Left-orderability of Dehn surgeries on knot complements

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ABSTRACT

Let M be a $\mathbb{Q}$-homology solid torus. In this thesis, we give a cohomological criterion for the existence of an interval of left orderable Dehn surgeries on M. We apply this criterion to prove that the two-bridge knot that corresponds to the continued fraction $[1, 1, 2, 2, 2j]$ for $j \geq 1$ and the $(-3, 3, 2j + 1)$-pretzel knot admit an interval of left orderable Dehn surgeries. These two families of knots gives some positive evidence for a question of Xinghua Gao.

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Chapter 1

INTRODUCTION

A non-trivial group G is *left-orderable* if it admits a strict total ordering on the group elements such that $g < h$ implies that $fg < fh$ for all elements $f, g, h \in G$. By convention, we say that a 3-manifold M is left-orderable if and only if $\pi_1(M)$ is left-orderable. Left orderability arises naturally in the study of low-dimensional topology, foliation theory, and group theory. Well-known examples of left orderable groups include torsion-free abelian groups, non-abelian free groups, surface groups and the group of orientation preserving homeomorphisms of the real line. In 3-manifold topology, left orderability is an important concept due to its role in the L-space conjecture.

Conjecture 1.1 (The L-space conjecture)**.** *For an irreducible $\mathbb{Q}$-homology 3-sphere M, the following are equivalent*

(i) $\pi_1(M)$ *is left orderable.*

(ii) M *is not an L-space.*

(iii) M *admits a coorientable taut foliation.*

An L-space is a $\mathbb{Q}$-homology 3-sphere with $\dim \widehat{HF}(M) = |H_1(M;\mathbb{Z})|$ where $\widehat{HF}(M)$ is the Heegaard Floer homology of M [27, Definition 1.1]. A foliation $\mathcal{F}$ of a 3-manifold M is a decomposition of M into 2-dimensional submanifolds locally modelled on $\mathbb{R}^2 \times \mathbb{R}$. A foliation of a 3-manifold M is said to be *taut* if for every leaf λ of $\mathcal{F}$ there is a circle S^1_λ transverse to $\mathcal{F}$ which intersects λ [7, Definition 4.25].

The equivalence between (i) and (ii) was conjectured in [4]. The equivalence between (ii) and (iii) was formulated as a question by Ozsváth and Szabó which was upgraded to a conjecture in a survey by Juhazs [23]. There has been a substantial amount of evidence in favor of Conjecture 1.1. For example, Conjecture 1.1 holds for all graph manifolds [3] and [17]. Recently, Dunfield has verified Conjecture 1.1 for at least 191,089 hyperbolic $\mathbb{Q}$-homology spheres [14, Theorem 1.6].

In view of Conjecture 1.1, there have been a lot of ideas developed to study left orderability of 3-manifold groups. It is a well-known fact that a countable group is left orderable if and only if it embeds in the group of orientation-preserving homeomorphisms of the real line [16, Theorem 6.8]. In the case of an irreducible compact 3-manifold, its fundamental group is left orderable if and only if it admits a non-trivial homomorphism onto a left orderable group [5, Theorem 1.1]. In particular, all 3-manifolds with positive first Betti number are left orderable. Therefore, it is interesting to construct left orders on $\mathbb{Q}$-homology spheres, for example those constructed by Dehn filling on and from taking cyclic branched covers of a $\mathbb{Q}$-homology solid torus.

A fruitful way to build left-orderings on $\mathbb{Q}$-homology spheres is by lifting $\mathrm{PSL}_2(\mathbb{R})$ representations to $\widetilde{\mathrm{PSL}_2}(\mathbb{R})$. This strategy has been employed with a lot of success, for example see [29, 30, 22]. Recently, Culler–Dunfield and Gao, building on the work of Culler and Dunfield, have introduced the idea of using the extension locus of a compact 3-manifold with torus boundary M to order families of $\mathbb{Q}$-homology spheres arising by doing Dehn filling on M [9, 15]. Furthermore, they gave several criteria implying the existence of intervals of left orderable Dehn fillings on M. To state their results, we need the following definition:

Definition 1.2. *A compact 3-manifold Y has* few characters *if each positive-dimensional component of the* $\mathrm{PSL}_2(\mathbb{C})$*-character variety $X(Y)$ consists entirely of characters of reducible representations. An irreducible $\mathbb{Q}$-homology solid torus M is called* longitudinally rigid *when $M(0)$ has few characters where $M(0)$ is the closed manifold* *obtained from M by doing Dehn filling along the homological longitude.*

We summarize their results in the following:

Theorem 1.3 ([9, Theorem 7.1] and [15, Theorem 5.1]). *Suppose that M is a longitudinally rigid irreducible $\mathbb{Z}$-homology solid torus. Then the following are true:*

1. *If the Alexander polynomial of M has a simple root $\xi \neq 1$ on the unit circle, then there exists $a > 0$ such that for every rational $r \in (-a, 0) \cup (0, a)$ the Dehn filling $M(r)$ is left orderable.*

2. *If the Alexander polynomial of M has a simple positive real root $\xi \neq 1$, then there exists $a > 0$ and a nonempty interval $(-a, 0]$ or $[0, a)$ such that for every rational r in the interval, the Dehn filling $M(r)$ is left orderable.*

Remark 1.4. *In fact, their techniques also apply to the case where M is a $\mathbb{Q}$-homology solid torus with some further hypothesis on ξ in the first statement. The full version of the second statement is stated below in Theorem 2.20.*

Culler and Dunfield also proved the following criterion for left orderability:

Theorem 1.5 ([9, Theorem 1.4]). *Suppose that M is a hyperbolic $\mathbb{Z}$-homology solid torus, whose trace field has a real embedding, then there exists $a > 0$ such that for every rational $r \in (-a, 0) \cup (0, a)$ the Dehn filling $M(r)$ is left orderable.*

In view of Theorem 1.3, it is natural to ask when a $\mathbb{Q}$-homology solid torus is longitudinally rigid. Since the character variety is notoriously hard to compute, (see for example [2, 8]), longitudinal rigidity is difficult to study in a general setting. Culler and Dunfield gave a topological condition which implies longitudinal rigidity. In particular, they introduced the following concept:

Definition 1.6. *Let M be a knot exterior in a $\mathbb{Q}$-homology sphere. We say that M is* lean *if the longitudinal Dehn filling $M(0)$ is prime and every closed essential surface in $M(0)$ is a fiber in a fibration over S^1.*

For example, the $(-2, 3, 2s+1)$-pretzel knots were shown to be lean for $s \geq 3$, so there is an interval of about 0 of left-orderable Dehn surgeries on these knot complements [26, Theorem 4]. However as remarked in [9, Section 1.6], this leanness condition is rather restrictive. In particular, for a knot complement K in S^3 being lean implies that K fibers. Nevertheless, the first statement of Theorem 1.3 was proved to be true without the condition of longitudinal rigidity by Herald and Zhang [18, Theorem 1]. Motivated by the result of Herald and Zhang, Xinghua Gao asked:

Question 1.7. *[15, Section 7] Can the longitudinal rigidity condition be dropped from the second statement of Theorem 1.3? Is it possible to prove that*

$$H^1(\pi_1(M(0)); \mathfrak{sl}_2(\mathbb{R})_\rho) = 0$$

where ρ is the non-abelian reducible representation of $\pi_1(M)$ coming from the root of the Alexander polynomial?

Following the suggestion in this question, we say that $\mathbb{Q}$-homology solid torus is locally longitudinally rigid at a root of the Alexander polynomial if

$$H^1(\pi_1(M(0)); \mathfrak{sl}_2(\mathbb{C})_\rho) = 0$$

where ρ is a non-abelian reducible representation coming from this root, see Definition 2.25 for a precise definition. In fact, we prove that the second item of Theorem 1.3 still holds true under this weakened hypothesis.

Theorem 1.8. *Suppose that M is an irreducible $\mathbb{Q}$-homology solid torus and that the Alexander polynomial of M has a simple positive real root ξ. Furthermore, suppose that M locally longitudinally rigid at ξ. Then there exists $a > 0$ and a nonempty interval $(-a, 0]$ or $[0, a)$ such that for every rational r in the interval, the Dehn filling $M(r)$ is left orderable.*

We will give a sketch of the proof of Theorem 1.8 to give some intuition behind the locally longitudinally rigid condition and to motivate the preliminary work in Chapter 2. For simplicity, let M be an exterior of a knot in S^3. To produce an interval of left-orderable Dehn surgeries on M, we show that there exists a path of representation $\rho_t : \pi_1(M) \to \mathrm{PSL}_2(\mathbb{R})$ such that:

- The entire path of representations lifts to a path $\widetilde{\rho}_t : \pi_1(M) \to \widetilde{\mathrm{PSL}_2}(\mathbb{R})$.
- The representations $\widetilde{\rho}_t$ factor through $\pi_1(M(r))$ for r in some interval near 0.

The path of representations ρ_t is obtained from deforming non-abelian reducible representations of $\pi_1(M)$. These non-abelian reducible representations are parametrized by the roots of the Alexander polynomial of the knot [6, 12]. The deformation theory of these representations over $\mathbb{C}$ has been studied in [20, 21]. We will use and adapt the work from [20, 21] to show that in fact when a root of the Alexander polynomial is simple, a path of real representations ρ_t starting from the corresponding non-abelian reducible representation exists. Moreover, this path of representations ρ_t is tranverse to the orbit of the non-abelian reducible representation under the conjugation action of $\mathrm{PSL}_2(\mathbb{R})$. Since the non-abelian reducible representation lifts to $\mathrm{PSL}_2(\mathbb{R})$, the path ρ_t lifts to a path $\widetilde{\rho}_t$. To show that the representations $\widetilde{\rho}_t$ factor through $\pi_1(M(r))$ for r in some interval near 0, we project this path to $H^1(\partial M; \mathbb{R})$ and get a path in the holonomy extension locus as defined in [15]. The intersection points between the holonomy extension locus and the lines with rational slope going through the origin in $H^1(\partial M; \mathbb{R})$ corresponds to left-orderable Dehn surgeries [15]. Finally, we need to ensure that the path in the holonomy extension locus on $H^1(\partial M; \mathbb{R})$ is not contained in the horizontal axis, which corresponds to representations factoring through $\pi_1(M(0))$. In contrast, we would like to see that the path in the holonomy extension locus on $H^1(\partial M; \mathbb{R})$ is transverse to the horizontal axis. The locally longitudinally rigid condition is a transversality condition. In particular, locally longitudinally rigidity implies that all representations in the neighborhood in $\mathrm{Hom}(\pi_1(M(0)); \mathrm{PSL}_2(\mathbb{R})$ of the non-abelian reducible representation must all be conjugate to the non-abelian reducible representation [31]. Since the path ρ_t is transverse to the orbit of the non-abelian reducible representation, locally longitudinally rigidity implies that the path in the holonomy extension locus must be transverse to the horizontal axis as well. This gives us the existence of an interval of left-orderable Dehn surgeries near 0.

As an application, we apply this result to produce an interval of left orderable Dehn surgeries on an infinite family of two-bridge knot complements.

Theorem 1.9. *For every two-bridge knot K_j corresponding to the continued fraction $[1,1,2,2,2j]$ where $j \geq 1$, there exists $a > 0$, possibly depending on j, and a nonempty interval $(-a, 0]$ or $[0, a)$ such that for every rational r in the interval, the Dehn filling $M(r)$ is left orderable.*

Remark 1.10. *As we will see in Lemma 3.2, the Alexander polynomial of K_j has all simple positive real roots, and is not monic for $j \geq 2$. In particular, the complement of K_j is not lean for $j \geq 2$. Furthermore, the trace field of K_j for $1 \leq j \leq 30$ has no real places, and it is likely that the trace fields of all knots in this family share this property. Therefore, Theorem 1.9 is not a direct consequence of Theorem 1.3 nor Theorem 1.5. The family of two-bridge knots $[1,1,2,2,2j]$ is a genuinely new family of knots with an interval left orderable Dehn surgeries which cannot be obtained from prior techniques.*

We will also investigate Question 1.7 among 3-generated knot groups. We provide further evidence to an affirmative answer to Question 1.7:

Theorem 1.11. *Let P_j be the $(-3, 3, 2j+1)$-pretzel knot where $j \geq 1$. Then the Alexander polynomial of P_j has the form*

$$(\tau - 2)(2\tau - 1)$$

Moreover, P_j is locally longitudinally rigid at 2. Consequently, there exists $a > 0$, possibly depending on j, and a nonempty interval $(-a, 0]$ or $[0, a)$ such that for every rational r in the interval, the Dehn filling $M(r)$ is left orderable.

The thesis is organized as follows. In Chapter 2, we start by recalling the work in [20] and [21] which is done for $\mathrm{PSL}_2(\mathbb{C})$-representations and adapt them to $\mathrm{PSL}_2(\mathbb{R})$-representations. In Section 2.4, we will also recall the construction and important properties of the holonomy extension locus from [15]. In Section 2.4.1, we illustrate the application of the holonomy extension locus to prove left-orderability of certain Dehn surgeries on a knot exterior using the example of the figure-8 knot complement. We will give a proof of Theorem 1.8 in Section 2.4.2. In Chapter 3, we will give the proof of Theorem 1.9 and Theorem 1.11. We note that Theorem 1.8 and Theorem 1.9 in the thesis appear in [24].

Chapter 2

PRELIMINARIES

In this chapter, we start by recalling the work in [20] and in [21] which studies non-abelian reducible representations in $\mathrm{Hom}(\pi_1(M); \mathrm{PSL}_2(\mathbb{C}))$, where M is a compact 3-manifold with torus boundary. We adapt these results to the real setting. In particular, we will use these results to construct a path of real representations starting at the non-abelian reducible representation.

To do this, we first consider the problem of constructing a representation into $\mathrm{PSL}_2(\mathbb{R}[[t]])$, where $\mathbb{R}[[t]]$ is the ring of formal power series, such that by specializing $t = 0$ we get the non-abelian reducible representation. The representation into $\mathrm{PSL}_2(\mathbb{R}[[t]])$ is called a formal deformation, see Definition 2.3. It turns out that the obstruction to constructing a formal deformation can be described by an infinite series of obstruction classes, see Proposition 2.8. Under the assumption that the root of the Alexander polynomial is positive and simple, we show that these obstruction classes vanish on the boundary torus and in fact vanish on the entire manifold, see Lemma 2.27. Therefore, we obtain a formal deformation of the non-abelian reducible representation. Using Theorem 2.10, we can convert a formal deformation into a path of representations, see Proposition 2.11.

Following our description of the work in [20] and in [21], we will recall the construction of the holomy extension locus in [15], see Definition 2.15, and the practical tool to prove left-orderability of Dehn surgery on the exterior of a knot, see Lemma 2.18. We will illustrate the application of Lemma 2.18 and the intuition behind the proof of Theorem 1.8 by using the example of the figure 8-knot complement Section 2.4.1. In Section 2.4.2, we will prove Theorem 1.8.

2.1 Group Cohomology And PSL_2-Representation Varieties

We set $G = \mathrm{PSL}_2(\mathbb{R})$ and $G_{\mathbb{C}} = \mathrm{PSL}_2(\mathbb{C})$ throughout this thesis. For a compact manifold M and a group H, we let $R_H(M) = \mathrm{Hom}(\pi_1(M), H)$ be the representation variety. When $H = G_{\mathbb{C}}$, we denote $R(M) := R_{G_{\mathbb{C}}}(M)$. Since $\pi_1(M)$ is finitely generated, $R(M)$ can be identified with an algebraic subset in some affine space $\mathbb{C}^N$. The group G acts on $R(M)$ by conjugation. Let us consider the minimal Hausdorff quotient $X(M) := R(M)//G_{\mathbb{C}}$ and the quotient map $\pi : R(M) \to X(M)$. Given a representation $\rho \in R(M)$, a character of ρ is the map $\chi_\rho : \pi(M) \to \mathbb{C}$ defined by $\chi_\rho(\gamma) = \mathrm{tr}^2(\rho(\gamma))$. By [19, Theorem 1.3], there exists a bijection between between the points of $X(M)$ and the characters of representations in $R(M)$ such that the point $\pi(\rho) = [\rho]$ corresponds to χ_ρ. Therefore, we refer to $X(M)$ as the $\mathrm{PSL}_2(\mathbb{C})$-character variety of M.

Let Γ be a group and $\rho : \Gamma \to G_{\mathbb{C}}$ be a representation. The Lie algebra of $G_{\mathbb{C}}$ can be identified with the set of trace-less 2-by-2 matrices over $\mathbb{C}$. Using the adjoint representation, the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ becomes a Γ-module by

$$\gamma \cdot v = \rho(\gamma) v \rho(\gamma)^{-1}.$$

We denote this Γ-module by $\mathfrak{sl}_2(\mathbb{C})_\rho$. The space of 1-cocycles is

$$Z^1(\Gamma; \mathfrak{sl}_2(\mathbb{C})_\rho) = \{z : \Gamma \to \mathfrak{sl}_2(\mathbb{C}) \mid z(\gamma\gamma') = z(\gamma) + \gamma \cdot z(\gamma') \ \forall \gamma, \gamma' \in \Gamma\}.$$

Alternatively when Γ is a finitely presented group, we can also describe the space of cocycles as maps $\Gamma \to \mathfrak{sl}_2(\mathbb{C})$ defined on a generating set that satisfy the group relations of Γ. In particular, suppose that $\Gamma = \langle \gamma_1, \ldots, \gamma_n \mid w_1(\gamma_i), \ldots, w_k(\gamma_i) \rangle$ is a finite presentation and that $z(\gamma_i) = v_i \in \mathfrak{sl}_2(\mathbb{C})$. Given any element $w \in \Gamma$, we can express w as a word $w(\gamma_i)$ in the generators γ_i's of Γ. The equation

$$z(\gamma\gamma') = z(\gamma) + \gamma \cdot z(\gamma') \quad \forall \gamma, \gamma' \in \Gamma \tag{2.1}$$

determines the image of $z(w)$. This gives us a well-defined cocycle on Γ if and only if $z(w_j) = 0$ for all relations w_j of Γ, see [31, Equation 4]. The space of 1-coboundaries is

$$B^1(\Gamma; \mathfrak{sl}_2(\mathbb{C})_\rho) = \{b : \Gamma \to \mathfrak{sl}_2(\mathbb{C}) \mid \exists\, v \in \mathfrak{sl}_2(\mathbb{C}),\ b(\gamma) = (\gamma - 1_\Gamma) \cdot v\}. \tag{2.2}$$

Finally, the group cohomology is defined by

$$H^1(\Gamma; \mathfrak{sl}(\mathbb{C})_\rho) = Z^1(\Gamma; \mathfrak{sl}_2(\mathbb{C})_\rho)/B^1(\Gamma; \mathfrak{sl}_2(\mathbb{C})_\rho).$$

Definition 2.1. *Suppose that V is an affine algebraic variety in $\mathbb{C}^n$. Let*

$$I(V) = \{f \in \mathbb{C}[x_1, \ldots, x_n] \mid f(x) = 0 \quad \forall x \in V\}$$

be the vanishing ideal of V. Define the Zariski tangent space to V at p to be the vector space of derivatives of polynomials.

$$T_p^{Zar}(V) = \left\{ \left.\frac{d\gamma}{dt}\right|_{t=0} \in \mathbb{C}^n \mid \gamma \in (\mathbb{C}[t])^n, \gamma(0) = p \text{ and } f \circ \gamma \in t^2\mathbb{C}[t]\ \forall f \in I(V) \right\}.$$

It was observed by Weil in [31] that for any Lie group H and $\rho \in R_H(M)$ the Zariski tangent space embed in the space of 1-cocycles $Z^1(\pi_1(M); \mathfrak{h})$ where $\mathfrak{h}$ is the Lie algerba of H. In particular, we have the following inequalities

$$\dim Z^1(\Gamma; \mathfrak{sl}_2(\mathbb{C})_\rho) \geq \dim T_\rho^{Zar}(R(\Gamma)) \geq \dim_\rho R(\Gamma).$$

Definition 2.2. *We say that a representation $\rho \in R(\Gamma)$ is a* regular point *if*

$$\dim Z^1(\Gamma; \mathfrak{sl}_2(\mathbb{C})_\rho) = \dim_\rho R(\Gamma).$$

We say that a representation $\rho \in R(\Gamma)$ is a smooth point *if*

$$\dim T_\rho^{Zar}(R(\Gamma)) = \dim_\rho R(\Gamma).$$

In particular, if a representation is a regular point then it is also a smooth point of $R(\Gamma)$.

2.2 Formal Deformation Of Representation

We will review some background materials on formal deformations of representations and integrability of cocycles. The concept of integrable cocycles will be important to building a certain path of representations required in the proof of Theorem 1.8, see also Lemma 2.27. For this discussion, let Γ be a finitely presented group, $A_\infty := \mathbb{R}[[t]]$ be the ring of formal power series over $\mathbb{R}$, and $G_\infty := \mathrm{PSL}_2(A_\infty)$.

Definition 2.3. *Let $\rho : \Gamma \to G$ be a representation. A* formal deformation of ρ *is a representation $\rho_\infty : \Gamma \to G_\infty$ such that $\rho = p_0 \circ \rho_\infty$ where $p_0 : G_\infty \to G$ is the homomorphism induced by evaluating the formal power series at $t = 0$.*

For any formal deformation $\rho_\infty : \Gamma \to G_\infty$ of ρ, we can write

$$\rho_\infty(\gamma) = \exp\left(\sum_{i=1}^{\infty} t^i u_i(\gamma)\right)\rho(\gamma) \tag{2.3}$$

where $u_i : \Gamma \to \mathfrak{sl}_2(\mathbb{R})_\rho$ is in $C^1(\Gamma; \mathfrak{sl}_2(\mathbb{R})_\rho)$. We observe that

Lemma 2.4. *The function $u_1 : \Gamma \to \mathfrak{sl}_2(\mathbb{R})$ defines a cocycle on Γ.*

Proof. Let $\gamma, \gamma' \in \Gamma$ be any two elements. Since $\rho_\infty : \Gamma \to G_\infty$ is a group homomorphism, we have $\rho_\infty(\gamma\gamma') = \rho_\infty(\gamma)\rho_\infty(\gamma')$, or equivalently,

$$\exp\left(\sum_{i=1}^{\infty} t^i u_i(\gamma\gamma')\right)\rho(\gamma\gamma') = \exp\left(\sum_{i=1}^{\infty} t^i u_i(\gamma)\right)\rho(\gamma)\exp\left(\sum_{i=1}^{\infty} t^i u_i(\gamma')\right)\rho(\gamma')$$

for all $\gamma, \gamma' \in \Gamma$. Applying $\frac{d}{dt}|_{t=0}$ on both sides, we get

$$\begin{aligned} u_1(\gamma\gamma')\rho(\gamma\gamma') &= u_1(\gamma)\rho(\gamma)\rho(\gamma') + \rho(\gamma)u_1(\gamma')\rho(\gamma') \\ &= [u_1(\gamma) + \gamma \cdot u_1(\gamma')]\rho(\gamma)\rho(\gamma') \end{aligned}$$

for all $\gamma, \gamma' \in \Gamma$. Therefore, $u_1 \in Z^1(\Gamma; \mathfrak{sl}_2(\mathbb{R})_\rho)$. □

Conversely, we have the following definition:

Definition 2.5. *A cocycle $u_1 \in Z^1(\Gamma; \mathfrak{sl}_2(\mathbb{R})_\rho)$ is* integrable *if there exists a formal deformation ρ_∞ of ρ so that u_1 is the linear term in Equation* (2.3)*. In this case, we say that ρ_∞ is a formal deformation of ρ with leading term u_1.*

Given a representation $\rho : \Gamma \to G$ and a cocycle $u_1 \in Z^1(\Gamma; \mathfrak{sl}_2(\mathbb{R})_\rho)$, the existence of a formal deformation of ρ with leading term u_1 is equivalent to the vanishing of a series of obstruction classes in $H^2(\Gamma; \mathfrak{sl}_2(\mathbb{R})_\rho)$ [21, Proposition 3.1 and Corollary 3.2]. Before constructing this series of obstruction classes, we set up some notations. Consider the ring $A_k := \mathbb{R}[[t]]/(t^{k+1})$ for $k \in \mathbb{Z}_{\geq 0}$, the Lie group $G_k := \mathrm{PSL}_2(A_k)$, and its Lie algebra $\mathfrak{g}_k := \mathfrak{sl}_2(A_k) = \{\sum_{i=0}^k t^i X_i \mid X_i \in \mathfrak{sl}_2(\mathbb{R})\}$. Note that $G_0 = G$ and $\mathfrak{g}_0 = \mathfrak{sl}_2(\mathbb{R})$. Suppose that we have a representation

$$\rho_k : \Gamma \to G_k$$

then $\mathfrak{g}_k$ becomes a Γ-module by letting Γ act on $\mathfrak{g}_k$ via conjugation by $\rho_k(\Gamma)$. We will write $\mathfrak{g}_k^{\rho_k}$ when we view $\mathfrak{g}_k$ as a Γ-module. For every $k > l$, we have the following projection maps:

$$\pi_{k,l} : G_k \to G_l,\ \pi_k : G_{k+1} \to G_k,\ \text{and } p_k : G_\infty \to G_k.$$

Let $\rho : \Gamma \to G_0$ and $u_i : \Gamma \to \mathfrak{g}_0$ for $i = 1, \ldots, k$. We can define a function $\widetilde{\rho}_k := \widetilde{\rho}_k^{(\rho; u_1, \ldots, u_k)} : \Gamma \to G_\infty$ by

$$\widetilde{\rho}_k(\gamma) := \exp\left(\sum_{i=1}^k t^i u_i(\gamma)\right) \rho(\gamma).$$

For all $j \geq 0$, we get a function $\rho_j := \rho_j^{(\rho; u_1, \ldots, u_k)} : \Gamma \to G_j$ given by $\rho_j := p_j \circ \widetilde{\rho}_k$. Note that both $\widetilde{\rho}_k$ and ρ_j are not necessarily a homomorphism. The following lemma gives a necessary and sufficient condition to extend a homomorphism ρ_k to a homomorphism ρ_{k+1}. More precisely, we have

Lemma 2.6 ([21, Lemma 3.3]). *Given $u_1, \ldots, u_{k+1} : \Gamma \to \mathfrak{g}_0$, suppose that*

$$\rho_k := p_k \circ \widetilde{\rho}_{k+1} : \Gamma \to G_k$$

is a homomorphism. Then $\rho_{k+1} := p_{k+1} \circ \widetilde{\rho}_{k+1} : \Gamma \to G_{k+1}$ is homomorphism if and only if $U_k : \Gamma \to \mathfrak{g}_k^{\rho_k}$, defined by

$$U_k(\gamma) = u_1(\gamma) + 2tu_2(\gamma) + \cdots + (k+1)t^k u_{k+1}(\gamma),$$

is a 1-cocycle.

Proof. The proof is similar to that of Lemma 2.4. The function ρ_{k+1} is a homomorphism if and only if

$$\widetilde{\rho}_{k+1}(\gamma)\widetilde{\rho}_{k+1}(\gamma') = \widetilde{\rho}_{k+1}(\gamma\gamma') \mod t^{k+2}.$$

We observe that:

$$\begin{aligned} \frac{d}{dt}\widetilde{\rho}_{k+1}(\gamma) &= \frac{d}{dt}\exp\left(\sum_{i=1}^{k+1} t^i u_i(\gamma)\right)\rho(\gamma) \\ &= U_k(\gamma)\exp\left(\sum_{i=1}^{k+1} t^i u_i(\gamma)\right)\rho(\gamma). \end{aligned}$$

Applying $\frac{d}{dt}$ on both sides, we get

$$U_k(\gamma)\widetilde{\rho}_{k+1}(\gamma)\widetilde{\rho}_{k+1}(\gamma') + \widetilde{\rho}_{k+1}(\gamma)U_k(\gamma')\widetilde{\rho}_{k+1}(\gamma') = U_k(\gamma\gamma')\widetilde{\rho}_{k+1}(\gamma\gamma') \mod t^{k+1}.$$

Since $A_k = \mathbb{R}[[t]]/(t^{k+1})$, the equality above is equivalent to

$$[U_k(\gamma) + \rho_k(\gamma)U_k(\gamma')\rho_k(\gamma)^{-1})]\rho_k(\gamma)\rho_k(\gamma') = U_k(\gamma\gamma')\rho_k(\gamma\gamma').$$

Since ρ_k is a group homomorphism, the equality above is equivalent to the condition that $U_k : \Gamma \to \mathfrak{g}_k^{\rho_k}$ is a cocycle in $Z^1(\Gamma; \mathfrak{g}_k^{\rho_k})$. $\square$

In view of the above lemma, we have the following notation. Given a collection $u_i : \Gamma \to \mathfrak{g}_0$ for $i = 1, \cdots, k$, we define $\widetilde{U}_{k-1} := \widetilde{U}_{k-1}^{(u_1,\ldots,u_k)} : \Gamma \to \mathfrak{g}_\infty$ by

$$\widetilde{U}_{k-1}(\gamma) = u_1(\gamma) + 2tu_2(\gamma) + \cdots + kt^{k-1}u_k(\gamma).$$

For all $j \geq 0$, we have a function $U_j := U_j^{(u_1,\ldots,u_k)} := p_j \circ \widetilde{U}_{k-1}$.

Now fix a homomorphism $\rho : \Gamma \to G_0$. Let $\rho_k := \rho_k^{(\rho;u_1,\ldots,u_k)} : \Gamma \to G_k$ be a homomorphism. The existence of a cochain $u_{k+1} : \Gamma \to \mathfrak{g}_0$ such that $\rho_{k+1}^{(\rho;u_1,\ldots,u_{k+1})}$ is homomorphism is governed by an obstruction class in the second cohomology. Consider the following short exact sequence of Γ-modules

$$0 \to \mathfrak{sl}_2(\mathbb{R})_\rho \xrightarrow{\alpha_k} \mathfrak{g}_k^{\rho_k} \xrightarrow{\pi_{k-1}} \mathfrak{g}_{k-1}^{\rho_{k-1}} \to 0$$

where $\alpha_k(X) = t^k X$. This short exact sequence induces the long exact sequence in cohomology

$$\cdots \to H^1(\Gamma; \mathfrak{g}_k^{\rho_k}) \xrightarrow{(\pi_{k-1})_*} H^1(\Gamma; \mathfrak{g}_{k-1}^{\rho_{k-1}}) \xrightarrow{\beta_{k-1}} H^2(\Gamma; \mathfrak{sl}_2(\mathbb{R})_\rho) \to \cdots$$

Since ρ_k is a homomorphism, $U_{k-1}^{(u_1,\ldots,u_k)} \in Z^1(\Gamma; \mathfrak{g}_{k-1}^{\rho_{k-1}})$ by Lemma 2.6. We can define an obstruction class as follows:

Definition 2.7. *[21, Definition 3.4] Given $u_1, \ldots, u_k : \Gamma \to \mathfrak{sl}_2(\mathbb{R})$ such that*

$$\rho_k := \rho_k^{(\rho;u_1,\ldots,u_k)} : \Gamma \to G_k$$

is a homomorphism. We define the following obstruction class*:*

$$\zeta_{k+1} := \zeta_{k+1}^{(u_1,\ldots,u_k)} := \beta_{k-1}([U_{k-1}^{(u_1,\ldots,u_k)}]) \in H^2(\Gamma; \mathfrak{sl}_2(\mathbb{R})_\rho).$$

In particular, we have the following proposition from [21]:

Proposition 2.8 ([21, Proposition 3.1])**.** *Let $\rho \in R_G(\Gamma)$ and $u_i \in C^1(\Gamma; \mathfrak{sl}_2(\mathbb{R})_\rho)$ for $1 \leq i \leq k$ be given. Suppose that we have constructed a representation*

$$\rho_k := \rho_k^{(\rho;u_1,\ldots,u_k)} : \Gamma \to G_k$$

given by

$$\rho_k(\gamma) = \exp\left(\sum_{i=1}^{k} t^i u_i(\gamma)\right) \rho(\gamma) \mod t^{k+1}.$$

There exists an obstruction class $\zeta_{k+1} := \zeta_{k+1}^{(u_1,\ldots,u_k)} \in H^2(\Gamma; \mathfrak{sl}_2(\mathbb{R})_\rho)$ with the following properties:

1. *There is a cochain u_{k+1} such that $\rho_{k+1}^{(\rho;u_1,\ldots,u_{k+1})} : \Gamma \to G_{k+1}$ given by*

   $$\rho_{k+1}^{(\rho;u_1,\ldots,u_{k+1})} = \exp\left(\sum_{i=1}^{k+1} t^i u_i(\gamma)\right) \rho(\gamma) \mod t^{k+2}$$

   is a homomorphism if and only if $\zeta_{k+1} = 0$.

2. *The obstruction ζ_{k+1} is natural in the following sense: if $f : \Gamma' \to \Gamma$ is a homomorphism then*

   $$f^* \rho_k^{(\rho;u_1,\ldots,u_k)} = \rho_k^{(f^*\rho;f^*u_1,\ldots,f^*u_k)}$$

   is a homomorphism and $f^* \zeta_{k+1}^{(u_1,\ldots,u_k)} = \zeta_{k+1}^{(f^*u_1,\ldots,f^*u_k)}$.

Consequently, an infinite sequence $\{u_i\}_{i=1}^{\infty} \subset C^1(\Gamma; \mathfrak{sl}_2(\mathbb{R})_\rho)$ *defines a formal deformation of* ρ*,* $\rho_\infty : \Gamma \to G_\infty$ *via Equation* (2.3) *if and only if* u_1 *is a cocycle and* $\zeta_{k+1}^{(u_1,\ldots,u_k)} = 0$ *for all* $k \geq 1$*.*

Proof. Since ρ_k is a group homomorphism, the function $U_{k-1}^{u_1,\ldots,u_k}$ is a cocycle by Lemma 2.6. Let $\zeta_{k+1} = \beta_{k-1}([U_{k-1}^{(u_1,\ldots,u_k)}])$. Suppose that a cochain u_{k+1} exists so that $\rho_{k+1}^{(\rho;u_1,\ldots,u_{k+11})} : \Gamma \to G_{k+1}$ is a homomorphism. By Lemma 2.6, $U_k^{(u_1,\ldots,u_{k+1})}$ is a cocycle. Since $\pi_{k-1} \circ U_k^{(u_1,\ldots,u_{k+1})} = U_{k-1}^{(u_1,\ldots,u_k)}$, $[U_{k-1}] \in \mathrm{Im}((\pi_{k-1})_*)$. By the long exact sequence of cohomology, we see that $\zeta_{k+1} = 0$.

In the other direction, $\zeta_{k+1} = 0$ implies that there exists $U_k \in Z^1(\Gamma; \mathfrak{g}_k^{\rho_k})$ such that $\pi_{k-1} \circ U_k = U_{k-1}^{(u_1,\ldots,u_k)}$. Therefore, there exists a cochain u_{k+1} such that

$$U_k = U_k^{(u_1,\ldots,u_{k+1})}$$

and $\rho_{k+1}^{(\rho;u_1,\ldots,u_{k+1})}$ is a homomorphism.

The naturality follows from the commutativity of the diagram

$$\begin{array}{ccc} H^1(\Gamma; \mathfrak{g}_{k-1}^{\rho_{k-1}}) & \xrightarrow{\beta_{k-1}} & H^2(\Gamma; \mathfrak{sl}_2(\mathbb{R})_\rho) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ H^1(\Gamma'; \mathfrak{g}_{k-1}^{\rho_{k-1}}) & \xrightarrow{\beta_{k-1}} & H^2(\Gamma'; \mathfrak{sl}_2(\mathbb{R})_\rho) \end{array}$$

□

Remark 2.9. *Proposition 2.8 was stated over* $\mathbb{C}$ *in [21]. Since the construction of the obstruction* ζ_{k+1}*, see [21, Definition 3.4], and the proof of Proposition 2.8 is purely homological, it remains true over* $\mathbb{R}$*.*

2.3 From Formal Deformation To Convergent Deformation

We denote by $\mathbb{R}\{t\} \subset \mathbb{R}[[t]]$ the ring of power series with positive radius of convergence. To construct a path of representations from a formal deformation, we need to obtain a *convergent* deformation from a formal deformation. The key ingredient is the following theorem of Artin:

Theorem 2.10. *[1, Theorem 1.2] Consider an arbitrary system of real analytic equations*

$$f(\vec{x},\vec{y}) = (f_1(\vec{x},\vec{y}),\dots,f_l(\vec{x},\vec{y})) = 0$$

where $f_1,\dots,f_l$ are convergent series in the variables $x_1,\dots,x_m$ and $y_1,\dots,y_n$ for k,m,n non-negative integers. Suppose that $\vec{y}(\vec{x}) = (y_1(\vec{x}),\dots,y_n(\vec{x}))$ are formal power series without constant term such that $f(\vec{x},\vec{y}(\vec{x})) = 0$. For any positive integer k, there exists a convergent series solution

$$\vec{y}^*(\vec{x}) = (y_1^*(\vec{x}),\dots,y_n^*(\vec{x}))$$

of $f(\vec{x},\vec{y}) = 0$ such that $\vec{y}(\vec{x}) = \vec{y}^*(\vec{x}) \mod \mathfrak{m}^k$ where $\mathfrak{m}$ is the maximal ideal of $\mathbb{R}[[\vec{x}]]$.

Proposition 2.11. *[21, Proposition 3.6] Let $\rho_\infty : \Gamma \to G_\infty$ be a formal deformation of $\rho : \Gamma \to G$. Then for every $k \in \mathbb{N}$, there exists a convergent deformation*

$$\widehat{\rho}_\infty : \Gamma \to \mathrm{PSL}_2(\mathbb{R}\{t\})$$

such that $\widehat{\rho}_\infty(\gamma) = \rho_\infty(\gamma) \mod t^k$ for all $\gamma \in \Gamma$

Proof. Let $\Gamma = \langle S_1,\dots,S_m \mid R_1,\dots,R_\ell\rangle$ be a finite presentation. Since

$$\mathrm{PSL}_2(\mathbb{R}) \cong \mathrm{SO}_o(2,1) \subset \mathbb{R}^9,$$

we can identify $R_G(\Gamma)$ with an algebraic subset of $\mathbb{R}^n$ for some integer n. Therefore, there exists a system of polynomial equations $\mathbf{F}(\mathbf{x}) = \mathbf{0}$ such that

$$R_G(\Gamma) \cong V(\mathbf{F}) := \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{F}(\mathbf{x}) = \mathbf{0}\}.$$

Without loss of generality, we may assume that the solution $\mathbf{F}(\mathbf{0}) = \mathbf{0}$ corresponds to ρ. A formal deformation of ρ corresponds to a formal solution $\mathbf{x}(t) \in \mathbb{R}[[t]]$. By Theorem 2.10, for a given $k \in \mathbb{N}$ there exists a convergent solution $\mathbf{x}^*(t) \in \mathbb{R}\{t\}$ such that $\mathbf{x}^*(t) \equiv \mathbf{x}(t) \mod t^k$. This gives us a convergent deformation $\widehat{\rho}_\infty$ as desired. $\square$

Lemma 2.12. *[21, Lemma 3.7] Let $\rho \in R_G(\Gamma)$ be regular and let $u_i \in C^1(\Gamma;\mathfrak{sl}_2(\mathbb{R})_\rho)$ be given such that $\rho_k^{(\rho;u_1,\dots,u_k)} : \Gamma \to G_k$ is a homomorphism. Then for every cocycle v in $Z^1(\Gamma;\mathfrak{sl}_2(\mathbb{R})_\rho)$ there exists a cochain $u_{k+1} \in C^1(\Gamma;\mathfrak{sl}_2(\mathbb{R})_\rho)$ such that*

$$\rho_{k+1}^{(\rho;u_1,\dots,u_k,u_{k+1}+v)} : \Gamma \to G_{k+1}$$

is a homomorphism.

Proof. Since ρ is a regular point in $R_G(\Gamma)$, $\dim_\rho(R(\Gamma)) = \dim Z^1(\Gamma;\mathfrak{sl}_2(\mathbb{R})_\rho)$ and ρ is a smooth point of $R_G(\Gamma)$. Let V be the irreducible component of $R_G(\Gamma)$ that contains the smooth point ρ. As in the proof of the previous proposition, we identify V with the vanishing set of a prime ideal in $\mathbb{R}[x_1,\ldots,x_n]$. By [25, Corollary 1.20], V is locally cut out by a system of polynomials $\mathbf{F} = (F_1,\ldots,F_r)$ such that the solution $\mathbf{F}(\mathbf{0}) = \mathbf{0}$ corresponds to the representation ρ and $\ker D_{\mathbf{0}}\mathbf{F} = T_\rho R$.

The representation ρ_k corresponds to a polynomial vector $\mathbf{x}_k(t) \in (\mathbb{R}[t])^n$ of degree k such that $\mathbf{F}(\mathbf{x}_k(t)) \equiv \mathbf{0} \mod t^{k+1}$. The element $v \in Z^1(\Gamma;\mathfrak{sl}_2(\mathbb{R})_\rho)$ corresponds to a vector $\mathbf{v} \in T_{\mathbf{0}}(V)$. To obtain a homorphism ρ_{k+1}, we want to find $\mathbf{w}$ such that

$$\mathbf{F}(\mathbf{x}_k(t) + t^{k+1}(\mathbf{v}+\mathbf{w})) = \mathbf{0} \mod t^{k+2}. \tag{2.4}$$

Note that $\mathbf{F}(\mathbf{x}_k(t) + t^{k+1}(\mathbf{v}+\mathbf{w})) = \mathbf{F}(\mathbf{x}_k(t)) = \mathbf{0} \mod t^{k+1}$. We observe that

$$\left.\frac{\partial^{k+1}}{\partial t^{k+1}}\right|_{t=0} F_i(\mathbf{x}_k(t) + t^{k+1}(\mathbf{v}+\mathbf{w})) = \sum_{j=1}^{N} \left.\frac{\partial F_i}{\partial x_j}\right|_{\mathbf{x}=\mathbf{0}} (v_j + w_j) + G_i$$

where $G_i \in \mathbb{R}$. Therefore, Equation (2.4) is satisfied if and only if

$$D_{\mathbf{0}}\mathbf{F}(\mathbf{v}+\mathbf{w}) = -\mathbf{G}$$

where $\mathbf{G} = (G_1,\ldots,G_N)$ and $D_{\mathbf{0}}\mathbf{F}$ is the Jacobian of $\mathbf{F}$ at $\mathbf{0}$. Since

$$\mathbf{v} \in T_{\mathbf{0}}(V) = \ker(D_{\mathbf{0}}\mathbf{F}),$$

the equation above is equivalent to

$$D_{\mathbf{0}}\mathbf{F}(\mathbf{w}) = -\mathbf{G}.$$

Since $\mathbf{0}$ is a smooth point of $V(\mathbf{F})$, the Jacobian $D_{\mathbf{0}}\mathbf{F}$ is full rank. Thus, we can find $\mathbf{w}$ so that Equation (2.4) is satisfied. This gives us a representation

$$\rho_{k+1}^{(\rho;u_1,\ldots,u_k,u_{k+1}+v)} : \Gamma \to G_{k+1}$$

as desired. $\square$

2.4 Holonomy Extension Locus

Now we recall some definitions and results about the holonomy extension locus from [15]. The group G acts on $P^1_{\mathbb{C}}$ by Mobius transformation leaving $P^1_{\mathbb{R}}$ invariant. Any nontrivial abelian subgroup of G either contains only parabolic elements and has one fixed point in $P^1_{\mathbb{C}}$ or contains only hyperbolic or elliptic elements and has two fixed points in $P^1_{\mathbb{C}}$. Let $\widetilde{G} = \widetilde{\mathrm{PSL}_2}(\mathbb{R})$ be the universal covering group of G. The group $\widetilde{G}$ also acts on $P^1_{\mathbb{C}}$ by pulling back the action of G. We say that an element $\widetilde{g} \in \widetilde{G}$ is hyperbolic, parabolic, elliptic, or central if the image of $\widetilde{g}$ in G is hyperbolic, parabolic, elliptic, or trivial, respectively.

We denote by M a compact 3-manifold with a single torus boundary component and define the augmented representation variety $R^{\mathrm{aug}}_G(M)$ as follows. Since abelian subgroups of G act with global fixed points, we define the augmented representation variety $R^{\mathrm{aug}}_G(M)$ to be the subvariety of $R_G(M) \times P^1_{\mathbb{C}}$ consisting of pairs (ρ, z) where z is a fixed point of $\rho(\pi_1(\partial M))$. Since the action of $\widetilde{G}$ on $P^1_{\mathbb{C}}$ comes from pulling back the action of G, we can also define $R^{\mathrm{aug}}_{\widetilde{G}}(M)$ to be the real analytic subvariety of $R_{\widetilde{G}}(M) \times P^1_{\mathbb{C}}$ consisting of pairs (ρ, z) where z is a fixed point of $\rho(\pi_1(\partial M))$. Similarly, we define $R^{\mathrm{aug}}_{\widetilde{G}}(\partial M)$ to be the real analytic subvariety of $R_{\widetilde{G}}(\partial M) \times P^1_{\mathbb{C}}$ consisting of pairs (ρ, z) where z is a fixed point of $\rho(\pi_1(\partial M))$.

Given a hyperbolic, parabolic or central element $\tilde{g} \in \widetilde{G}$ with a fixed point $v \in P^1_{\mathbb{C}}$, let $g \in G$ be the image of $\widetilde{g}$ and a be a square root of the derivative of g at v. Since elements of G can be viewed as orientation-preserving homeomorphims of S^1, an element $\widetilde{g} \in \widetilde{G}$ is a lift of $g \in G$ to the universal cover $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$. It follows that $\widetilde{G}$ acts on $\mathbb{R}$ by orientation-preserving homeomophisms. We have the following lemma from [16]

Lemma 2.13. *[16, Section 5] Let $\widetilde{g} \in \widetilde{G}$. The following limit*

$$\mathrm{trans}(\widetilde{g}) = \lim_{n\to\infty} \frac{\widetilde{g}^n(0)}{n} \tag{2.5}$$

exists. Consequently, trans : $\widetilde{G} \to \mathbb{R}$ *is a well-defined function on $\widetilde{G}$. Furthermore,* trans *is a homomorphism when restricted to abelian subgroups of $\widetilde{G}$. The function* $\mathrm{trans}(\widetilde{g})$ *is refered to as the translation number of $\widetilde{g} \in \widetilde{G}$.*

Proof. Let $x > y$ be two real numbers and $\widetilde{f} \in \widetilde{G}$. We claim that

$$|\widetilde{f}(x) - \widetilde{f}(y) - x + y| < 2.$$

We write $x = k + y'$ such that $k \in \mathbb{N}$ and $|y - y'| < 1$. Since $\widetilde{f}$ is a lift of a homeomorphism of $\mathbb{R}/\mathbb{Z}$, we have $|\widetilde{f}(y) - \widetilde{f}(y')| < 1$. Therefore, we have

$$\begin{aligned} |\widetilde{f}(x) - \widetilde{f}(y) - x + y| &= |\widetilde{f}(y') + k - \widetilde{f}(y) - y' - k + y| \\ &= |\widetilde{f}(y') - \widetilde{f}(y) - y' + y| < 2 \end{aligned}$$

for any $x, y \in \mathbb{R}$ as claimed.

Let m, n be two positive integers. Applying the above inequality with $\widetilde{f} = \widetilde{g}^n$, we get

$$|\widetilde{g}^{mn}(0) - m\widetilde{g}^n(0)| = \left|\sum_{i=0}^{m-1} (\widetilde{g}^{(m-i)n}(0) - \widetilde{g}^{(m-i-1)n}(0) - \widetilde{g}^n(0))\right| < 2m.$$

Dividing both sides by mn, we get

$$\left|\frac{\widetilde{g}^{mn}(0)}{mn} - \frac{\widetilde{g}^n(0)}{n}\right| < \frac{2}{n}.$$

By a similar argument with the role of m and n switched, we have

$$\left|\frac{\widetilde{g}^{mn}(0)}{mn} - \frac{\widetilde{g}^m(0)}{m}\right| < \frac{2}{m}.$$

Finally, we see that

$$\left|\frac{\widetilde{g}^n(0)}{n} - \frac{\widetilde{g}^m(0)}{m}\right| < \frac{2}{m} + \frac{2}{n}$$

which implies that $\{\widetilde{g}^n(0)/n\}$ is a Cauchy sequence. It follows that $\mathrm{trans}(\widetilde{g})$ exists.

It remains to verify that trans is a group homomorphism when restricted abelian subgroups of $\widetilde{G}$. Let $\widetilde{g}$ and $\widetilde{h}$ be two commuting elements in $\widetilde{G}$. We have

$$\mathrm{trans}(\widetilde{g}\widetilde{h}) - \mathrm{trans}(\widetilde{g}) - \mathrm{trans}(\widetilde{h}) = \lim_{n\to\infty} \frac{(\widetilde{g}\widetilde{h})^n(0) - \widetilde{g}^n(0) - \widetilde{h}^n(0)}{n} = 0$$

since the numerator is uniformly bounded by 2. $\square$

We define

$$\mathrm{ev}(\widetilde{g}, v) := (\ln(|a|), \mathrm{trans}(\widetilde{g}))$$

where trans : $\widetilde{G} \to \mathbb{R}$ is the translation number given by Equation (2.5) and a is the square root of the derivative at v of the action of $\widetilde{g}$ on $P^1_{\mathbb{C}}$. An immediate consequence of Lemma 2.13 is that we get a group homomorphism

$$\mathrm{ev}(\widetilde{\rho}(-), v) : \pi_1(\partial M) \to \mathbb{R} \times \mathbb{Z}$$

for $\widetilde{\rho} \in R^{\mathrm{aug}}_{\widetilde{G}}(\partial M)$ whose image in $\widetilde{G}$ is hyperbolic, parabolic or central. In other words, we can view $\mathrm{ev}(\widetilde{\rho}(-), v)$ as an element of $\mathrm{Hom}(\pi_1(\partial M), \mathbb{R} \times \mathbb{Z})$. We are now ready to define the holonomy extension locus.

Definition 2.14. *[15, Definition 3.2] Let $PH_{\widetilde{G}}(M)$ be the subset of $R^{\mathrm{aug}}_{\widetilde{G}}(M)$ whose restriction to $\pi_1(\partial M)$ is either hyperbolic, parabolic or central. Consider the restriction map $i^* : R^{\mathrm{aug}}_{\widetilde{G}}(M) \to R^{\mathrm{aug}}_{\widetilde{G}}(\partial M)$ induced by the inclusion $i : \partial M \to M$. Define* $\mathrm{EV} : i^*(PH_{\widetilde{G}}(M)) \to H^1(\partial M; \mathbb{R}) \times H^1(\partial M; \mathbb{Z})$ *by*

$$(\widetilde{\rho}, v) \mapsto \mathrm{ev}((\widetilde{\rho}(-), v)).$$

Definition 2.15. *[15, Definition 3.3] Consider the composition*

$$PH_{\widetilde{G}}(M) \subset R^{\mathrm{aug}}_{\widetilde{G}}(M) \xrightarrow{i^*} R^{\mathrm{aug}}_{\widetilde{G}}(\partial M) \xrightarrow{\mathrm{EV}} H^1(\partial M; \mathbb{R}) \times H^1(\partial M; \mathbb{Z})$$

The closure of $EV \circ i^*(PH_{\widetilde{G}}(M))$ in $H^1(\partial M; \mathbb{R}) \times H^1(\partial M; \mathbb{Z})$ is called the holonomy extension locus of M and denoted $HL_{\widetilde{G}}(M)$.

Definition 2.16. *We call a point in $HL_{\widetilde{G}}(M)$ a hyperbolic, parabolic, or central point if it comes from a representation $\widetilde{\rho} \in PH_{\widetilde{G}}(M)$ such that $i^*(\widetilde{\rho})$ is hyperbolic, parabolic, or central, respectively. We call points in $HL_{\widetilde{G}}(M)$ but not in $EV \circ i^*(PH_{\widetilde{G}}(M))$ ideal points.*

To get concrete coordinates on the holonomy extension locus as well as the Dehn surgery space, let us pick a basis (μ, λ) for $H_1(\partial M; \mathbb{R})$ where λ is the homological longitude of M. In particular, λ is a generator of $\ker(H_1(\partial M; \mathbb{Z}) \to H_1(M; \mathbb{R}))$. We identify $H^1(\partial M; \mathbb{R})$ with $\mathbb{R}^2$ using the dual basis (μ^*, λ^*). Let L_r be the line through the origin in $\mathbb{R}^2$ of slope $-r$ where $r \in \mathbb{Q} \cup \{\infty\}$. In terms of the dual basis (μ^*, λ^*), the line L_r consists of linear functions that vanish on the primitive element γ representing the slope r in $\pi_1(\partial M)$ with respect to the basis (μ, λ). The structure of the holonomy extension locus is summarized as follows:

Theorem 2.17. *[15, Theorem 3.1] The holonomy extension locus*

$$HL_{\widetilde{G}}(M) = \bigsqcup_{i,j \in \mathbb{Z}} H_{i,j}(M)$$

is a locally finite union of analytic arcs and isolated points. Each component $H_{i,j}(M)$ contains at most one parabolic point and has finitely many ideal points. The locus $H_{0,0}(M)$ contains the horizontal axis L_0, which comes from representations to $\widetilde{G}$ with abelian image.

The holonomy extension locus gives a tool to detect left orderable Dehn surgeries. We have the following lemma:

Lemma 2.18. *[15, Lemma 3.8] If L_r intersects the component $H_{0,0}(M)$ of $HL_{\widetilde{G}}(M)$ at non-parabolic and non-ideal points, and assume that $M(r)$ is irreducible, then $M(r)$ is left orderable.*

2.4.1 The Figure-8 Knot

Lemma 2.18 is the main practical tool to produce an interval of left-orderable Dehn surgeries of a knot exterior. We will illustrate the application of this lemma in the case of the figure-8 knot by plotting $H_{0,0}$ for the figure-8 knot. This example serves as the motivation for the proof of Theorem 2.20 and Theorem 1.8. The example first appeared in [4, Proposition 10].

Let M be the complement of the figure-8 knot and $\Gamma = \pi_1(M)$ which has the following presentation

$$\Gamma = \langle a, b \mid aw = wb, \ w = ba^{-1}b^{-1}a \rangle.$$

We fix a peripheral subgroup of Γ generated by a and $\ell = ba^{-1}b^{-1}aab^{-1}a^{-1}b$.

Consider a $\mathrm{SL}_2(\mathbb{R})$-representation of Γ given by

$$a \mapsto \begin{pmatrix} x & 1 \\ 0 & x^{-1} \end{pmatrix}, \quad b \mapsto \begin{pmatrix} x & 0 \\ y & x^{-1} \end{pmatrix}. \tag{2.6}$$

Using the group relation, we see that the assignment in Equation (2.6) defines a representation of Γ if and only if

$$x^2y^2 + (x^4 - 3x^2 + 1)y - (x^4 - 3x^2 + 1) = 0 \text{ and } x, y \in \mathbb{R}$$

Or equivalently, we have

$$y(x) = \frac{-(x^4 - 3x^2 + 1) \pm \sqrt{(x^4 + x^2 + 1)(x^4 - 3x^2 + 1)}}{2x^2} \text{ and } x^4 - 3x^2 + 1 \geq 0.$$

The inequality $x^4 - 3x^2 + 1 \geq 0$ holds in the intervals

$$(-\infty, -\frac{1+\sqrt{5}}{2}\Bigg] \cup \Bigg[\frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\Bigg] \cup \Bigg[\frac{1+\sqrt{5}}{2}, \infty).$$

Note that $\mathrm{tr}(\rho(a)) = |x + x^{-1}| < 2$ if and only if x belongs to the bounded interval above. In other words, the representation is hyperbolic on the peripheral subgroup of Γ if and only if x belongs to the two unbounded intervals. Viewing y as a function of x, we obtain two continuous families ρ_x of $\mathrm{SL}_2(\mathbb{R})$-representations of Γ that corresponds to each interval $x \in (-\infty, -\frac{1+\sqrt{5}}{2}] \cup [\frac{1+\sqrt{5}}{2}, \infty)$ such that the restriction of ρ_x to the peripheral subgroup is hyperbolic. Since $H^2(\Gamma; \mathbb{Z}) = H^2(M; \mathbb{Z}) = 0$, the representations ρ_x lift to a family $\widetilde{\rho}_x$ of $\widetilde{G}$-reprensetations such that the restriction of $\widetilde{\rho}_x$ to the peripheral subgroup is hyperbolic or central. Since

$$\widetilde{\rho}_x(\langle a, \ell \rangle)$$

is contained in the upper triangular subgroup, $(\widetilde{\rho}_x, \infty)$ gives us a path in $PH_{\widetilde{G}}(M)$. Adjusting by an appropriate element of the center of $\widetilde{G}$, we may assume that the translation numbers of $\widetilde{\rho}_x(a)$ and of $\widetilde{\rho}_x(\ell)$ are zero. The path $(\widetilde{\rho}_x, \infty)$ projects under $\mathrm{EV} \circ i^*$ to a path in $H_{0,0}(M)$ which is shown in Figure 2.1.

Figure 2.1 shows the plane $H^1(\partial M; \mathbb{Z})$ which is identified with $\mathbb{R}^2$ by choosing the dual basis a^* and ℓ^*. In particular, $a^*(a) = \ell^*(\ell) = 1$ and $a^*(\ell) = \ell^*(a) = 0$.

Furthermore, the point $(s,t) \in \mathbb{R}^2$ is identified with $sa^* + t\ell^*$. Under this identification, the path $(\widetilde{\rho}_x, \infty)$ is mapped to the path

$$(\log|x|, \log|z(x)|)$$

where $z(x)$ is the top left entry of the matrix $\rho_x(\ell)$. The paths in blue and orange in Figure 2.1 correspond to different choices of $y(x)$ when we construct ρ_x. The paths in $H^1(\partial M; \mathbb{R})$ are asymptotic to the lines $L_{\pm 4}$ which are shown as two dashed lines in Figure 2.1. By Lemma 2.18, we conclude that $M(r)$ is left-orderable for $r \in (-4, 4)$.

Remark 2.19. *Recent work of Zung showed that the Dehn surgery $M(r)$ on the figure-8 knot exterior is left-orderable for all $r \in \mathbb{Q}$ [32].*

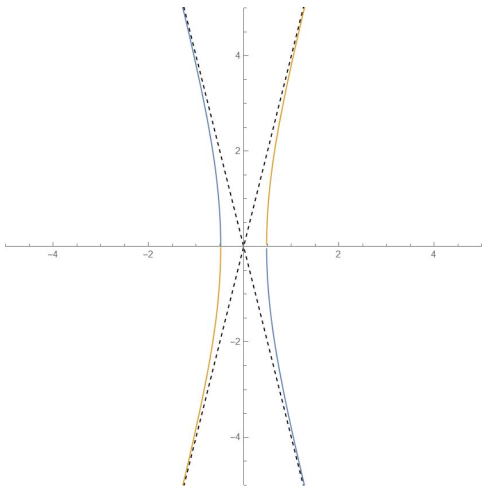

Figure 2.1: The holonomy extension locus of the figure-8 knot

2.4.2 The Main Theorem

Using Lemma 2.18, Xinghua Gao gives a criterion in terms of the $\mathrm{PSL}_2(\mathbb{R})$-character variety to produce an interval of left orderable Dehn surgeries around the 0-filling.

Theorem 2.20. *[15, Theorem 5.1] Suppose that M is a longitudinally rigid irreducible $\mathbb{Q}$-homology solid torus and that the Alexander polynomial of M has a simple positive real root ξ. Then there exists $a > 0$ and a nonempty interval $(-a, 0]$ or $[0, a)$ such that for every rational r in the interval, the Dehn filling $M(r)$ is left orderable.*

For completeness, we include the proof of this theorem. The key to the proof of Theorem 2.20 is to produce an arc in $H_{0,0}(M)$ transverse to the horizontal axis. By construction, this arc does not contain any parabolic or ideal points. Theorem 2.20 then follows from Lemma 2.18. To construct an arc in $H_{0,0}(M)$, we start by deforming abelian representations coming from the roots of the Alexander polynomial into irreducible representations which is studied in [20]. In particular, let ξ be a simple positive real root of the Alexander polynomial and $\alpha : \pi_1(M) \to \mathbb{R}_+$, the multiplicative group of the real numbers, such that α factors through $H_1(M;\mathbb{Z})_{\text{free}} \cong \mathbb{Z}$ and takes a generator of $H_1(M;\mathbb{Z})_{\text{free}}$ to ξ. We let $\rho_\alpha : \pi_1(M) \to G_{\mathbb{C}}$ be the associated diagonal representation given by

$$\rho_\alpha(\gamma) = \pm \begin{pmatrix} \alpha(\gamma)^{1/2} & 0 \\ 0 & \alpha(\gamma)^{-1/2} \end{pmatrix} \tag{2.7}$$

where $\alpha(\gamma)^{1/2}$ is either square root. The condition on the root of the Alexander polynomial allows one to deform $\rho_\alpha =: \rho_0$ into an analytic path of representations $\rho_t : \pi_1(M) \to G$ where $t \in [-1, 1]$, see [15, Lemma 5.1]. Furthermore this path of representations has the following properties.

Lemma 2.21. *[15, Lemma 5.1] The path $\rho_t : [-1, 1] \to R_G(M)$ satisfies:*

1. *The representations ρ_t are irreducible over $G_{\mathbb{C}}$ for $t \neq 0$.*

2. *The corresponding path $[\rho_t]$ of characters in $X_G(M)$ is also a non-constant analytic path.*

3. *The function $\operatorname{tr}^2(\gamma)$ is nonconstant in t for some $\gamma \in \pi_1(\partial M)$.*

Proof of Theorem 2.20. Let ρ_t be the path of representations from Lemma 2.21. Using this path, we can produce an arc in $H_{0,0}(M)$ as follows. Since ρ_0 factors through $H_1(M;\mathbb{Z})_{\text{free}} \cong \mathbb{Z}$, we can lift this representation to $\tilde{\rho}_0 : \pi_1(M) \to \widetilde{G}$. Since the obstruction of lifting a representation from G to $\widetilde{G}$ is a continuous function on $R_G(M)$ with value in $\mathbb{Z}$, it remains constant on each connected component of $R_G(M)$. Therefore, the obstruction of lifting a representation from G to $\widetilde{G}$ vanishes on the entire path ρ_t. We can lift the path ρ_t to a path $\widetilde{\rho}_t$ in $R_{\widetilde{G}}(M)$. Adjusting $\widetilde{\rho}_0$ by the appropriate central element of $\widetilde{G}$, we can assume that $\operatorname{trans}(\widetilde{\rho}_0(\mu)) = 0$. Since the image of ρ_0 is upper triangular and the image $\rho_0(\lambda)$ is trivial, we can choose a lift of ρ_0 such that $\operatorname{trans}(\widetilde{\rho}_0(\lambda)) = 0$. Therefore, $\widetilde{\rho}_0$ is mapped to a point on the horizontal axis of $H_{0,0}(M)$. Since $\xi \neq 1$, the x-coordinate of $\widetilde{\rho}_0$, $\ln(|\xi|)$, is nonzero.

Let k be the index of $[\mu]$ in $H_1(M;\mathbb{Z})_{\text{free}}$ and ξ be the positive real root of the Alexander polynomial that corresponds to ρ_0, see Remark 2.22. We have

$$\operatorname{tr}^2(\rho_0(\mu)) = \xi^k + 2 + \xi^{-k} > 4.$$

Therefore, $\rho_0(\mu)$ is hyperbolic and so is $\widetilde{\rho}_0(\mu)$. This implies that the image of the abelian subgroup $\langle \mu, \lambda \rangle$ under $\widetilde{\rho}_0$ contains only hyperbolic elements and that $\widetilde{\rho}_0$ is in $PH_{\widetilde{G}}(M)$. Since being hyperbolic is an open condition, there exists $\varepsilon > 0$ such that $\widetilde{\rho}_t \in PH_{\widetilde{G}}(M)$ for all $t \in [-\varepsilon, \varepsilon]$. The path $\widetilde{\rho}_t$ restricted to $[-\varepsilon, \varepsilon]$ projects to a path A in $H_{0,0}(M)$. The projection A must be non-constant since $[\rho_t]$ is a non-constant path on the character variety $X_G(M)$.

Finally, we use the hypothesis that M is longitudinally rigid to argue that A is not contained in the horizontal axis of $H_{0,0}(M)$. If A is contained in L_0, then $\ln|a| = 0$ where a is the eigenvalue of $\rho_t(\lambda)$. Since $\rho_t(\lambda)$ is either hyperbolic or trivial, we must have $\rho_t(\lambda)$ is trivial in G. Therefore, the representation ρ_t factors through $M(0)$. We get a non-constant path $[\rho_t]$ in $X(M(0))$ of irreducible characters when $t \neq 0$. However, this contradicts the assumption that M is longitudinally rigid. $\square$

Remark 2.22. *We note that k is not necessarily 1 in this case. Here, λ is a generator of $\ker(H_1(\partial M;\mathbb{Z}) \to H_1(M;\mathbb{R}))$. In particular, λ can represent a torsion class of order $k \geq 1$ in $H_1(M;\mathbb{Z})$ where $k = 1$ means that $[\lambda] = 0$ in $H_1(M;\mathbb{Z})$. The generator of*

$$H_2(M;\partial M;\mathbb{Z}) \cong H^1(M;\mathbb{Z}) \cong \mathbb{Z}$$

can be represented by a surface S such that $[\partial S] = k\lambda$ in $H_1(\partial M;\mathbb{Z})$. Therefore, μ has algebraic intersection number $\pm k$ with S. It follows that the element

$$i_*(\mu) \in H_1(M;\mathbb{Z})_{free} \cong H^1(M;\mathbb{Z}) \cong H_2(M,\partial M;\mathbb{Z}) \cong \mathbb{Z}$$

generate a subgroup of index k.

Remark 2.23. *The condition that M is longitudinally rigid ensures that the path of representations ρ_t obtained by deforming the abelian representation ρ_0 does not factor through the longitudinal filling. We can weaken this hypothesis by a local condition at the non-abelian reducible representation ρ_ξ^+ that corresponds to a root ξ of the Alexander polynomial.*

We have the following theorem of Burde and de Rham which connects the Alexander polynomial and non-abelian reducible representations:

Theorem 2.24 ([6] and [12])**.** *Let $\alpha : \pi_1(M) \to \mathbb{C}^*$ be a representation and define ρ_α as in Equation* (2.7)*. Then there exists a reducible, non-abelian representation $\rho_\xi^+ : \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$ such that $[\rho_\xi^+] = [\rho_\alpha]$ in $X(M)$ if and only if α factors through $H_1(M;\mathbb{Z})_{free} \cong \mathbb{Z}$ sending a generator to the root ξ of the Alexander polynomial of M.*

Definition 2.25. *Suppose that M be an irreducible $\mathbb{Q}$-homology solid torus. Let ξ be a root of the Alexander polynomial of M and ρ_ξ^+ be a non-abelian reducible representation associated to ξ. We say that M is* locally longitudinally rigid at ξ *if*

$$H^1(M(0);\mathfrak{sl}_2(\mathbb{C})_{\rho_\xi^+}) = 0.$$

Before proving Theorem 1.8, we need the following lemmas from [20] in the real setting. We include the proofs of these lemmas for completeness.

Lemma 2.26. *[20, Lemma 7.5] Let ξ be a simple positive real root of the Alexander polynomial and*

$$\phi := \rho_\xi^+ : \pi_1(M) \to \mathrm{PSL}_2(\mathbb{R})$$

be a non-abelian reducible representation that corresponds to ξ. Then the map

$$H^2(\pi_1(M);\mathfrak{sl}_2(\mathbb{R})_\phi) \to H^2(\pi_1(\partial M);\mathfrak{sl}_2(\mathbb{R})_\phi)$$

induced by the inclusion $\pi_1(\partial M) \hookrightarrow \pi_1(M)$ is injective.

Proof. We have $\phi|_{\pi_1(\partial M)}$ is nontrivial since $\mathrm{tr}^2(\phi(\mu)) = \xi^k + 2 + \xi^{-k} > 4$ where k is the index $\langle[\mu]\rangle$ in $H_1(M;\mathbb{Z})_{\mathrm{free}}$. Since ∂M is aspherical, we have

$$H^*(\partial M;\mathfrak{sl}_2(\mathbb{R})_\phi) \cong H^*(\pi_1(\partial M);\mathfrak{sl}_2(\mathbb{R})_\phi).$$

Since $\phi|_{\pi_1(\partial M)}$ is non-trivial, we have

$$H^0(\partial M;\mathfrak{sl}_2(\mathbb{R})_\phi)\cong \mathfrak{sl}_2(\mathbb{R})^{\phi(\pi_1(\partial M))}\cong \mathbb{R}.$$

By duality for cohomology with local coefficients and Euler characteristic, see [11, Chapter 3 and 5] and [13, Proposition 2.5.4], we have

$$H^2(\partial M;\mathfrak{sl}_2(\mathbb{R})_\phi)\cong \mathbb{R} \quad \text{and} \quad H^1(\partial M;\mathfrak{sl}_2(\mathbb{R})_\phi)\cong \mathbb{R}^2$$

Since ξ is a simple root of the Alexander polynomial, [20, Corollary 5.4] gives that

$$H^1(M;\mathfrak{sl}_2(\mathbb{R})_\phi)\cong H^1(\pi_1(M);\mathfrak{sl}_2(\mathbb{R})_\phi)\cong \mathbb{R}.$$

By duality for cohomology with local coefficient and universal coefficient for a local system, we have

$$H^2(M,\partial M;\mathfrak{sl}_2(\mathbb{R})_\phi)\cong H_1(M;\mathfrak{sl}_2(\mathbb{R})_\phi),$$

see [11, Chapter 3 and 5] and the discussion that follows [13, Proposition 2.5.4]. Therefore, the segment

$$H^1(M;\mathfrak{sl}_2(\mathbb{R})_\phi)\to H^1(\partial M;\mathfrak{sl}_2(\mathbb{R})_\phi)\to H^2(M,\partial M;\mathfrak{sl}_2(\mathbb{R})_\phi)$$

of the long exact sequence for the pair $(M,\partial M)$ is short exact. From the long exact sequence of pair for $(M,\partial M)$ we see that the map

$$H^2(M;\mathfrak{sl}_2(\mathbb{R})_\phi)\to H^2(\partial M;\mathfrak{sl}_2(\mathbb{R})_\phi)$$

is injective. The conclusion of the lemma follows from the commutative diagram

$$\begin{tikzcd}
H^2(M;\mathfrak{sl}_2(\mathbb{R})_\phi) \arrow[r, hook] & H^2(\partial M;\mathfrak{sl}_2(\mathbb{R})_\phi) \\
H^2(\pi_1(M);\mathfrak{sl}_2(\mathbb{R})_\phi) \arrow[u] \arrow[r] & H^2(\pi_1(\partial M);\mathfrak{sl}_2(\mathbb{R})_\phi) \arrow[u, "\cong"']
\end{tikzcd}$$

and the fact that the natural map between group cohomology and manifold cohomology $H^i(\pi_1(M);\mathfrak{sl}_2(\mathbb{R})_\phi)\to H^i(M;\mathfrak{sl}_2(\mathbb{R})_\phi)$ is injective for $i=2$, see also [20, Lemma 3.1]. $\square$

Lemma 2.27. *[20, Lemma 7.5] Let ξ be a simple positive real root of the Alexander polynomial and*

$$\phi := \rho_\xi^+ : \pi_1(M) \to \mathrm{PSL}_2(\mathbb{R})$$

be a non-abelian reducible representation that corresponds to ξ. All cocycles in the space $Z^1(\pi_1(M); \mathfrak{sl}_2(\mathbb{R})_\phi)$ are integrable.

Proof. As noted in the proof of Lemma 2.26, $\phi|_{\pi_1(\partial M)}$ is non-trivial and contains only hyperbolic elements. Since

$$\phi(\pi_1(\partial M)) \subset \mathrm{PSL}_2(\mathbb{R}),$$

the image $\phi(\pi_1(\partial M))$ cannot be the Klein 4-group. Since the image of $\phi|_{\pi_1(\partial M)}$ is hyperbolic and abelian, the elements in the image of $\phi|_{\pi_1(\partial M)}$ act by translations along a unique geodesic axis α in $\mathbb{H}^2$. In a small enough neighborhood of $\phi|_{\pi_1(\partial M)}$, the representations of $\pi_1(\partial M)$ are parametrized by a choice of a geodesic in $\mathbb{H}^2$ near α and two choices of translation lengths. Since the space of geodesic in $\mathbb{H}^2$ is two-dimensional, $\phi|_{\pi_1(\partial M)}$ is a smooth point of an irreducible component of $R_G(\mathbb{Z}^2)$ with local dimension four.

Let $i : \pi_1(\partial M) \to \pi_1(M)$ be an inclusion map and $u_1 : \pi_1(M) \to \mathfrak{sl}_2(\mathbb{R})$ be a cocycle. Suppose we have cochains $u_2, \ldots, u_k : \pi_1(M) \to \mathfrak{sl}_2(\mathbb{R})$ such that

$$\phi_k(\gamma) = \exp\left(\sum_{i=1}^{k} t^i u_i(\gamma)\right)\phi(\gamma)$$

is a homomorphism modulo t^{k+1}. From Proposition 2.8, we get an obstruction class

$$\zeta_{k+1}^{(u_1,\ldots,u_k)} \in H^2(\pi_1(M); \mathfrak{sl}(\mathbb{R})_\phi),$$

which vanishes if and only if ϕ_k can be extended to a homomorphism modulo t^{k+2}.

The restriction $\phi_k \circ i$ is a homomorphism modulo t^{k+1}. Since $\phi \circ i$ is a smooth point of $R_G(\mathbb{Z}^2)$, $\phi_k \circ i$ extends to a homomorphism modulo t^{k+2} by Lemma 2.12. Therefore, the order $k+1$ obstruction vanishes on the boundary:

$$i^* \zeta_{k+1}^{(u_1,\ldots,u_k)} = \zeta_{k+1}^{(i^*u_1,\ldots,i^*u_k)} = 0.$$

By Lemma 2.26, i^* is injective, and so the obstruction $\zeta_{k+1}^{(u_1,\ldots,u_k)}$ vanishes for $\pi_1(M)$ as well. Iterating this process starting with u_1, we get an infinite sequence of cochains $\{u_i\}_{i=1}^{\infty}$ such that u_1 is a cocycle and the obstruction

$$\zeta_{k+1}^{(u_1,\ldots,u_k)} = 0$$

for all $k \geq 1$. By Proposition 2.8, we get a representation $\phi_\infty : \pi_1(M) \to \mathrm{PSL}_2(\mathbb{R}[[t]])$

$$\phi_\infty(\gamma) = \exp\left(\sum_{i=1}^{\infty} t^i u_i(\gamma)\right)\phi(\gamma)$$

for all cocycles u_1. Therefore, all cocycles of $Z^1(\pi_1(M); \mathfrak{sl}_2(\mathbb{R}))$ are integrable. $\square$

Proof of Theorem 1.8. Following the proof of Theorem 2.20, it suffices to prove that the arc A constructed in the proof of Theorem 2.20 is not contained in L_0. Arguing by contradiction, suppose this arc is contained in L_0. As in the proof of Theorem 2.20, this would imply that the path of representations ρ_t factors through $M(0)$. Since ρ_t is irreducible for all $t \neq 0$, we obtain an arc of irreducible characters in $X(M(0))$ that contains $[\rho_\alpha] = [\rho_\xi^+]$.

On the other hand, we claim that there exists a path $\phi_t : [-1, 1] \to R_G(M)$ that is transverse to the orbit of ρ_ξ^+ at $\phi_0 = \rho_\xi^+$, the non-abelian reducible representation that corresponds to ξ. For convenience, we let $\phi := \rho_\xi^+$. We have the following isomorphism of cohomology groups

$$H^1(\pi_1(M); \mathfrak{sl}_2(\mathbb{C})_\phi) = H^1(\pi_1(M); \mathfrak{sl}_2(\mathbb{R})_\phi) \otimes_{\mathbb{R}} \mathbb{C}.$$

Since ξ is a simple root of the Alexander polynomial, [20, Corollary 5.4] gives that $H^1(\pi_1(M); \mathfrak{sl}_2(\mathbb{R})_\phi)$ is one-dimensional. Therefore, $Z^1(\pi_1(M); \mathfrak{sl}_2(\mathbb{R})_\phi)$ has dimension four. By Lemma 2.27, all cocycles in $Z^1(\pi_1(M); \mathfrak{sl}_2(\mathbb{R})_\phi)$ are integrable. Therefore, ϕ is a smooth point of $R_G(M)$ with local dimension 4. Integrating a cocycle that generates $H^1(\pi_1(M); \mathfrak{sl}_2(\mathbb{R})_\phi)$, we obtain a path $\phi_t : [-1, 1] \to R_G(M)$ which has $\phi_0 = \phi$ and is transverse to the orbit of ϕ at $t = 0$.

We note that since ϕ is a smooth point of $R_G(M)$, it is contained in a unique irreducible component of $R_G(M)$. Since the abelian representations of $\pi_1(M)$ form an irreducible component of dimension 3, ϕ is locally four-dimensional implies that the path ϕ_t cannot contain any abelian representation. Consider the path $[\phi_t]$ in $X_G(M)$ which contains the character $[\phi_0] = [\rho_0]$ coming from an abelian representation. By [20, Proposition 10.2], $[\rho_0]$ is contained in precisely two real curves of characters. One of the curves is associated with abelian representations, and the other one with irreducible representations. Since ϕ_t is non-abelian representation for all t, the path $[\phi_t]$ is contained in the curve of irreducible characters. Therefore, for $t \neq 0$, the character $[\phi_t]$ is the character of some irreducible representation. Up to shrinking and reparametrizing either ρ_t or ϕ_t, we may assume that $[\phi_t] = [\rho_t]$ for all t. Since ϕ_t has the same character as an irreducible representation ρ_t for $t \neq 0$, the representation ϕ_t is conjugate to ρ_t for all $t \neq 0$ by [10, Proposition 1.5.2]. Since ρ_t factors through $M(0)$ for all t and ϕ_t is conjugate to ρ_t, we also get that ϕ_t factors through $M(0)$ for all t. We obtain a path $[\phi_t]$ in $R_G(M(0))$ going through $\phi = \rho_\xi^+$ that is transverse to the orbit of ρ_ξ^+ under the action of G by conjugation. The existence of this path implies that

$$\begin{aligned} \dim_{\mathbb{R}} Z^1(\Gamma(0), \mathfrak{sl}_2(\mathbb{R})_{\rho_\xi^+}) &\geq \dim_{\mathbb{R}} T^{Zar}_{\rho_\xi^+}(R(\Gamma(0))) \\ &\geq 1 + \dim_{\mathbb{R}} B^1(\Gamma(0), \mathfrak{sl}_2(\mathbb{R})_{\rho_\xi^+}) = 4. \end{aligned}$$

This would imply that $\dim_{\mathbb{R}} H^1(\Gamma(0), \mathfrak{sl}_2(\mathbb{R})_{\rho_\xi^+}) \geq 1$. Since $\mathfrak{sl}_2(\mathbb{C}) = \mathfrak{sl}_2(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$, we have

$$H^1(\Gamma(0), \mathfrak{sl}_2(\mathbb{C})_{\rho_\xi^+}) = H^1(\Gamma(0), \mathfrak{sl}_2(\mathbb{R})_{\rho_\xi^+}) \otimes_{\mathbb{R}} \mathbb{C}.$$

Therefore, the dimension of $H^1(\Gamma(0), \mathfrak{sl}_2(\mathbb{C})_{\rho_\xi^+})$ is at least 1. This gives a desired contradiction to the assumption that M is locally longitudinally rigid at ξ. $\square$

Chapter 3

APPLICATIONS

In this chapter, we apply Theorem 1.8 to study left orderability on two infinite families of knots, namely the family of two-bridge knots K_j associated to the continued fraction $[1, 1, 2, 2, 2j]$ for $j \geq 1$ and the family of $(-3, 3, 2j + 1)$-pretzel knots P_j. In particular, we will prove Theorem 1.9 and Theorem 1.11.

We first make some remarks about the family of two-bridge knot complements. These knot complements are obtained by doing $1/j$ Dehn filling on the unknot component of the link L^2_{25}, see Figure 3.1. The first two members of the family are the knots 8_{12} and 10_{13} in Rolfsen's table. As we will see in Lemma 3.2, the Alexander polynomial of K_j has all simple positive real roots, and is not monic for $j \geq 2$. In particular, the complement of K_j is not lean for $j \geq 2$. Furthermore, the trace field of K_j for $1 \leq j \leq 30$ has no real places, and it is most likely that the trace fields of all knots in this family share this property. Therefore, Theorem 1.9 is not a direct consequence of Theorem 1.3 nor Theorem 1.5. The family of two-bridge knots $[1, 1, 2, 2, 2j]$ is a new family of knots with an interval of left orderable Dehn surgeries which cannot be obtained from prior techniques.

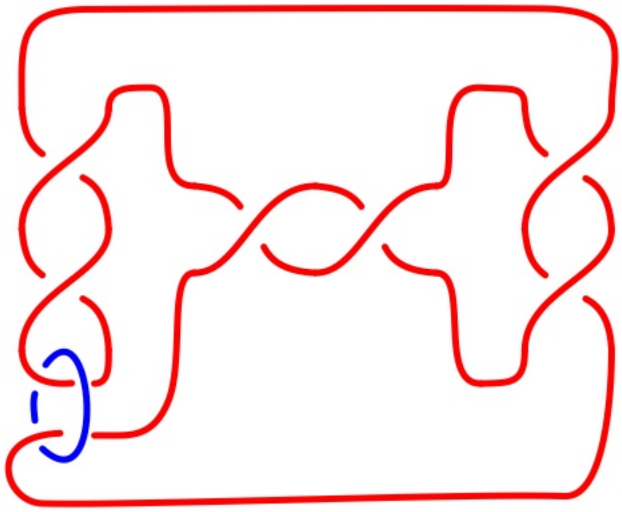

Figure 3.1: The link complement L_{25}^2

3.1 The $[1, 1, 2, 2, 2j]$ Two-bridge Knots

3.1.1 Group Presentation

We will denote by Γ the fundamental group of the complement of the knot K_j. The knot corresponding to the continued fraction $[1, 1, 2, 2, 2j]$ has the associated fraction

$$[1, 1, 2, 2, 2j] = 1 + \cfrac{1}{1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2j}}}} = \frac{24j + 5}{14j + 3}.$$

By [28, Proposition 1], the knot group Γ has the presentation $\Gamma = \langle x, y \mid xw = wy \rangle$. The word w is given by

$$w = y^{e_1} x^{e_2} \dots y^{e_{24j+3}} x^{e_{24j+4}} \tag{3.1}$$

where $e_i = (-1)^{\lfloor i(14j+3)/(24j+5)\rfloor}$. Also by [28, Proposition 1], the homological longitude of K_j that commutes with x is given by $\ell = wv$ where

$$v = x^{e_{24j+4}} y^{e_{24j+3}} \dots x^{e_2} y^{e_1}. \tag{3.2}$$

We first give an explicit description of w in terms of x and y by giving a formula for the right-hand sides of Equation (3.1) and Equation (3.2). We have the following lemma.

Lemma 3.1. *In the terms of the generators x, y of Γ, the word w has the form*

$$w = (yx^{-1}y^{-1}x)u^j \quad \textit{and} \quad v = s^j(xy^{-1}x^{-1}y) \tag{3.3}$$

where

$$u = (yx^{-1}yx)(y^{-1}x^{-1}yx^{-1})(y^{-1}xyx^{-1})(y^{-1}xy^{-1}x^{-1})(yxy^{-1}x)(yx^{-1}y^{-1}x)$$

and s is u spelled backwards.

Proof. Since v is w spelled backwards, it suffices to prove the lemma for w. Let us consider

$$k_{i,j} = \frac{i(14j+3)}{24j+5}$$

for $1 \leq i \leq 24j+4$. We first claim that

$$\lfloor k_{i,j} \rfloor = \lfloor k_{i,m} \rfloor = \left\lfloor \frac{7i}{12} \right\rfloor \tag{3.4}$$

for all $j \geq m$ and $\varepsilon_m \leq i \leq 24m+4$ where $\varepsilon_m = \max\{1, 24(m-1)+5\}$. Fixing i, we can view $k_{i,j}$ as a continuous function in the variable j. Since $i \geq 1$, the derivative of $k_{i,j}$ with respect to j is

$$\frac{dk_{i,j}}{dj} = -\frac{2i}{(24j+5)^2} < 0.$$

The function $k_{i,j}$ is strictly decreasing and has a horizontal asymptote at $7i/12$ as $j \to +\infty$. Therefore, we have the following chain of inequalities

$$\frac{7i}{12} < k_{i,j} < k_{i,m} = \frac{(14m+3)i}{24m+5}$$

for all $j \geq m$ and $\varepsilon_m \leq i \leq 24m+4$. We have

$$0 < k_{i,m} - \left\lfloor \frac{7i}{12} \right\rfloor \leq k_{i,m} - \frac{7i}{12} + \frac{11}{12} = \frac{264m+55+i}{288m+60} < 1$$

for all $\varepsilon_m \leq i \leq 24m+4$. It follows that $k_{i,j}$ is contained in the interval $(\left\lfloor \frac{7i}{12} \right\rfloor, \left\lfloor \frac{7i}{12} \right\rfloor + 1]$ for all $j \geq m$ and $\varepsilon_m \leq i \leq 24m+4$. To verify Equation (3.4), it remains to show that $k_{i,m}$ is not an integer for all $\varepsilon_m \leq i \leq 24m+4$. Since $14m+3$ and $24m+5$ are relatively prime, $k_{i,m}$ is an integer if and only if $24m+5$ divides i. But this is not possible since $1 \leq i < 24m+5$.

By a direct computation, we can verify Equation (3.3) when $j=1$. From Equation (3.4), we see that the right-hand side of Equation (3.1) has prefix $w_1 = yx^{-1}y^{-1}xu$ for all $j \geq 1$. We write $w = w_1 w'_j = (yx^{-1}y^{-1}x)uw'_j$. It remains to show that $w'_j = u^{j-1}$. Using Equation (3.4), we have

$$\lfloor k_{i+24n,j} \rfloor = \left\lfloor \frac{7i}{12} + 14n \right\rfloor = \left\lfloor \frac{7i}{12} \right\rfloor + 14n = \lfloor k_{i,j} \rfloor + 14n$$

for all $5 \leq i \leq 28$ and $5 \leq i + 24n \leq 24j+4$. We have

$$\lfloor k_{i,j} \rfloor \equiv \lfloor k_{i+24n,j} \rfloor \mod 2 \tag{3.5}$$

for all $5 \leq i \leq 28$ and $5 \leq i+24n \leq 24j+4$. Equation (3.5) implies that the parity of $\lfloor k_{i,j} \rfloor$ repeats with period 24 when $i \geq 5$. Since the word for w in x and y only depends on this parity, the word w is given by Equation (3.3) as claimed. This completes the proof of the lemma. □

3.1.2 The Alexander Polynomial Of K_j

Now we will compute the Alexander polynomial of K_j using non-abelian reducible representations. Let $\rho : \Gamma \to \mathrm{SL}_2(\mathbb{C})$ be a non-abelian reducible representation of Γ. Since Γ is generated by two conjugate meridians x and y, the representation ρ can be conjugated to have the form

$$x \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \quad \text{and} \quad y \mapsto \begin{pmatrix} t & 1 \\ 0 & t^{-1} \end{pmatrix} \tag{3.6}$$

where $t \neq \pm 1$. By Theorem 2.24, for a knot group Γ the assignment in Equation (3.6) defines a representation of Γ if and only if t^2 is a root of the Alexander polynomial $\Delta(\tau) \in \mathbb{Z}[\tau^{\pm 1}]$. Consequently, we can use this fact to compute the Alexander polynomial of the knot K_j as follows.

Let F_2 be the free group on two letters X and Y. Consider the representation $P : F_2 \to \mathrm{SL}_2(\mathbb{Z}[\tau^{\pm 1}])$

$$X \mapsto \begin{pmatrix} \tau & 0 \\ 0 & \tau^{-1} \end{pmatrix}, \quad \text{and} \quad Y \mapsto \begin{pmatrix} \tau & 1 \\ 0 & \tau^{-1} \end{pmatrix}.$$

Let W be the word in X and Y given by Equation (3.3). A direct calculation shows that

$$P(W) = \begin{pmatrix} 1 & -j\tau^3 + (5j+1)\tau - (5j+1)\tau^{-1} + j\tau^{-3} \\ 0 & 1 \end{pmatrix}.$$

The representation P factors through the natural projection $F_2 \to \Gamma$ if and only if $P(XW) = P(WY)$. Or equivalently, we have

$$j\tau^4 - (6j+1)\tau^2 + (10j+3) - (6j+1)\tau^{-2} + j\tau^{-4} = 0.$$

The expression above is the Alexander polynomial of K_j evaluated at τ^2. As a convention, we will normalize the Alexander polynomial so that the lowest term of $\Delta(\tau)$ is a non-zero constant term. We have the following lemma.

Lemma 3.2. *The Alexander polynomial of K_j has the form*

$$\Delta(\tau) = j\tau^4 - (6j+1)\tau^3 + (10j+3)\tau^2 - (6j+1)\tau + j. \tag{3.7}$$

Furthermore, $\Delta(\tau)$ has exactly 4 simple real roots.

Proof. The discussion prior to the lemma implies that

$$\Delta(\tau^2) = j\tau^8 - (6j+1)\tau^6 + (10j+3)\tau^4 - (6j+1)\tau^2 + j.$$

This gives us Equation (3.7) as claimed. For the claim about the roots of Δ, we consider $\delta(\tau) = \Delta(\tau)/j$. We note that

$$\delta_j(0) = 1,\ \delta_j(1/2) = \frac{-3j+2}{16j},\ \delta_j(1) = \frac{1}{j},\ \delta_j(2) = \frac{-3j+2}{j},\ \delta_j(5) = \frac{96j-55}{j}.$$

For all $j \geq 1$, we see that δ_j changes signs 4 times in the interval $[0, 5]$. By continuity, $\delta_j(\tau)$ has 4 distinct real roots in the interval $[0, 5]$. Therefore, $\Delta(\tau)$ has at least 4 positive real roots. Since Δ has degree 4, Δ has precisely 4 simple positive real roots for all $j \geq 1$. $\square$

3.1.3 The Group Cohomology $H^1(\Gamma(0); \mathfrak{sl}_2(\mathbb{C})_\rho)$

In this section, we will prove that the knots K_j are locally longitudinal rigid by directly computing the group cohomology with coefficients in $\mathfrak{sl}_2(\mathbb{C})$. We first identify $\mathfrak{sl}_2(\mathbb{C})$ with $\mathbb{C}^3$ by choosing the following basis

$$v_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad v_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{and} \quad v_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{3.8}$$

With respect to this basis, the adjoint representation $\mathrm{Ad} : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SL}_3(\mathbb{C})$ becomes

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^2 & -2ab & -b^2 \\ -ac & ad+bc & bd \\ -c^2 & 2cd & d^2 \end{pmatrix}.$$

By Lemma 3.2, we can choose $t \in \mathbb{R}$ such that t^2 is a simple root of the Alexander polynomial $\Delta(\tau)$. Since the longitude ℓ belongs to the second commutator subgroup of Γ, any non-abelian reducible representation on Γ factors through $\Gamma(0)$. We get a non-abelian reducible representation $\rho : \Gamma(0) \to \mathrm{SL}_2(\mathbb{C})$ given by Equation (3.6). For convenience, we will write

$$\rho(w) = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}$$

where $f = -jt^3 + (5j+1)t - (5j+1)t^{-1} + jt^{-3}$. The action of $\Gamma(0)$ on $\mathfrak{sl}_2(\mathbb{C})$ is given by

$$x \mapsto \begin{pmatrix} t^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t^{-2} \end{pmatrix} \quad \text{and} \quad y \mapsto \begin{pmatrix} t^2 & -2t & -1 \\ 0 & 1 & t^{-1} \\ 0 & 0 & t^{-2} \end{pmatrix}.$$

Using Equation (2.2), we see that the space of coboundaries can be parametrized by $d : \Gamma(0) \to \mathfrak{sl}_2(\mathbb{C})_\rho$ such that

$$d(x) = \begin{pmatrix} (t^2-1)a \\ 0 \\ (t^{-2}-1)c \end{pmatrix} \quad \text{and} \quad d(y) = \begin{pmatrix} (t^2-1)a - 2tb - c \\ t^{-1}c \\ (t^{-2}-1)c \end{pmatrix}. \tag{3.9}$$

Proposition 3.3. *Any cohomology class in $H^1(\Gamma(0); \mathfrak{sl}_2(\mathbb{C})_\rho)$ can be represented by a 1-cocycle $z \in Z^1(\Gamma(0); \mathfrak{sl}_2(\mathbb{C})_\rho)$ such that*

$$z(x) = \begin{pmatrix} 0 \\ \alpha \\ \beta \end{pmatrix} \quad \textit{and} \quad z(y) = \begin{pmatrix} 0 \\ \alpha \\ 0 \end{pmatrix}. \tag{3.10}$$

Proof. Let $z \in Z^1(\Gamma(0); \mathfrak{sl}_2(\mathbb{C})_\rho)$ be a 1-cocycle. Since $t^2 \neq 1$, by an appropriate choice of $a, b, c \in \mathbb{C}$ for a coboundary d in Equation (3.9), we can assume that

$$z(x) = \begin{pmatrix} 0 \\ \alpha \\ \beta \end{pmatrix} \quad \text{and} \quad z(y) = \begin{pmatrix} 0 \\ \delta \\ 0 \end{pmatrix}.$$

The relation $z(xw) = z(wy)$ implies that

$$z(x) + (x-1) \cdot z(w) - w \cdot z(y) = 0.$$

Or equivalently, we have

$$\begin{pmatrix} 0 \\ \alpha \\ \beta \end{pmatrix} + \begin{pmatrix} t^2-1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & t^{-2}-1 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} - \begin{pmatrix} 1 & -2f & -f^2 \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \delta \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The second coordinate of the previous equation implies that $\delta = \alpha$. □

We will need the following lemma

Lemma 3.4. *Let $z \in Z^1(\Gamma(0); \mathfrak{sl}_2(\mathbb{C})_\rho)$ be given by Equation (3.10). Suppose that*

$$z(w) = \omega_1 v_+ + \omega_2 v_0 + \omega_3 v_- \quad \textit{and} \quad z(v) = \nu_1 v_+ + \nu_2 v_0 + \nu_3 v_-.$$

Then

$$\begin{aligned}
\omega_1 =&\alpha(-4jt^3 + (10j+2)t - 2jt^{-3}) + \beta h \\
\omega_2 =&\left(\frac{1}{2}j(j+1)t^7 - 5j(j+1)t^5 + \frac{1}{2}(35j^2+31j+2)t^3 - (26j^2+14j+2)t \right.\\
&\left. + \frac{1}{2}j(35j-3)t^{-1} - j(5j-3)t^{-3} + \frac{1}{2}j(j-1)t^{-5}\right)\beta \\
\nu_2 =&\left(\frac{1}{2}j(j+1)t^7 - 5j(j+1)t^5 + \frac{1}{2}(35j^2+27j+2)t^3 - (26j^2+6j)t \right.\\
&\left. + \frac{1}{2}j(35j-7)t^{-1} - j(5j-3)t^{-3} + \frac{1}{2}j(j-1)t^{-5}\right)\beta \\
\omega_3 =& -\nu_3 = tf\beta
\end{aligned}$$

where $h \in \mathbb{C}$.

Proof. The proof of this lemma is a direct calculation. By a repeat application of the cocycle relation in Equation (2.1), we have

$$z(w) = z(yx^{-1}y^{-1}x) + (yx^{-1}y^{-1}x) \cdot \sum_{i=0}^{j-1} u^i \cdot z(u),$$

$$z(v) = s^j \cdot z(xy^{-1}x^{-1}y) + \sum_{i=0}^{j-1} s^i \cdot z(s).$$

We also have

$$\sum_{i=0}^{j-1} (\mathrm{Ad} \circ \rho)(u^i) = \begin{pmatrix} j & (j^2-j)(t^3-5t+5t^{-1}-t^{-3}) & -\frac{1}{6}(2j^3-3j^2+j)(t^3-5t+5t^{-1}-t^{-3})^2 \\ 0 & j & -\frac{1}{2}(j^2-j)(t^3-5t+5t^{-1}-t^{-3}) \\ 0 & 0 & j \end{pmatrix},$$

$$\sum_{i=0}^{j-1} (\mathrm{Ad} \circ \rho)(s^i) = \begin{pmatrix} j & -(j^2-j)(t^3-5t+5t^{-1}-t^{-3}) & -\frac{1}{6}(2j^3-3j^2+j)(t^3-5t+5t^{-1}-t^{-3})^2 \\ 0 & j & \frac{1}{2}(j^2-j)(t^3-5t+5t^{-1}-t^{-3}) \\ 0 & 0 & j \end{pmatrix},$$

and

$$z(u) = \begin{pmatrix} (-4t^3 + 10t - 2t^{-3})\alpha + h'\beta \\ (t^7 - 9t^5 + 27t^3 - 30t + 10t^{-1} - t^{-3})\beta \\ (-t^4 + 5t^2 - 5 + t^{-2})\beta \end{pmatrix},$$

$$z(s) = \begin{pmatrix} (4t^3 - 10t + 2t^{-3})\alpha + h''\beta \\ (t^7 - 9t^5 + 25t^3 - 22t + 8t^{-1} - t^{-3})\beta \\ (t^4 - 5t^2 + 5 - t^{-2})\beta \end{pmatrix}$$

for some $h', h'' \in \mathbb{C}$. The lemma will follow once we note that

$$z(yx^{-1}y^{-1}x) = \begin{pmatrix} 2t\alpha - (t^4 - 3t^2 + 1)\beta \\ (t^3 - 2t)\beta \\ (t^2 - 1)\beta \end{pmatrix},$$

$$z(xy^{-1}x^{-1}y) = \begin{pmatrix} (-6t + 2t^{-1})\alpha + t^4\beta \\ t^3\beta \\ (-t^2 + 1)\beta \end{pmatrix}.$$

□

Now we are ready to show that $H^1(\Gamma(0); \mathfrak{sl}_2(\mathbb{C})_\rho) = 0$.

Proof of Theorem 1.9. Now we will show that K_j is locally longitudinally rigid for all $j \geq 1$ at any root of the Alexander polynomial. Let $[z] \in H^1(\Gamma(0); \mathfrak{sl}_2(\mathbb{C})_\rho)$. By Proposition 3.3, we can assume that z satisfies Equation (3.10). Since $z(\ell) = 0$, we must have

$$z(w) + w \cdot z(v) = 0.$$

Let us write $z(w) = \omega_1 v_+ + \omega_2 v_0 + \omega_3 v_-$ and $z(v) = \nu_1 v_+ + \nu_2 v_0 + \nu_3 v_-$. The second coordinate of this equation implies that $\omega_2 + \nu_2 + f\nu_3 = 0$. Using Lemma 3.4, we have

$$\frac{(t^4 - 1)(t^4 + j(t^4 - 4t^2 + 1)^2)}{t^5}\beta = 0.$$

Suppose that $\beta \neq 0$. Since $t \in \mathbb{R} - \{\pm 1\}$, we have

$$t^4 + j(t^4 - 4t^2 + 1)^2 = 0.$$

Since $t \in \mathbb{R}$ and $j \geq 1$, the above equation holds if and only if

$$t = 0 \quad \text{and} \quad t^4 - 4t^2 + 1 = 0.$$

This is the desired contradiction. Therefore, we must have $\beta = 0$.

From the first coordinate of the relation $z(xw) = z(wy)$, we have

$$(t^2 - 1)\omega_1 + 2f\alpha = 0.$$

Using $\beta = 0$ and Lemma 3.4, this equation is equivalent to

$$(t^4 - 1)(2jt^4 - (6j + 1)t^2 + 2j))\alpha = 0.$$

Similarly, if $\alpha \neq 0$, we must have $(2jt^4 - (6j + 1)t^2 + 2j)) = 0$. It follows that t^2 is a root of both

$$\Delta(\tau) \quad \text{and} \quad h(\tau) := (2j\tau^2 - (6j + 1)\tau + 2j).$$

Note that the roots of $h(\tau)$ are reciprocal of each other. Since $\Delta(\tau)$ is a reciprocal polynomial, the roots of $\Delta(\tau)$ come in reciprocal pairs. Therefore, $h(\tau)$ divides $\Delta(\tau)$. By Gauss's lemma, we can write $\Delta(\tau) = h(\tau)k(\tau)$ for $k(\tau) \in \mathbb{Z}[\tau]$. This implies that $\Delta(0) = h(0)k(0)$ or $j = 2jk(0)$. This contradicts the fact that $j \geq 1$ and $k(0) \in \mathbb{Z}$. Therefore, $\alpha = 0$ and z can only be the zero cocycle. Consequently, $H^1(\Gamma(0); \mathfrak{sl}_2(\mathbb{C})_\rho) = 0$ where ρ is any non-abelian reducible representation of $\Gamma(0)$. In other words, the knot K_j is locally longitudinally rigid at any root of the Alexander polynomial. By Theorem 1.8, there exists an interval of left orderable Dehn surgeries near 0. $\square$

3.2 The $(-3, 3, 2j + 1)-$ Pretzel Knot

Let P_j be the $(-3, 3, 2j + 1)-$ pretzel knot for $j \geq 1$.

3.2.1 The Group Presentation

We will denote by Γ the fundamental group of the complement of the pretzel knot P_j. We first compute a presentation of Γ from the knot diagram. Consider the crossing

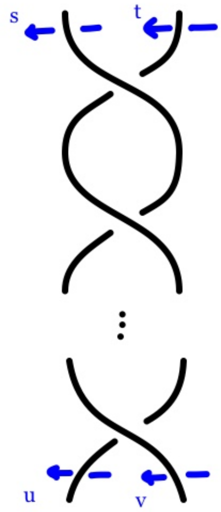

Figure 3.2: A twist region with n positive half-twists.

region with n crossings as in Figure 3.2. By our convention, the crossings in Figure 3.2 are all positive. Let us fix a basepoint lying above the diagram. Let s and t be the meridians at the top, u and v be the meridians at the bottom as in Figure 3.2.

We will denote ${}^{g}h := ghg^{-1}$. In this notation, we have the following relations among the meridians. When $n = 2k+1 > 0$, we have

$$u = {}^{(s^{-1}t^{-1})^k s^{-1}}t, \text{ and } v = {}^{(s^{-1}t^{-1})^k}s.$$

When $n = 2k > 0$, we have

$$u = {}^{(s^{-1}t^{-1})^k}s, \text{ and } v = {}^{(s^{-1}t^{-1})^{k-1}s^{-1}}t.$$

Let s_1, s_2 and s_3 be the meridians as in Figure 3.3. The knot group Γ has the following presentation:

$$\begin{aligned}\Gamma &= \langle s_1, s_2, s_3 \mid {}^{s_2^{-1}s_1s_2^{-1}}s_1 = {}^{s_2^{-1}s_3s_2^{-1}}s_3, {}^{s_2^{-1}s_3}s_2 = {}^{(s_3^{-1}s_1)^j s_3^{-1}}s_1, {}^{(s_3^{-1}s_1)^j}s_3 = {}^{s_2^{-1}s_1}s_2 \rangle \\ &= \langle s_1, s_2, s_3 \mid w_1s_1 = s_2w_1, w_2s_2 = s_3w_2, w_3s_3 = s_1w_3, \rangle\end{aligned} \tag{3.11}$$

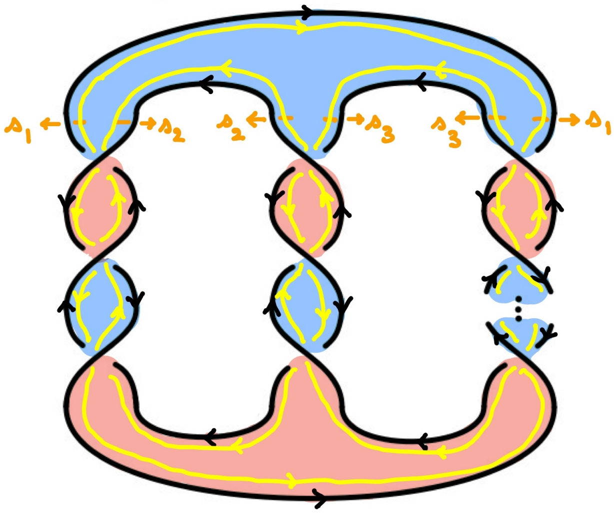

Figure 3.3: The (-3,3,2j+1)-pretzel knot with a Seifert surface and its longitude

where $w_1 = s_3^{-1}s_2(s_3^{-1}s_1)^j s_3^{-1}$, $w_2 = (s_1^{-1}s_3)^j s_2^{-1}s_1$, and $w_3 = s_2s_1^{-1}s_3s_2^{-1}$.

To compute the homological longitude of the knot, we first construct a Seifert surface using Seifert's algorithm. As a result, we obtain the surface in Figure 3.3 with each side colored in red and blue. Tracing out the boundary of this surface, we obtain the longitude of the knot. In terms of the meridians s_i, the homological longitude is given by

$$\ell = s_1^{-1}w_1^{-1}w_2^{-1}w_3^{-1}. \tag{3.12}$$

3.2.2 The Alexander Polynomial Of P_j

Now we will use the presentation of Γ to compute the Alexander polynomial of P_j using Fox calculus. For a reference regarding this material, we will follow [6, Chapter 9]. We first recall the definition of the Alexander polynomial of a knot via Fox calculus. Let

$$\Gamma_K = \langle S_1, \ldots, S_n \mid R_1, \ldots, R_k \rangle$$

be the knot group and $\phi : \Gamma_K \to \mathbb{Z}$ be the abelianization of Γ_k. In particular, ϕ sends all meridians of the knot K to the generator τ of $\mathbb{Z} \cong \langle \tau \rangle$.

Let X be a connected 2-complex with one 0-cells x, n 1$-$cells each corresponds to a generator S_i, and k 2$-$cells each corresponds to a relation R_j. Let $(\widehat{X}, \widehat{x}) \to (X, x)$ be the infinite cyclic cover of X corresponds to the kernel of ϕ and $\widehat{X}_0$ be the 0-skeleton of $\widehat{X}$. The cover $(\widehat{X}, \widehat{x})$ has the structure of a 2-complex equipped with a covering action of $\langle \tau \rangle$. Using the covering action of τ, we get an action of $\mathbb{Z}[\tau^{\pm 1}]$ on $H_1(\widehat{X}, \widehat{X}_0)$. The Alexander module of Γ_K is the $\mathbb{Z}[\tau^{\pm 1}]-$module $H_1(\widehat{X}, \widehat{X}_0)$.

The module $H_1(\widehat{X}, \widehat{X}_0)$ is generated by lifts of the (oriented) 1-cells in X to $\widehat{X}$ based at $\widehat{x}$ and in particular, is finitely generated. The lifts of the boundary $R_j = \partial e_j$ of the 2-cells are the defining relations of the Alexander module. As a consequence, we get a $n \times k$ presentation matrix J_ϕ over $\mathbb{Z}[\tau^{\pm 1}]$ of the Alexander module. The i^{th}-elementary ideal of J_ϕ is the ideal generated by all $(n-i) \times (n-i)$ minors of J_ϕ. Since $\mathbb{Z}[\tau^{\pm 1}]$ is a P.I.D, the Alexander polynomial of Γ_K is defined to be the generators of the first elementary ideal of J_ϕ, which is defined up to ± 1 and powers of τ, the units of $\mathbb{Z}[\tau^{\pm 1}]$.

To compute the Alexander polynomial Γ_K, we will write down the presentation matrix of the Alexander module by lifting R_j to $\widehat{X}$. Let us first consider a closed path γ in the 1-skeleton of X based at x. We denote by $\widehat{\gamma}$ the lift of γ to $\widehat{X}$ based at $\widehat{x}$. Now suppose that $\gamma = \gamma_1\gamma_2$ that is γ can be decomposed into two closed paths γ_1 and γ_2 also based at x. Then the lift of γ based at $\hat{x}$ to $\hat{X}$ can be written as 1-cycle in $(\widehat{X}, \widehat{x})$ as

$$\widehat{\gamma} = \widehat{\gamma_1} + \phi(\gamma_1)\widehat{\gamma_2}.$$

Geometrically, the previous equation means that travelling along $\widehat{\gamma}$ is the same as first travelling along $\widehat{\gamma_1}$ from $\hat{x}$ to $\phi(\gamma_1)(\hat{x})$ and then travelling along the lift of γ_2 based at $\phi(\gamma_1)(\hat{x})$, which is $\phi(\gamma_1)(\widehat{\gamma_2})$. This geometric intuition motivates the definition of Fox derivatives

Definition 3.5. *Let $F_n = \langle S_1, \ldots, S_n \rangle$ be the free group on n letters. The Fox derivative with respect to S_i is the map $\frac{\partial}{\partial S_i} : \mathbb{Z}[F_n] \to \mathbb{Z}[F_n]$ that satisfies the following property:*

1. $\dfrac{\partial S_j}{\partial S_i} = \delta_{ij}$, *the Kronecker delta*

2. $\dfrac{\partial e}{\partial S_i} = 0$

3. $\dfrac{\partial}{\partial S_i}(ab) = \dfrac{\partial a}{\partial S_i} + a\dfrac{\partial b}{\partial S_i}$

Definition 3.6. *Let $\Gamma_K = \langle S_1, \ldots, S_n \mid R_1, \ldots, R_k \rangle$ be the knot group and $\phi : \Gamma_K \to \mathbb{Z}$ be the abelianization of Γ_K. The map ϕ extends linearly to a map between the integral group rings $\phi : \mathbb{Z}\Gamma_K \to \mathbb{Z}[\tau^{\pm 1}]$. The presentation matrix of the Alexander module of Γ_K is given by*

$$J_\phi = \left(\phi\left(\frac{\partial R_j}{\partial S_i}\right)\right)$$

where $1 \leq i \leq n$, $1 \leq j \leq k$.

Remark 3.7. *Technically, the Fox derivative is only defined for free group. To make sense of the formula in the previous definition, we can think about S_i as generators of the free group F_n and R_j as words in S_i in F_n and compute the Fox derivatives in $\mathbb{Z}[F_n]$. Now, we can precompose the natural surjection $\psi : \mathbb{Z}[F_n] \to \mathbb{Z}[\Gamma_K]$ with ϕ to get the entries of the presentation matrix over $\mathbb{Z}[\tau^{\pm 1}]$.*

We will need the following to compute the Jacobian matrix J_ϕ.

$$\begin{aligned}
\phi\left(\frac{\partial w_1}{\partial s_1}\right) &= \phi\left(\sum_{k=0}^{j-1} s_3^{-1}s_2(s_3^{-1}s_1)^k s_3^{-1}\right) = j\tau^{-1} \\
\phi\left(\frac{\partial w_1}{\partial s_2}\right) &= \phi(s_3^{-1}) = \tau^{-1} \\
\phi\left(\frac{\partial w_1}{\partial s_3}\right) &= \phi\left(-s_3^{-1} - \sum_{k=0}^{j} s_3^{-1}s_2(s_3^{-1}s_1)^k s_3^{-1}\right) = -(j+2)\tau^{-1}
\end{aligned}$$

$$\begin{aligned}
\phi\left(\frac{\partial w_3}{\partial s_1}\right) &= \phi(-s_2s_1^{-1}) = 1 \\
\phi\left(\frac{\partial w_3}{\partial s_2}\right) &= \phi(1 - w_3) = 0 \\
\phi\left(\frac{\partial w_3}{\partial s_3}\right) &= \phi(s_2s_1^{-1}) = 1
\end{aligned}$$

$$\phi\left(\frac{\partial r_1}{\partial s_1}\right) = \phi\left(\frac{\partial w_1}{\partial s_1} + w_1 - w_1 s_1 w_1^{-1}\frac{\partial w_1}{\partial s_1}\right) = (1-\tau)j\tau^{-1} + \tau^{-1}$$
$$\phi\left(\frac{\partial r_1}{\partial s_2}\right) = \phi\left(\frac{\partial w_1}{\partial s_2} - w_1 s_1 w_1^{-1}\frac{\partial w_1}{\partial s_2} - w_1 s_1 w_1^{-1} s_2^{-1}\right) = (1-\tau)\tau^{-1} - 1$$
$$\phi\left(\frac{\partial r_1}{\partial s_3}\right) = \phi\left(\frac{\partial w_1}{\partial s_3} - w_1 s_1 w_1^{-1}\frac{\partial w_1}{\partial s_3}\right) = -(1-\tau)(j+2)\tau^{-1}$$

$$\phi\left(\frac{\partial r_3}{\partial s_1}\right) = \phi\left((1 - w_3 s_3 w_3^{-1})\frac{\partial w_3}{\partial s_1} - r_3\right) = \tau - 2$$
$$\phi\left(\frac{\partial r_3}{\partial s_2}\right) = \phi\left((1 - w_3 s_3 w_3^{-1})\frac{\partial w_3}{\partial s_2}\right) = 0$$
$$\phi\left(\frac{\partial r_3}{\partial s_3}\right) = \phi\left((1 - w_3 s_3 w_3^{-1})\frac{\partial w_3}{\partial s_3} + w_3\right) = -\tau + 2$$

Lemma 3.8. *The Alexander polynomial of P_j, up to normalization, has the form*

$$\Delta(\tau) = (\tau - 2)(2\tau - 1) \tag{3.13}$$

In particular, $\Delta(\tau)$ has a simple real root that is larger than 1.

Proof. The prensetation matrix of the Alexander module of Γ is

$$J_\phi = \begin{pmatrix} (j+1)\tau^{-1} - j & \tau^{-1} - 2 & (j+2)(-\tau^{-1}+1) \\ \tau - 2 & 0 & -\tau + 2 \end{pmatrix}.$$

All 2×2 minors of this matrix are the same up to signs. Therefore, the Alexander polynomial of P_j is the determinant of the matrix

$$\begin{pmatrix} (j+1)\tau^{-1} - j & \tau^{-1} - 2 \\ \tau - 2 & 0 \end{pmatrix}$$

which has the form

$$-(\tau - 2)(\tau^{-1} - 2).$$

Up to some power of τ, the Alexander polynomial of P_j is $\Delta(\tau)$. $\square$

3.2.3 The Group Cohomology $H^1(\Gamma(0); \mathfrak{sl}_2(\mathbb{C})_\rho)$

We will now prove that the knots P_j are locally longitudinal rigid by directly computing the group cohomology with coefficients in $\mathfrak{sl}_2(\mathbb{C})$. We will use the same identification of $\mathfrak{sl}_2(\mathbb{C})$ with $\mathbb{C}^3$ as in Equation (3.8).

We now choose a non-abelian reducible representation of $\Gamma(0)$ that corresponds to a root of the Alexander polynomial. Consider the assignment $\rho : \Gamma \to \mathrm{SL}_2(\mathbb{C})$ given by

$$s_1 \mapsto \begin{pmatrix} 2^{1/2} & 1 \\ 0 & 2^{-1/2} \end{pmatrix}, \quad s_2 \mapsto \begin{pmatrix} 2^{1/2} & \beta \\ 0 & 2^{-1/2} \end{pmatrix}, \quad s_3 \mapsto \begin{pmatrix} 2^{1/2} & 0 \\ 0 & 2^{-1/2}. \end{pmatrix} \tag{3.14}$$

Using the group relations in Equation (3.11), we deduce that ρ defines a group homomorphism of Γ if and only if $\beta = (-j+1)/3$. Since the longitude ℓ lies in the second commutator subgroup of Γ, the representation ρ factors through $\Gamma(0)$ and gives us $\rho : \Gamma(0) \to \mathrm{SL}_2(\mathbb{C})$ defined by Equation (3.14).

Using Equation (2.2), we parametrize the space of coboundaries $B^1(\Gamma(0); \mathfrak{sl}_2(\mathbb{C})_\rho)$ by $d : \Gamma(0) \to \mathfrak{sl}_2(\mathbb{C})_\rho$ such that

$$d(s_1) = \begin{pmatrix} a - 2\sqrt{2}b - c \\ c/\sqrt{2} \\ -c/2 \end{pmatrix} d(s_1) = \begin{pmatrix} a - 2\sqrt{2}\beta b - \beta^2 c \\ \beta c/\sqrt{2} \\ -c/2 \end{pmatrix} d(s_3) = \begin{pmatrix} a \\ 0 \\ -c/2 \end{pmatrix} \tag{3.15}$$

Proposition 3.9. *Any cohomology class in $H^1(\Gamma(0); \mathfrak{sl}_2(\mathbb{C})_\rho)$ can be represented by a 1-cocycle $z \in Z^1(\Gamma(0); \mathfrak{sl}_2(\mathbb{C})_\rho)$ such that*

$$z(s_1) = \begin{pmatrix} 0 \\ b_1 \\ 0 \end{pmatrix} \quad z(s_2) = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} \quad z(s_3) = \begin{pmatrix} 0 \\ b_1 \\ 0 \end{pmatrix}. \tag{3.16}$$

Proof. Let $z \in Z^1(\Gamma(0); \mathfrak{sl}_2(\mathbb{C})_\rho)$ be a 1-cocycle. Adjusting z by an appropriate coboundary d as in Equation (3.15), we may assume that

$$z(s_1) = \begin{pmatrix} 0 \\ b_1 \\ c_1 \end{pmatrix} \quad z(s_2) = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} \quad z(s_3) = \begin{pmatrix} 0 \\ b_3 \\ 0 \end{pmatrix}. \tag{3.17}$$

By a direct calculation, we have

$$z(w_3) = z(s_2 s_1^{-1} s_3 s_2^{-1}) = \begin{pmatrix} \frac{2\sqrt{2}}{3}(j+2)(-b_1+b_3) - 2\sqrt{2}b_2 + \frac{2}{9}(2+j)^2 c_1 + 2c_2 \\ -b_1 + b_3 + (j+2)\frac{\sqrt{2}}{3}c_1 + \sqrt{2}c_2 \\ -c_1 \end{pmatrix}.$$

From the group relation, we must have $z(w_3 s_3) = z(s_1 w_3)$ which is equivalent to

$$\begin{pmatrix} \frac{2\sqrt{2}}{3}(j-1)b_1 + 2\sqrt{2}b_2 + \frac{2\sqrt{2}}{3}(-j+4)b_3 + \frac{-2j^2+4j+7}{9}c_1 + 2c_2 \\ -b_1 + b_3 + \frac{c_1}{\sqrt{2}} \\ -\frac{3}{2}c_1 \end{pmatrix} = 0.$$

Therefore, $c_1 = 0$ and $b_3 = b_1$. $\square$

Lemma 3.10. *Let $z \in Z^1(\Gamma(0); \mathfrak{sl}_2(\mathbb{C})_\rho)$ be given by Equation (3.16). Suppose that*

$$z(w_i) = w_{i,1}v_+ + w_{i,2}v_0 + w_{i,3}v_-.$$

Then we have

$$\begin{aligned} w_{1,1} &= \frac{1}{2}a_2 + \frac{\sqrt{2}}{3}(2j+1)b_1,\ w_{1,2} = -2b_1 + b_2,\ w_{1,3} = 2c_2 \\ w_{2,2} &= b_1 - b_2 + \frac{\sqrt{2}}{3}(-7j+1)c_2,\ w_{2,3} = -2c_2 \\ w_{3,1} &= -2\sqrt{2}b_2 + 2c_2,\ w_{3,2} = \sqrt{2}c_2,\ w_{3,3} = 0 \end{aligned}$$

Proof. The proof of the lemma is a direct calculation. It is helpful to note that

$$z(s_3^{-1}s_1) = z(s_1^{-1}s_3) = 0.$$

Using this identity, we have

$$z(w_1) = z(s_3^{-1}s_2) - w_1 \cdot z(s_3) = \begin{pmatrix} \frac{1}{2}a_2 + \frac{\sqrt{2}}{3}(2j+1)b_1 \\ -2b_1 + b_2 \\ 2c_2 \end{pmatrix},$$

$$z(w_2) = (s_1^{-1}s_3) \cdot z(s_2^{-1}z_1) = \begin{pmatrix} * \\ b_1 - b_2 + \frac{\sqrt{2}}{3}(-7j+1)c_2 \\ -2c_2 \end{pmatrix},$$

$$z(w_3) = z(s_2) - w_3 \cdot z(s_2) = \begin{pmatrix} -2\sqrt{2}b_2 + 2c_2 \\ \sqrt{2}c_2 \\ 0 \end{pmatrix}.$$

□

Proof of Theorem 1.11. Now we will show that P_j is locally longitudinally rigid for all $j \geq 1$ at the root $\tau = 2$ of the Alexander polynomial. Let $[z] \in H^1(\Gamma(0); \mathfrak{sl}_2(\mathbb{C})_\rho)$. By Proposition 3.9, we may assume that z is given by Equation (3.16). By Lemma 3.10, the second coordinate of the equation $z(\ell) = 0$ is

$$\sqrt{2}(3j-2)c_2 = 0,$$

which implies that $c_2 = 0$. Letting $c_2 = 0$, the relation $z(w_3 s_3) = z(s_1 w_3)$ becomes

$$2\sqrt{2}(b_1 + b_2) = 0.$$

By Lemma 3.10, the first and second coordinates of the equation $z(w_1 s_1) = z(s_2 w_1)$ become

$$2b_1 = 0 \quad \text{and} \quad -\frac{3}{2}a_2 + \frac{2\sqrt{2}}{3}(j-4)b_1 = 0.$$

This implies that $a_2 = b_1 = 0$ and that z must be the zero cocycle. Consequently, $H^1(\Gamma(0); \mathfrak{sl}_2(\mathbb{C})_\rho) = 0$ where ρ is any non-abelian reducible representation of $\Gamma(0)$. In other words, the knot P_j is locally longitudinally rigid at the root $\tau = 2$ of the Alexander polynomial. By Theorem 1.8, there exists an interval of left orderable Dehn surgeries near 0. □

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