1}{\ell} \rceil} \right) q_\nu^{-n} \\
&= 1 + \sum_{n=1}^{d(\nu)\ell} \left(q_\nu^{-\lceil \frac{n}{\ell} \rceil} - q_\nu^{-1-\lceil \frac{n-1}{\ell} \rceil} \right) \\
&= 1 + \sum_{n=1}^{d(\nu)\ell} q_\nu^{-\lceil \frac{n}{\ell} \rceil} - q_\nu^{-1} \sum_{n=0}^{d(\nu)\ell-1} q_\nu^{-\lceil \frac{n}{\ell} \rceil} \\
&= 1 + \sum_{n=1}^{d(\nu)\ell} q_\nu^{-\lceil \frac{n}{\ell} \rceil} - q_\nu^{-1} \sum_{n=1}^{d(\nu)\ell} q_\nu^{-\lceil \frac{n}{\ell} \rceil} - q_\nu^{-1} + q_\nu^{-d(\nu)-1} \\
&= 1 + (1 - q_\nu^{-1}) \sum_{n=1}^{d(\nu)\ell} q_\nu^{-\lceil \frac{n}{\ell} \rceil} - q_\nu^{-1} + q_\nu^{-d(\nu)-1}.
\end{aligned}$$

But since $\displaystyle\sum_{n=1}^{d(\nu)\ell} q_\nu^{-\lceil \frac{n}{\ell} \rceil} = \sum_{m=1}^{d(\nu)} \ell q_\nu^{-m}$, $\displaystyle\left(1 - q_\nu^{-1}\right) \sum_{n=1}^{d(\nu)\ell} q_\nu^{-\lceil \frac{n}{\ell} \rceil} = \ell \left(q_\nu^{-1} - q_\nu^{-d(\nu)-1} \right)$.

Therefore

$$\sum_{\substack{\Phi(\chi_\nu)=(\pi_\nu^n) \\ n \le d(\nu)\ell}} |\Phi(\chi_\nu)|^{-1} = 1 + (\ell - 1)q_\nu^{-1} - (\ell - 1)q_\nu^{-d(\nu)-1}. \tag{4.4}$$

We will next determine $\displaystyle\sum_{\Phi(\chi_\nu)=(\pi_\nu^{d(\nu)\ell+1})} |\Phi(\chi_\nu)|^{-1}$. To this end, we need to determine $\left|\ker \varphi_{d(\nu)\ell+1}\right|$. If $ord_{\pi_\nu}(x-1) = m < d(\nu)$, then $ord_{\pi_\nu}(x^\ell - 1) = m\ell < d(\nu)\ell + 1$. Hence, $x \notin \ker \varphi_{d(\nu)\ell+1}$.

If $ord_{\pi_\nu}(x-1) = m > d(\nu)$, then $ord_{\pi_\nu}(x^\ell - 1) \geq d(\nu)\ell + 1$. Hence $x \in \ker\varphi_{d(\nu)\ell+1}$, that is, $\ker\varphi_{d(\nu)\ell+1}$ contains $1+\pi_\nu^{d(\nu)+1}\mathcal{O}_\nu$. It remains to determine $\left|\ker\varphi_{d(\nu)\ell+1}/\left(1+\pi_\nu^{d(\nu)+1}\mathcal{O}_\nu\right)\right|$. That is, we want to determine

$$|\{x \in 1+\pi_\nu^{d(\nu)}\mathcal{O}_\nu : x^\ell \equiv 1 \, mod \, \pi_\nu^{d(\nu)\ell+1}\}/\left(1+\pi_\nu^{d(\nu)+1}\mathcal{O}_\nu\right)|.$$

Observe that since $ord_{\pi_\nu}(x-1) \geq d(\nu)$, $ord_{\pi_\nu}(x-\zeta^i) \geq d(\nu)$ for every $i = 0,1,\ldots,\ell-1$. Therefore, $ord_{\pi_\nu}(x^\ell-1) \geq d(\nu)\ell+1$ if and only if $ord_{\pi_\nu}(x-\zeta^i) \geq d(\nu)+1$ for some $i = 0,1,\ldots,\ell-1$. But for all $i \neq j$, $ord_{\pi_\nu}(\zeta^i - \zeta^j) = d(\nu)$. Hence $\zeta^i \not\equiv \zeta^j \, mod \, \pi_\nu^{d(\nu)+1}$. Consequently, $x \in \ker\varphi_{d(\nu)\ell+1}$ if and only if $x \equiv \zeta^i \, mod \, \pi_\nu^{d(\nu)+1}$ for some $i = 0,1,\ldots,\ell-1$. Hence the number of such $x \, mod \, \pi_\nu^{d(\nu)+1}$ is ℓ. Therefore

$$\begin{aligned} |\ker\varphi_n| &= \left|\ker\varphi_{d(\nu)\ell+1}/\left(1+\pi_\nu^{d(\nu)+1}\mathcal{O}_\nu\right)\right|\left|\left(1+\pi_\nu^{d(\nu)+1}\mathcal{O}_\nu\right)/\left(1+\pi_\nu^{d(\nu)\ell+1}\mathcal{O}_\nu\right)\right| \\ &= \ell q_\nu^{d(\nu)(\ell-1)}. \end{aligned}$$

Thus the number of characters χ_ν with $\Phi(\chi_\nu) = (\pi_\nu^{d(\nu)\ell+1})$ is

$$\ell q_\nu^{d(\nu)(\ell-1)} - q_\nu^{d(\nu)\ell-d(\nu)} = (\ell-1)q_\nu^{d(\nu)(\ell-1)}.$$

Therefore in this case

$$\begin{aligned} \sum_{\chi_\nu \in C_\ell(\mathcal{O}_\nu^*)} |\Phi(\chi_\nu)|^{-1} &= 1 + (\ell-1)q_\nu^{-1} - (\ell-1)q_\nu^{-d(\nu)-1} + (\ell-1)q_\nu^{d(\nu)(\ell-1)-d(\nu)\ell-1} \\ &= 1+(\ell-1)q_\nu^{-1}. \end{aligned}$$

Case 2: Suppose $\nu \mid \ell$ and $\mathfrak{p}_\nu$ does not split completely in k_z. In this case $\zeta^i \notin k_\nu$ for $i = 1,2,3,...,\ell-1$.

By Lemma 4.1, $\Phi(\chi_\nu) = (\pi_\nu^n)$, where $n = 0,1,\ldots,d(\nu)\ell + r_0(\nu)+1$. To count the number of characters of conductor (π_ν^n), consider

$$G_n = \left(1+\pi_\nu\mathcal{O}_\nu\right)/\left(1+\pi_\nu^n\mathcal{O}_\nu\right).$$

As before, a character $\chi_\nu \in C_\ell(\mathcal{O}_\nu^*)$ satisfying $\chi_\nu|_{1+\pi_\nu^n\mathcal{O}_\nu} = 1$ is also a character on G_n/G_n^ℓ, and by conducting the same analysis as before, we deduce that the number of such characters is $|\ker \varphi_n|$, where $\varphi_n : G_n \mapsto G_n$ is given by $\varphi_n(x) = x^\ell$.

First, let us determine

$$ord_{\pi_\nu}(x^\ell - 1).$$

Recall that for $\omega|\nu$, $(k_z)_\omega = k_\nu(\zeta)$. Let π_ω be a uniformizer of the prime ideal of $\mathcal{O}_\omega$. Then

$$(\pi_\nu) = (\pi_\omega^{e_z}) \text{ where } e_z = e(\omega/\nu).$$

This implies that $(\ell) = (\pi_\omega^{e(\nu)e_z})$. But since $(\ell) = (1-\zeta)^{\ell-1}$, $(\ell - 1)$ divides $e(\nu)e_z$, that is $e(\nu)e_z = d_{z\nu}(\ell - 1)$, for some positive integer $d_{z\nu}$.

Let $ord_{\pi_\omega}(x-1) = m_z$. Then if $m_z < d_{z\nu}$, then $ord_{\pi_\omega}(x^\ell - 1) = \ell m_z$. Moreover, since $ord_{\pi_\omega}(x^\ell - 1) = e_z ord_{\pi_\nu}(x^\ell - 1)$, then for $m_z < d_{z\nu}$, $ord_{\pi_\nu}(x^\ell - 1) = \ell\, ord_{\pi_\nu}(x-1)$.

If $ord_{\pi_\nu}(x - 1) = m$, then $m_z = e_z m$. Then $m_z < d_{z\nu} = \dfrac{e(\nu)e_z}{\ell - 1}$ if and only if $m < \dfrac{e(\nu)}{\ell - 1} = d(\nu) + \dfrac{r_0(\nu)}{\ell - 1}$. If $r_0(\nu) \geq 1$, then for all $m \leq d(\nu)$, $ord_{\pi_\nu}(x-1) = m$ implies that $ord_{\pi_\nu}(x^\ell - 1) = m\ell$. If $r_0(\nu) = 0$, then for all $m < d(\nu)$, $ord_{\pi_\nu}(x-1) = m$ implies that $ord_{\pi_\nu}(x^\ell - 1) = \ell m$.

Now suppose $r_0(\nu) = 0$ and $m = d(\nu)$. In this case, $e(\nu) = d(\nu)(\ell - 1)$ and $d_{z\nu} = e_z d(\nu)$. If we set $ord_{\pi_\nu}(x-1) = d(\nu)$, then $ord_{\pi_\omega}(x-1) = e_z d(\nu) = d_{z\nu}$. Note that $ord_{\pi_\omega}(x^\ell - 1) > \ell d_{z\nu}$ if for some i, $i \neq 0 \, mod\, \ell$, $ord_{\pi_\omega}(x - \zeta^i) > d_{z\nu}$. But for $j \neq i$, $ord_{\pi_\omega}(\zeta^j - \zeta^i) = d_{z\nu}$. Hence for all $j \neq i \, mod\, \ell$, $ord_{\pi_\omega}(x - \zeta^j) = d_{z\nu}$. Therefore for all $j \neq i \mod \ell$

$$|x - \zeta^i|_\omega < |\zeta^j - \zeta^i|_\omega.$$

Then by Krasner's Lemma, $k_\nu(\zeta^i) \subset k_\nu(x) = k_\nu$. Therefore $\zeta^i \in k_\nu$, which is impossible. Hence for $m = d(\nu)$, $ord_{\pi_\omega}(x^\ell - 1) = \ell d_{z\nu}$ and $ord_{\pi_\nu}(x^\ell - 1) = \dfrac{\ell d_{z\nu}}{e_z} = \ell d(\nu)$.

Note that $d(\nu) + 1 > \dfrac{e(\nu)}{\ell - 1} = d(\nu) + \dfrac{r_0(\nu)}{\ell - 1}$. Hence if $m \geq d(\nu) + 1$, $m_z = e_z m \geq e_z(d(\nu)+1) > \dfrac{e(\nu)e_z}{\ell - 1} = d_{z\nu}$. Therefore for $m \geq d(\nu)+1$, $ord_{\pi_\omega}(x^\ell - 1) = d_{z\nu}(\ell - 1) + m_z$, and

$$ord_{\pi_\nu}(x^\ell - 1) = \frac{ord_{\pi_\omega}(x^\ell - 1)}{e_z} = \frac{m_z + (\ell - 1)d_{z\nu}}{e_z} = e(\nu) + m.$$

Thus for $m \geq d(\nu)+1$ and $e(\nu) = d(\nu)(\ell - 1) + r_0(\nu)$, we get $ord_{\pi_\nu}(x^\ell - 1) \geq d(\nu)\ell + r_0(\nu) + 1$. But since $n = 0, 1 \ldots, d(\nu)\ell + r_0(\nu) + 1$, then $ord_{\pi_\nu}(x^\ell - 1) \geq d(\nu)\ell + r_0(\nu) + 1 \geq n$ for any n that occurs in the conductor of $\chi_\nu \in C_\ell(\mathcal{O}_\nu^*)$. Hence if $\zeta \notin k_\nu$, and $ord_{\pi_\nu}(x - 1) = m$, then

$$ord_{\pi_\nu}(x^\ell - 1) = m\ell \qquad \text{if } m \leq d(\nu)$$

and

$$ord_{\pi_\nu}(x^\ell - 1) \geq d(\nu)\ell + r_0(\nu) + 1 \qquad \text{if } m \geq d(\nu) + 1.$$

The above discussion implies that if $x \in G_n$, $0 \leq n \leq d(\nu)\ell + r_0(\nu) + 1$ and $ord_{\pi_\nu}(x - 1) = m \leq n$, then $x \in \ker \varphi_n$ if and only if $m\ell \geq n$. Thus

$$|ker\varphi_n| = q_\nu^{n - \lceil \frac{n}{\ell} \rceil}.$$

Therefore

$$
\begin{aligned}
\sum_{\chi_\nu \in C_\ell(\mathcal{O}_\nu^*)} |\Phi(\chi_\nu)|^{-1} &= 1 + \sum_{n=1}^{d(\nu)\ell + r_0(\nu)+1} \left(q_\nu^{n - \lceil \frac{n}{\ell} \rceil} - q_\nu^{n-1-\lceil \frac{n-1}{\ell} \rceil} \right) q_\nu^{-n} \\
&= 1 + \sum_{n=1}^{d(\nu)\ell + r_0(\nu)+1} q_\nu^{-\lceil \frac{n}{\ell} \rceil} - q_\nu^{-1} \sum_{n=0}^{d(\nu)\ell + r_0(\nu)} q_\nu^{-\lceil \frac{n}{\ell} \rceil} \\
&= 1 - q_\nu^{-1} + (1 - q_\nu^{-1}) \sum_{n=1}^{d(\nu)\ell + r_0(\nu)+1} q_\nu^{-\lceil \frac{n}{\ell} \rceil} + q_\nu^{-1} \cdot q_\nu^{-d(\nu)-1} \\
&= (1 - q_\nu^{-1}) + (1 - q_\nu^{-1}) \sum_{n=1}^{d(\nu)\ell} q_\nu^{-\lceil \frac{n}{\ell} \rceil} \\
&+ (1 - q_\nu^{-1}) \sum_{n=d(\nu)\ell+1}^{d(\nu)\ell + r_0(\nu)+1} q_\nu^{-\lceil \frac{n}{\ell} \rceil} + q_\nu^{-d(\nu)-2} \\
&= (1 - q_\nu^{-1}) + (1 - q_\nu^{-1})\ell \sum_{m=1}^{d(\nu)} \left[q_\nu^{-m} + (1 - q_\nu^{-1})(r_0(\nu)+1) q_\nu^{-d(\nu)-1} \right. \\
&\left. + q_\nu^{-d(\nu)-2} \right] \\
&= 1 + (\ell - 1) q_\nu^{-1} - \left[\ell - (r_0(\nu)+1)(1 - q_\nu^{-1}) - q_\nu^{-1} \right] q_\nu^{-d(\nu)-1} \\
&= 1 + (\ell - 1) q_\nu^{-1} - \left[\ell - 1 - r_0(\nu)(1 - q_\nu^{-1}) \right] q_\nu^{-d(\nu)-1}.
\end{aligned}
$$

We summarize the results of this section in the following proposition:

Proposition 4.1 *Suppose* $\nu|\ell$, $(\ell) = (\pi_\nu)^{e(\nu)}$ *and* $e(\nu) = d(\nu)(\ell - 1) + r_0(\nu)$ *where* $0 \le r_0(\nu) \le \ell - 2$. *Then*

1. $\displaystyle\sum_{\chi_\nu \in C_\ell(\mathcal{O}_\nu^*)} |\Phi(\chi_\nu)|^{-1} = 1 + (\ell - 1) q_\nu^{-1}$ *if* $\mathfrak{p}_\nu$ *splits completely in* k_z.

2. $\displaystyle\sum_{\chi_\nu \in C_\ell(\mathcal{O}_\nu^*)} |\Phi(\chi_\nu)|^{-1} = 1 + (\ell - 1) q_\nu^{-1} - \left[\ell - 1 - r_0(\nu)(1 - q_\nu^{-1}) \right] q_\nu^{-d(\nu)-1}$ *if* $\mathfrak{p}_\nu$ *does not split completely in* k_z.

Applying the result of Proposition (4.1) to (4.3), we obtain:

$$P_0 = \prod_{\substack{\mathfrak{p}_\nu | \ell \\ \mathfrak{p}_\nu \notin \mathscr{D}}} \left(1 + (\ell-1)q_\nu^{-1} - \left[\ell - 1 - r_0(\nu)(1-q_\nu^{-1})\right] q_\nu^{-d(\nu)-1}\right)$$
$$\prod_{\substack{\mathfrak{p}_\nu | \ell \\ \mathfrak{p}_\nu \in \mathscr{D}}} \left(1 + (\ell-1)q_\nu^{-1}\right) \prod_{q_\nu \equiv 1 mod\, \ell} \left(1 + (\ell-1)q_\nu^{-1}\right)\left(1 - q_\nu^{-1}\right)^{\ell-1}$$
$$\prod_{q_\nu \not\equiv 1 mod\, \ell} \left(1 - q_\nu^{-f_\nu}\right)^{q_z g_\nu}.$$

Putting all these details together and using Theorem 3.9 and its corollary in Chapter III of [7], we state the main theorem of the thesis.

Theorem 4.1 *Let k be a number field of signature (r_1, r_2). Let $\mathscr{D}$ be the set of prime ideals of k which are totally split in k_z/k. Then*

$$\begin{aligned} \sum_{m \le X^{\ell-1}} N(k, C_\ell; m) &\sim \frac{c(k, C_\ell)(\ell-1)}{(q_z-1)!}(\ell-1)^{q_z-1} X(\log X)^{q_z-1} \\ &\sim \frac{\zeta_{k_z}(1)^{q_z}}{d_z \ell^{r_2+r_z} q_z!} P_0\, X(\log X)^{q_z-1} \quad as \quad X \to \infty. \end{aligned}$$

Here,

$$P_0 = \prod_{\substack{\mathfrak{p}_\nu | \ell \\ \mathfrak{p}_\nu \notin \mathscr{D}}} \left(1 + (\ell-1)q_\nu^{-1} - \left[\ell - 1 - r_0(\nu)(1-q_\nu^{-1})\right] q_\nu^{-d(\nu)-1}\right)$$
$$\prod_{\substack{\mathfrak{p}_\nu | \ell \\ \mathfrak{p}_\nu \in \mathscr{D}}} \left(1 + (\ell-1)q_\nu^{-1}\right) \prod_{q_\nu \equiv 1 mod\, \ell} \left(1 + (\ell-1)q_\nu^{-1}\right)\left(1 - q_\nu^{-1}\right)^{\ell-1}$$
$$\prod_{q_\nu \not\equiv 1 mod\, \ell} \left(1 - q_\nu^{-f_\nu}\right)^{q_z g_\nu},$$

$r_z = 0$ if $\zeta \in k$ and $r_z = r_1 - 1$ otherwise, $d(\nu) = \left\lfloor \frac{e(\nu/\ell)}{\ell - 1} \right\rfloor$, $r_0(\nu)$ is the least nonnegative residue of $e(\nu/\ell)$ modulo $(\ell-1)$ and $\zeta_{k_z}(1)$ denotes the residue of the Dedekind zeta function $\zeta_{k_z}(s)$ at $s = 1$.

4.3 Verifying the result of Theorem 4.1

In this section, we will show that the result in Theorem 4.1 is identical to that of Theorem 1.1 of [1] (also stated below in Theorem 4.2).

For reader's convenience, we restate the theorem:

Theorem 4.2 *Let k be a number field of signature (r_1, r_2). Let $\mathscr{R}$(resp.,$\mathscr{D}$) be the set of prime ideals of k which are ramified (resp., totally split) in k_z/k. Then*

$$\sum_{m \le X^{\ell-1}} N(k, C_\ell; m) \sim c_1 c_2 c_3 c_4 X \log^{q_z - 1} X$$

with

$$\begin{aligned}
c_1 &= \frac{\left(\prod_{d|d_z} \zeta_{k_z[d]}(d)^{\mu(d)}\right)^{q_z}}{d_z \ell^{r_2+r_z} q_z!}, \\
c_2 &= \prod_{\mathfrak{p}\in\mathscr{D}} \left(\left(1 + \frac{\ell-1}{N\mathfrak{p}}\right) \prod_{d|d_z} \left(1 - \frac{1}{N\mathfrak{p}^d}\right)^{(\ell-1)\mu(d)/d} \right), \\
c_3 &= \left(\prod_{\mathfrak{p}\in\mathscr{R}} \prod_{d|d_z} \left(1 - \frac{1}{N\mathfrak{p}^{df(\mathfrak{p}_d/\mathfrak{p})}}\right)^{g(\mathfrak{p}_d/\mathfrak{p})\mu(d)} \right)^{q_z}, \\
c_4 &= \prod_{\substack{\mathfrak{p}|\ell \\ \mathfrak{p}\notin\mathscr{D}}} \left(1 + \frac{\ell-1}{N\mathfrak{p}} - \frac{\ell - 1 - r(e(\mathfrak{p}))(1-1/N\mathfrak{p})}{N\mathfrak{p}^{\lceil e(\mathfrak{p})/(\ell-1)\rceil}}\right);
\end{aligned}$$

here $r_z = 0$ if $\zeta_\ell \in k$, while $r_z = r_1 - 1$ otherwise, and by abuse of notation, for any number field L we write $\zeta_L(1)$ for the residue of the Dedekind zeta function $\zeta_L(s)$ at $s = 1$.

Proposition 4.2 *If c_1, c_2, c_3, c_4 are the constants of Theorem 4.2, then*

$$c_1^* c_2 c_3 c_4 = \zeta_{k_z}(1)^{q_z} P_0$$

where $c_1^* = d_z \ell^{r_2+r_z} q_z! c_1 = \left(\prod_{d|d_z} \zeta_{k_z[d]}(d)^{\mu(d)}\right)^{q_z}$

Proof:

First, by Proposition 4.1,

$$\prod_{\substack{\mathfrak{p}_\nu | \ell \\ \mathfrak{p}_\nu \notin \mathscr{D}}} \sum_{\chi_\nu \in C_\ell(\mathcal{O}_\nu^*)} |\Phi(\chi_\nu)|^{-1} = \prod_{\substack{\mathfrak{p}_\nu | \ell \\ \mathfrak{p}_\nu \notin \mathscr{D}}} \left(1 + (\ell-1)q_\nu^{-1} - \left[\ell - 1 - r_0(\nu)(1-q_\nu^{-1})\right] q_\nu^{-d(\nu)-1}\right).$$

For $\nu \in M(k)_0$, let $\mathfrak{p}$ be the prime ideal of $\mathcal{O}_k$ corresponding to ν. Then $N(\mathfrak{p}) = q_\nu$. Recall that $e(\nu) \equiv r_0(\nu) \, mod \, (\ell - 1)$, where $0 \le r_0(\nu) \le \ell - 2$. Also, recall that $e(\nu) = e(\mathfrak{p}) \equiv r(e(\mathfrak{p})) \, mod \, (\ell-1)$, where $1 \le r(e(\mathfrak{p})) \le \ell - 1$.

If $0 < r_0(\nu) \le \ell - 2$, $r_0(\nu) = r(e(\mathfrak{p}))$ and

$$\left\lceil \frac{e(\nu)}{\ell-1} \right\rceil = \left\lceil d(\nu) + \frac{r_0(\nu)}{\ell-1} \right\rceil = d(\nu) + 1.$$

Hence

$$\begin{aligned} \sum_{\chi_\nu \in C_\ell(\mathcal{O}_\nu^*)} |\Phi(\chi_\nu)|^{-1} &= 1 + (\ell-1)q_\nu^{-1} - \left[\ell - 1 - r_0(\nu)(1-q_\nu^{-1})\right] q_\nu^{-d(\nu)-1} \\ &= \left(1 + \frac{(\ell-1)}{N\mathfrak{p}} - \frac{\ell - 1 - r_0(\nu)(1-1/N\mathfrak{p})}{N\mathfrak{p}^{\lceil e(\mathfrak{p})/(\ell-1)\rceil}}\right). \end{aligned}$$

If $r_0(\nu) = 0$, and consequently $r(e(\mathfrak{p})) = \ell - 1$, then

$$\left\lceil \frac{e(\nu)}{\ell-1} \right\rceil = d(\nu)$$

and

$$\begin{aligned} \sum_{\chi_\nu \in C_\ell(\mathcal{O}_\nu^*)} |\Phi(\chi_\nu)|^{-1} &= 1 + (\ell-1)q_\nu^{-1} - (\ell-1)q_\nu^{-d(\nu)-1} \\ &= 1 + \frac{(\ell-1)}{N\mathfrak{p}} - \frac{\ell-1}{(N\mathfrak{p})^{\lceil \frac{e(\mathfrak{p})}{\ell-1} \rceil + 1}} \\ &= 1 + \frac{(\ell-1)}{N\mathfrak{p}} - \frac{(\ell-1) - r(e(\mathfrak{p}))\left(1 - \frac{1}{N\mathfrak{p}}\right)}{N\mathfrak{p}^{\lceil \frac{e(\mathfrak{p})}{(\ell-1)} \rceil}}. \end{aligned}$$

Therefore,

$$c_4 = \prod_{\substack{\mathfrak{p}_\nu | \ell \\ \mathfrak{p}_\nu \notin \mathscr{D}}} \sum_{\chi_\nu \in C_\ell(\mathcal{O}_\nu^*)} |\Phi(\chi_\nu)|^{-1}.$$

Next note that $\zeta_{k_z}\left((\ell-1)s\right) = \prod_{\nu \in M(k)_0} \left(1 - q_\nu^{-f_\nu(\ell-1)s}\right)^{-g_\nu}.$

Also, $\mathfrak{p} \in \mathscr{D}$ if and only if $\nu \nmid \ell$ and $q_\nu \equiv 1\,(mod\,\ell)$ or $\nu \mid \ell$ and $\zeta \in k_\nu$.

Now

$$c_1^* c_2 c_3 c_4 = \zeta_{k_z}(1)^{q_z} \left(\prod_{\substack{d|d_z \\ d>1}} \zeta_{k_z[d]}(d)^{\mu(d)} \right)^{q_z} \prod_{\mathfrak{p} \in \mathscr{D}} \left(1 + \frac{(\ell-1)}{N\mathfrak{p}}\right) \left(1 - \frac{1}{N\mathfrak{p}}\right)^{\ell-1} \prod_{\substack{\mathfrak{p} \in \mathscr{D} \\ d|d_z \\ d \geq 2}} \left(1 - \frac{1}{N\mathfrak{p}^d}\right)^{(\ell-1)\frac{\mu(d)}{d}} c_3\, c_4.$$

Hence

$$c_1^* c_2 c_3 c_4 = \zeta_{k_z}(1)^{q_z} P_0 \left(\prod_{\substack{d|d_z \\ d>1}} \zeta_{k_z[d]}(d)^{\mu(d)} \right)^{q_z} \prod_{\substack{\mathfrak{p} \in \mathscr{D} \\ d|d_z \\ d \geq 2}} \left(1 - \frac{1}{N\mathfrak{p}^d}\right)^{(\ell-1)\frac{\mu(d)}{d}} \prod_{\mathfrak{p} \notin \mathscr{D}} \left(1 - \frac{1}{N\mathfrak{p}^{f(\mathfrak{p}_1/\mathfrak{p})}}\right)^{-g(\mathfrak{p}_1/\mathfrak{p})q_z} c_3$$

where $\mathfrak{p}_1$ is an ideal in $k_z[1] = k_z$ that divides $\mathfrak{p}$.

We now claim that

$$\left(\prod_{\substack{d|d_z\\d>1}}\zeta_{k_z[d]}(d)^{\mu(d)}\right)^{q_z}\prod_{\substack{\mathfrak{p}\in\mathscr{D}\\d|d_z\\d\geq 2}}\left(1-\frac{1}{N\mathfrak{p}^d}\right)^{(\ell-1)\frac{\mu(d)}{d}}\prod_{\mathfrak{p}\notin\mathscr{D}}\left(1-\frac{1}{N\mathfrak{p}^{f(\mathfrak{p}_1/\mathfrak{p})}}\right)^{-g(\mathfrak{p}_1/\mathfrak{p})q_z}c_3=1. \tag{4.5}$$

Note that

$$\left(\prod_{\substack{d|d_z\\d>1}}\zeta_{k_z[d]}(d)^{\mu(d)}\right)^{q_z}=\left(\prod_{\mathfrak{p}}\prod_{\substack{d|d_z\\d\geq 2}}\left(1-\frac{1}{N\mathfrak{p}^{df(\mathfrak{p}_d/\mathfrak{p})}}\right)^{-g(\mathfrak{p}_d/\mathfrak{p})\mu(d)}\right)^{q_z}.$$

If $\mathfrak{p}\in\mathscr{D}$, $f(\mathfrak{p}_d/\mathfrak{p})=1$, $g(\mathfrak{p}_d/\mathfrak{p})=d_z/d$ and $\dfrac{q_zd_z}{d}=\dfrac{\ell-1}{d}$. Hence these factors (in c_1^*) cancel with the factors

$$\prod_{\mathfrak{p}\in\mathscr{D}}\prod_{\substack{d|d_z\\d\geq 2}}\left(1-\frac{1}{N\mathfrak{p}^d}\right)^{(\frac{\ell-1}{d})\mu(d)}$$

in c_2.

For $\mathfrak{p}\notin\mathscr{D}$, the left side of (4.5) contains

$$\prod_{d|d_z}\left(1-\frac{1}{N\mathfrak{p}^{df(\mathfrak{p}_d/\mathfrak{p})}}\right)^{-g(\mathfrak{p}_d/\mathfrak{p})\mu(d)q_z}.$$

If $\mathfrak{p}\in\mathscr{R}$, these factors cancel with

$$\left(\prod_{\mathfrak{p}\in\mathscr{R}}\prod_{d|d_z}\left(1-\frac{1}{N\mathfrak{p}^{df(\mathfrak{p}_d/\mathfrak{p})}}\right)^{g(\mathfrak{p}_d/\mathfrak{p})\mu(d))}\right)^{q_z}\quad\text{in}\quad c_3.$$

It remains to show that

$$\prod_{\mathfrak{p}\notin\mathscr{D}\cup\mathscr{R}}\prod_{d|d_z}\left(1-q^{-df(\mathfrak{p}_d/\mathfrak{p})}\right)^{-g(\mathfrak{p}_d/\mathfrak{p})\mu(d)}=1$$

where $q=N\mathfrak{p}$.

Proposition 4.3 *Let k be a number field and $k_z = k(\zeta)$. If $\mathfrak{p}$ is a prime ideal in k such that $\mathfrak{p}$ is unramified in k_z/k and $\mathfrak{p}$ is not totally split, then*

$$\prod_{d|d_z} \left(1 - q^{-df(\mathfrak{p}_d/\mathfrak{p})}\right)^{-g(\mathfrak{p}_d/\mathfrak{p})\mu(d)} = 1.$$

Proof: Let $f = f(\mathfrak{p}_1/\mathfrak{p})$ and let $G_\mathfrak{p}$ be the decomposition group of $\mathfrak{p}$ in $Gal(k_z/k)$. Note that $\mid G_\mathfrak{p} \mid = f(\mathfrak{p}_1/\mathfrak{p})$ and that $G_\mathfrak{p} = < \sigma >$ is a cyclic group generated by σ where $\sigma = (\mathfrak{p}, k_z/k)$ is the Artin symbol of $\mathfrak{p}$. Consider a tower of field extensions

$$k \subset k_z[d] \subset k_z$$

and a corresponding chain of prime ideals

$$\mathfrak{p} \subset \mathfrak{p}_d \subset \mathfrak{p}_1.$$

Then

$$G_\mathfrak{p} = \{\tau \in Gal(k_z/k) : \tau(\mathfrak{p}_1) = \mathfrak{p}_1\}$$

and

$$G_{\mathfrak{p}_d} = \{\tau \in Gal(k_z/k_z[d]) : \tau(\mathfrak{p}_1) = \mathfrak{p}_1\} = G_{\mathfrak{p}_1} \cap Gal(k_z/k_z[d]).$$

Note that $Gal(k_z/k)$ is isomorphic to a subgroup of $(\mathbf{Z}/\ell\mathbf{Z})^*$. Hence $G = Gal(k_z/k)$ is a cyclic group of order d_z. Consequently $Gal(k_z/k_z[d])$ is a cyclic subgroup G of order d. Also $G_\mathfrak{p}$ is a cyclic subgroup of G of order $f = f(\mathfrak{p}_1/\mathfrak{p})$. Hence

$$G_\mathfrak{p} \cap Gal(k_z/k_z[d])$$

is the cyclic group of order (f, d), which implies that

$$f(\mathfrak{p}_1/\mathfrak{p}_d) = |G_{\mathfrak{p}_d}| = (f, d).$$

But

$$f(\mathfrak{p}_1/\mathfrak{p}_d) f(\mathfrak{p}_d/\mathfrak{p}) = f(\mathfrak{p}_1/\mathfrak{p}) = f.$$

Hence $f(\mathfrak{p}_d/\mathfrak{p}) = \dfrac{f}{(f,d)}$.

Moreover,

$$e(\mathfrak{p}_d/\mathfrak{p})f(\mathfrak{p}_d/\mathfrak{p})g(\mathfrak{p}_d/\mathfrak{p}) = [k_z[d]:k] = \frac{d_z}{d},$$

and since $e(\mathfrak{p}_d/\mathfrak{p}) = 1$,

$$g(\mathfrak{p}_d/\mathfrak{p}) = \frac{d_z(f,d)}{fd} = \frac{d_z}{[f,d]}.$$

Therefore the product in the proposition becomes

$$\prod_{d|d_z}\left(1 - q^{\frac{-df}{(f,d)}}\right)^{-\frac{d_z(f,d)}{df}\mu(d)} = \prod_{d|d_z}\left(1 - q^{-[f,d]}\right)^{-\frac{d_z\mu(d)}{[f,d]}}.$$

Since $f|d_z$ and $d|d_z$, $[f,d]|d_z$. Therefore the above product can be written as

$$\prod_{m|d_z}\prod_{[f,d]=m}\left(1 - q^{-m}\right)^{-\frac{d_z\mu(d)}{m}}.$$

To complete the proposition, we only need to show that if $f > 1$, then

$$\sum_{[f,d]=m}\mu(d) = 0.$$

Consider the prime factorization of f, d and m : $f = \prod p^{\alpha(p)}$, $m = \prod p^{\gamma(p)}$ and $d = \prod p^{\beta(p)}$. $[f,d] = m$ if and only if for all p, $max\{\alpha(p), \beta(p)\} = \gamma(p)$. This means that if $\alpha(p) = \gamma(p)$, then $0 \leq \beta(p) \leq \gamma(p)$ and if $\alpha(p) < \gamma(p)$, then $\beta(p) = \gamma(p)$. We note that μ is multiplicative and the above conditions are independent of each other for all primes p. Hence

$$\sum_{[f,d]=m}\mu(d) = \prod_p \begin{cases} \sum_{\beta=0}^{\alpha(p)}\mu(p^\beta) & \text{if } \alpha(p) = \gamma(p) \\ \mu(p^{\gamma(p)}) & \text{if } \alpha(p) < \gamma(p). \end{cases} \tag{4.6}$$

But since $f > 1$ there exists a prime p such that $\alpha(p) \geq 1$. Then $\displaystyle\sum_{\beta=0}^{\alpha(p)}\mu(p^\beta) = 0$.

Also, if $\gamma(p) > \alpha(p)$, then $\gamma(p) \geq 2$ and $\mu(p^{\gamma(p)}) = 0$. In any event, one of the factors in the product that gives $\sum_{[f,d]=m} \mu(d)$ is 0 and therefore

$$\sum_{[f,d]=m} \mu(d) = 0.$$

This concludes the proof of the proposition.

REFERENCES

[1] Cohen, H., Diaz y Diaz, F., and Oliver, M., On the density of discriminants of cyclic extensions of prime degree, J. Reine Angew. Math. **550** (2002), 169-209.

[2] Cohen, H., Diaz y Diaz, F., and Oliver, M., Cyclotomic extensions of number fields, Indag Math (NS). **14** (2003), 183 - 196.

[3] Cohn, H., The density of abelian cubic fields, Proc. Amer. Math. Soc. **5** (1954), 476-477.

[4] Hasse, H., Führer, Disckriminante und Verzweigungskörper relativ-Abelscher Zahlkörper. J. Reine Angew. Math. **162** (1930), 169-184.

[5] Lang, S., Algebraic Number Theory, 2nd edition, Springer- Verlag, Berlin - Heidelberg - New York, 1994.

[6] Mäki, S., On the Density of Abelian Number Fields, Thesis, University of Helsinki, 1985.

[7] Narkiewicz, W., Number Theory, World Scientific, Singapore, 1983.

[8] Narkiewicz, W., Elementary and Analyic Theory of Algebraic Numbers, Springer-Verlag, Berlin - Heidelberg - New York, 1990.

[9] Tatuzawa, T., On the conductor-discriminant formula, J. Reine Angew. Math. **262/263** (1973), 436-440

[10] Taylor, M., On the equidistribution of Frobenius in cyclic extensions of a number field, J. London Math. Soc. **29** (1984), 211-223.

[11] Urazbaev, B., On the density of distribution of cyclic fields of prime order, Izv. Akad. Nauk. Kazakh. SSR Ser. Math. Meh. 5, **62** (1951), 37-52.

[12] Weil, A., Basic Number Theory, Classics in Mathematics, Springer-Verlag, Berlin - Heidelberg - New York, 1990.

[13] Wright, D., Distribution of discriminants of abelian extensions, Proc. London Math. Soc. **58** (1989), 17-50.