QUANTIFICATION OF STABILITY OF ANALYTIC CONTINUATION WITH APPLICATIONS TO ELECTROMAGNETIC THEORY

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ABSTRACT

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Analytic functions in a domain Ω are uniquely determined by their values on any curve $\Gamma \subset \Omega$. We provide sharp quantitative version of this statement. Namely, let f be of order ϵ on Γ relative to its global size in Ω (measured in some Hilbert space norm). How large can f be at a point z away from the curve? We give a sharp upper bound on $|f(z)|$ in terms of a solution of a linear integral equation of Fredholm type and demonstrate that the bound behaves like a power law: $\epsilon^{\gamma(z)}$. In special geometries, such as the upper half-plane, annulus or ellipse the integral equation can be solved explicitly, giving exact formulas for the optimal exponent $\gamma(z)$. Our methods can be applied to non-Hilbertian settings as well.

Further, we apply the developed theory to study the degree of reliability of extrapolation of the complex electromagentic permittivity function based on its analyticity properties. Given two analytic functions, representing extrapolants of the same experimental data, we quantify how much they can differ at an extrapolation point outside of the experimentally accessible frequency band.

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To my parents...

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CHAPTER 1

INTRODUCTION

Many inverse problems reduce to analytic continuation questions when solutions of direct problems are known to possess analyticity in a domain in the complex plane, but can be measured only on a subset (often a part of the boundary) of this domain. For example, if one wants to recover a signal corrupted by a low-pass convolution filter, then one needs to recover an entire function from its measured values on an interval [21, 3]. Another large class of inverse problems can be termed "Dehomogenization" [16, 56], where one wants to reconstruct some details of microgeometry from measurements of effective properties of the composite. The idea of reconstruction is based on the analytic properties of effective moduli [7, 54, 35] of composites. See e.g. [55] for an extensive bibliography in this area. Another example that we will study in detail in Chapter 5 is the question of stability of extrapolation of complex electromagnetic permittivity of materials as a function of frequency [46, 28]. An underlying mathematical problem here is about identifying a Herglotz function—a complex analytic function in the upper half-plane $\mathbb{H}_+$ that has nonnegative imaginary part, given its values at specific points in the upper half-plane or on its boundary. Such functions, and their variants, are ubiquitous in physics. For example, the complex impedance of an electrical circuit as a function of frequency has a similar property. Yet another example, is the dependence of effective moduli of composites on the moduli of its constituents

[5, 53]. These functions appear in areas as diverse as optimal design problems [47, 48], nuclear physics [12, 13] and medical imaging [26].

The method of recovery via analytic continuation is a tempting proposition in view of the uniqueness properties of analytic functions. Unfortunately, analyticity is a local property "stored" at an infinite depth within the continuum of function values and can be represented by delicate cancellation properties responsible for the validity of Carleman and Carleman type extrapolation formulas [14, 36, 1]. Adding small errors to the exact values of analytic functions destroys these local properties. Instead we want to accumulate the remnants of analyticity and use global properties of analytic functions to achieve analytic continuation. This is only possible under some additional regularizing constraints, such as global boundedness [22, 11, 67, 33, 69]. Taking this idea to the extreme, any bounded entire function is a constant by Liouville's theorem, so that the effect of boundedness depends strongly on the geometry of the domain of analyticity.

In order to quantify the degree to which analytic continuation is possible, consider an analytic function F in a domain Ω. Assume that F is measured on a curve $\Gamma \Subset \Omega$ with a relative error ϵ, with respect to some norm $\|F\|_\Gamma$. Can one perform an analytic continuation of F from Γ to Ω in the presence of measurement errors? Without discussing specific analytic continuation algorithms we would like to examine theoretical feasibility of such a procedure. For example, if two different algorithms are deployed matching F on Γ with relative precision ϵ how much their outputs could possibly differ at a given point $z \in \Omega \setminus \Gamma$? To answer this question we consider the difference f of the two purported analytic continuations. Such a difference will be small on Γ, and we want to quantify how large such a function can possibly be at some point $z \in \Omega$ relative to its global size on Ω.

Based on established upper and lower bounds, exact and numerical results [19, 11, 17, 52, 61, 31, 70, 32, 20, 69] a general *power law principle* emerges, whereby the relative precision of analytic continuation decays as power law $\epsilon^{\gamma(z)}$, where the exponent $\gamma(z) \in (0, 1)$ decreases to 0, as we move further away from the source of data. How fast $\gamma(z)$ decays depends strongly on the geometry of the domain and the data source. We believe that such power law transition from well-posedness to practical ill-posedness is a general property of analytic continuation, quantifying the tug-of-war between their rigidity (unique continuation property) and flexibility (as in the Riesz density theorem [59]).

The lower bounds on $\gamma(z)$ can be obtained by exhibiting bounded analytic functions that are small on a curve Γ, but not quite as small at a particular extrapolation point. The upper bounds are harder to prove but there is ample literature where such results are achieved [19, 11, 17, 52, 61, 31, 70, 32, 20, 69]. The most common setting considered in the literature is that of bounded analytic functions $H^\infty(\Omega)$, where the size of a function on Γ is measured in L^∞-norm. The power law estimates are then derived from a maximum modulus principle, the classical Hadamard three-circles theorem [49] being the prime example. Taking an example from [69], the modulus of the function $e^{\zeta \ln \epsilon} f(\zeta)$ does not exceed ϵ on the boundary of the infinite strip $\mathfrak{Re}\zeta \in (0, 1)$, provided $|f(\zeta)| \le 1$ in the strip and $|f(iy)| \le \epsilon$. The maximum modulus principle (or rather its Phragmén-Lindelöf version) then implies that $|f(z)| \le \epsilon^{1-\mathfrak{Re}z}$. The estimate is optimal, since $f(\zeta) = \epsilon e^{-\zeta \ln \epsilon}$ satisfies the constraints and achieves equality in the maximum modulus principle. In fact, it was observed in [69] that upper and lower bounds of the form $\epsilon^{\gamma(z)}$ on the extrapolation error do hold for *all geometries*. However, with few exceptions the upper and lower bounds do not match. In those examples where they do match [20, 69] the optimality of the bounds are concluded a posteriori.

In Chapter 2 we develop the theory of the worst case behavior and reveal the mechanism by which the power laws $\epsilon^{\gamma(z)}$ arise. Namely, we introduce a new method for characterizing analytic functions in reproducing kernel Hilbert spaces $\mathcal{H} = \mathcal{H}(\Omega)$ attaining the optimal upper bound in the extrapolation error in terms of a solution of an integral equation of the second kind with compact, positive, self-adjoint operator $\mathscr{K}$ on $\mathcal{H}$. The error maximization problem is reformulated as maximization of a linear objective functional subject to quadratic constraints, permitting us to use convex duality methods. The optimality conditions take the form of a linear integral equation of Fredholm type, where the integral operator $\mathscr{K}$ is expressed in terms of the reproducing kernel of $\mathcal{H}$. This operator occurs frequently in the context of reproducing kernel Hilbert spaces (e.g. [19]) and is related to the restriction operator $\mathscr{R} : \mathcal{H} \to L^2(\Gamma)$ (if we assume that the measurements on Γ are done w.r.t. $L^2(\Gamma)$ norm). Namely, $\mathscr{K} = \mathscr{R}^*\mathscr{R}$. The optimal exponent $\gamma(z)$ in the power law asymptotics can then be expressed in terms of the rates of exponential decay of eigenvalues of the integral operator $\mathscr{K}$ and its eigenfunctions at the extrapolation point $z \in \Omega$ (cf. Section 2.2). For certain classes of restriction operators the exponential decay of the eigenvalues of $\mathscr{K}$ has been known for a long time, and their exact asymptotics has been established in [58] (see also [72, 57, 41, 63]). Alternatively, the exponent $\gamma(z)$ can be read off the explicit solution of the integral equation in cases where such an explicit solution is available (Section 4.2). In some applications (e.g. Section 3.2) additional $\mathbb{C}$-linear constraints are imposed on the analytic functions. In Section 2.3 we discuss such constraints and the relation of the problems with and without them.

In Chapters 3 and 4 we present applications of the theory developed in Chapter 2, where the integral equation can be solved; the exponential decay of the eigenvalues and eigenfunctions of $\mathscr{K}$ can be seen explicitly. As a result we obtain explicit formulas for $\gamma(z)$ in a number of special cases. In Section 3.1 we consider the case when Γ is a concentric circle in an annulus. In Section 3.2 we present a somewhat unexpected [1] application of the annulus result to the problem of analytic continuation in a Bernstein ellipse [8], studied in [20]. When the extrapolation point z lies on the real line inside the Bernstein ellipse we recover the optimal exponent $\gamma(z)$ obtained in [20]. However, our approach also gives the formula for the exponent $\gamma(z)$ for arbitrary points z inside the ellipse. Moreover, it also shows that the exponent $\gamma(z)$ is the same both in

[1] Since the annulus is not conformally equivalent to the ellipse one would not expect a direct relation.

H^∞ and (weighted) H^2 spaces of the ellipse, suggesting that the exponents must be robust and not very sensitive to the choice of specific norms in the spaces of analytic functions. This phenomenon could be related to the fact that functions with worst extrapolation error can be analytically continued into much larger domains, as is evident from our integral equation, and hence satisfy the required constraints in all L^p or H^p norms. Section 4.1 deals with the case when Γ is a circle in the upper half-plane $\mathbb{H}_+$. In Section 4.2 we analyze the case when $\Gamma = [-1, 1]$ lies on the boundary of $\mathbb{H}_+$. We show that the error maximizer again solves an integral equation, but with a singular, non-compact integral operator. This singular equation is then solved explicitly and the exponent $\gamma(z)$ is computed. Examining the formula for $\gamma(z)$ we find a beautiful geometric interpretation of this exponent: it is the angular size of the interval $[-1, 1]$ as viewed from z, measured in units of π. Conformal mappings between domains can be used to "transplant" the exponent estimates from one geometry to a different one (e.g. Remark 4.3).

In Chapter 5 we study the feasibility of extrapolation of the complex electromagnetic permittivity function , which was the motivating problem of this dissertation. Given the experimentally measured data on a band of frequencies, the unavoidable random noise makes the measured values mathematically inconsistent with the analyticity of the complex electromagnetic permittivity function. In Section 5.4 we analyze the least squares problem: find the closest admissible function to the experimental data. We show that the least squares problem has a unique solution, which yields a mathematically stable extrapolant. It turns out that the minimizer must be a rational function and we derive the necessary and sufficient conditions for its optimality. Surprisingly, the class of physically admissible functions is also "flexible" in the sense that the data can be matched up to a given precision by two functions that are very different away from the interval, where the data is available. In Section 5.5 we quantify this phenomenon by giving an optimal upper bound on the possible discrepancy between any two approximate extrapolants. This is done by reducing this question to the stability of analytic continuation in the upper half-plane from the interval $\Gamma = [-1,1] + ih$ in presence of the symmetry constraint $\overline{f(-\overline{\zeta})} = f(\zeta)$. This constraint is only $\mathbb{R}$-linear (in f) and the discussions of Section 2.3 do not apply. Therefore, the analysis of the optimal bound in presence of this constraint is not straightforward. In Section 5.6 we show that adding this symmetry constraint has no effect on the asymptotic behavior of the error of analytic continuation (in fact, for arbitrary curves Γ and not just the interval $[-1,1] + ih$). Consequently, we ignore the symmetry constraint and arrive at the problem of quantifying the error of analytic continuation in $\mathbb{H}_+$ from the interval $\Gamma = [-1,1] + ih$ to a given point $z \notin \Gamma$. General theory developed in Chapter 2 demonstrates the power law behavior $\epsilon^{\gamma(z)}$ for this error. In Section 5.7, invoking results of Chapter 2 we solve the resulting integral equation numerically and produce a plot of the exponent $\gamma(z)$. Motivated by the explicit results of Section 4.1, we construct a test function that gives an upper bound on $\gamma(z)$ in terms of an analytical expression, which is then checked numerically to be in an excellent agreement with it for $h > 0.6$.

It is worth mentioning that in most cases, where we obtain explicit formulas for $\gamma(z)$ it coincides with the harmonic measure of Γ relative to the region Ω. Namely, it is the harmonic function in $\Omega \backslash \Gamma$ [2] that takes value 1 on Γ and value 0 on $\partial\Omega$. However, as Section 5.7.1 and Figure 5.5 show $\gamma(z)$ in general is different from the harmonic measure.

The content of Chapters 2 and 3 is based on [39], Chapter 4 is based on [40] and Chapter 5 on [38].

[2]When Γ is part of the boundary of Ω, the harmonic measure is the harmonic function in Ω taking value 1 on Γ and value 0 on $\partial\Omega \backslash \Gamma$

CHAPTER 2

QUANTIFYING STABILITY OF ANALYTIC CONTINUATION

Notation 2.1. *We will write $A \lesssim B$, if there exists a constant c such that $A \leq cB$ and likewise the notation $A \gtrsim B$ will be used. If both $A \lesssim B$ and $A \gtrsim B$ are satisfied, then we will write $A \simeq B$. Throughout the paper all the implicit constants will be independent of the parameter ϵ.*

The main results of this chapter are Theorems 2.1 and 2.2. In the first theorem, we reduce the quantification problem of analytic continuation to a solution of an integral equation of Fredholm type. In the second one, we express the optimal exponent $\gamma(z)$ in the power law (describing the relative error of analytic continuation) in terms of exponential decay rates of the eigenvalues and eigenfunctions of the integral operator.

2.1 Problem formulation and reduction to an integral equation

Our goal is to characterize how large a function f analytic in a domain Ω can be at a point $z \in \Omega$, provided that it is small on a finite curve $\Gamma \Subset \Omega$, relative to its global size in Ω. If some norms $\|f\|_\Gamma$ and $\|f\|_\mathcal{H}$ are used to measure the magnitude of f on Γ and on Ω, respectively, then we are looking at the problem

$$\begin{cases} |f(z)| \to \max \\ \|f\|_\mathcal{H} \leq 1 \\ \|f\|_\Gamma \leq \epsilon \end{cases} \tag{2.1}$$

Assume that the global norm is induced by an inner product $(\cdot,\cdot)$ and that the point evaluation functional $f \mapsto f(z)$ is continuous (for any point $z \in \Omega$), then by the Riesz representation theorem, there exists an element $p_z \in \mathcal{H}$ such that $f(z) = (f, p_z)$. Now inner products with the function $p(\zeta, z) := p_z(\zeta)$ reproduce values of a function in $\mathcal{H}$. In this case $\mathcal{H}$ is called a a reproducing kernel Hilbert space (RKHS) with kernel p (cf. [60]) . Examples of such spaces include the Hardy spaces H^2 over the unit disk, annulus or upper half-plane (cf. Chapters 3 and 4). From now on we will drop the subscript $\mathcal{H}$ for the Hilbert space norm in $\mathcal{H}$.

Lemma 2.1. *Suppose that $\mathcal{H}$ is a RKHS whose elements are continuous functions on a metric space Ω. Then the function $\Omega \ni \tau \mapsto \|p_\tau\|$ is bounded on compact subsets of Ω.*

Proof. Assume the contrary. Suppose $S \subset \Omega$ is compact, but there exists a sequence $\{\tau_k\}_{k=1}^\infty \subset S$, such that $\|p_{\tau_k}\| \to \infty$ as $k \to \infty$. Since S is compact we can extract a convergent subsequence (without relabeling it) $\tau_k \to \tau_*$, then for any $f \in \mathcal{H}$ we have $f(\tau_k) = (f, p_{\tau_k}) \to f(\tau_*) = (f, p_{\tau_*})$, by continuity of f. Thus, $p_{\tau_k} \rightharpoonup p_{\tau_*}$ in $\mathcal{H}$, but this implies boundedness of $\|p_{\tau_k}\|$, leading to a contradiction. ∎

Corollary 2.1. *Under the assumption of Lemma 2.1 the function* $p(\zeta,\tau)$ *is bounded on compact subsets of* $\Omega\times\Omega$*, since* $|p(\zeta,\tau)| = |(p_\tau, p_\zeta)| \le \|p_\tau\|\|p_\zeta\|$.

Assume that the smallness on Γ is measured in $L^2 := L^2(\Gamma, |d\tau|)$-norm, where $|d\tau|$ is the arc length measure. So $\|\cdot\|_\Gamma$ and $(\cdot,\cdot)_\Gamma$ denote the norm and the inner product of L^2. Then, there is a constant $c > 0$ such that

$$\|f\|_\Gamma \le c\|f\|, \qquad \forall f \in \mathcal{H}. \tag{2.2}$$

Indeed, for all $\tau \in \Gamma$ we have $|f(\tau)| = |(f, p_\tau)| \le \|p_\tau\|\|f\|$. Since Γ lies in a compact subset of Ω and has finite length we conclude by Lemma 2.1 that (2.2) holds.

In order to analyze problem (2.1) we consider a Hermitian symmetric form

$$B : \mathcal{H}\times\mathcal{H} \to \mathbb{C}, \qquad B(f,g) = (f,g)_\Gamma.$$

By (2.2) $B(f,g)$ is continuous, and thus there exists a positive, self-adjoint and bounded operator $\mathscr{K} : \mathcal{H}\to\mathcal{H}$ with $B(f,g) = (\mathscr{K} f, g)$. Moreover we can write an explicit formula for $\mathscr{K}$ in terms of the kernel p:

$$(\mathscr{K} f, g) = (f,g)_\Gamma = \int_\Gamma f(\tau)(p_\tau, g)|d\tau| = \left(\int_\Gamma f(\tau)p_\tau|d\tau|, g\right). \tag{2.3}$$

Thus, for every $f \in \mathcal{H}$

$$(\mathscr{K} f)(\zeta) = \int_\Gamma p(\zeta,\tau) f(\tau)|d\tau|, \qquad \zeta\in\Omega. \tag{2.4}$$

This formula permits to define a new operator $\mathscr{K} : L^2(\Gamma) \to \mathcal{H}$. However, in doing so we may lose injectivity, which underlies uniqueness of analytic continuation[1]. Therefore, we restrict the domain of $\mathscr{K}$ to a closed subspace of $L^2(\Gamma)$

$$\mathcal{W} = \mathrm{cl}_{L^2}\left(\mathcal{H}|_\Gamma\right) \subset L^2(\Gamma). \tag{2.5}$$

[1]It is this property that forces us to restrict attention to reproducing kernel Hilbert spaces of analytic functions.

In fact, in many cases $\mathcal{W} = L^2(\Gamma)$. The density in the context of Hardy spaces is known as the Riesz theorem (see e.g. [59]). If Ω is bounded it is usually proved using density of polynomials in $L^2(\Gamma)$, which always holds if all polynomials are in $\mathcal{H}$ (and Γ is not a closed curve).

We note that the operator $\mathscr{K} : \mathcal{W} \to \mathcal{H}$ is bounded. Indeed, by Corollary 2.1 the function $\Gamma \ni \tau \mapsto p(\zeta, \tau)$ is bounded for each $\zeta \in \Omega$ and by (2.3) we have

$$\|\mathscr{K} f\|^2 = (\mathscr{K} f, \mathscr{K} f) = (f, \mathscr{K} f)_\Gamma \le \|\mathscr{K} f\|_\Gamma \|f\|_\Gamma \le c\|\mathscr{K} f\| \|f\|_\Gamma, \tag{2.6}$$

where we have used (2.2) in the last inequality. It follows that $\|\mathscr{K} f\| \le c\|f\|_\Gamma$.

The outcome of our constructions is the ability to write the two inequalities in (2.1) as quadratic constraints for $f \in \mathcal{H}$:

$$(f, f) \le 1, \qquad (\mathscr{K} f, f) \le \epsilon^2. \tag{2.7}$$

The final observation is that the objective functional $|f(z)|$ in (2.1) can be replaced by a (real) linear functional $\mathfrak{Re}(f, p_z)$. Indeed,

$$|f(z)| = \sup_{|\lambda|=1} \mathfrak{Re}(\lambda f(z)) = \sup_{|\lambda|=1} \mathfrak{Re}(\lambda f, p_z).$$

It remains to notice that if f satisfies (2.7) then so does λf for every $\lambda \in \mathbb{C}$, $|\lambda| = 1$. Thus we arrive at the problem

$$\begin{cases} \mathfrak{Re}(f, p_z) \to \max \\ (f, f) \le 1 \\ (\mathscr{K} f, f) \le \epsilon^2 \end{cases} \tag{2.8}$$

Lemma 2.2. *The operator $\mathscr{K} : \mathcal{H} \to \mathcal{H}$ is compact, positive definite and self-adjoint.*

Proof. Self-adjointness and positivity of $\mathscr{K}$ on $\mathcal{H}$ are immediate consequences of (2.3). To prove compactness, let $\{f_k\}_{k=1}^\infty \subset \mathcal{H}$ be a bounded sequence. Extract a weakly convergent subsequence (without relabeling it) $f_k \rightharpoonup f$. Then for every $\tau \in \Omega$ we have $f_k(\tau) = (f_k, p_\tau) \to (f, p_\tau) = f(\tau)$. In addition, for every $\tau \in \Gamma$ we have $|f_k(\tau)| = |(f_k, p_\tau)| \leq \|f_k\| \|p_\tau\|$. The sequence $\|f_k\|$ is bounded, since f_k is weakly convergent, while $\|p_\tau\|$ is bounded on Γ by Lemma 2.1. Thus, $f_k(\tau)$ is uniformly bounded on Γ. Then $f_k|_\Gamma \to f|_\Gamma$ in the L^2 norm. But then by the estimate $\|\mathscr{K}(f_k - f)\| \leq c\|f_k - f\|_\Gamma$ (see (2.6)) we conclude that $\mathscr{K} f_k \to \mathscr{K} f$ in $\mathcal{H}$.

□

Theorem 2.1. *Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a RKHS of functions analytic in domain Ω, with kernel p and norm $\|\cdot\|$. Let $\Gamma \Subset \Omega$ be a rectifiable curve of finite length and $\|\cdot\|_\Gamma$ be the $L^2 := L^2(\Gamma, |d\tau|)$ norm. Fix a point $z \in \Omega \backslash cl(\Gamma)$ and let*

$$A_z(\epsilon) = \sup\{|f(z)| \ : f \in \mathcal{H} \text{ and } \|f\| \leq 1, \ \|f\|_\Gamma \leq \epsilon\}$$

then the following hold true:

(i) Let $\eta_* = \eta_*(\epsilon, z) > 0$ be the unique solution of $\|(\mathscr{K} + \eta_*)^{-1} p_z\|_\Gamma = \epsilon\|(\mathscr{K} + \eta_*)^{-1} p_z\|$, then

$$A_z(\epsilon) = \frac{u^*(z)}{\|u^*\|}, \tag{2.9}$$

where $u^* = u^*_{\epsilon,z}$ solves the integral equation $(\mathscr{K} + \eta_*)u^* = p_z$. In particular, the maximizer function is $f^* = u^*/\|u^*\|$. Alternatively,

$$A_z(\epsilon) = C \exp\left\{-\int_\epsilon^1 \frac{tdt}{t^2 + \eta_*(t, z)}\right\}, \tag{2.10}$$

where C is a constant independent of ϵ, namely $C = A_z(1)$.

(ii) Let $u = u_{\epsilon,z}$ solve

$$(\mathscr{K} + \epsilon^2)u = p_z \tag{2.11}$$

and

$$M_{\epsilon,z}(\zeta) = u_{\epsilon,z}(\zeta) \min\left\{\frac{1}{\|u_{\epsilon,z}\|}, \frac{\epsilon}{\|u_{\epsilon,z}\|_\Gamma}\right\}. \tag{2.12}$$

Then

$$M_{\epsilon,z}(z) \le A_z(\epsilon) \le \frac{3}{2} M_{\epsilon,z}(z) \tag{2.13}$$

The theorem is proved in Section 2.4. Let us make some remarks.

1. Finding the asymptotics of $\eta_*(\epsilon)$ as $\epsilon \to 0$ lies beyond the capabilities of classical asymptotic analysis, but as our insight shows $\eta_*(\epsilon) \simeq \epsilon^2$ (see the proof of Theorem 2.1). Note that this insight is also confirmed by (2.13), which is an optimal bound up to the constant 3/2. Its advantage over the exact formula (2.9) is that it is accessible numerically. A deeper insight into the asymptotic behavior of η_* shows that $\epsilon^{-2}\eta_*(\epsilon)$ oscillates as $\epsilon \to 0$ (hence has no limit) and can be approximated by an elliptic function (cf. Section 2.5).

   The function (2.12) is obviously in $\mathcal{H}$ and satisfies the constraints in (2.1). Hence, the lower bound in (2.13) is trivial. Only the upper bound itself requires a proof. Further, we expect the two quantities under the minimum in (2.12) to be comparable, which is just a restatement of $\eta_*(\epsilon) \simeq \epsilon^2$.

2. An obvious thing to do is to set $\epsilon = 0$ in (2.11). If $p_z \in \mathscr{K}(\mathcal{W})$, where $\mathcal{W}$ is given by (2.5), then $u_{\epsilon,z} \to u_0 = \mathscr{K}^{-1}p_z$, as $\epsilon \to 0$. In which case the upper bound (2.13) is simply

$$|f(z)| \le C\epsilon, \qquad C = \frac{3u_0(z)}{2\|u_0\|_\Gamma}. \tag{2.14}$$

   In other words we have numerically stable analytic continuation. Examples where this happens are mentioned in Remarks 3.2 and 4.1 of

Chapters 3 and 4, respectively. This case will be referred to as the *trivial case.*

3. The upper bound in (2.13) is not an explicit function of ϵ and z. Its asymptotics as $\epsilon \to 0$ depends on fine properties of the operator $\mathscr{K}$ (e.g. the exponential decay rates of its eigenvalues). This will be discussed in Section 2.2. In specific examples of Chapters 3 and 4, the equation (2.11) is solved explicitly and the power law behavior $M_{\epsilon,z}(z) \simeq \epsilon^{\gamma(z)}$ is exhibited.

4. The precise asymptotics of the exponential decay of eigenvalues of $\mathscr{K}$ is known for certain classes of spaces. For example, assume $\mathcal{H}$ coincides with the Smirnov class $E^2(\Omega)$ [23]. If the domain Ω is bounded and simply connected and $\Gamma \Subset \Omega$ is a closed Jordan rectifiable curve of class $C^{1+\alpha}$ for $\alpha > 0$, with Ω' denoting the domain bounded by it, then the eigenvalues of $\mathscr{K}$ satisfy the asymptotic relation [58]

$$\lambda_n(\mathscr{K}) \sim \rho^{2n+1}, \qquad \text{as } n \to +\infty, \tag{2.15}$$

where $\rho < 1$ is the Riemann invariant , whereby the domain $\Omega \backslash cl(\Omega')$ is conformally equivalent to the annulus $\{\omega \in \mathbb{C} : \rho < |\omega| < 1\}$.

5. The eigenvalues λ_n are also connected to Kolmogorov n-widths [62] , since they are squares of singular values of the restriction operator $\mathscr{R} : \mathcal{H} \to L^2(\Gamma)$, because $\mathscr{K} = \mathscr{R}^*\mathscr{R}$. Specifically (cf. [29, Theorem 6.1]), $\sqrt{\lambda_{n+1}}$ is the Kolmogorov n-width of the restriction to $L^2(\Gamma)$ of closed unit ball in $\mathcal{H}$. The relation of the Kolmogorov n-widths of restrictions of various classes of analytic functions to corresponding Riemann invariants have been know in many cases [27, 73, 30].

2.2 Solving the integral equation

We begin by making several observations about a priori properties of the solution $u = u_\epsilon$ (we suppress its dependence on z) of (2.11) in the non-trivial case $p_z \notin \mathscr{K}(\mathcal{W})$. The most immediate consequence of the non-triviality is that $\|u_\epsilon\|_\Gamma$ blows up as $\epsilon \to 0$. If it did not, we would be able to extract a weakly convergent subsequence $u_{\epsilon_k} \rightharpoonup u_0 \in \mathcal{W}$ and passing to the weak limits in (2.11) obtained that $(\mathscr{K}u_0)(\zeta) = p_z(\zeta)$, for $\zeta \in \Gamma$. However, since $\mathscr{K}(\mathcal{W}) \subset \mathcal{H}$ we get a contradiction with the non-triviality.

The definition of u implies $u(z) = (u, p_z) = (u, \mathscr{K}u + \epsilon^2 u) = (u, \mathscr{K}u) + \epsilon^2(u, u)$, i.e.

$$u(z) = \|u\|_\Gamma^2 + \epsilon^2\|u\|^2. \tag{2.16}$$

Let us show that equation (2.16) implies that $M_{\epsilon,z}(z) \gg \epsilon$, as $\epsilon \to 0$. On the one hand, dividing equation (2.16) by $\|u_\epsilon\|_\Gamma$ we obtain

$$\frac{u_\epsilon(z)}{\|u_\epsilon\|_\Gamma} \geq \|u_\epsilon\|_\Gamma.$$

On the other, we have $\|u_\epsilon\|_\Gamma^2 + \epsilon^2\|u_\epsilon\|^2 \geq 2\epsilon\|u_\epsilon\|_\Gamma\|u_\epsilon\|$ and therefore

$$\frac{u_\epsilon(z)}{\epsilon\|u_\epsilon\|} \geq 2\|u_\epsilon\|_\Gamma,$$

proving that $\epsilon^{-1}M_{\epsilon,z}(z) \geq \|u_\epsilon\|_\Gamma \to +\infty$. This means that one cannot expect full numerical stability of analytic continuation.

Finally, we prove the "mathematical well-posedness" of analytic continuation: $M_{\epsilon,z}(z) \to 0$ as $\epsilon \to 0$. This is a consequence of the weak convergence of $u_\epsilon/\|u_\epsilon\|$ to 0. If we divide (2.11) by $\|u_\epsilon\|$ and pass to weak limits, using the fact that $\|u_\epsilon\| \geq c^{-1}\|u_\epsilon\|_\Gamma \to +\infty$ we obtain that the weak limit $\hat{u}$ of $u_\epsilon/\|u_\epsilon\|$ satisfies $\mathscr{K}\hat{u} = 0$. But if $\mathscr{K}\hat{u} = 0$, then $\|\hat{u}\|_\Gamma^2 = (\mathscr{K}\hat{u}, \hat{u}) = 0$. It follows that the analytic function $\hat{u} = 0$ on Γ and hence must vanish everywhere in Ω. This shows that the operator $\mathscr{K}$ has a trivial null-space and that $M_{\epsilon,z}(z) \leq (u_\epsilon/\|u_\epsilon\|, p_z) \to 0$, as $\epsilon \to 0$.

A consequence of the just established strict positivity of $\mathscr{K}$ is separability of the Hilbert space $\mathcal{H}$. This should not be surprising, since $\mathcal{H}$ consists of analytic functions each of which can be completely described by a countable set of numbers.

Lemma 2.3. *The Hilbert space $\mathcal{H}$ is always separable.*

Proof. We saw that $\mathscr{K} : \mathcal{H} \to \mathcal{H}$ given by (2.4) is a self-adjoint, compact operator. We have just seen that $\mathscr{K}$ has a trivial null-space. In this case the Hilbert space $\mathcal{H}$ is the orthogonal sum of countable number of finite dimensional eigenspaces of $\mathscr{K}$ with positive eigenvalues. Thus, $\mathcal{H}$ has a countable complete orthonormal set and is therefore separable. □

In applications of our theory in Chapters 3 and 4 we solve the equation (2.11) exactly by finding all eigenvalues and eigenfunctions of $\mathscr{K}$. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal eigenbasis of $\mathcal{H}$ with $\mathscr{K} e_n = \lambda_n e_n$. In this basis the equation (2.11) diagonalizes:

$$\lambda_n(u, e_n) + \epsilon^2(u, e_n) = (p_z, e_n),$$

therefore we find

$$u_\epsilon(\zeta) = \sum_n \frac{\overline{e_n(z)}}{\lambda_n + \epsilon^2} e_n(\zeta). \tag{2.17}$$

Using this expansion, formula $\|u\|_\Gamma^2 = (\mathscr{K} u, u)$, and (2.16) we find that

$$u_\epsilon(z) = \sum_n \frac{|e_n(z)|^2}{\lambda_n + \epsilon^2}, \qquad \|u_\epsilon\|^2 = \sum_n \frac{|e_n(z)|^2}{(\lambda_n + \epsilon^2)^2}, \qquad \|u_\epsilon\|_\Gamma^2 = \sum_n \frac{\lambda_n |e_n(z)|^2}{(\lambda_n + \epsilon^2)^2}. \tag{2.18}$$

It follows that

$$\sum_n \frac{|e_n(z)|^2}{\lambda_n} = \infty, \tag{2.19}$$

since if the series had a finite sum then formula (2.18) for $\|u_\epsilon\|_\Gamma$ would imply

$$\|u_\epsilon\|_\Gamma^2 \le \sum_n \frac{|e_n(z)|^2}{\lambda_n},$$

contradicting to the blow up of $\|u_\epsilon\|_\Gamma$.

In our examples where the eigenvalues λ_n and eigenfunctions $e_n(\zeta)$ can be found explicitly they are seen to decay exponentially fast to 0 (see also (2.15)). Let us show that this implies the power law principle

$$M_{\epsilon,z}(z) \simeq \epsilon^{\gamma(z)}, \qquad \text{as} \qquad \epsilon \to 0, \tag{2.20}$$

where $\gamma(z) \in (0,1)$ can be expressed in terms of the rates of exponential decay of spectral data for $\mathscr{K}$.

Theorem 2.2. *Let* $\{e_n\}_{n=1}^{\infty}$ *be an orthonormal eigenbasis of* $\mathcal{H}$ *with* $\mathscr{K} e_n = \lambda_n e_n$. *Let* $u = u_{\epsilon,z}$ *and* $M_{\epsilon,z}$ *be given by* (2.11) *and* (2.12) *respectively. Assume*

$$\lambda_n \simeq e^{-\alpha n}, \qquad |e_n(z)|^2 \simeq e^{-\beta n}, \qquad 0 < \beta < \alpha, \tag{2.21}$$

with implicit constants independent of n *(so that* (2.19) *holds). Then,*

$$\|u_{\epsilon,z}\|_\Gamma \simeq \epsilon \|u_{\epsilon,z}\| \simeq \epsilon^{\frac{\beta}{\alpha}-1} \qquad \textit{and} \qquad u_{\epsilon,z}(z) \simeq \epsilon^{2\left(\frac{\beta}{\alpha}-1\right)},$$

with implicit constants independent of ϵ. *In particular, this implies the power law principle (2.20) with exact exponent:*

$$M_{\epsilon,z}(z) \simeq \epsilon^{\frac{\beta}{\alpha}}.$$

To prove the above theorem let us first investigate the limiting behavior of the function

$$\phi(\eta) = \sum_{n=0}^{\infty} \frac{a^n}{\eta + b^n}, \qquad 0 < b < |a| < 1,$$

as $\eta \to 0^+$, where $a \in \mathbb{C}$. The asymptotics of this seemingly nice series is surprisingly irregular. Let ln be any branch of the logarithm (whose choice is independent of a, b and η), as long as a^p is always understood as $e^{p \ln a}$.

Lemma 2.4. *Let* $\eta_j \to 0$, *as* $j \to \infty$, *such that the fractional parts* $\left\{\frac{\ln \eta_j}{\ln b}\right\} \to t \in [0,1]$. *Then, as* $j \to \infty$

$$\phi(\eta_j) \sim \phi_0 \eta_j^{-\gamma},$$

where

$$\phi_0 = \frac{b^t}{a^t} \sum_{k\in\mathbb{Z}} \frac{a^k}{b^t + b^k}, \qquad \gamma = 1 - \frac{\ln a}{\ln b}.$$

Proof. We first notice that unlike $\phi(\eta)$, the function

$$\psi(\eta) = \sum_{n=1}^{\infty} \frac{a^{-n}}{\eta + b^{-n}}$$

is regular at $\eta = 0$. In fact, $\psi(0) = b/(a-b)$. We therefore define a new function

$$F(\eta) = \sum_{n\in\mathbb{Z}} \frac{a^n}{\eta + b^n} = \phi(\eta) + \psi(\eta),$$

which obviously satisfies

$$\lim_{j\to\infty} F(\eta_j)\eta_j^{\gamma} = \lim_{j\to\infty} \phi(\eta_j)\eta_j^{\gamma},$$

whenever $\eta_j \to 0^+$ and the limit on the right-hand side exists. Introducing the integer and fractional parts

$$N(\eta) = \left[\frac{\ln \eta}{\ln b}\right], \qquad \alpha(\eta) = \left\{\frac{\ln \eta}{\ln b}\right\}$$

we make a change of index of summation $k = n - N(\eta)$ and obtain, using

$$N(\eta) = \frac{\ln \eta}{\ln b} - \alpha(\eta),$$

after a short calculation, that

$$F(\eta)\eta^{\gamma} = \sum_{k\in\mathbb{Z}} \frac{a^{k-\alpha(\eta)}}{1 + b^{k-\alpha(\eta)}} = \frac{b^{\alpha(\eta)}}{a^{\alpha(\eta)}} \sum_{k\in\mathbb{Z}} \frac{a^k}{b^{\alpha(\eta)} + b^k}.$$

The statement of the lemma is now apparent. □

Immediately from the above lemma we obtain:

Corollary 2.2. *Let* $\{a_n, b_n\}_{n=1}^{\infty}$ *be nonnegative numbers such that* $a_n \simeq e^{-\alpha n}$ *and* $b_n \simeq e^{-\beta n}$ *with* $0 < \beta < \alpha$*, where the implicit constants do not depend on* n*. Let* $\eta > 0$ *be a small parameter, then*

$$\sum_{n=1}^{\infty} \frac{b_n}{a_n + \eta} \simeq \eta^{\frac{\beta}{\alpha}-1}, \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{b_n}{(a_n + \eta)^2} \simeq \eta^{\frac{\beta}{\alpha}-2}, \tag{2.22}$$

where the implicit constants do not depend on η*.*

The proof of Theorem 2.2 now follows from (2.18) and Corollary 2.2.

2.3 Linear constraints

In one of the examples of Chapter 3 we encounter a situation where additional linear constraints are imposed on a previously solved problem. In general all linear constraints on analytic functions will simply be incorporated into the definition of the RKHS $\mathcal{H}$. The question is whether we can use the already found solution of a problem if additional linear constraints are imposed. Let $L \subset \mathcal{H}$ be a closed, $\mathbb{C}$-linear subspace. Then L with the inner product from $\mathcal{H}$ is still a RKHS with the reproducing kernel $\mathcal{P}_L p_z$, where $\mathcal{P}_L$ denotes the orthogonal projection onto L. If we restrict f and g in (2.3) to elements from L, then the operator $\mathscr{K}$ can be written as $\mathcal{P}_L \mathscr{K} \mathcal{P}_L$. Then equation (2.11) can be written (in the language of the original RKHS $\mathcal{H}$) as

$$\mathcal{P}_L \mathscr{K} \mathcal{P}_L u + \epsilon^2 u = \mathcal{P}_L p_z, \qquad u \in \mathcal{H}, \tag{2.23}$$

whose unique solution u necessarily belongs to L. In general, one's ability to solve the original problem (2.11) would be of little help for solving (2.23), except in the special case when L is an invariant subspace of $\mathscr{K}$. In this case $\mathcal{P}_L$ commutes with $\mathscr{K}$ and if u solves (2.11), then $\mathcal{P}_L u$ solves (2.23).

The requirement that L be a $\mathbb{C}$-linear subspace is important, because the linearization argument taking the objective functional $|f(z)|$ in (2.1) to the one in (2.8) requires all the constraints to be invariant under multiplication by a phase factor $\lambda \in \mathbb{C}$, $|\lambda| = 1$. In some applications, like the analytic continuation of the complex electromagnetic permittivity function, the constraints may be just $\mathbb{R}$-linear, in which case other techniques have to be applied (cf. Section 5.6 and Chapter 5).

2.4 Proof of Theorem 2.1

We start by analyzing the trivial case.

Lemma 2.5. *Assume the setting of Theorem 2.1, let $p_z \in \mathscr{K}(\mathcal{W})$, then*

$$|f(z)| \leq c\epsilon.$$

Proof. Let $v \in \mathcal{W} \subset L^2$ satisfy $\mathscr{K} v = p_z$, (note that v does not depend on ϵ), then using (2.3) we have

$$f(z) = (f, p_z) = (f, \mathscr{K} v) = (f, v)_\Gamma.$$

It remains to use the Cauchy-Schwartz inequality to conclude the desired inequality with $c = \|v\|_\Gamma$.

□

Let us now turn to the case $p_z \notin \mathscr{K}(\mathcal{W})$. For every f, satisfying (2.7) and for every nonnegative numbers μ and ν ($\mu^2 + \nu^2 \neq 0$) we have the inequality

$$((\mu + \nu \mathscr{K})f, f) \leq \mu + \nu\epsilon^2 \tag{2.24}$$

obtained by multiplying (2.8)(b) by μ and (2.8)(c) by ν and adding. Also, for any uniformly positive definite self-adjoint operator T on $\mathcal{H}$ we have

$$\mathfrak{Re}(f, g) - \frac{1}{2}(T^{-1}g, g) \leq \frac{1}{2}(Tf, f),$$

valid for all functions $f, g \in \mathcal{H}$ (expand $(T(T^{-1}g - f), (T^{-1}g - f)) \geq 0$). The uniform positivity of T ensures that T^{-1} is defined on all of $\mathcal{H}$. This is an example of convex duality (cf. [24]) applied to the convex function $F(f) = (Tf, f)/2$. Then we also have for $\mu > 0$

$$\mathfrak{Re}(f, p_z) - \frac{1}{2}\left((\mu + \nu\mathscr{K})^{-1}p_z, p_z\right) \le \frac{1}{2}\left((\mu + \nu\mathscr{K})f, f\right) \le \frac{1}{2}\left(\mu + \nu\epsilon^2\right), \quad (2.25)$$

so that

$$\mathfrak{Re}(f, p_z) \le \frac{1}{2}\left((\mu + \nu\mathscr{K})^{-1}p_z, p_z\right) + \frac{1}{2}\left(\mu + \nu\epsilon^2\right), \quad (2.26)$$

which is valid for every f, satisfying (2.7) and all $\mu > 0$, $\nu \ge 0$. In order for the bound to be optimal we must have equality in (2.25), which holds if and only if

$$p_z = (\mu + \nu\mathscr{K})f,$$

giving the formula for optimal vector f:

$$f = (\mu + \nu\mathscr{K})^{-1}p_z. \quad (2.27)$$

The goal is to choose the Lagrange multipliers μ and ν so that the constraints in (2.8) are satisfied by f, given by (2.27). Let us first consider special cases.

- if $\nu = 0$, then $f = \frac{p_z}{\mu}$ and optimality implies that the first inequality constraint of (2.8) must be attained, i.e. $\|f\| = 1$. Thus, $f = \frac{p_z}{\|p_z\|}$ does not depend on the small parameter ϵ, which leads to a contradiction, because the second constraint $(\mathscr{K}f, f) \le \epsilon^2$ is violated if ϵ is small enough.

- if $\mu = 0$, then $\mathscr{K}f = \frac{1}{\nu}p_z$. But this equation has no solutions in $\mathcal{H}$ according to the assumption $p_z \notin \mathscr{K}(\mathcal{W})$.

Thus we are looking for $\mu > 0$, $\nu > 0$, so that equalities in (2.8) hold (these are the complementary slackness relations in Karush-Kuhn-Tucker conditions), i.e.

$$\begin{cases} ((\mu + \nu\mathscr{K})^{-1}p_z, (\mu + \nu\mathscr{K})^{-1}p_z) = 1, \\ (\mathscr{K}(\mu + \nu\mathscr{K})^{-1}p_z, (\mu + \nu\mathscr{K})^{-1}p_z) = \epsilon^2. \end{cases} \quad (2.28)$$

Let $\eta = \frac{\mu}{\nu}$, we can solve either the first or the second equation in (2.28) for ν

$$\nu^2 = \|(\mathscr{K} + \eta)^{-1} p_z\|^2, \tag{2.29}$$

or

$$\nu^2 = \epsilon^{-2} \left(\mathscr{K}(\eta + \mathscr{K})^{-1} p_z, (\eta + \mathscr{K})^{-1} p_z\right). \tag{2.30}$$

The two analysis paths stemming from using one or the other representation for ν lead to two versions of the upper bound on $|f(z)|$, however we cannot prove the optimality of either of those versions. However, the minimum of the two upper bounds is still an upper bound and its optimality is then apparent. At first glance both expressions for ν should be equivalent and not lead to different bounds. Indeed, their equivalence can be stated as an equation

$$\Phi(\eta) := \frac{(\mathscr{K}(\mathscr{K} + \eta)^{-1} p_z, (\mathscr{K} + \eta)^{-1} p_z)}{\|(\mathscr{K} + \eta)^{-1} p_z\|^2} = \epsilon^2 \tag{2.31}$$

for η. We will prove that this equation has a unique solution $\eta_* = \eta_*(\epsilon)$, but we will be unable to prove that $\eta_*(\epsilon) \simeq \epsilon^2$, as $\epsilon \to 0$, which would follow from the purported strict exponential decay of λ_n and $|e_n(z)|$ (cf. (2.21)). Thus, we will take $\eta_*(\epsilon) = \epsilon^2$ without justification, observing that *any* choice of η gives a valid upper bound. But then the two expressions (2.29) and (2.30) for ν give non-identical upper bounds, whose combination will achieve our goal.

We observe that

$$\lim_{\eta \to \infty} \Phi(\eta) = \lim_{\eta \to \infty} \frac{(\mathscr{K}(\eta^{-1}\mathscr{K} + 1)^{-1} p_z, (\eta^{-1}\mathscr{K} + 1)^{-1} p_z)}{\|(\eta^{-1}\mathscr{K} + 1)^{-1} p_z\|^2} = \frac{(\mathscr{K} p_z, p_z)}{\|p_z\|^2} < +\infty.$$

Using the spectral representation of $\mathscr{K}$ in its eigenbasis, we have

$$\left(\mathscr{K}(\mathscr{K} + \eta)^{-1} p_z, (\mathscr{K} + \eta)^{-1} p_z\right) = \sum_n \frac{\lambda_n |e_n(z)|^2}{(\lambda_n + \eta)^2}, \tag{2.32}$$

and

$$\|(\mathscr{K} + \eta)^{-1} p_z\|^2 = \sum_n \frac{|e_n(z)|^2}{(\lambda_n + \eta)^2}. \tag{2.33}$$

Note that $\sum_n \frac{|e_n(z)|^2}{\lambda_n^2} = +\infty$, since otherwise the function $v = \sum_n \frac{\overline{e_n(z)}}{\lambda_n} e_n \in \mathcal{H}$ would satisfy $\mathscr{K} v = p_z$, contradicting the assumption $p_z \notin \mathscr{K}(\mathcal{W})$. Now

Fatou's Lemma implies that

$$\lim_{\eta\to 0}\|(\mathscr{K}+\eta)^{-1}p_z\|^2=+\infty.$$

Let $\delta>0$ be arbitrary. Let K be such that $\lambda_n<\delta$ for all $n>K$. Then

$$\Phi(\eta)=\Phi_K(\eta)+\Psi_K(\eta),$$

where

$$\Phi_K(\eta)=\frac{\sum_{n\le K}\frac{\lambda_n|e_n(z)|^2}{(\lambda_n+\eta)^2}}{\|(\mathscr{K}+\eta)^{-1}p_z\|^2},\qquad \Psi_K(\eta)=\frac{\sum_{n>K}\frac{\lambda_n|e_n(z)|^2}{(\lambda_n+\eta)^2}}{\|(\mathscr{K}+\eta)^{-1}p_z\|^2}.$$

Then

$$\overline{\lim_{\eta\to 0}}\,\Phi_K(\eta)=0.$$

We also have

$$\Psi_K(\eta)\le\frac{\sum_{n>K}\frac{\lambda_n|e_n(z)|^2}{(\lambda_n+\eta)^2}}{\sum_{n>K}\frac{|e_n(z)|^2}{(\lambda_n+\eta)^2}}\le\lambda_{K+1}<\delta.$$

Thus,

$$\overline{\lim_{\eta\to 0}}\,\Phi(\eta)\le\overline{\lim_{\eta\to 0}}\,\Phi_K(\eta)+\overline{\lim_{\eta\to 0}}\,\Psi_K(\eta)\le\delta.$$

Since $\delta>0$ was arbitrary we conclude that $\Phi(0^+)=0$. Thus, for every $\epsilon<\sqrt{(\mathscr{K}p_z,p_z)}/\|p_z\|$ equation (2.31) has at least one solution $\eta>0$.

Let us prove that this solution is unique by showing that $\Phi(\eta)$ is a monotone increasing function. To prove this we only need to write the numerator $N(\eta)$ of $\Phi'(\eta)$, obtained by the quotient rule. Using the formula

$$\frac{d}{d\eta}(\mathscr{K}+\eta)^{-1}=-(\mathscr{K}+\eta)^{-2}$$

and denoting $u=(\mathscr{K}+\eta)^{-1}p_z$ we obtain

$$N(\eta)=2((\mathscr{K}+\eta)^{-1}u,u)(\mathscr{K}u,u)-2(\mathscr{K}(\mathscr{K}+\eta)^{-1}u,u)\|u\|^2.$$

Using the formula $\mathscr{K}(\mathscr{K}+\eta)^{-1}=1-\eta(\mathscr{K}+\eta)^{-1}$ we also have

$$N(\eta)=2((\mathscr{K}+\eta)^{-1}u,u)((\mathscr{K}+\eta)u,u)-2(u,u)^2.$$

Since the operator $\mathscr{K} + \eta$ is positive definite we can use the inequality

$$(Ax, y)^2 \leq (Ax, x)(Ay, y)$$

for $A = \mathscr{K} + \eta$, $x = (\mathscr{K} + \eta)^{-1}u$ and $y = u$, showing that $N(\eta) \geq 0$. The equality occurs if and only if $x = \lambda y$. In our case this would correspond to p_z being an eigenfunction of $\mathscr{K}$, which is never true, due to the fact that $p_z \notin \mathscr{K}(\mathcal{W})$. Thus, $N(\eta) > 0$ and (2.31) has a unique solution $\eta_* > 0$. Finding the asymptotics of $\eta_*(\epsilon)$, as $\epsilon \to 0$ lies beyond capabilities of classical asymptotic methods because $\Phi(\eta)$ has an essential singularity at $\eta = 0$. Indeed, it is not hard to show[2] that $\Phi'(-\lambda_n) = 0$ for all $n \geq 1$. Thus $\eta = 0$ is neither a pole nor a removable singularity of $\Phi(\eta)$.

We can avoid the difficulty by observing that since the bound (2.26) is valid for *any* choice of μ and ν, we can choose $\eta = \mu/\nu$ based on a non-rigorous analysis of what η_* should be, and then choose ν according to (2.29) or (2.30), while still obtaining an upper bound.

In accordance with (2.21) we postulate that

$$|e_n(z)|^2 \simeq e^{-\beta n}, \qquad \lambda_n \simeq e^{-\alpha n}$$

for some $0 < \beta < \alpha$ and implicit constants independent of n. Hence using equations (2.32) and (2.33) in (2.31) we obtain that for small η

$$\epsilon^2 = \Phi(\eta) = \frac{\displaystyle\sum_{n=1}^{\infty} \frac{\lambda_n |e_n(z)|^2}{(\lambda_n + \eta)^2}}{\displaystyle\sum_{n=1}^{\infty} \frac{|e_n(z)|^2}{(\lambda_n + \eta)^2}} \simeq \frac{\eta^{\frac{\alpha+\beta}{\alpha} - 2}}{\eta^{\frac{\beta}{\alpha} - 2}} = \eta, \tag{2.34}$$

where the implicit constants are independent of η and we used Corollary 2.2 to estimate the sums. Thus, we see that

$$\eta_* \simeq \epsilon^2. \tag{2.35}$$

[2]Specifically $\eta = -\lambda_n$ is a pole of order 4 of $\|u\|^4$, while it is a pole of order 3 of $N(\eta)$.

With this motivation let us choose $\eta = \epsilon^2$. With this and formulas (2.29) and (2.30) for ν we obtain the two forms of the upper bound (2.26) conveniently written in terms of $u = (\mathscr{K} + \epsilon^2)^{-1} p_z$:

$$\mathfrak{Re}(f, p_z) \leq \frac{(u, p_z)}{2\|u\|} + \epsilon^2 \|u\|, \qquad \mathfrak{Re}(f, p_z) \leq \frac{\epsilon(u, p_z)}{2\|u\|_\Gamma} + \epsilon \|u\|_\Gamma.$$

The formula (2.16) implies the inequalities

$$\epsilon^2 \|u\| \leq \frac{u(z)}{\|u\|}, \qquad \|u\|_\Gamma \leq \frac{u(z)}{\|u\|_\Gamma}.$$

Therefore, we have both

$$|f(z)| = \mathfrak{Re}(f, p_z) \leq \frac{3}{2} \frac{u(z)}{\|u\|}, \qquad |f(z)| \leq \frac{3\epsilon}{2} \frac{u(z)}{\|u\|_\Gamma}.$$

This concludes the proof of part (ii) of the theorem.

Note that the equation (2.31) for the optimal choice $\eta_*(\epsilon)$ can be written as $\|u^*\|_\Gamma = \epsilon \|u^*\|$, where $u^* = (\mathscr{K} + \eta_*)^{-1} p_z$. In this case we get

$$\frac{\epsilon u^*}{\|u^*\|_\Gamma} = \frac{u^*}{\|u^*\|} = A_z(\epsilon),$$

which proves (2.9).

Thus it remains to establish (2.10). The definition of u^* implies $u^*(z) = (u^*, p_z) = (u^*, \mathscr{K} u^* + \eta_* u^*) = (u^*, \mathscr{K} u^*) + \eta_*(u^*, u^*)$, i.e.

$$u^*(z) = \|u^*\|_\Gamma^2 + \eta_* \|u^*\|^2 = (\epsilon^2 + \eta_*) \|u^*\|^2, \tag{2.36}$$

where the last step follows from the definition of η_*. In particular we find that $A_z(\epsilon) = (\epsilon^2 + \eta_*) \|u^*\|$, therefore it is enough to derive a formula for $\|u^*\|$ in terms of η_*. Let us write u^*_ϵ instead of u^* to show its dependence on ϵ. The key observation is the relation between $\partial_\epsilon u^*_\epsilon(z)$ and $\|u^*_\epsilon\|$ which we are going to use in (2.36) to deduce the desired formula. The integral equation for u^*_ϵ diagonalizes in the eigenbasis and we find $(e_n, u^*_\epsilon) = e_n(z)/(\lambda_n + \eta_*(\epsilon))$. Therefore,

$$u_\epsilon^*(z) = \sum_{n=1}^{\infty} \frac{|e_n(z)|^2}{\lambda_n + \eta_*(\epsilon)}, \qquad \|u_\epsilon^*\|^2 = \sum_{n=1}^{\infty} \frac{|e_n(z)|^2}{\left(\lambda_n + \eta_*(\epsilon)\right)^2}.$$

These formulas readily imply

$$\partial_\epsilon u_\epsilon^*(z) = -\eta_*'(\epsilon)\|u_\epsilon^*\|^2. \tag{2.37}$$

Differentiating (2.16) with respect to ϵ and using the relation (2.37) we find

$$(2\epsilon + \eta_*'(\epsilon))\,\|u_\epsilon^*\|^2 + 2\|u_\epsilon^*\|\left(\epsilon^2 + \eta_*(\epsilon)\right)\partial_\epsilon\|u_\epsilon^*\| = -\eta_*'(\epsilon)\|u_\epsilon^*\|^2,$$

which then gives

$$\frac{\partial_\epsilon\|u_\epsilon^*\|}{\|u_\epsilon^*\|} = -\frac{\epsilon + \eta_*'(\epsilon)}{\epsilon^2 + \eta_*(\epsilon)} = -\frac{2\epsilon + \eta_*'(\epsilon)}{\epsilon^2 + \eta_*(\epsilon)} + \frac{\epsilon}{\epsilon^2 + \eta_*(\epsilon)}. \tag{2.38}$$

Integrating (2.38) we find

$$\|u_\epsilon^*\| = \frac{C}{\epsilon^2 + \eta_*(\epsilon)} \exp\left\{-\int_\epsilon^1 \frac{tdt}{t^2 + \eta_*(t)}\right\},$$

which concludes the proof.

2.5 Insight into the asymptotics of $\eta_*(\epsilon)$

The discussions in preceding sections show that we expect $A_z(\epsilon) \simeq \epsilon^\gamma$, therefore let us consider the quantity

$$\gamma(z) := \underline{\lim}_{\epsilon\to 0} \frac{\ln A_z(\epsilon)}{\ln \epsilon}.$$

Combining (2.9) with (2.36) on one hand and using (2.10) on the other hand (where we change the variables in the integral), we obtain two different representations for the power law exponent:

$$\gamma(z) = \underline{\lim}_{\epsilon\to 0} \frac{\ln\left((\epsilon + \frac{\eta_*}{\epsilon})\|u^*\|_\Gamma\right)}{\ln \epsilon} = \underline{\lim}_{t\to+\infty} \frac{1}{t}\int_0^t \frac{dx}{1 + e^{2x}\eta_*(e^{-x})}. \tag{2.39}$$

Thus, understanding the asymptotic behavior of $\eta_*(\epsilon)$ as $\epsilon \to 0$ is crucial for unraveling the above formulas. Recall that $\eta = \eta_* > 0$ is the unique solution of the following equation (cf. (2.34))

$$\Phi(\eta) = \frac{\displaystyle\sum_{n=1}^{\infty} \frac{\lambda_n |e_n(z)|^2}{(\lambda_n + \eta)^2}}{\displaystyle\sum_{n=1}^{\infty} \frac{|e_n(z)|^2}{(\lambda_n + \eta)^2}} = \epsilon^2. \tag{2.40}$$

Under the purported exponential decay (2.21) of eigenvalues and eigenfunctions (at the point z) of $\mathscr{K}$ we proved in Section 2.4 that $\Phi(\eta) \simeq \eta$ with implicit constants independent of η, leading to $\eta_*(\epsilon) \simeq \epsilon^2$ with implicit constants independent of ϵ. Moreover, in Theorem 2.2 we also showed that $\|u^*\|_\Gamma \simeq \epsilon^{\frac{\beta}{\alpha}-1}$, which then implies that the ratio inside the first liminf in (2.39) converges as $\epsilon \to 0$ and gives the formula $\gamma = \beta/\alpha$.

On the other hand, substituting $\lambda_n, |e_n(z)|$ in (2.40) with their corresponding exponentials from (2.21), and applying (a version) of Lemma 2.4 we can approximate

$$\Phi(\eta) \approx \eta L\left(\ln\left(\frac{1}{\eta}\right)\right), \qquad L(\tau) = \frac{e^\tau \displaystyle\sum_{k\in\mathbb{Z}} \frac{e^{(\alpha+\beta)k}}{(e^{\alpha k}+e^{-\tau})^2}}{\displaystyle\sum_{k\in\mathbb{Z}} \frac{e^{\beta k}}{(e^{\alpha k}+e^{-\tau})^2}}. \tag{2.41}$$

Note that $L(\tau)$ is an elliptic function with periods α and $2\pi i$, further it has symmetries $\overline{L(\tau)} = L(\overline{\tau})$ and $L(\beta - \tau) = L(\tau)$. Figure 2.1 shows the plot of L. Therefore, we expect $\epsilon^{-2}\eta_*(\epsilon)$ to be oscillatory and periodic as $\epsilon \to 0$, more precisely

$$\epsilon^{-2}\eta_*(\epsilon) \sim \frac{1}{L(-2\ln\epsilon)}.$$

So the integral averages of the function $r(x) = (1+e^{2x}\eta_*(e^{-x}))^{-1}$ in the second formula of (2.39) converge to the integral (over one period) of its periodic approximation, namely

$$\frac{\beta}{\alpha} = \gamma = \lim_{t\to+\infty} \frac{1}{t} \int_0^t r(x)dx = \lim_{t\to+\infty} \int_0^1 r(tx)dx = \int_0^1 \frac{L(2x)}{1+L(2x)}dx.$$

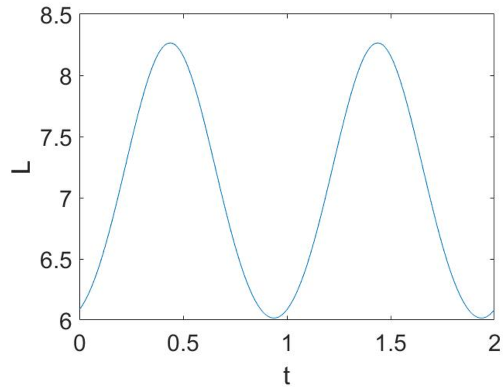


Figure 2.1: The graph of $L(t)$ for $\alpha = 4$ and $\beta = 3.5$.

CHAPTER 3

THE ANNULUS AND THE BERNSTEIN ELLIPSE

Let us use the Notation 2.1 defined in the beginning of Chapter 2. In this chapter we consider applications of Theorems 2.1 and 2.2 to the settings when Ω is an annulus and Γ is a concentric circle; and Ω is the Bernstein ellipse and $\Gamma = [-1, 1]$ is the interval between its foci. We obtain explicit formulas for the exponent $\gamma(z)$ and the maximizer function $M = M_{\epsilon,z}$ attaining the optimal bound in both cases (cf. Theorems 3.1 and 3.2).

Surprisingly, the result in ellipse follows from the one in annulus. The trick we use, inspired by [20], is to map the Bernstein ellipse cut along $[-1, 1]$ onto the annulus using the inverse of the Joukowski function . Then, functions analytic in the ellipse are distinguished from functions analytic in the cut ellipse by their continuity across the cut. After the conformal transformation the image of functions analytic in the entire ellipse would consist of functions analytic in the annulus with a reflection symmetry on the unit circle. Our Hilbert space-based approach can easily incorporate linear constraints by making an appropriate choice of the underlying Hilbert space. However, the question is about the relation between the problems with and without such constraints. In the case of the Bernstein ellipse and the annulus, we discover that the subspace of functions analytic in the annulus corresponding to functions analytic in the Bernstein ellipse is invariant with respect to the integral operator $\mathscr{K}$. It is this invariance that permits us to solve the problem with additional linear constraints using the known solution of the original problem as explained in Section 2.3.

3.1 The annulus

For $0 < \rho < 1$ and $r > 0$ let

$$A_\rho = \{\zeta \in \mathbb{C} : \rho < |\zeta| < 1\}, \qquad \Gamma_r = \{\zeta \in \mathbb{C} : |\zeta| = r\}. \tag{3.1}$$

Consider the Hardy space (e.g. [23])

$$H^2(A_\rho) = \{f \text{ is analytic in } A_\rho : \|f\|_{H^2(A_\rho)} = \sup_{\rho<r<1} \|f\|_{L^2(\Gamma_r)} < \infty\}, \tag{3.2}$$

where for a curve $\Gamma \subset \mathbb{C}$ the space $L^2(\Gamma)$ denotes the space of square-integrable functions on Γ with respect to the arc length measure $|d\tau|$ on Γ.

Theorem 3.1 (Annulus)**.** *Let $\Gamma = \Gamma_r$ with $r \in (\rho, 1)$ fixed and $z \in A_\rho \backslash \Gamma$. Then there exists $C > 0$, such that for any $\epsilon > 0$ and any $f \in H^2(A_\rho)$ with $\|f\|_{H^2(A_\rho)} \le 1$ and $\|f\|_{L^2(\Gamma)} \le \epsilon$, we have*

$$|f(z)| \le C\epsilon^{\gamma(z)}, \tag{3.3}$$

where

$$\gamma(z) = \begin{cases} \dfrac{\ln|z|}{\ln r}, & \text{if } r < |z| < 1 \\ \dfrac{\ln(|z|/\rho)}{\ln(r/\rho)}, & \text{if } \rho < |z| < r \end{cases} \tag{3.4}$$

Moreover, (3.3) is asymptotically optimal in ϵ and the function attaining the bound is

$$M(\zeta) = \epsilon^{2-\gamma(z)} \sum_{n\in\mathbb{Z}} \frac{(\overline{z}\zeta)^n}{r^{2n} + \epsilon^2(1+\rho^{2n})}, \qquad \zeta \in A_\rho. \tag{3.5}$$

In addition M is analytic in the closure of A_ρ and $\|M\|_{H^\infty(\overline{A_\rho})}$ is bounded uniformly in ϵ.

Remark 3.1. *The statement that M attains the bound in* (3.3) *means that* $\|M\|_{H^2(A_\rho)} \lesssim 1$, $\|M\|_{L^2(\Gamma)} \lesssim \epsilon$ *and* $|M(z)| \simeq \epsilon^{\gamma(z)}$, *with all implicit constants independent of* ϵ.

It is somewhat surprising that the worst case function, which was required to be analytic only in A_ρ is in fact analytic in a larger annulus $\{|z_\rho^*| < |\zeta| < |z_1^*|\}$, where $z_1^* = 1/\overline{z}$ is the point symmetric to z w.r.t the circle Γ_1 and $z_\rho^* = \rho^2/\overline{z}$ is the point symmetric to z w.r.t the circle Γ_ρ. In particular, $M \in H^\infty(A_\rho)$ the space of analytic and bounded functions in A_ρ. Hence, $M(\zeta)$ also maximizes $|M(z)|$, asymptotically, as $\epsilon \to 0$, if the constraints were given in $H^\infty(A_\rho)$ and $L^\infty(\Gamma)$, instead of $H^2(A_\rho)$ and $L^2(\Gamma)$, respectively.

Remark 3.2. *The limiting case as* $\rho \to 0^+$ *corresponds to the analytic continuation from the circle* Γ_r *into the unit disk* D. *The limiting value of the exponent is* $\gamma(z) = \frac{\ln|z|}{\ln r}$ *for* $|z| > r$, *and* $\gamma(z) = 1$, *for* $|z| < r$. *The numerical stability of extrapolation inside* Γ_r *can be seen directly from Cauchy's integral formula. The same formula for* $\gamma(z)$ *has been obtained in [69] for* $H^\infty(D)$.

Proof of Theorem 3.1. Note that if we replace the H^2-norm in Theorem 3.1 by another equivalent norm, this will only change the constant C in the inequality (3.3). In order to apply our theory we need a norm, induced by an inner product, with respect to which the reproducing kernel of the space H^2 is as simple as possible. To define such an inner product we use the Laurent expansion

$$f(\zeta) = \sum_{n\geq 0} f_n\zeta^n + \sum_{n<0} f_n\zeta^n =: f_+(\zeta) + f_-(\zeta), \tag{3.6}$$

then $f \in H^2(A_\rho)$ if and only if $f_+ \in H^2(\{|\zeta| < 1\})$ and $f_- \in H^2(\{|\zeta| > \rho\})$ (cf. [66]). So we define

$$(f, g) = \tfrac{1}{2\pi}(f_+, g_+)_{L^2(\Gamma_1)} + \tfrac{1}{2\pi\rho}(f_-, g_-)_{L^2(\Gamma_\rho)}. \tag{3.7}$$

The norm in $H^2(A_\rho)$ induced by (3.7) is equivalent to the norm (3.2) (e.g. [66, 42]). Now the functions $\{\zeta^n\}_{n\in\mathbb{Z}}$ form a basis in $H^2(A_\rho)$, let us normalize them:

$$e_n(\zeta) = \begin{cases} \zeta^n, & n \geq 0 \\ (\zeta/\rho)^n, & n < 0, \end{cases} \tag{3.8}$$

then $\{e_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis of $H^2(A_\rho)$. Definition of the reproducing kernel implies that $p(\zeta,\tau) = \sum_n \overline{e_n(\tau)} e_n(\zeta)$. Computing this sum, or by adding kernels of the spaces $H^2(\{|\zeta| < 1\})$ and $H^2(\{|\zeta| > \rho\})$, we find the reproducing kernel of $H^2(A_\rho)$:

$$p(\zeta,\tau) = \frac{1}{1-\zeta\overline{\tau}} + \frac{\rho^2}{\zeta\overline{\tau}-\rho^2}. \tag{3.9}$$

Note that $p_z \notin \mathscr{K}(\mathcal{W})$ (cf. (2.5)). Indeed, the function p_z has simple poles at $\overline{z}^{-1}, \rho^2\overline{z}^{-1}$. At the same time, for any $f \in \mathcal{W} \subset L^2(\Gamma)$ the function $\mathscr{K}f$ may have singularities only in the set $S = \cup_{\tau\in\Gamma}\{\overline{\tau}^{-1}, \rho^2\overline{\tau}^{-1}\}$. If $\overline{z}^{-1} \in S$, then $z \in \Gamma \cup \rho^{-2}\Gamma$. If $\rho^2\overline{z}^{-1} \in S$, then $z \in \Gamma \cup \rho^2\Gamma$. But since $z \notin \Gamma$ and curves $\rho^{\pm 2}\Gamma$ are outside of the annulus A_ρ, the equation $\mathscr{K}f(\zeta) = p(\zeta, z)$ for $\zeta \in A_\rho$ cannot have any solutions in $\mathcal{W}$.

We observe that for any orthonormal basis $\{e_n : n \in \mathbb{Z}\}$ of $\mathcal{H}$ we have, using (2.3),

$$\mathscr{K}f = \sum_{n\in\mathbb{Z}} (\mathscr{K}f, e_n)e_n = \sum_{n\in\mathbb{Z}} (f, e_n)_{L^2(\Gamma)}\ e_n. \tag{3.10}$$

It is easy to verify that when Γ is a circle centered at the origin, the functions $\{e_n\}$, given by (3.8) are also orthogonal in $L^2(\Gamma)$ and hence, taking $f = e_m$ in (3.10) we conclude that $\mathscr{K}e_m = \|e_m\|^2_{L^2(\Gamma)} e_m$. So we have proved

Lemma 3.1. *Let* $\{e_n\}_{n\in\mathbb{Z}}$ *be given by* (3.8) *and* $\mathscr{K}$ *given by* (3.10), *then*

$$\mathscr{K}e_n = \lambda_n e_n, \qquad n \in \mathbb{Z},$$

where

$$\lambda_n = 2\pi r \begin{cases} r^{2n}, & n \geq 0 \\ (r/\rho)^{2n}, & n < 0 \end{cases} \tag{3.11}$$

We see that λ_n and $|e_n(z)|$ approach to zero along two different sequences and have two different asymptotic behaviors, which are distinguished by the location of z relative to Γ. Therefore, to apply Theorem 2.2 we need to consider two cases. Assume that z lies outside of Γ, i.e. $|z| \in (r, 1)$. The function u from (2.11) is given by

$$u(\zeta) = \sum_{n \in \mathbb{Z}} \frac{\overline{e_n(z)} e_n(\zeta)}{\lambda_n + \epsilon^2}. \tag{3.12}$$

Note that, for any $n \in \mathbb{Z}$

$$\frac{|e_n(z)|^2}{\lambda_n} = \frac{1}{2\pi r} \left(\frac{|z|}{r} \right)^{2n}.$$

By assumption the above quantity is summable over $n < 0$, this implies that in analyzing $u(z)$ the sum over negative indices is $O(1)$, as $\epsilon \to 0$, and hence can be ignored. The dominant part is the sum over $n \geq 0$. Analogously, in quantities $\|u\|_{H^2(A_\rho)}, \|u\|_{L^2(\Gamma)}$ as well, the sum can be restricted to $n \geq 0$. This determines the behaviors $\lambda_n \simeq r^{2n}$ and $|e_n(z)| \simeq |z|^n$, therefore Theorem 2.2 implies that the exponent is $\gamma(z) = \frac{\ln |z|}{\ln r}$. The case $|z| \in (\rho, r)$ is done analogously and (3.4) now follows.

Next, we can rewrite (3.12) as

$$u(\zeta) = \sum_{n \geq 0} \frac{\overline{z}^n \zeta^n}{2\pi r r^{2n} + \epsilon^2} + \sum_{n < 0} \frac{\overline{z}^n \zeta^n}{2\pi r r^{2n} + \epsilon^2 \rho^{2n}}. \tag{3.13}$$

Let us consider the function

$$\tilde{u}(\zeta) = \sum_{n \in \mathbb{Z}} \frac{\overline{z}^n \zeta^n}{r^{2n} + \epsilon^2 (1 + \rho^{2n})}, \tag{3.14}$$

clearly for negative indices $\rho^{2n} \ll 1$ and hence can be ignored, and for positive indices 1 can be ignored from the denominator in the definition of $\tilde{u}$. Therefore, values of $\tilde{u}, u$ at z and their H^2 and L^2-norms have the same behavior in ϵ. Thus, we may consider $\tilde{u}$ instead, which then gives rise to the maximizer function M in (3.5). Finally, the fact that $\|M\|_{H^\infty(\overline{A_\rho})}$ is bounded uniformly in ϵ follows from the application of Corollary 2.2.

□

3.2 The Bernstein ellipse

Let E_R be the open ellipse with foci at ± 1 and the sum of semi-minor and semi-major axes equal to $R > 1$. The axes lengths of such an ellipse are therefore $(R \pm R^{-1})/2$. E_R is called the Bernstein ellipse [8, 68]. Its boundary is an image of a circle of radius R centered at the origin under the Joukowski map $J(\omega) = (\omega + \omega^{-1})/2$. Let $H^\infty(E_R)$ be the space of bounded analytic functions in E_R, with the usual supremum norm.

Theorem 3.2 (Ellipse)**.** *Let $z \in E_R \backslash [-1, 1]$. Then there exists $C > 0$, such that for every $\epsilon > 0$ and $F \in H^\infty(E_R)$ with $\|F\|_{H^\infty(E_R)} \leq 1$ and $\|F\|_{L^\infty(-1,1)} \leq \epsilon$, we have*

$$|F(z)| \leq C\epsilon^{\alpha(z)}, \tag{3.15}$$

where

$$\alpha(z) = 1 - \frac{\ln |J^{-1}(z)|}{\ln R} \in (0, 1), \qquad J^{-1}(z) = z + (z - 1)\sqrt{\frac{z + 1}{z - 1}}. \tag{3.16}$$

Moreover, (3.15) *is asymptotically optimal in ϵ and function attaining the bound is*

$$M(\zeta) = \epsilon^{2-\alpha(z)} \sum_{n=1}^{\infty} \frac{(\overline{J^{-1}(z)})^n T_n(\zeta)}{1 + \epsilon^2 R^{2n}}, \tag{3.17}$$

where T_n is the Chebyshev polynomial of degree n: $T_n(x) = \cos(n \cos^{-1} x)$ for $x \in [-1, 1]$.

Several remarks are now in order.

(i) $J^{-1}(\zeta)$ is the branch of an inverse of the Joukowski map J, that is analytic in the slit ellipse $E_R \backslash [-1, 1]$ and satisfies the inequalities $1 < |J^{-1}(\zeta)| < R$.

(ii) Chebyshev polynomials T_n play the same role in the ellipse as monomials ζ^n play in the annulus, i.e. they are the building blocks of analytic functions. In fact $J^{-1} \circ T_n \circ J = \zeta^n$.

(iii) The same bound (3.15) was obtained in [20] when $z \in E_R \cap \mathbb{R}$, where it was shown that the bound (up to logarithmic factors) could be attained by a polynomial

$$g(\zeta) = \epsilon T_{K(\epsilon)}(\zeta), \qquad K = K(\epsilon) = \lfloor \ln(1/\epsilon)/\ln R \rfloor. \tag{3.18}$$

We observe that the terms in (3.17) increase exponentially fast from $n = 1$ to $n = K(\epsilon)$ and then decrease exponentially fast for $n > K(\epsilon)$. Hence, asymptotically (up to logarithmic factors) we can say that

$$|M(\zeta)| \approx \epsilon^{2-\alpha(z)} \frac{|J^{-1}(z)|^{K(\epsilon)} \, |T_{K(\epsilon)}(\zeta)|}{1 + \epsilon^2 R^{2K(\epsilon)}} \approx \epsilon |T_{K(\epsilon)}(\zeta)|,$$

in agreement with (3.18).

The proof of Theorem 3.2 is given in the three subsections below.

3.2.1 From the ellipse to the annulus

The ellipse E_R is conformally equivalent to a disk or the upper half-plane. The conformal mapping effecting the equivalence can be written explicitly in terms of the Weierstrass ζ-function, but the image of the interval $[-1, 1]$ will then be a curve that would not permit any kind of explicit solution of the resulting integral equation. Instead we use a much simpler Joukowski function $J(\omega) = \frac{\omega + \omega^{-1}}{2}$ that will convert the problem in the ellipse to the problem in an annulus with Γ being a concentric circle inside the annulus. We observe that $J(\omega)$ maps the annulus $\{R^{-1} < |\omega| < R\}$ onto the Bernstein ellipse E_R in 2-1 fashion, meaning that each point in E_R has exactly two (if we count the multiplicity) preimages in the annulus (note that $J(\omega) = J(\omega^{-1})$). Moreover, the unit circle gets mapped onto $[-1, 1] \subset E_R$ under J. So given a function

$F \in H^\infty(E_R)$, the function $f(\zeta) := F(J(R\zeta))$ is analytic in A_ρ defined in (3.1), with $\rho = R^{-2}$, has the same H^∞ norm, and satisfies the symmetry property

$$f(\overline{\zeta}) = f(\zeta) \qquad \forall |\zeta| = r = \frac{1}{R}. \tag{3.19}$$

Conversely, any function $f \in H^\infty(A_\rho)$, satisfying (3.19) defines an analytic function in a Bernstein ellipse (with the same H^∞ norm). This is so because (3.19) can also be written as

$$f\left(\frac{1}{R^2\zeta}\right) = f(\zeta) \qquad \forall |\zeta| = r. \tag{3.20}$$

The Schwarz reflection principle then guarantees that (3.20) holds for all $\zeta \in A_\rho$. This implies that $F(t) = f(R^{-1}J^{-1}(t))$ gives the same value for each of the two branches of J^{-1} and hence defines an analytic function in E_R. Thus, the analytic continuation problem in ellipse reduces to the one in the annulus, but with an additional symmetry constraint (3.19).

3.2.2 The annulus with symmetry

Let us now define

$$\mathcal{H} = \{f \in H^2(A_\rho) : f(\overline{\zeta}) = f(\zeta) \quad \forall |\zeta| = \sqrt{\rho}\} \tag{3.21}$$

and let the curve Γ be the circle Γ_r centered at the origin of radius $r = \sqrt{\rho}$.

Lemma 3.2 (Annulus with symmetry)**.** *Let $0 < \rho < 1$ and let $z \in \mathbb{C}$ be such that $r < |z| < 1$. Then there exists $C > 0$, such that for every $\epsilon > 0$ and every $f \in \mathcal{H}$ with $\|f\|_{H^2(A_\rho)} \leq 1$ and $\|f\|_{L^2(\Gamma_r)} \leq \epsilon$ we have the bound*

$$|f(z)| \leq C\epsilon^{\gamma(z)}, \tag{3.22}$$

where the exponent $\gamma(z)$ is the same as in Theorem 3.1, i.e.

$$\gamma(z) = \frac{\ln|z|}{\ln r}. \tag{3.23}$$

Moreover, (3.22) is asymptotically optimal as $\epsilon \to 0$ and the function attaining the bound is

$$M(\zeta) = \epsilon^{2-\gamma(z)} \sum_{n=1}^{\infty} \frac{\overline{z}^n + (\rho/\overline{z})^n}{\rho^n + \epsilon^2}[\zeta^n + (\rho/\zeta)^n], \qquad \zeta \in A_\rho. \tag{3.24}$$

Proof of Lemma 3.2. We note that the maximization problem in Lemma 3.2 differs from the one in Theorem 3.1 by the requirement of symmetry (3.19). Hence, following the theory in Section 2.3 we define the subspace

$$L = \{f \in H^2(A_\rho) : f(\zeta) = f(\overline{\zeta}) \qquad \forall \zeta \in \Gamma_r\}, \qquad r = \sqrt{\rho}.$$

Then, the orthogonal projection onto L will be given by

$$\mathcal{P}_L f(\zeta) = \frac{f(\zeta) + f(\rho/\zeta)}{2}. \tag{3.25}$$

Lemma 3.3. *The integral operator* $\mathscr{K}$ *given by* (2.4) *with kernel (3.9) and* $\Gamma = \Gamma_r$ *commutes with* $\mathcal{P}_L$*.*

Proof. It is straightforward to show that the commutation $\mathcal{P}_L \mathscr{K} = \mathscr{K} \mathcal{P}_L$ is equivalent to

$$\int_{\Gamma_r} p(\zeta, \tau) u(r^2/\tau) |d\tau| = \int_{\Gamma_r} p(r^2/\zeta, \tau) u(\tau) |d\tau|,$$

which, after change of variables on the left-hand side reduces to

$$p(\zeta, \rho/\tau) = p(\rho/\zeta, \tau) \qquad\qquad \forall \zeta \in A_\rho,\ \forall \tau \in \Gamma_r.$$

Substituting the definition of p from (3.9) into this formula we easily verify it. □

According to the theory in Section 2.3 the solution of (2.23) is $u_L = \mathcal{P}_L u$, where u is given by (3.12). We observe that in the case $r^2 = \rho$ we have $\lambda_n = \lambda_{-n}$ and $e_n(\rho/\zeta) = e_{-n}(\zeta)$, so that

$$u_L = \mathcal{P}_L u(\zeta) = \frac{1}{1+\epsilon^2} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{\overline{e_n(z)} + \overline{e_{-n}(z)}}{\lambda_n + \epsilon^2} [e_n(\zeta) + e_{-n}(\zeta)].$$

Substituting the expressions for λ_n, e_n from (3.11), (3.8), respectively, and ignoring the first $O(1)$ term and some constants, which affect the asymptotics of u_L by constant factors, we arrive at the function

$$u_L(\zeta) = \sum_{n=1}^{\infty} \frac{\overline{z}^n + (\rho/\overline{z})^n}{\rho^n + \epsilon^2} [\zeta^n + (\rho/\zeta)^n].$$

We note that

$$e_n^L = \frac{1}{2}\left(\zeta^n + (\rho/\zeta)^n\right), \quad n \geq 0,$$

is the orthonormal eigenbasis of L with respect to $\mathcal{P}_L \mathscr{K} \mathcal{P}_L$. The corresponding eigenvalues are $\lambda_n = 2\pi\sqrt{\rho}\rho^n$, and for $|z| \in (r,1)$ we have $|e_n^L(z)| \simeq |\overline{z}^n + (\rho/\overline{z})^n| \simeq |z|^n$. Then, Theorem 2.2 gives the formula (3.23) as well as the maximizer function (3.24).

□

3.2.3 From the annulus to the ellipse

In this section we will show that Theorem 3.2 follows from Lemma 3.2. Let $F \in H^\infty(E_R)$ be such that $\|F\|_{H^\infty} \leq 1$ and $|F(x)| \leq \epsilon$ for all $x \in [-1,1]$. As discussed in Section 3.2.1, the function $f(\zeta) := F(J(R\zeta))$ is analytic in A_ρ, with $\rho = R^{-2}$ and has the symmetry $f(\overline{\zeta}) = f(\zeta) \quad \forall |\zeta| = r,$ where $r = R^{-1}$. It also satisfies

$$\|f\|_{H^2(A_\rho)} \lesssim \|F\|_{H^\infty(E_R)} \leq 1$$

as well as

$$\|f\|^2_{L^2(\Gamma_r)} = \frac{1}{R}\int_0^{2\pi} |F(J(e^{it}))|^2 dt \leq \frac{2\pi\epsilon^2}{R}.$$

Let $z \in E_R \setminus [-1,1]$. Let $z_a \in A_\rho$ be the unique solution of $J(Rz_a) = z$, satisfying $|z_a| > r$. Then by Lemma 3.2 (with $\rho = R^{-2}$ and $r = R^{-1}$) we have

$$|F(z)| = |f(z_a)| \leq C\epsilon^{-\frac{\ln|z_a|}{\ln R}} = C\epsilon^{1-\frac{\ln\left|J^{-1}(z)\right|}{\ln R}} = C\epsilon^{\alpha(z)},$$

where $\alpha(z)$ is given by (3.16). This proves (3.15).

In order to prove the optimality of the bound (3.15) we use Corollary 2.2 to show that $M(\zeta)$ given by (3.24) satisfies

$$\begin{cases} |M(\zeta)| \lesssim \epsilon, & |\zeta| = r, \\ |M(\zeta)| \lesssim 1, & r < |\zeta| < 1. \end{cases}$$

Using the Joukowski function to map this to a function on the Bernstein ellipse we obtain

$$M_{\text{ellipse}}(t) = M\left(R^{-1}J^{-1}(t)\right) = \epsilon^{2-\alpha(z)} \sum_{n=1}^{\infty} \frac{\overline{T_n(z)}T_n(t)}{1+\epsilon^2 R^{2n}}, \tag{3.26}$$

where T_n is the Chebyshev polynomial of degree n. Chebyshev polynomials are just monomials ζ^n in the annulus after the Joukowski transformation:

$$J^{-1} \circ T_n \circ J = \zeta \mapsto \zeta^n, \qquad \forall \zeta \neq 0.$$

We note that due to the choice of the branch of J^{-1} to correspond to a point in the exterior of the unit disk we can neglect $1/(J^{-1}(z))^n$ in

$$T_n(z) = \frac{1}{2}\left((J^{-1}(z))^n + \frac{1}{(J^{-1}(z))^n}\right).$$

Thus, the function in (3.17) is asymptotically equivalent to (3.26). Theorem 3.2 is now proved.

CHAPTER 4

THE UPPER HALF-PLANE

Let us use the Notation 2.1 defined in the beginning of Chapter 2. In this chapter we consider further applications of Theorems 2.1 and 2.2 in the setting of Hardy functions over the upper half-plane. So let $\mathbb{H}_+ = \{\zeta \in \mathbb{C} : \mathfrak{Im}(\zeta) > 0\}$ denote the complex upper half-plane and consider the Hardy space

$$H^2 := H^2(\mathbb{H}_+) = \{f \text{ is analytic in } \mathbb{H}_+ : \sup_{y>0} \|f(\cdot + iy)\|_{L^2(\mathbb{R})} < \infty\}. \quad (4.1)$$

It is well known [42] that these functions have L^2-boundary data, and that $\|f\| := \|f\|_{H^2} = \|f\|_{L^2(\mathbb{R})}$ defines a norm in $H^2(\mathbb{H}_+)$. Moreover, H^2 is a RKHS with the inner product $(f, g) = (f, g)_{L^2(\mathbb{R})}$ and by Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(x)dx}{x - z} = (f, p_z),$$

where p_z, the reproducing kernel of H^2, is given by

$$p_\tau(\zeta) = p(\zeta, \tau) = \frac{i}{2\pi(\zeta - \overline{\tau})}. \quad (4.2)$$

Now the operator $\mathscr{K}$ (2.4) takes the form

$$\mathscr{K} u(\zeta) = \frac{1}{2\pi} \int_{\Gamma} \frac{iu(\tau)|d\tau|}{\zeta - \overline{\tau}}. \quad (4.3)$$

Note that $p_z \notin \mathscr{K}(\mathcal{W})$ (cf. (2.5)) for $z \notin \Gamma$. Indeed, the function p_z is analytic everywhere in $\mathbb{C}$, except at $\overline{z}$, where it has a pole. At the same time for any $f \in \mathcal{W} \subset L^2(\Gamma)$ the function $\mathscr{K}f$ is analytic everywhere in $\mathbb{C}$ outside of $\overline{\Gamma}$. But $\overline{z} \notin \overline{\Gamma}$, since z lies outside of Γ. Therefore, the equation $\mathscr{K}f = p_z$ has no solutions in $\mathcal{W}$.

According to Theorem 2.1, the maximizer function in

$$\sup\left\{|f(z)| : f \in H^2 \text{ with } \|f\| \leq 1,\ \|f\|_{L^2(\Gamma)} \leq \epsilon\right\} \tag{4.4}$$

is given by

$$M_{\epsilon,z}(\zeta) = u_{\epsilon,z}(\zeta) \min\left\{\frac{1}{\|u_{\epsilon,z}\|}, \frac{\epsilon}{\|u_{\epsilon,z}\|_{L^2(\Gamma)}}\right\}, \tag{4.5}$$

where $u = u_{\epsilon,z}$ solves the integral equation $\mathscr{K}u + \epsilon^2 u = p_z$. Our goal is to establish the estimate

$$M_{\epsilon,z}(z) \simeq \epsilon^{\gamma(z)} \tag{4.6}$$

and describe the exponent $\gamma(z)$.

In Section 4.1 we consider the case when Γ is a circle in $\mathbb{H}_+$. In this case the eigenvalues and eigenfunctions of $\mathscr{K}$ are computed explicitly and are seen to decay exponentially, so that the assumption (2.21) of Theorem 2.2 is satisfied. As a result, (4.6) follows with explicit formulas for $\gamma(z)$ and the maximizer function $M_{\epsilon,z}$.

In Section 4.2 we consider the case $\Gamma = [-1, 1]$, which lies on the boundary of $\mathbb{H}_+$. We first show that the error maximizer again solves an integral equation, but with a singular, non-compact integral operator. This singular equation is then solved explicitly and the exponent $\gamma(z)$ is computed. Examining the formula for $\gamma(z)$ we find a beautiful geometric interpretation of this exponent.

4.1 The circle

Notation: Let $D(c,r)$ and $C(c,r)$ denote respectively the closed disk and the circle centered at c and of radius r in the complex plane.

Assume that the data curve $\Gamma \Subset \mathbb{H}_+$ is a circle. By considering affine automorphisms $\zeta \mapsto a\zeta + b$, $a > 0$, $b \in \mathbb{R}$, of $\mathbb{H}_+$ we may "translate" Γ to be centered at i.

Theorem 4.1. *Let $\Gamma = C(i,r)$ with $r < 1$ and let $z \in \mathbb{H}_+$ be a point outside of Γ. Then there exists $C > 0$, such that for any $\epsilon > 0$ and any $f \in H^2$ with $\|f\|_{H^2} \leq 1$ and $\|f\|_{L^2(\Gamma)} \leq \epsilon$, we have*

$$|f(z)| \leq C\epsilon^{\gamma(z)}, \tag{4.7}$$

where

$$\gamma(z) = \frac{\ln|m(z)|}{\ln\rho}, \qquad \rho = \frac{1-\sqrt{1-r^2}}{r}, \tag{4.8}$$

and

$$m(\zeta) = \frac{\zeta - z_0}{\zeta + z_0}, \qquad z_0 = i\sqrt{1-r^2}$$

is the Möbius map transforming the upper half-plane into the unit disc and the circle Γ into a concentric circle, whose radius has to be ρ. Moreover, (4.7) is asymptotically optimal in ϵ and the function attaining the bound can be written as a convergent in the upper half-plane "power" series

$$M(\zeta) = \frac{\epsilon^{2-\gamma(z)}}{\zeta + z_0} \sum_{n=1}^{\infty} \frac{\left(\overline{m(z)}m(\zeta)\right)^n}{\epsilon^2 + \rho^{2n}}, \qquad \zeta \in \mathbb{H}_+. \tag{4.9}$$

Remark 4.1. *When z is inside Γ we have complete stability, indeed Cauchy's integral formula implies that*

$$|f(z)| \leq c\epsilon$$

for a constant c independent of ϵ.

Lemma 4.1. *Let $\Gamma = C(i,r)$ with $r \in (0,1)$ and let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal eigenbasis of $\mathscr{K}$ in H^2, with eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$. Then*

$$\lambda_n = \frac{r\rho^{2n}}{1+\sqrt{1-r^2}}, \qquad e_n(\zeta) = \frac{\sqrt[4]{1-r^2}}{\sqrt{\pi}}\frac{m(\zeta)^n}{\zeta+z_0}, \tag{4.10}$$

where $\rho, z_0, m(\zeta)$ are as in Theorem 4.1.

Before proving this lemma, let us see that it concludes the proof of Theorem 4.1 upon the application of Theorems 2.1 and 2.2. Indeed, $\lambda_n \simeq \rho^{2n}$ and $|e_n(z)| \simeq |m(z)|^n$, then the formula (4.8) for the exponent $\gamma(z)$ follows. From (2.17), the function $u = u_{\epsilon,z}$ is then given by

$$u(\zeta) = \frac{\pi^{-1}\sqrt{1-r^2}}{(\overline{z}+\overline{z}_0)(\zeta+z_0)}\sum_{n=1}^{\infty}\frac{\overline{m(z)}^n m(\zeta)^n}{\rho^{2n+1}+\epsilon^2}. \tag{4.11}$$

As in the case of the annulus (cf. Section 3.1), ignoring the constants that do not affect the asymptotics of the function as $\epsilon \to 0$ we obtain the maximizer (4.9).

Proof of Lemma 4.1. Let $\mathscr{K}w(\zeta) = \lambda w(\zeta)$, then w must be analytic in the extended complex plane with the closed disk $D(-i,r)$ removed. In particular, it is analytic in $D(i,r)$. Thus, we can evaluate the operator $\mathscr{K}$ explicitly in terms of values of w.

$$\mathscr{K}w(\zeta) = \frac{1}{2\pi}\int_0^{2\pi}\frac{irw(i+re^{it})dt}{\zeta+i-re^{-it}} = \frac{1}{2\pi}\int_{C(0,r)}\frac{rw(i+\tau)d\tau}{(\zeta+i)\tau-r^2}.$$

We note that $r^2/|\zeta+i| < r$ precisely when ζ is outside of the closed disk $D(-i,r)$. In addition $w(i+\tau)$ is analytic in $D(0,r)$, hence

$$\mathscr{K}w(\zeta) = \frac{ir}{\zeta+i}w\left(i+\frac{r^2}{\zeta+i}\right).$$

Next we note that the Möbius transformation

$$\sigma(\zeta) = i + \frac{r^2}{\zeta+i}$$

maps $D(-i,r)$ onto the exterior of $D(i,r)$. In particular there is a disk $D_1 \subset D(-i,r)$ such that $\sigma(D_1) = D(-i,r)$. Then $\mathscr{K}w$ is analytic in the exterior of

D_1, since w is analytic outside of $D(-i, r)$. But w is an eigenfunction of $\mathscr{K}$, hence it must also be analytic outside of D_1. Repeating the argument using the fact that w is analytic in the larger domain $\mathbb{C} \setminus D_1$ we conclude that it must also be analytic outside of $D_2 \subset D_1$, such that $\sigma(D_2) = D_1$. We can continue like this indefinitely, showing that the only possible singularity of w must be at the fixed point $\zeta_0 \in D(-i, r)$ of $\sigma(\zeta)$. We find

$$\zeta_0 = -i\sqrt{1 - r^2}.$$

Since w is analytic at infinity the transformation $\eta = 1/(\zeta - \zeta_0)$ will map the extended complex plane with ζ_0 removed to the entire complex plane (without the infinity). The eigenfunction w will then be an entire function in the η-plane. Let $v(\eta) = w(\eta^{-1} + \zeta_0)$. Then

$$w(\zeta) = v\left(\frac{1}{\zeta - \zeta_0}\right).$$

The relation $\mathscr{K} w = \lambda w$ now reads

$$\lambda v(\eta) = \frac{ir\eta}{\eta(\zeta_0 + i) + 1} v\left(\frac{\eta(\zeta_0 + i) + 1}{i - \zeta_0}\right).$$

One corollary of this equation is that $v(0) = 0$. Hence, $\phi(\eta) = \eta^{-1} v(\eta)$ is also an entire function, satisfying

$$\lambda\phi(\eta) = \frac{ir}{i - \zeta_0} \phi\left(\frac{\eta(\zeta_0 + i) + 1}{i - \zeta_0}\right).$$

We see that $\phi(\eta)$ is an entire function with the property that $\phi(a\eta + b)$ is a constant multiple of $\phi(\eta)$, with $b = \frac{1}{i - \zeta_0}$ and $a = \rho^2$, where ρ is given by (4.8). It remains to observe that such a property holds for functions $\phi_n(\eta) = (\eta - \eta_0)^n$, provided

$$\frac{\eta_0 - b}{a} = \eta_0 \quad \Longleftrightarrow \quad \eta_0 = \frac{b}{1 - a}.$$

Indeed,

$$(a\eta + b - \eta_0)^n = a^n \left(\eta - \frac{\eta_0 - b}{a}\right)^n = a^n (\eta - \eta_0)^n.$$

In our case we get $\eta_0 = -\frac{1}{2\zeta_0}$ and conclude that $\phi_n(\eta) = \left(\eta + \frac{1}{2\zeta_0}\right)^n$ and λ_n is given by (4.10). Converting the formula back to $w_n(\zeta)$ we obtain (up to a constant multiple)

$$w_n(\zeta) = \frac{1}{\zeta - \zeta_0}\left(\frac{\zeta + \zeta_0}{\zeta - \zeta_0}\right)^n = \frac{m(\zeta)^n}{\zeta - \zeta_0}.$$

It remains to normalize the eigenfunctions w_n. For that we compute

$$\|w_n\|_{H^2}^2 = \int_{\mathbb{R}} |w_n|^2 dx = \int_{\mathbb{R}} \frac{dx}{|x - \zeta_0|^2} = \frac{\pi}{\sqrt{1 - r^2}}.$$

□

4.2 The interval on the boundary

We recall that functions in the Hardy space H^2 (4.1) are determined uniquely not only by their values on any curve $\Gamma \subset \mathbb{H}_+$, but also on $\Gamma \subset \mathbb{R}$. Indeed, if $f = 0$ on $\Gamma \subset \mathbb{R}$, the Cauchy integral representation formula implies

$$f(\zeta) = \frac{1}{2\pi i}\int_{\Gamma^c} \frac{f(x)dx}{t - \zeta}, \qquad \zeta \in \mathbb{H}_+,$$

where $\Gamma^c = \mathbb{R} \setminus \Gamma$. Then $f(\zeta)$ has analytic extension to $\mathbb{C} \setminus \Gamma^c$, which vanishes on a curve Γ inside its domain of analyticity and therefore $f \equiv 0$. This rigidity property suggests that we should expect the same power law behavior of the analytic continuation error as for the curves in the interior of $\mathbb{H}_+$.

We will consider the most basic case when $\Gamma \subset \mathbb{R}$ is an interval. By rescaling and translation we may assume, without loss of generality, that $\Gamma = [-1, 1]$. We proceed by representing Γ as a limit of interior curves $\Gamma_h = [-1, 1] + ih$ as $h \downarrow 0$. For curves Γ_h, Theorem 2.1 can be applied and in the resulting upper bound and the integral equation, limits, as $h \downarrow 0$, can be taken. As a result we obtain

Theorem 4.2 (Boundary). *Let $z = z_r + iz_i \in \mathbb{H}_+$ and $\epsilon \in (0, 1)$. Assume $f \in H^2$ is such that $\|f\| \leq 1$ and $\|f\|_{L^2(-1,1)} \leq \epsilon$, then*

$$|f(z)| \leq C\epsilon^{\gamma(z)} \tag{4.12}$$

where $C^{-2} = \frac{z_i}{9}\left(\arctan\frac{z_r+1}{z_i} - \arctan\frac{z_r-1}{z_i}\right)$ *and*

$$\gamma(z) = -\frac{1}{\pi}\arg\frac{z+1}{z-1} \in (0,1) \tag{4.13}$$

is the angular size of $[-1,1]$ *as seen from* z*, measured in units of* π *radians. Moreover, the upper bound* (4.12) *is asymptotically (in* ϵ*) optimal and the maximizer that attains the bound up to a multiplicative constant independent of* ϵ *is*

$$M(\zeta) = \epsilon\frac{p_z(\zeta)}{\|p_z\|_{L^2(-1,1)}}e^{\frac{i}{\pi}\ln\epsilon\ln\frac{1+\zeta}{1-\zeta}}, \qquad \zeta \in \mathbb{H}_+ \tag{4.14}$$

where $p_z(\zeta) = i/2\pi(\zeta - \overline{z})$ *and* $\ln$ *denotes the principal branch of logarithm.*

Remark 4.2.

1. *Our explicit formulas show that the problem of predicting the value of a function at* $z = z_0 \in \mathbb{R}\setminus[-1,1]$ *is ill-posed in every sense. Indeed, in the optimal bound* (4.12) $C \to +\infty$ *and* $\gamma(z) \to 0$ *as* $z \to z_0$.

2. *The set of points* $z \in \mathbb{H}_+$ *for which* $\gamma(z)$ *is the same is an arc of a circle passing through* -1 *and* 1 *that lies in the upper half-plane.*

Remark 4.3. *Conformal mappings between domains can be used to "transplant" the exponent estimates from one geometry to a different one. For example, we can transplant the exponent* $\gamma(z)$ *in* (4.13) *for the half-plane to the half-strip* $\mathfrak{Re}\omega > 0$, $|\mathfrak{Im}\omega| < 1$*, considered in [69]. The analytic function* $f(\omega)$ *is assumed to be bounded in the half-strip and also of order* ϵ *on the interval* $[-i,i]$ *on the imaginary axis. Then any such function must satisfy* $|f(x)| \leq C\epsilon^{\gamma(x)}$, $x > 0$*, where* $\gamma(x) = (2/\pi)\mathrm{arccot}(\sinh(\pi x/2))$*. Moreover, the estimate is sharp, since it is attained by the function* $M(-i\sinh(\pi\omega/2))$*, where* $M(\zeta)$ *is given by* (4.14)*. This result follows from the observation that*

$\zeta = -i\sinh(\pi\omega/2)$ *is a conformal map from the half-strip to the upper half-plane, mapping interval* $[-i, i]$ *to the interval* $[-1, 1]$.

The proof of Theorem 4.2 is given in the two subsections below.

4.2.1 The integral equation

Let us first establish an analogous result to Theorem 2.1, i.e. below we formulate the upper bound in the case $\Gamma = [-1, 1]$ via the solution to an integral equation.

Theorem 4.3. *Let* $z \in \mathbb{H}_+$ *and* $\epsilon > 0$. *Assume* $f \in H^2$ *is such that* $\|f\| \leq 1$ *and* $\|f\|_{L^2(-1,1)} \leq \epsilon$, *then*

$$|f(z)| \leq \frac{3}{2}\epsilon \frac{u_{\epsilon,z}(z)}{\|u_{\epsilon,z}\|_{L^2(-1,1)}}, \tag{4.15}$$

where $u = u_{\epsilon,z}$ *solves the integral equation*

$$\frac{1}{2}(Ku + u) + \epsilon^2 u = p_z, \qquad \text{on } (-1, 1) \tag{4.16}$$

with p_z *as in Theorem 4.2 and*

$$Ku(x) = \frac{i}{\pi}\int_{-1}^{1} \frac{u(y)}{x - y} dy, \tag{4.17}$$

where the integral is understood in the principal value sense.

Proof. It is enough to prove the inequality (4.15) for $\|f\| \leq 1$ and $\|f\|_{L^2(-1,1)} < \epsilon$, because when $\|f\|_{L^2(-1,1)} = \epsilon$ we can consider the sequence $f^n := (1 - \frac{1}{n})f$ and take limits in the inequality for f^n as $n \to \infty$.

Since $f(\cdot + ih) \to f$ as $h \downarrow 0$ in $L^2(-1, 1)$ (a well-known property of H^2 functions, see [42]), the assumption $\|f\|_{L^2(-1,1)} < \epsilon$ implies that $\|f(\cdot + ih)\|_{L^2(-1,1)} \leq \epsilon$ for h small enough. In other words $\|f\|_{L^2(\Gamma_h)} \leq \epsilon$, where $\Gamma_h = [-1, 1] + ih$, so we can apply Theorem 2.1 and conclude

$$|f(z)| \leq \frac{3}{2}\epsilon \frac{u_h(z)}{\|u_h\|_{L^2(\Gamma_h)}}, \qquad \forall h \text{ small enough}$$

where u_h solves the integral equation $\mathscr{K}u + \epsilon^2 u = p_z$, which reads

$$\frac{1}{2\pi}\int_{\Gamma_h} \frac{iu(\tau)}{\zeta - \overline{\tau}}|d\tau| + \epsilon^2 u(\zeta) = \frac{i}{2\pi(\zeta - \overline{z})}, \qquad \zeta \in \mathbb{H}_+.$$

Let us set $v(\omega) = u(\omega + ih)$, then the above integral equation can be rewritten as

$$\mathscr{K}_h v(\omega) + \epsilon^2 v(\omega) = q_h(\omega), \qquad \mathfrak{Im}\omega > -h \tag{4.18}$$

with

$$\mathscr{K}_h v(\omega) = \frac{1}{2\pi}\int_{-1}^{1} \frac{iv(y)dy}{\omega - y + 2ih}, \qquad q_h(\omega) = \frac{i}{2\pi(\omega + ih - \overline{z})}. \tag{4.19}$$

Let us denote this solution by v_h to indicate its dependence on the small parameter h, namely $v_h = \left(\mathscr{K}_h + \epsilon^2\right)^{-1} q_h$. Then the upper bound on f becomes

$$|f(z)| \leq \frac{3}{2}\epsilon \frac{v_h(z - ih)}{\|v_h\|_{L^2(-1,1)}}, \qquad \forall h \text{ small enough} \tag{4.20}$$

Our goal is to take limits in this upper bound as $h \downarrow 0$.

Lemma 4.2. *Let $\mathscr{K}_h$ and K be defined by* (4.19) *and* (4.17)*, respectively. Then for any $g \in L^2(-1,1)$*

$$\mathscr{K}_h g \to \tfrac{1}{2}(K+1)g, \qquad \text{as } h \downarrow 0, \text{ in } L^2(-1,1). \tag{4.21}$$

Proof.

• $\{\mathscr{K}_h\}_{h>0}$ is uniformly bounded in the operator norm on $L^2(-1,1)$. To prove this we observe that $\mathscr{K}_h g = k * \chi_1 g$, where $\chi_1 := \chi_{(-1,1)}$ and

$$k(t) = \frac{i}{2\pi(t + 2ih)}.$$

With the definition $\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-i\xi x}dx$ we can compute $\widehat{k}(\xi) = e^{-2h\xi}\chi_{>0}(\xi)$, where $\chi_{>0}(\xi) = \chi_{(0,+\infty)}(\xi)$. In particular $|\widehat{k}| \leq 1$, but then

$$\|\mathscr{K}_h g\|_{L^2(-1,1)} \leq \|\mathscr{K}_h g\|_{L^2(\mathbb{R})} = \tfrac{1}{\sqrt{2\pi}}\|\widehat{k} \cdot \widehat{\chi_1 g}\|_{L^2(\mathbb{R})} \leq \tfrac{1}{\sqrt{2\pi}}\|\widehat{\chi_1 g}\|_{L^2(\mathbb{R})} =$$
$$= \|\chi_1 g\|_{L^2(\mathbb{R})} = \|g\|_{L^2(-1,1)}$$

which immediately implies $\|\mathscr{K}_h\| \leq 1$ for any $h > 0$.

• By uniform boundedness of $\|\mathscr{K}_h\|$, it is enough to show convergence $\mathscr{K}_h g \to \frac{1}{2}(K+1)g$ in $L^2(-1,1)$ for a dense set of functions g. We will now show convergence for all $g \in C_0^\infty(-1,1)$. Since by Sokhotski-Plemelj formula this convergence holds a.e. in $(-1,1)$, to achieve the desired conclusion it is enough to show that the family of functions $|\mathscr{K}_h g|^2$ is equiintegrable in $(-1,1)$. Vitali convergence theorem [65, p. 133, exercise 10(b)] then implies convergence of $\mathscr{K}_h g$ in $L^2(-1,1)$. We recall the definition of equiintegrability:

$$\sup_{|A|\leq\delta}\sup_{h>0}\int_A |\mathscr{K}_h g(x)|^2 dx \to 0, \qquad \text{as } \delta \to 0, \tag{4.22}$$

where the first supremum is taken over measurable subsets $A \subset (-1,1)$. We compute

$$\int_A |\mathscr{K}_h g(x)|^2 dx = \|\chi_A \mathscr{K}_h g\|^2_{L^2(\mathbb{R})} = \|\widehat{\chi_A} * \widehat{\mathscr{K}_h g}\|^2_{L^2(\mathbb{R})} \leq \|\widehat{\chi_A}\|^2_{L^2(\mathbb{R})}\|\widehat{\mathscr{K}_h g}\|^2_{L^1(\mathbb{R})},$$

where we have used Young's inequality. Now (4.22) follows from uniform boundedness of $\|\widehat{\mathscr{K}_h g}\|_{L^1(\mathbb{R})}$. We compute

$$\widehat{\mathscr{K}_h g}(\xi) = e^{-2h\xi}\chi_{>0}(\xi)\widehat{\chi_1 g}(\xi)$$

hence

$$\|\widehat{\mathscr{K}_h g}\|_{L^1(\mathbb{R})} \leq \|\widehat{\chi_1 g}\|_{L^1(\mathbb{R})} = \|\widehat{g}\|_{L^1(\mathbb{R})} < \infty,$$

since for $g \in C_0^\infty(-1,1)$ we have $\widehat{\chi_1 g} = \widehat{g} \in L^1(\mathbb{R})$. Thus,

$$\int_A |\mathscr{K}_h g(x)|^2 dx \leq \|\widehat{\chi_A}\|^2_{L^2(\mathbb{R})}\|\widehat{g}\|^2_{L^1(\mathbb{R})} = |A|\|\widehat{g}\|^2_{L^1(\mathbb{R})} \to 0, \qquad \text{as } \delta \to 0$$

□

Since $\mathscr{K}_h$ is a positive operator for any h, we see that so is $K+1$ and hence the inverse of $\frac{1}{2}(K+1)+\epsilon^2$ is well-defined on $L^2(-1,1)$. We now see that, as $h \downarrow 0$

$$v_h = \left(\mathscr{K}_h + \epsilon^2\right)^{-1} q_h \longrightarrow \left(\tfrac{1}{2}(K+1) + \epsilon^2\right)^{-1} p_z =: w, \qquad \text{in } L^2(-1,1) \tag{4.23}$$

Indeed, using the resolvent identity

$$\left(\mathscr{K}_h + \epsilon^2\right)^{-1} - \left(\mathscr{K}_0 + \epsilon^2\right)^{-1} = \left(\mathscr{K}_h + \epsilon^2\right)^{-1} \left(\mathscr{K}_0 - \mathscr{K}_h\right) \left(\mathscr{K}_0 + \epsilon^2\right)^{-1},$$

where $\mathscr{K}_0 = \frac{1}{2}(K+1)$, we conclude that

$$\left(\mathscr{K}_h + \epsilon^2\right)^{-1} g \to \left(\mathscr{K}_0 + \epsilon^2\right)^{-1} g$$

for any $g \in L^2(-1,1)$, since all operators above are uniformly bounded as $h \to 0$. Relation (4.23) then easily follows.

We now observe that because of the convergence (4.21) $w \in L^2(-1,1)$ represents boundary values of an analytic function in the upper half-plane (in fact an H^2 function), hence we can extend w to $\mathbb{H}_+$, more specifically

$$\epsilon^2 w(\zeta) := p_z(\zeta) - \frac{i}{2\pi}\int_{-1}^{1} \frac{w(y)}{\zeta - y} dy, \qquad \zeta \in \mathbb{H}_+$$

defines the extension. But then, from the integral equation for v_h we see that

$$\epsilon^2 v_h(z - ih) = \frac{i}{2\pi(z - \overline{z})} - \frac{i}{2\pi}\int_{-1}^{1} \frac{v_h(y)}{z - y + ih} dy \longrightarrow \epsilon^2 w(z)$$

and thus we conclude

$$|f(z)| \leq \frac{3}{2}\epsilon \frac{w(z)}{\|w\|_{L^2(-1,1)}}$$

It remains to relabel w by $u_{\epsilon,z}$ and conclude the proof. $\square$

4.2.2 Solution of the integral equation

The goal of this section is to find the function u appearing in the upper bound (4.15). Recall that u solves the integral equation

$$Ku + \lambda u = 2p, \qquad \text{on } (-1, 1)$$

where $\lambda = 1 + 2\epsilon^2$, K is the truncated Hilbert transform given by (4.17), and we dropped the subscript from p_z to simplify the notation.

The reason that makes it possible to solve this integral equation, is the spectral representation of K obtained in [43]. Below we state the results of [43]. For $x, \zeta \in (-1, 1)$ let

$$\sigma(x, \zeta) = \frac{\exp\left\{\frac{i}{2\pi} L(x) L(\zeta)\right\}}{\pi\sqrt{(1 - x^2)(1 - \zeta^2)}}, \qquad L(x) = \ln\left(\frac{1 + x}{1 - x}\right) \tag{4.24}$$

Theorem 4.4. *The formulae*

$$f(x) = \int_{-1}^{1} g(\zeta)\sigma(x, \zeta) d\zeta, \qquad g(\zeta) = \int_{-1}^{1} f(x)\overline{\sigma(x, \zeta)} dx$$

are inversion formulae which represent isometries from the space $L^2(-1, 1)$ to itself.

Theorem 4.5. *If $f(x)$ corresponds to $g(\zeta)$, then $Kf(x)$ corresponds to $\zeta g(\zeta)$ (w.r.t. the above transformation).*

Remark 4.4. *Integrals are understood in a limiting sense as the Fourier transform of an L^2 function, namely as the limit of $\int_{-1+\delta}^{1-\delta}$ when $\delta \downarrow 0$ in the sense of $L^2(-1, 1)$.*

Let $(\cdot, \cdot)$ denote the inner product of $L^2(-1, 1)$, using the stated result we can write

$$u(x) = \int_{-1}^{1} (u, \sigma(\cdot, \zeta))\, \sigma(x, \zeta) d\zeta, \qquad p(x) = \int_{-1}^{1} (p, \sigma(\cdot, \zeta))\, \sigma(x, \zeta) d\zeta$$

$$Ku(x) = \int_{-1}^{1} \zeta\,(u, \sigma(\cdot,\zeta))\,\sigma(x,\zeta)d\zeta$$

then the integral equation gives

$$(\lambda + \zeta)\,(u, \sigma(\cdot,\zeta)) = 2\,(p, \sigma(\cdot,\zeta))$$

and therefore

$$u(x) = \int_{-1}^{1} \frac{2\,(p, \sigma(\cdot,\zeta))\,\sigma(x,\zeta)}{\lambda + \zeta} d\zeta \tag{4.25}$$

Let us compute $(p, \sigma(\cdot,\zeta))$ explicitly by changing variables $y = \tanh(t)$, in which case $L(y) = 2t$. We obtain

$$(p, \sigma(\cdot,\zeta)) = \frac{i}{2\pi^2\sqrt{1-\zeta^2}} \int_{\mathbb{R}} \frac{e^{-itL(\zeta)/\pi}}{\sinh t - \overline{z}\cosh t} dt$$

let $\alpha \in \mathbb{C}$ be such that $\coth\alpha = \overline{z}$, then

$$(p, \sigma(\cdot,\zeta)) = -\frac{i\sinh\alpha}{2\pi^2\sqrt{1-\zeta^2}} \int_{\mathbb{R}} \frac{e^{-itL(\zeta)/\pi}}{\cosh(t-\alpha)} dt$$

We observe that

$$\coth\alpha = \frac{e^{2\alpha}+1}{e^{2\alpha}-1} = \frac{w+1}{w-1}, \qquad\qquad w = e^{2\alpha}.$$

The fractional linear map $w \mapsto \frac{w+1}{w-1}$ maps lower half-plane into the upper half-plane and therefore, $w = w(\overline{z})$ is in the upper half-plane. Hence, $\mathfrak{Im}\alpha \in (0, \pi/2)$. It follows that there are no zeros of $\cosh(t-\alpha)$ in the strip bounded by $\mathbb{R}$ and $\mathfrak{Im}t = \mathfrak{Im}\alpha$. Taking into account that

$$\lim_{R\to\infty} \int_{0}^{\mathfrak{Im}\alpha} \frac{e^{-i(i\tau \pm R)L(\zeta)/\pi}}{\cosh(i\tau \pm R - \alpha)} i d\tau = 0$$

we conclude that

$$(p, \sigma(\cdot,\zeta)) = -\frac{ie^{-i\alpha L(\zeta)/\pi}\sinh\alpha}{2\pi^2\sqrt{1-\zeta^2}} \int_{\mathbb{R}} \frac{e^{-itL(\zeta)/\pi}}{\cosh(t)} dt = -\frac{ie^{-i\alpha L(\zeta)/\pi}\sinh\alpha}{2\pi\sqrt{1-\zeta^2}\cosh(L(\zeta)/2)}.$$

Simplifying we obtain

$$(p, \sigma(\cdot, \zeta)) = -\frac{i}{2\pi} e^{-i\alpha L(\zeta)/\pi} \sinh \alpha.$$

We now use this formula in (4.25)

$$u(x) = -\frac{i \sinh \alpha}{\pi^2 \sqrt{1-x^2}} \int_{-1}^{1} \frac{e^{iL(\zeta)[L(x)-2\alpha]/2\pi}}{(\lambda+\zeta)\sqrt{1-\zeta^2}} d\zeta,$$

once again changing the variables $\zeta = \tanh s$ we obtain

$$u(x) = -\frac{i \sinh \alpha}{\pi^2 \sqrt{1-x^2}} \int_{\mathbb{R}} \frac{e^{is[L(x)-2\alpha]/\pi}}{\sinh s + \lambda \cosh s} ds.$$

Let $\beta = \beta(\lambda)$ be such that $\coth \beta = \lambda$, then $\beta(\lambda) > 0$ and $\beta(\lambda) \to +\infty$, as $\lambda \to 1$. Now

$$u(x) = -\frac{i \sinh \alpha \sinh \beta}{\pi^2 \sqrt{1-x^2}} \int_{\mathbb{R}} \frac{e^{is[L(x)-2\alpha]/\pi}}{\cosh(s+\beta)} ds = -\frac{i \sinh \alpha \sinh \beta}{\pi \sqrt{1-x^2}} \frac{e^{-i\beta[L(x)-2\alpha]/\pi}}{\cosh(L(x)/2-\alpha)}.$$

Next we simplify

$$\cosh\left(\tfrac{L(x)}{2} - \alpha\right) = \cosh\left(\tfrac{L(x)}{2}\right) \cosh \alpha - \sinh\left(\tfrac{L(x)}{2}\right) \sinh \alpha = \frac{\cosh \alpha - x \sinh \alpha}{\sqrt{1-x^2}}.$$

Thus we obtain the final answer

$$u(x) = \frac{i \sinh \beta}{\pi(x-\overline{z})} e^{-i\frac{\beta}{\pi}[L(x)-2\alpha]} = 2p(x) \sinh(\beta) e^{-i\frac{\beta}{\pi}[L(x)-2\alpha]}, \tag{4.26}$$

where (with ln denoting the principal branch of logarithm)

$$\beta = \tfrac{1}{2} \ln\left(1 + \epsilon^{-2}\right), \qquad \alpha = \tfrac{1}{2} \ln \frac{\overline{z}+1}{\overline{z}-1}.$$

We see that

$$\|u\|_{L^2(-1,1)} = 2\|p\|_{L^2(-1,1)} \sinh(\beta) e^{-2\frac{\beta}{\pi}\mathfrak{Im}\alpha}.$$

Because $\mathfrak{Re} L(z) = 2\mathfrak{Re}\alpha$ and $e^{\beta} \sim \epsilon^{-1}$ as $\epsilon \to 0$, we find that

$$\epsilon \frac{u(z)}{\|u\|_{L^2(-1,1)}} = \epsilon \frac{p(z) e^{\frac{\beta}{\pi} \Im\mathfrak{m} L(z)}}{\|p\|_{L^2(-1,1)}} \sim \frac{p(z) \epsilon^{\frac{1}{\pi}[\pi - \Im\mathfrak{m} L(z)]}}{\|p\|_{L^2(-1,1)}} =: B, \qquad \text{as } \epsilon \to 0 \tag{4.27}$$

Since $\frac{1+z}{1-z} \in \mathbb{H}_+$ we see that $\pi - \arg \frac{1+z}{1-z} = -\arg \frac{z+1}{z-1}$ and with $z = z_r + iz_i$ we obtain

$$B = \frac{\epsilon^{-\frac{1}{\pi} \arg \frac{z+1}{z-1}}}{2\sqrt{z_i}\sqrt{\arctan \frac{z_r+1}{z_i} - \arctan \frac{z_r-1}{z_i}}} \tag{4.28}$$

when $\epsilon < 1$ we can replace the asymptotic equivalence to B in (4.27) by $\leq \sqrt{2}B$ and conclude the proof of (4.12). To prove the optimality of this upper bound we consider the function

$$M(\zeta) = \epsilon \frac{p(\zeta)}{\|p\|_{L^2(-1,1)}} e^{\frac{i}{\pi} \ln \epsilon \ln \frac{1+\zeta}{1-\zeta}}, \qquad \zeta \in \mathbb{H}_+$$

clearly this is an analytic function in the upper half-plane and belongs to H^2, $\|M\|_{L^2(-1,1)} = \epsilon$ and

$$\|M\|_{H^2}^2 = \epsilon^2 + \frac{\|p\|^2_{L^2((-1,1)^C)}}{\|p\|^2_{L^2(-1,1)}} = \epsilon^2 - 1 + \frac{\pi}{\arctan \frac{z_r+1}{z_i} - \arctan \frac{z_r-1}{z_i}} \leq c,$$

where $c > 0$ is independent of ϵ, therefore W is an admissible function. Further,

$$|M(z)| = B$$

that is, $M(\zeta)$ attains the bound (4.12) up to a constant independent of ϵ.

CHAPTER 5

EXTRAPOLATION OF THE COMPLEX ELECTROMAGNETIC PERMITTIVITY FUNCTION

5.1 Introduction

Properties of linear, time-invariant, causal systems are characterized by functions analytic in a complex half-plane. Examples include transfer functions of digital filters [34], complex impedance and admittance functions of electrical circuits [9], complex magnetic permeability and complex dielectric permittivity functions [46, 28]. Arising from the world of real-valued fields, these functions also possess specific symmetries. The underlying mathematical structure is the Fourier (or Laplace) transforms of real-valued functions that vanish on negative semi-axis. More generally, the analyticity arises from the analyticity of resolvents of linear operators, while their symmetries reflect that these operators are very often real and self-adjoint.

In a typical situation we can measure the values of such analytic functions on a compact subset of the boundary of their half-plane of analyticity. The real and imaginary parts of such a function are not independent, but are Hilbert transforms of one another. In the context of the complex dielectric permittivity this fact is expressed by the Kramers-Kronig relations [45]. It is therefore tempting to use these relations in order to reconstruct the analytic functions from their measured values. Unfortunately, such a reconstruction problem is ill-posed (e.g. [52]), and one needs to place additional constraints on the set of admissible analytic functions for the extrapolation problem to be mathematically well-posed.

In this work we propose a physically natural regularization that implies that the underlying analytic functions can be analytically continued into a larger complex half-plane. In that case, the idea is to exploit the fact that complex analytic functions possess a large degree of rigidity, being uniquely determined by values at any infinite set of points in any finite interval. This rigidity also implies that even very small measurement errors will produce data *mathematically* inconsistent with values of an analytic function. In such cases the least squares approach [18, 17, 12, 13] that treats all data points equally is the most natural one. In the first part of this chapter we prove that the least squares problem has a unique solution, that yields a mathematically stable extrapolant. We show that the minimizer must be a rational function and derive the necessary and sufficient conditions for its optimality.

Previous chapters show that surprisingly, the space of analytic functions is also "flexible" in the sense that the data can often be matched up to a given precision by two physically admissible functions that are very different away from the interval, where the data is available. The second part of this chapter quantifies this phenomenon by giving an optimal upper bound on the possible discrepancy between any two approximate extrapolants. We show that this discrepancy behaves like a power law: $\Delta_z(\epsilon) \approx \epsilon^\gamma$ for some $\gamma \in (0, 1)$, where γ can be computed numerically from an integral equation. We also give an upper bound on γ in terms of an analytical expression, which is checked numerically to be in excellent agreement with γ.

5.2 Preliminaries

When the electromagnetic wave passes through the material the incident electric field $\boldsymbol{E}(\boldsymbol{x}, t)$ interacts with charge carriers inside the matter. We assume that the induced polarization field $\boldsymbol{P}(\boldsymbol{x}, t)$ depends on the incident electric field linearly and locally. This is expressed by the constitutive relation

$$\boldsymbol{P}(\boldsymbol{x}, t) = \int_0^{+\infty} \boldsymbol{E}(\boldsymbol{x}, t-s) a(s) ds, \tag{5.1}$$

indicating that the polarization field depends only on the past values of $\boldsymbol{E}(\boldsymbol{x}, t)$. The function $a(t)$ is called the impulse response or a memory kernel, which is assumed to decay exponentially. Its decay rate, $a(t) \sim e^{-t/\tau_0}$, $t \to \infty$, indicates how fast the system "forgets" the past values of the incident field. The parameter $\tau_0 > 0$ is called the relaxation time, which can be measured for many materials.

Let

$$a_0(t) = \begin{cases} a(t), & t \geq 0, \\ 0, & t < 0. \end{cases}$$

Then we can extend the integral in (5.1) to the entire real line and apply the Fourier transform to convert the convolution into a product:

$$\hat{\boldsymbol{P}}(\boldsymbol{x}, \zeta) = \hat{a}_0(\zeta) \hat{\boldsymbol{E}}(\boldsymbol{x}, \zeta),$$

where

$$\hat{f}(\zeta) = \int_{\mathbb{R}} f(x) e^{i\zeta x} dx$$

is the Fourier transform. In physics, the function $\varepsilon(\zeta) = \varepsilon_0 + \hat{a}_0(\zeta)$ is called the complex dielectric permittivity of the material, where ε_0 is the dielectric permittivity of the vacuum. Mathematically, it is more convenient to study $\hat{a}_0(\zeta)$, rather than $\varepsilon(\zeta)$. From now on, we will denote

$$f(\zeta) = \hat{a}_0(\zeta),$$

and refer to it as the complex electromagnetic permittivity , in a convenient abuse of terminology. Let us recall the well-known analytic properties of isotropic complex electromagnetic permittivity as a function of frequency ζ of the incident electromagnetic wave [46, 28]:

(a) $\overline{f(\zeta)} = f(-\bar{\zeta})$;

(b) $f(\zeta)$ is analytic in the complex upper half-plane $\mathbb{H}_+$;

(c) $\mathfrak{Im}\, f(\zeta) > 0$ for ζ in the first quadrant $\mathfrak{Re}(\zeta) > 0$, $\mathfrak{Im}(\zeta) > 0$;

(d) $f(\zeta) = -A\zeta^{-2} + O(\zeta^{-3})$, $A > 0$ as $\zeta \to \infty$.

Property (a) expresses the fact that physical fields are real. Property (b) is the consequence of the causality principle i.e. independence of $\boldsymbol{P}(\boldsymbol{x}, t)$ of the future values of $E(\boldsymbol{x}, \tau)$, $\tau > t$. Property (c) comes from the fact that the electromagnetic energy gets absorbed by the material as the electromagnetic wave passes through. Property (d) is called the plasma limit, where at very high frequencies the electrons in the medium may be regarded as free. Complex analytic functions with properties (a)–(d) and their variants, are ubiquitous in physics. The complex impedance of electrical circuits as a function of frequency has similar properties. Yet another example is the dependence of effective moduli of composites on the moduli of its constituents [6, 53]. These functions appear in areas as diverse as optimal design problems [47, 48] and nuclear physics [51, 50, 10]. Typically[1] only the values of such a function on a real line can be measured. In the case of complex electromagnetic permittivity the measurements are usually made either on a finite interval or at a discrete set of frequencies. However, the requirements (a)–(d) do not place any regularity requirements on $f(\zeta)$, when ζ is real. For example, the function

$$f(\zeta) = \frac{1}{z^2 - \zeta^2}, \quad z > 0$$

satisfies properties (a)–(d), but blows up at the frequency $z > 0$. We exclude such examples by assuming that the memory kernel $a(t)$ decays exponentially

[1]In the context of viscoelastic composites measurements corresponding to values of $f(\zeta)$ in the upper half-plane are also possible.

with relaxation time $\tau_0 > 0$. In this case $f(\zeta)$ will have an analytic extension into the larger half-plane

$$\mathbb{H}_h = \{\zeta \in \mathbb{C} : \mathfrak{Im}\,\zeta > -h\}, \tag{5.2}$$

where $h = 1/\tau_0 > 0$. In general, the analytic continuation of $f(\zeta)$ need not have positive imaginary part when $\mathfrak{Im}(\zeta) > -h$ and $\mathfrak{Re}(\zeta) > 0$. For example, $f(\zeta) = -\frac{\zeta+i}{(\zeta+3i)^3}$ satisfies conditions (a)–(d), is analytic in $\mathbb{H}_3$, but $\mathfrak{Im}\,f(x - i\epsilon)$ takes negative values for any $\epsilon \in (0, 3)$ for some $x > 0$. We therefore make an additional regularizing assumption that positivity property (c) continues to hold in the larger half-plane $\mathbb{H}_h$. In fact, under the additional assumption that $f'(0) \neq 0$, the positivity condition can be guaranteed in some possibly smaller half-plane $\mathbb{H}_{h'}$, $0 < h' \leq h$ (see Proposition A.1). The quantity $-if'(0) > 0$ is the analog of the Elmore delay (at zero frequency) in electronic circuits [25]. Thus, the class of physically admissible complex dielectric permittivity functions is narrowed in a natural way to one of the classes $\mathcal{K}_h$ defined as follows.

Definition 5.1. *A complex analytic function $f : \mathbb{H}_h \to \mathbb{C}$ belongs to the class $\mathcal{K}_h$ if it has the following list of physically justified properties.*

(S) Symmetry: $\overline{f(\zeta)} = f(-\overline{\zeta})$;

(P) Passivity: $\mathfrak{Im}(f(\zeta)) > 0$, when $\mathfrak{Im}(\zeta) > -h, \quad \mathfrak{Re}(\zeta) > 0$;

(L) Plasma limit: $f(\zeta) = -A\zeta^{-2} + O(\zeta^{-3})$, $A > 0$ as $\zeta \to \infty$.

Functions in the set $\mathcal{K}_h$ are closely related to an important class of functions called Stieltjes functions .

Definition 5.2. *A non-constant function analytic in the complex upper half-plane is said to be of Stieltjes class $\mathfrak{S}$ if its imaginary part is positive, and it is analytic on the negative real axis, where it takes real and nonnegative values. Such functions together with all nonnegative constant functions form the Stieltjes class $\mathfrak{S}$.*

It is well-known that a Stieltjes function $F(\omega)$ is uniquely determined by a constant $\rho \geq 0$ and a Borel-regular positive measure σ by the representation

$$F(\omega) = \rho + \int_0^\infty \frac{d\sigma(\lambda)}{\lambda - \omega}, \qquad \int_0^\infty \frac{d\sigma(\lambda)}{\lambda + 1} < +\infty. \tag{5.3}$$

The measure σ is often referred to as *spectral measure* [45, 46]. Let us show that every function $f \in \mathcal{K}_h$ can be represented by

$$f(\zeta) = F((\zeta + ih)^2), \quad F \in \mathfrak{S}, \quad \rho = 0, \quad \int_0^\infty d\sigma(\lambda) = A < +\infty, \tag{5.4}$$

where σ is the spectral measure for $F(\omega)$.

For any $f \in \mathcal{K}_h$ consider the function $g(z) = f(z - ih)$ which is analytic in $\mathbb{H}_+$, $\overline{g(z)} = g(-\overline{z})$, $\mathfrak{Im} g > 0$ in the first quadrant and $g(z) \sim -Az^{-2}$ as $z \to \infty$ for some $A > 0$.

Unfolding the first quadrant in the z-plane into the upper half-plane in the ω-plane via $\omega = z^2$ we obtain a function $F(\omega) = g(\sqrt{\omega})$, which is analytic in $\mathbb{H}_+$ and has a positive imaginary part there. The symmetry of g implies that it is real on $i\mathbb{R}_{>0}$, but then F is real on $\mathbb{R}_{<0}$. Clearly, analyticity of g on $i\mathbb{R}_{>0}$ implies that of F on $\mathbb{R}_{<0}$. The plasma limit assumption implies that $F(-x) \geq 0$ for x large enough, which is enough to conclude that F is a Stieltjes function (see the proof of [44, Theorem A.4]). Thus, F admits the representation (5.3). But then, the asymptotic relation $F(\omega) \sim -A\omega^{-1}$ as $\omega \to \infty$ implies that $\rho = 0$ and $\int_0^\infty d\sigma(\lambda) = A < \infty$. Thus, $f(\zeta) = g(\zeta + ih) = F((\zeta + ih)^2)$. Conversely, if f is given by (5.4) then it is straightforward to check that it satisfies all the required properties of class $\mathcal{K}_h$.

5.3 Main results

Let us assume that the experimentally measured data $f_{\text{exp}}(\zeta)$ is known on a band of frequencies $\Gamma = [0, B]$. The unavoidable random noise makes the measured values mathematically inconsistent with the analyticity of the complex dielectric permittivity function. The standard way to deal with the noise is to use the "least squares" approach by looking for a function $f \in \mathcal{K}_h$ that is closest to the experimental data $f_{\rm exp}(\zeta)$ in the L^2 norm on Γ. Thus, after rescaling the frequency interval Γ to the interval $[0,1]$ we arrive at the following least squares problem

$$\inf_{f \in \mathcal{K}_h} \|f - f_{\rm exp}\|_{L^2(0,1)}. \tag{5.5}$$

One approach [15, 16] is to ignore the positivity requirement, while retaining the spectral representation (5.4). The resulting problem constrains f to a vector space, but becomes ill-posed. It is then solved by Tikhonov regularization techniques. Unfortunately, such an approach cannot guarantee that the solution possesses the required positivity.

We will see in Section 5.4 that the positivity property of functions in $\mathcal{K}_h$ plays a regularizing role, making the least squares problem (5.5) well-posed. So the solution to (5.5) exists, is unique and lies in the closure $\mathscr{S}_h = \overline{\mathcal{K}_h}$ with respect to the topology of the space $H(\mathbb{H}_h)$ of analytic functions on $\mathbb{H}_h$. We then characterize the set $\mathscr{S}_h$ and obtain stability of analytic continuation: if $\{f_n\}, f \subset \mathscr{S}_h$ are such that $f_n \to f$ in $L^2(0,1)$, then $f_n \to f$ as $n \to \infty$ in $H(\mathbb{H}_h)$. In Section 5.4.2 we study the properties of the minimizer of (5.5).

Even though we have established well-posedness and stability of the extrapolation problem, the above-mentioned results are not quantitative, since they do not give rates of convergence of the extrapolation errors. The Figure 5.1 below (corresponding to a small value of the natural regularization parameter) shows two perfectly admissible functions in $\mathcal{K}_h$ that are extremely close on $[0,1]$, but diverge almost immediately beyond the data window. It suggests that in practice the degree of mathematical well-posedness needs to be quantified. While there is no shortage of proposed algorithms for extrapolation of experimental data in the vast literature on the subject, there is no mathematically rigorous quantitative analysis of uncertainty inherent in any such extrapolation procedure.

We therefore consider two different extrapolants f and g in $\mathcal{K}_h$, constructed by some algorithms, that match the same experimental data on the frequency

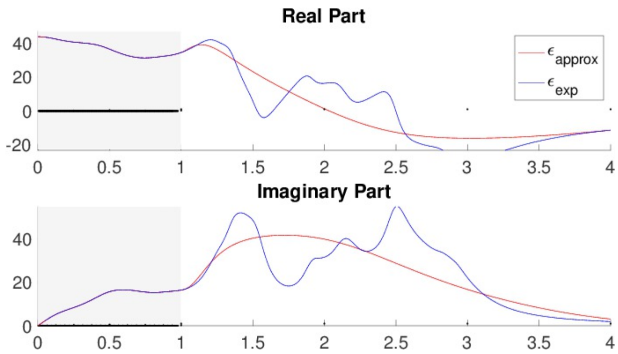


Figure 5.1: Apparent ill-posedness of the extrapolation process.

band $[0, 1]$ with relative precision ϵ. As Figure 5.1 shows there is uncertainty inherent in such extrapolation algorithms. Without discussing specific algorithms we would like to examine theoretical feasibility of such a procedure: how much f and g can differ at a given point $z > 1$?

Remark 5.1. *All the results of this chapter hold for* $z \in \overline{\mathbb{H}}_+ \backslash [-1, 1]$, *however we concentrate on the case of practical interest and assume* $z > 1$.

Recall that to a function $f \in \mathcal{K}_h$ corresponds a measure σ_f via (5.4), such that

$$\|\sigma_f\|_* := \int_0^\infty \frac{d\sigma_f(\lambda)}{\lambda + 1}$$

is finite and can be interpreted as the "total norm" of f. The relative extrapolation error at the point z is then given by

$$\Delta_{z,h}(\epsilon) = \sup\left\{ \frac{|f(z) - g(z)|}{\max(\|\sigma_f\|_*, \|\sigma_g\|_*)} : (f, g) \in \mathcal{K}_h \quad \text{and} \quad \frac{\|f - g\|_{L^2(0,1)}}{\max(\|\sigma_f\|_*, \|\sigma_g\|_*)} \le \epsilon \right\}. \tag{5.6}$$

Observe that the admissible functions in Δ are those pairs that are close to each other on $(0, 1)$ in L^2 sense, relative[2] to their "total norms". The two fundamental questions regarding the extrapolation procedure are

[2]Note that it is important to consider relative errors, because otherwise the condition $\|f - g\|_{L^2(0,1)} \le \epsilon$ allows pairs (f, g) that have small L^2-norms and are not at all relatively close to each other on $(0, 1)$.

1. The well-posedness: Is it true that $\Delta_{z,h}(\epsilon) \to 0$ as $\epsilon \to 0$?

2. What is the exact convergence rate of $\Delta_{z,h}(\epsilon)$ to 0?

The first insight is the realization that, in fact, these questions are about the difference $\phi = f - g$, rather than the pair (f, g). The difference ϕ has the same spectral representation (5.3), (5.4) as f and g, except the spectral measure is no longer positive. Our next observation is that the asymptotic behavior of $\Delta_{z,h}(\epsilon)$, as $\epsilon \to 0$ is insensitive to certain restrictions on the spectral measures σ, as long as the set of admissible measures is dense (in the weak-* topology) in the space of measures (5.3). For example, we may work only with absolutely continuous measures with densities in $L^2(0, +\infty)$, permitting us to use the theory of Hardy functions $H^2(\mathbb{H}_h)$ and Hilbert space methods to obtain exact asymptotic behavior of $\Delta_{z,h}(\epsilon)$. Namely let

$$A_{z,h}(\epsilon) = \sup \left\{ |f(z)| \ : f \in H^2(\mathbb{H}_h) \text{ and } \|f\|_{H^2(\mathbb{H}_h)} \le 1, \ \|f\|_{L^2(-1,1)} \le \epsilon \right\} \tag{5.7}$$

We will prove that (cf. Corollary 5.2) $A_h(\epsilon) \lesssim \Delta_h(\epsilon) \lesssim A_{h'}(\epsilon)$ for any $h' \in (0, h)$ with implicit constants depending only on h and h' (where we suppressed the dependence on z from the notation). We prove $\epsilon^{r_1} \lesssim A_h(\epsilon) \lesssim \epsilon^{r_0}$, with explicit $r_0, r_1 \in (0, 1)$ depending on z, h (see Appendix A.2). In particular this answers positively to the first question, with an explicit power law estimate $\Delta_{z,h}(\epsilon) \lesssim \epsilon^{r_0}$. To answer the second question, let

$$\gamma(z, h) = \varliminf_{\epsilon \to 0} \frac{\ln A_{z,h}(\epsilon)}{\ln \epsilon}. \tag{5.8}$$

From the above bounds we in particular obtain

$$\gamma(z, h') \le \varliminf_{\epsilon \to 0} \frac{\ln \Delta_{z,h}(\epsilon)}{\ln \epsilon} \le \gamma(z, h), \qquad \forall h' \in (0, h) \tag{5.9}$$

It is clear that continuity of $\gamma(z, h)$ in h will imply that $\Delta_{z,h}(\epsilon)$ also has power law exponent $\gamma(z, h)$. In fact, the same conclusion follows under continuity of $\gamma(z, h)$ in z, because

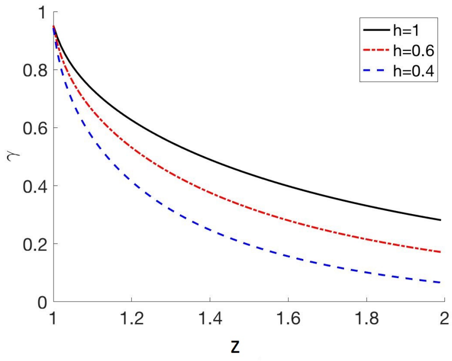


Figure 5.2: Power law exponent γ as a function of z for several values of h.

$$\gamma(z, h') \geq \gamma\left(\frac{h}{h'}z, h\right). \tag{5.10}$$

To prove inequality (5.10), let $f^*_{\epsilon,z,h'}(\zeta)$ be the maximizer function for $A_{z,h'}(\epsilon)$ and consider the function $g(\zeta) = \sqrt{\frac{h'}{h}} f^*\left(\frac{h'}{h}\zeta\right)$. Note that $\|g\|_{H^2(\mathbb{H}_h)} = \|f^*\|_{H^2(\mathbb{H}_{h'})} = 1$ and $\|g\|_{L^2(-1,1)} \leq \|f^*\|_{L^2(-1,1)} = \epsilon$. Therefore, g is an admissible function for $A_{\frac{hz}{h'},h'}(\epsilon)$, hence

$$A_{\frac{hz}{h'},h'}(\epsilon) \geq g\left(\frac{hz}{h'}\right) = \sqrt{\frac{h'}{h}} f^*(z) = \sqrt{\frac{h'}{h}} A_{z,h'}(\epsilon),$$

which concludes the proof. In particular, inequalities (5.9) and (5.10) imply that $\gamma(z, h)$ is a non-increasing function of z.

Numerical computations of $\gamma(z, h)$ shown in Figure 5.2 indicate that it is indeed a continuous function of z. In Appendix A.2 we prove that $\gamma(z, h)$ is also a non-decreasing function of h, satisfying $\gamma(z, h) \in (0, 1)$ for any $h > 0$ and that $\lim_{h\to 0^+} \gamma(z, h) = 0$. Figure 5.2 also shows how rapidly $\gamma(z, h)$ decays to 0, as z moves further away from Γ for several values of h. The larger the regularization parameter h is, the better behaved is the extrapolation problem.

Further, Theorem 2.1 shows that γ can be computed from a linear integral equation of Fredholm type. Namely, let

$$(\mathscr{K}u)(\zeta) = \int_{-1}^{1} p_x(\zeta)u(x)dx, \qquad p_z(\zeta) = \frac{i}{2\pi(\zeta - \overline{z} + 2ih)}, \tag{5.11}$$

and let $u = u_{\epsilon,z,h}$ be the unique solution of the integral equation

$$\mathscr{K}u + \epsilon^2 u = p_z, \tag{5.12}$$

then (cf. Section 5.7)

$$\gamma(z,h) = 1 - \lim_{\epsilon \to 0^+} \frac{\ln \|u\|_{L^2(-1,1)}}{\ln(1/\epsilon)}. \tag{5.13}$$

Finally, in Section 5.7.1 we give an upper bound on γ in terms of an analytical expression γ_1 (cf. (5.48)), which is checked numerically to be in excellent agreement with γ, when $h > 0.6$ (see Figure 5.5).

5.4 The least squares problem

5.4.1 Existence and uniqueness

Let us begin by examining the existence and uniqueness questions before identifying the necessary and sufficient conditions that the minimizer has to satisfy. Let $f_n \in \mathcal{K}_h$ be a minimizing sequence in (5.5). Then it has to be bounded in the $L^2(0,1)$ norm. We will show that this implies existence of a subsequence converging uniformly on compact subsets of $\mathbb{H}_h$ to an analytic function. In general, this limit does not need to be in $\mathcal{K}_h$. We will, therefore, need to characterize the closure $\overline{\mathcal{K}_h}$ of $\mathcal{K}_h$.

We recall that for an open subset $G \subset \mathbb{C}$ convergence in the space $H(G)$ of analytic functions on G is uniform convergence on compact subsets. We also recall that a family of functions in $H(G)$ is called normal, if every sequence has a convergent in $H(G)$ subsequence. In other words, normal families of functions are exactly the precompact subsets in $H(G)$.

Theorem 5.1.

(i) The closure of $\mathcal{K}_h$ in $H(\mathbb{H}_h)$ is $\mathscr{S}_h = \{f(\zeta) = F((\zeta + ih)^2) : F \in \mathfrak{S}\}$.

(ii) For any $M > 0$ the family of functions $\mathscr{S}_h^M = \{f \in \mathscr{S}_h : \|f\|_{L^2(0,1)} \leq M\}$ is normal.

Proof. The proof is based on the representation (5.3), where we interpret the measure σ as an element of the Banach space $\mathscr{B}^*$ dual to

$$\mathscr{B} = \left\{\phi \in C([0, +\infty)) : \lim_{\lambda\to\infty} \lambda\phi(\lambda) = 0\right\},$$

with the norm

$$\|\phi\|_{\mathscr{B}} = \max_{\lambda\geq 0}(\lambda + 1)|\phi(\lambda)|.$$

If we define the action of the measure σ on $\phi \in \mathscr{B}$ by

$$\langle\phi, \sigma\rangle = \int_0^\infty \phi(\lambda)d\sigma(\lambda),$$

then

$$\|\sigma\|_* = \int_0^\infty \frac{d\sigma(\lambda)}{\lambda + 1}, \tag{5.14}$$

when the measure σ is nonnegative.

The conclusion of the theorem then follows easily from the fundamental estimate in the lemma below.

Lemma 5.1. *There exists $c_h > 0$ and $C_h > 0$ depending only on h, such that for every $f \in \mathscr{S}_h$*

$$c_h\|f\|_{L^2(0,1)} \leq \rho + \|\sigma\|_* \leq C_h\|f\|_{L^2(0,1)}$$

Proof. Let us start by proving the second inequality. Hölder's inequality implies

$$\|f\|_{L^2(0,1)} \geq \left(\int_0^1 |\mathfrak{Re}(f)|^2 d\zeta\right)^{\frac{1}{2}} \geq \left|\int_0^1 \mathfrak{Re}(f)d\zeta\right|.$$

Applying Fubini's theorem we then compute

$$\int_0^1 \mathfrak{Re}(f)d\zeta = \rho+\int_0^1\int_0^\infty \mathfrak{Re}\left(\frac{1}{\lambda - (\zeta + ih)^2}\right)d\sigma(\lambda)d\zeta = \rho+\int_0^\infty \varphi(\sqrt{\lambda})\frac{d\sigma(\lambda)}{\lambda + 1},$$

where

$$\varphi(x) = \frac{x^2 + 1}{4x}\ln\left(1 + \frac{4x}{(x - 1)^2 + h^2}\right).$$

Note that $\varphi(x) > 0$ for $x > 0$, and because $\ln(1+x) \sim x$ as $x \to 0$ we get

$$\lim_{x\to 0} \varphi(x) = \frac{1}{1+h^2} > 0, \qquad \lim_{x\to\infty} \varphi(x) = 1 > 0.$$

Thus $\inf_{[0,\infty)} \varphi(x) = c_h > 0$, which implies the desired estimate.

Let us now turn to the first inequality. Again, by Hölder's inequality

$$\begin{aligned}\frac{1}{2}\|f\|^2_{L^2(0,1)} - \rho^2 \leq & \int_0^1 \left(\int_0^\infty \frac{d\sigma(\lambda)}{|\lambda - (\zeta + ih)^2|} \right)^2 d\zeta \leq \\ \leq & \int_0^\infty \frac{d\sigma(\lambda)}{\lambda+1} \cdot \int_0^1 \int_0^\infty \frac{\lambda+1}{|\lambda - (\zeta+ih)^2|^2} d\sigma(\lambda) d\zeta = \\ = & \|\sigma\|_* \cdot \int_0^\infty \psi(\lambda) d\sigma(\lambda),\end{aligned}$$

where

$$\psi(\lambda) = \int_0^1 \frac{\lambda+1}{|\lambda - (\zeta+ih)^2|^2} d\zeta = \frac{\varphi(\sqrt{\lambda})}{\lambda+h^2} + \frac{\lambda+1}{4h(\lambda+h^2)} \left(\arctan \tfrac{\sqrt{\lambda}+1}{h} - \arctan \tfrac{\sqrt{\lambda}-1}{h} \right).$$

Note that $(\lambda+1)\psi(\lambda)$ is bounded in $[0,\infty)$, because φ is a bounded function and the difference of arctangents can be bounded by $\frac{2h}{\lambda-1}$ for $\lambda > 1$, by the mean value theorem. But then the desired inequality follows from the estimate

$$\int_0^\infty \psi(\lambda) d\sigma(\lambda) \leq C_h \int_0^\infty \frac{d\sigma(\lambda)}{\lambda+1} = C_h \|\sigma\|_*.$$

□

Obviously $\mathcal{K}_h \subset \mathscr{S}_h$ and Theorem 5.1 follows from the next lemma.

Lemma 5.2.

(i) $\mathscr{S}_h$ *is closed in* $H(\mathbb{H}_h)$.

(ii) $\mathscr{S}_h \subset \overline{\mathcal{K}_h}$

Proof. (i) Let $\{f_n\} \subset \mathscr{S}_h$ be a sequence such that $f_n \to f$ in $H(\mathbb{H}_h)$. Then according to Lemma 5.1 the sequences $\{\rho_n\} \subset \mathbb{R}$ and $\{\sigma_n\} \subset \mathscr{B}^*$ are bounded. By the Banach-Alaoglu theorem the closed unit ball in $\mathscr{B}^*$ is compact in the weak-* topology. Note that the space $\mathscr{B}$ is separable. Indeed, it is well known that the space $C_0[0,\infty)$ of continuous functions on $[0,\infty)$ that vanish at infinity, equipped with the sup norm, is separable (e.g. by the Stone–Weierstrass theorem for locally compact spaces). If $\{\phi_n(\lambda)\}$ is a dense subset of $C_0[0,\infty)$, then $\{\frac{\phi_n(\lambda)}{\lambda+1}\}$ is dense in $\mathscr{B}$. Separability of $\mathscr{B}$ then implies that the weak-* topology on the closed unit ball of $\mathscr{B}^*$ is metrizable (cf. [64]) and hence we can extract a convergent subsequence from $\{\sigma_n\}$.

Thus, there exist subsequences (which we do not relabel) $\rho_n \to \rho$ and $\sigma_n \overset{*}{\rightharpoonup} \sigma$ weakly-* in $\mathscr{B}^*$. Let us write

$$f_n(\zeta) = \rho_n + \|\sigma_n\|_* + \int_0^\infty G(\zeta,\lambda)d\sigma_n(\lambda),$$

where

$$G(\zeta,\lambda) = \frac{1}{\lambda-(\zeta+ih)^2} - \frac{1}{\lambda+1} = \frac{1+(\zeta+ih)^2}{(\lambda-(\zeta+ih)^2)\,(\lambda+1)}.$$

It is now evident that $G(\zeta,\cdot) \in \mathscr{B}$ for each fixed $\zeta \in \mathbb{H}_h$. Upon extracting convergent subsequence of the bounded sequence $\{\|\sigma_n\|_*\}$, with limit denoted by a, we obtain that

$$f(\zeta) = \lim_{n\to\infty} f_n(\zeta) = \rho + a + \int_0^\infty G(\zeta,\lambda)d\sigma(\lambda) = \rho + a - \|\sigma\|_* + \int_0^\infty \frac{d\sigma(\lambda)}{\lambda-(\zeta+ih)^2}.$$

By lower semicontinuity of the norm $a \geq \|\sigma\|_*$, hence we conclude that $f \in \mathscr{S}_h$.

(ii) 1. Let us start by showing that for any constant $\rho \geq 0$, there exists $\{g_n\} \subset \mathcal{K}_h$ such that $g_n \to \rho$ in $H(\mathbb{H}_h)$ as $n \to \infty$. Indeed, define

$$g_n(\zeta) = \rho \int_n^{n+1} \frac{\lambda d\lambda}{\lambda-(\zeta+ih)^2}.$$

Clearly, $g_n \in \mathcal{K}_h$ and

$$g_n(\zeta) - \rho = \rho(\zeta + ih)^2 \int_n^{n+1} \frac{d\lambda}{\lambda - (\zeta + ih)^2},$$

which approaches to zero, as $n \to \infty$, uniformly on compact subsets of $\mathbb{H}_h$.

2. Let now $f \in \mathscr{S}_h$ and let ρ and σ be as in its definition. Consider the functions

$$h_n(\zeta) = \int_0^n \frac{d\sigma(\lambda)}{\lambda - (\zeta + ih)^2}.$$

Note that $h_n \in \mathcal{K}_h$, since its corresponding measure is $d\sigma_n = \chi_{(0,n)} d\sigma$ and

$$\int_0^\infty d\sigma_n(\lambda) = \int_0^n d\sigma(\lambda) \leq (n+1) \int_0^n \frac{d\sigma(\lambda)}{\lambda + 1} < \infty.$$

Now

$$f(\zeta) - h_n(\zeta) = \rho + \int_n^\infty \frac{d\sigma(\lambda)}{\lambda - (\zeta + ih)^2}$$

and by dominated convergence the above difference tends to ρ uniformly on compact subsets of $\mathbb{H}_h$. It remains to use the sequence $\{g_n\}$ from part 1 to get that $g_n + h_n$ is the desired sequence in $\mathcal{K}_h$ converging to f in $H(\mathbb{H}_h)$.

□

□

A corollary of Theorem 5.1 is stability of analytic continuation.

Corollary 5.1. *Let $\{f_n\}, f \subset \mathscr{S}_h$ be such that $f_n \to f$ in $L^2(0,1)$, then $f_n \to f$ as $n \to \infty$ in $H(\mathbb{H}_h)$.*

Indeed, if $f_n \to f$ in $L^2(0,1)$, then $\|f_n\|_{L^2(0,1)}$ is bounded. Then any converging subsequence $f_{n_k} \to g$ in $H(\mathbb{H}_h)$ must also converge to g in $L^2(0,1)$. But then $f = g$ on $(0,1)$. Since both f and g are analytic in $\mathbb{H}_h$, then $f = g$ everywhere. Since the set of limits of converging subsequences of f_n consists of a single element $\{f\}$, we conclude that $f_n \to f$ in $H(\mathbb{H}_h)$.

Let us now return to the least squares problem (5.5) .

Theorem 5.2. *For a given $f_{\exp} \in L^2(0,1)$, the least squares problem*

$$\mathfrak{E} = \mathfrak{E}(f_{\exp}) = \min_{f \in \mathscr{S}_h} \|f - f_{\exp}\|_{L^2(0,1)} \tag{5.15}$$

has a unique solution. Moreover,

$$\inf_{f \in \mathcal{K}_h} \|f - f_{\exp}\|_{L^2(0,1)} = \mathfrak{E}(f_{\exp}).$$

Proof. To prove existence, let $\{f_n\}_{n=1}^{\infty} \in \mathscr{S}_h$ be a minimizing sequence, then it is bounded in $L^2(0,1)$. Let us extract a weakly convergent subsequence, not relabeled, $f_n \rightharpoonup f_0$ in $L^2(0,1)$, as $n \to \infty$. The limiting function f_0 is in $\mathscr{S}_h$. By the convexity of the L^2-norm we have

$$\mathfrak{E} = \varliminf_{n\to\infty} \|f_n - f_{\exp}\|_{L^2(0,1)} \geq \|f_0 - f_{\exp}\|_{L^2(0,1)}.$$

Hence, f_0 is a minimizer . To prove that the infimum in (5.15) stays the same if we replace $\mathscr{S}_h$ by $\mathcal{K}_h$ we note that if $f_0 \in \mathscr{S}_h$ is a minimizer, then there exists a sequence $\{g_n\} \subset \mathcal{K}_h$ converging to f_0 strongly in $L^2(0,1)$.

To prove uniqueness, let f_1 and f_2 be two different solutions. Then $\|f_j - f_{\exp}\|_{L^2(0,1)} = \mathfrak{E}$ for $j = 1,2$. Observe that the function $f_t = tf_1 + (1-t)f_2$ is also admissible and therefore

$$\mathfrak{E} \leq \|f_t - f_{\exp}\|_{L^2(0,1)} \leq t\|f_1 - f_{\exp}\|_{L^2(0,1)} + (1-t)\|f_2 - f_{\exp}\|_{L^2(0,1)} = \mathfrak{E},$$

therefore $\|f_t - f_{\exp}\|_{L^2(0,1)} = \mathfrak{E}$ for all $t \in [0,1]$. However,

$$\|f_t - f_{\exp}\|^2_{L^2(0,1)} = t^2\|f_1 - f_2\|^2_{L^2(0,1)} + 2t\Re(f_1 - f_2, f_2 - f_{\exp}) + \|f_2 - f_{\exp}\|^2_{L^2(0,1)},$$

which cannot be constant, since the coefficient at t^2 is non-zero by our assumption $f_1 \neq f_2$. The obtained contradiction, concludes the theorem. □

5.4.2 Properties of the minimizer

In this section we will prove that if the minimum in (5.15) is nonzero, then the minimizer must be a rational function in $\mathbb{C}$ with poles (and zeros) on the line $\mathfrak{Im}(\zeta) = h$. We use the method of Caprini [12] to prove the statement.

The method for finding the necessary and sufficient conditions for a minimizer in (5.15) is based on our ability to compute the effect of the change of ρ and spectral measure σ in representation (5.3) on the value of the functional we want to minimize. Suppose that

$$f_*(\zeta) = \rho + \int_0^\infty \frac{d\sigma(\lambda)}{\lambda - (\zeta + ih)^2}$$

is the minimizer and

$$f(\zeta) = \widetilde{\rho} + \int_0^\infty \frac{d\widetilde{\sigma}(\lambda)}{\lambda - (\zeta + ih)^2} \tag{5.16}$$

is a competitor. The variation $\phi = f - f_*$ can then be written as

$$\phi(\zeta) = \Delta\rho + \int_0^\infty \frac{d\nu(\lambda)}{\lambda - (\zeta + ih)^2}, \qquad \nu = \widetilde{\sigma} - \sigma, \quad \Delta\rho = \widetilde{\rho} - \rho.$$

We then compute

$$\|f_* + \phi - f_{\exp}\|_{L^2}^2 - \|f_* - f_{\exp}\|_{L^2}^2 = \Delta\rho \lim_{t\to\infty} tC(t) + \int_0^\infty C(t)d\nu(t) + \|\phi\|_{L^2}^2, \tag{5.17}$$

where

$$C(t) = 2\mathfrak{Re} \int_0^1 \frac{f_*(\zeta) - f_{\exp}(\zeta)}{t - (\zeta - ih)^2} d\zeta, \qquad t \geq 0 \tag{5.18}$$

is the Caprini function of $f_*(\zeta)$.

Theorem 5.3. *Suppose the infimum in* (5.5) *is nonzero, then the minimizer* $f_* \in \mathscr{S}_h$ *in* (5.15) *is given by*

$$f_*(\zeta) = \rho + \sum_{j=1}^{N} \frac{\sigma_j}{t_j - (\zeta + ih)^2} \tag{5.19}$$

for some $\sigma_j > 0$, $t_j \geq 0$ *and* $\rho \geq 0$. *Moreover,* f_*, *given by* (5.19) *is the minimizer if and only if its Caprini function* $C(t)$ *is nonnegative and vanishes at* $t = t_j$, $j = 1, \ldots, N$, *and "at infinity", in the sense that*

$$2\mathfrak{Re} \int_0^1 (f_{\exp}(\zeta) - f_*(\zeta))d\zeta = \lim_{t\to\infty} tC(t) = 0, \tag{5.20}$$

provided $\rho > 0$.

Proof. If $\rho > 0$, then we can consider the competitor (5.16) with $\widetilde{\sigma} = \sigma$. Formula (5.17) then implies that

$$\Delta\rho \lim_{t\to\infty} tC(t) + (\Delta\rho)^2 > 0,$$

where $\Delta\rho$ can be either positive or negative and can be chosen as small in absolute value as we want. This implies (5.20).

Next, suppose $t_0 \in [0, +\infty)$ is in the support of σ. For every $\epsilon > 0$ we define $I_\epsilon(t_0) = \{t \geq 0 : |t - t_0| < \epsilon\}$. Saying that t_0 is in the support of σ is equivalent to $m(t_0, \epsilon) = \sigma(I_\epsilon(t_0)) > 0$ for all $\epsilon > 0$. Then, there are two possibilities. Either

(i) $\lim_{\epsilon\to 0} m(t_0, \epsilon) = 0$, or

(ii) $\lim_{\epsilon\to 0} m(t_0, \epsilon) = \sigma_0 > 0$

Let us first consider case (i). Then we construct a competitor measure

$$\sigma_\epsilon(\lambda) = \sigma(\lambda) - \sigma|_{I_\epsilon(t_0)} + \theta m(t_0, \epsilon)\delta_{t_0}(\lambda),$$

where $\theta > 0$ is an arbitrary constant. We then define

$$f_\epsilon(\zeta) = \rho + \int_0^\infty \frac{d\sigma_\epsilon(\lambda)}{\lambda - (\zeta + ih)^2}. \tag{5.21}$$

Formula (5.17) then implies

$$\lim_{\epsilon\to 0} \frac{\|f_{\exp} - f_\epsilon\|^2_{L^2(0,1)} - \|f_{\exp} - f_*\|^2_{L^2(0,1)}}{m(t_0, \epsilon)} = (1 - \theta)C(t_0).$$

If f_* is a minimizer, then we must have $(1 - \theta)C(t_0) \geq 0$ for all $\theta > 0$, which implies that $C(t_0) = 0$.

In the case (ii) we have $\sigma(\{t_0\}) = \sigma_0 > 0$. Then, for every $|\epsilon| < \sigma_0$ we construct a competitor measure

$$\sigma_\epsilon(\lambda) = \sigma(\lambda) - \epsilon\delta_{t_0}(\lambda),$$

as well as the corresponding f_ϵ, given by (5.21). We then compute

$$\lim_{\epsilon\to 0} \frac{\|f_{\exp} - f_\epsilon\|^2_{L^2(0,1)} - \|f_{\exp} - f_*\|^2_{L^2(0,1)}}{\epsilon} = C(t_0). \tag{5.22}$$

Since in this case ϵ can be both positive and negative we conclude that $C(t_0) = 0$.

Hence, we have shown that $C(t_0) = 0$ whenever $t_0 \in [0, +\infty)$ is in the support of the spectral measure σ of the minimizer f_*. It remains to observe that for any $t \in \mathbb{R}$

$$C(t) = \int_0^1 \frac{f_{\rm exp}(\zeta) - f_*(\zeta)}{t - (\zeta - ih)^2} d\zeta + \int_0^1 \frac{\overline{f_{\rm exp}(\zeta)} - \overline{f_*(\zeta)}}{t - (\zeta + ih)^2} d\zeta$$

Thus, $C(t)$ is a restriction to the real line of a complex analytic function on the neighborhood of the real line in the complex t-plane. By assumption, $f_{\rm exp} \neq f_*$, and therefore $C(t)$ is not identically zero. In particular, the zeros of $C(t)$ cannot have an accumulation point on the real line. We can see that the sequence of zeros of $C(t)$ cannot go to infinity by considering

$$B(s) = C\left(\frac{1}{s}\right) = s \int_0^1 \frac{f_{\rm exp}(\zeta) - f_*(\zeta)}{1 - s(\zeta + ih)^2} d\zeta + s \int_0^1 \frac{\overline{f_{\rm exp}(\zeta)} - \overline{f_*(\zeta)}}{1 - s(\zeta - ih)^2} d\zeta,$$

which is analytic in a neighborhood of 0, and hence cannot have a sequence of zeros $s_n \to 0$, as $n \to \infty$. We conclude that the support of the spectral measure of the minimzer f_* must be finite:

$$\sigma(\lambda) = \sum_{j=1}^{N} \sigma_j \delta_{t_j}(\lambda),$$

and the minimizer must be a rational function.

Now let us consider the competitor (5.16) defined by $\widetilde{\rho} = \rho$ and $\nu(\lambda) = \epsilon \delta_{t_0}(\lambda)$, where $\epsilon > 0$ and $t_0 \notin \{t_1, \ldots, t_N\}$. Formula (5.17) then implies that

$$\epsilon C(t_0) + C\epsilon^2 \geq 0$$

for all sufficiently small ϵ, which implies that $C(t) \geq 0$ for all $t \geq 0$. The necessity of the stated properties of the Caprini function $C(t)$ is now established. Sufficiency is a direct consequence of formula (5.17), since we can write

$$\nu(\lambda) = \widetilde{\sigma}(\lambda) - \sigma(\lambda) = \sum_{j=1}^{N} \Delta\sigma_j \delta_{t_j}(\lambda) + \widetilde{\nu}(\lambda),$$

where $\widetilde{\nu}(\lambda)$ is a positive Radon measure without any point masses at $\lambda = t_j$, $j = 1, \dots, N$. We then compute, via formula (5.17), taking into account that $C(t_j) = 0$

$$\|f_* + \phi - f_{\exp}\|_{L^2}^2 - \|f_* - f_{\exp}\|_{L^2}^2 = \Delta\rho \lim_{t\to\infty} tC(t) + \int_0^\infty C(t)d\widetilde{\nu}(t) + \|\phi\|_{L^2}^2 \geq 0,$$

since the first term on the right-hand side is either nonnegative, if $\rho = 0$ or zero, if $\rho > 0$. □

We observe that if $t_j > 0$, then we must also have $C'(t_j) = 0$, since $t = t_j$ is a local minimizer of $C(t)$. If we write formula (5.19) in the form

$$f_*(\zeta) = \rho - \frac{\sigma_0}{(\zeta + ih)^2} + \sum_{j=1}^{N} \frac{\sigma_j}{t_j - (\zeta + ih)^2}, \qquad \rho \geq 0,\ \sigma_0 \geq 0,\ t_j > 0,\ \sigma_j > 0,\ j = 1, \dots, N, \tag{5.23}$$

then we have exactly $2(N + 1)$ equations for $2(N + 1)$ unknowns ρ, σ_0, t_j, σ_j, $j = 1, \dots, N$:

$$\rho \lim_{t\to\infty} tC(t) = 0, \quad \sigma_0 C(0) = 0, \quad C(t_j) = 0, \quad C'(t_j) = 0, \quad j = 1, \dots, N.$$

Obviously, these equations do not enforce the nonnegativity of $C(t)$ and may very well be satisfied when $C(t)$ is not nonnegative. Taken together with their highly nonlinear nature and unknown value of N, their practical utility for finding f_* is dubious. Instead, Theorem 5.3 could be used to verify that a particular $f_*(\zeta)$ is the minimizer.

5.5 Worst case error analysis: Reduction to the error of analytic continuation with a symmetry constraint

Let us assume the Notation 2.1 from the beginning of Chapter 2 and let

$$Sf(\zeta) := \overline{f(-\bar{\zeta})}. \tag{5.24}$$

In this section we analyze the quantity $\Delta_{z,h}(\epsilon)$, given by (5.6) and show that the two questions posed in Section 5.3 can be entirely restated in terms of the difference $f - g$. The main result of this section is Theorem 5.4.

To analyze $\Delta_{z,h}(\epsilon)$ we examine the difference $\phi = f - g$. First observe that ϕ also has an integral representation (5.4) with a signed measure $\sigma = \sigma_f - \sigma_g$. Let now $\sigma = \sigma^+ - \sigma^-$ be the unique Hahn decomposition of σ as a difference of two mutually orthogonal positive measures $\sigma^\pm$. Then we may write $\phi = \phi^+ - \phi^-$, where $\phi^\pm \in \mathcal{K}_h$ are given by

$$\phi^\pm(\zeta) := \int_0^\infty \frac{d\sigma^\pm(\lambda)}{\lambda - (\zeta + ih)^2}. \tag{5.25}$$

So we arrive at the quantity

$$\sup\left\{\frac{|\phi(z)|}{\max\|\sigma^\pm\|_*} : \phi \in \mathcal{K}_h - \mathcal{K}_h \quad \text{and} \quad \frac{\|\phi\|_{L^2(0,1)}}{\max\|\sigma^\pm\|_*} \le \epsilon\right\}, \tag{5.26}$$

where we have abbreviated $\max\|\sigma^\pm\|_* := \max(\|\sigma^+\|_*, \|\sigma^-\|_*)$. The next idea comes from the realization that the asymptotics of the worst possible error is not very sensitive to specific norms and spaces. The reason, as we have seen in Chapter 4 for a similar problem, is that the analytic function delivering the largest error at z is analytic in a larger half-space $\mathbb{H}_{2h}$ and is therefore bounded in a wide variety of norms. Our idea is therefore to prove asymptotic equivalence of $\Delta_{z,h}(\epsilon)$ to a quadratic optimization problem in a Hilbert space, permitting us to express the asymptotics of $\Delta_{z,h}(\epsilon)$ in terms of the solution of the integral equation (5.12). So let us first recall the definition of the Hardy class

$$H^2(\mathbb{H}_h) = \left\{f \text{ is analytic in } \mathbb{H}_h : \sup_{y>-h} \|f\|_{L^2(\mathbb{R}+iy)} < \infty\right\}.$$

It is well known [42] that functions in H^2 have L^2 boundary data and that $\|f\|_{H^2(\mathbb{H}_h)} = \|f\|_{L^2(\mathbb{R}-ih)}$ defines a norm in H^2. We describe the relation between the Hardy space $H^2(\mathbb{H}_h)$ and $\mathcal{K}_h - \mathcal{K}_h$ more precisely in the following lemma.

Lemma 5.3. *Let* $f \in H^2(\mathbb{H}_h)$ *with* $Sf = f$ *and* $\int_0^\infty x|\mathfrak{Im} f(x-ih)| < \infty$, *then* $f \in \mathcal{K}_h - \mathcal{K}_h$ *with*

$$d\sigma(\lambda) = \frac{1}{\pi}\mathfrak{Im} f(\sqrt{\lambda} - ih)d\lambda. \tag{5.27}$$

Moreover, $f^\pm \in \mathcal{K}_h$ *and*

$$\max \|\sigma_{f^\pm}\|_* \leq \frac{1}{2\sqrt{\pi}}\|f\|_{H^2(\mathbb{H}_h)}. \tag{5.28}$$

Proof. We observe that it is enough to prove the lemma for $h = 0$ and then apply it to functions $f(\omega - ih) \in H^2(\mathbb{H}_+)$, where $f \in H^2(\mathbb{H}_h)$ and $\omega \in \mathbb{H}_+$.

For Hardy functions the following representation formula holds (cf. [42] p. 128)

$$f(\omega) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\mathfrak{Im} f(x)}{x - \omega}dx, \qquad \omega \in \mathbb{H}_+. \tag{5.29}$$

Passing to limits in the symmetry relation $Sf(\omega) = f(\omega)$ as $\mathfrak{Im}\omega \downarrow 0$, and taking imaginary parts we see that $-\mathfrak{Im} f(x) = \mathfrak{Im} f(-x)$. The formula (5.29) now gives

$$\pi f(\omega) = \int_0^\infty \frac{\mathfrak{Im} f(x)}{x - \omega}dx + \int_0^\infty \frac{\mathfrak{Im} f(-x)}{-x - \omega}dx = \int_0^\infty \frac{2x\mathfrak{Im} f(x)dx}{x^2 - \omega^2} = \int_0^\infty \frac{\mathfrak{Im} f(\sqrt{\lambda})d\lambda}{\lambda - \omega^2},$$

which implies (5.27).

Next, consider the functions

$$f^\pm(\omega) = \int_0^\infty \frac{d\sigma^\pm(\lambda)}{\lambda - \omega^2}, \qquad d\sigma^\pm(\lambda) = \frac{1}{\pi}\left(\mathfrak{Im} f\right)^\pm(\sqrt{\lambda})d\lambda,$$

where $(\mathfrak{Im} f)^\pm$ denote the positive and negative parts of the real valued function $\mathfrak{Im} f$. Then $f = f^+ - f^-$ and since $\int_0^\infty x|\mathfrak{Im} f(x)|dx < \infty$, the measures $\sigma^\pm$ are finite and so $f^\pm \in \mathcal{K}_0$.

Finally, we prove the inequality (5.28). We compute

$$\|\sigma^\pm\|_* = \frac{2}{\pi}\int_0^\infty \frac{x(\mathfrak{Im} f)^\pm(x)}{1 + x^2}dx.$$

Applying the Cauchy-Schwarz inequality we obtain

$$\|\sigma^{\pm}\|_* \leq \frac{1}{\sqrt{\pi}}\|(\mathfrak{Im} f)^{\pm}\|_{L^2(0,+\infty)} \leq \frac{1}{\sqrt{\pi}}\|\mathfrak{Im} f\|_{L^2(0,+\infty)} = \frac{1}{2\sqrt{\pi}}\|f\|_{H^2(\mathbb{H}_+)},$$

where we have used the symmetry and the fact that the real part of a Hardy function is the Hilbert transform of its imaginary part [42], and therefore,

$$\|f\|^2_{H^2(\mathbb{H}_+)} = 2\|\mathfrak{Im} f\|^2_{L^2(\mathbb{R})} = 4\|\mathfrak{Im} f\|^2_{L^2(0,+\infty)}.$$

□

In order to complete the transition from $\mathcal{K}_h$ to Hardy spaces we need to replace the norm $\|\sigma\|_*$ in (5.26) with an equivalent Hilbert space norm. This is accomplished in our next Lemma.

Lemma 5.4. *Let $h' \in (0, h)$, then for any $f \in \mathcal{K}_h$*

$$\|f\|_{h'} := \left\|\frac{f}{\zeta + ih}\right\|_{H^2(\mathbb{H}_{h'})} \simeq \|\sigma\|_*, \tag{5.30}$$

where the implicit constants depend only on $h - h'$.

Proof. Since $\mathbb{H}_{h'} \subset \mathbb{H}_h$, it is clear that the function $f(\zeta)/(\zeta + ih)$ is analytic in $\mathbb{H}_{h'}$. Next letting $\delta = h - h'$, using the integral representation (5.4) for f and Fubini's theorem we compute

$$\|f\|^2_{h'} = \int_{\mathbb{R}} \frac{1}{x^2 + \delta^2} \int_0^\infty \int_0^\infty \frac{d\sigma(\lambda)d\sigma(t)}{[\lambda - (x + i\delta)^2][t - (x - i\delta)^2]} dx =$$

$$= \int_0^\infty \int_0^\infty I(\lambda, t)\frac{d\sigma(\lambda)}{\lambda + 1}\frac{d\sigma(t)}{t + 1},$$

where

$$I(\lambda, t) = \frac{\pi(\lambda + 1)(t + 1)}{\delta(\lambda + 4\delta^2)(t + 4\delta^2)} \cdot \frac{(\lambda - t)^2 + 12\delta^2(\lambda + t) + 96\delta^4}{(\lambda - t)^2 + 8\delta^2(\lambda + t) + 16\delta^4}.$$

This concludes the proof, since it is clear that the function $I(\lambda, t)$ is bounded above and below by two positive constants depending only on δ. □

Now we are ready to give the desired Hilbert space reformulation of our problem. For any $h > 0$ we define

$$A_h^S(\epsilon) = \sup\left\{|f(z)| \ : f \in H^2(\mathbb{H}_h) \text{ and } Sf = f,\ \|f\|_{H^2(\mathbb{H}_h)} \leq 1,\ \|f\|_{L^2(-1,1)} \leq \epsilon\right\}. \tag{5.31}$$

Notice that for convenience we suppressed the dependence on z and also replaced interval from $[0, 1]$ by a symmetric interval $[-1, 1]$, resulting in an equivalent formulation due to the symmetry $Sf = f$ of the functions in $\mathcal{K}_h$.

Theorem 5.4 (Equivalence of A^S and Δ)**.** *For any $h' \in (0, h)$*

$$A_h^S(\epsilon) \lesssim \Delta_h(\epsilon) \lesssim A_{h'}^S(\epsilon), \tag{5.32}$$

as $\epsilon \to 0$, where the implicit constants depend only on h and h'.

Proof. We first observe that

$$\Delta_h(\epsilon) = \sup\{|f(z)-g(z)| : \{f, g\} \subset \mathcal{K}_h,\ \max\{\|\sigma_f\|_*, \|\sigma_g\|_*\} = 1,\ \|f-g\|_{L^2(-1,1)} \leq \epsilon\}.$$

To prove the first inequality in (5.32), let $\{f, g\} \subset \mathcal{K}_h$ be such that

$$\max\{\|\sigma_f\|_*, \|\sigma_g\|_*\} = 1, \qquad \|f - g\|_{L^2(-1,1)} \leq \epsilon.$$

Let

$$\phi(\zeta) = \frac{i(f(\zeta) - g(\zeta))}{\zeta + ih}.$$

Then, $S\phi = \phi$. Moreover, by Lemma 5.4, for any $h' \in (0, h)$ we estimate

$$\|\phi\|_{H^2(\mathbb{H}_{h'})} = \|f - g\|_{h'} \leq \|f\|_{h'} + \|g\|_{h'} \lesssim \|\sigma_f\|_* + \|\sigma_g\|_* \leq 2.$$

We conclude that there exists a constant $c > 0$, depending only on h and h', such that $c\phi$ is admissible for $A_{h'}^S(\epsilon)$. Therefore,

$$A_{h'}^S(\epsilon) \geq c|\phi(z)| = \frac{c|f(z) - g(z)|}{|z + ih|}.$$

Taking supremum over all such pairs (f, g) we conclude that

$$\Delta_h(\epsilon) \leq CA_{h'}^S(\epsilon)$$

for some constant $C > 0$, that depends on h and h', but not on ϵ.

To prove the other inequality, let $\phi \in H^2(\mathbb{H}_h)$ be admissible for $A_h^S(\epsilon)$. The idea is to construct a pair of functions $\{f, g\} \subset \mathcal{K}_h$ that are admissible for $\Delta_h(\epsilon)$. Since ϕ might not decay sufficiently fast at infinity to be in $\mathcal{K}_h - \mathcal{K}_h$ we modify it and define

$$\psi(\zeta) = \frac{\phi(\zeta)}{(\zeta + ih)^2}.$$

This modification preserves the symmetry ($S\psi = \psi$) and ensures the required decay, so that Lemma 5.3 is applicable. So that $\psi^\pm \in \mathcal{K}_h$ and $\|\sigma_{\psi^\pm}\|_* \lesssim 1$. Now, let $\psi_0(\zeta) \in \mathcal{K}_h$ be such that $\|\sigma_{\psi_0}\|_* = 1$. We define

$$F(\zeta) = \psi^+(\zeta) + \psi_0(\zeta), \qquad G(\zeta) = \psi^-(\zeta) + \psi_0(\zeta).$$

We observe that there exists a constant $C > 0$, such that

$$1 = \|\sigma_{\psi_0}\|_* \leq \|\sigma_F\|_* \leq C, \qquad 1 = \|\sigma_{\psi_0}\|_* \leq \|\sigma_G\|_* \leq C.$$

Thus, the pair (f, g) given by

$$f(\zeta) = \frac{F(\zeta)}{M}, \quad g(\zeta) = \frac{G(\zeta)}{M}, \quad M = \max\{\|\sigma_F\|_*, \|\sigma_G\|_*\} \geq 1$$

is admissible for $\Delta_h(\epsilon)$. Thus,

$$\Delta_h(\epsilon) \geq |f(z) - g(z)| = \frac{|\phi(z)|}{(z^2 + h^2)M} \geq \frac{|\phi(z)|}{C}.$$

Taking supremum over all admissible ϕ we obtain the remaining inequality in (5.32).

□

5.6 The effect of the symmetry constraint

Let $H^2 = H^2(\mathbb{H}_+)$ with $\|\cdot\|$ denoting its norm. Theorem 5.4 implies that for the asymptotic behavior of $\Delta(\epsilon)$ it is enough to analyze the following quantity (after shifting the function space up by ih in (5.31))

$$A_z^S(\epsilon) := \sup\left\{|f(z)| : f \in H^2 \text{ and } Sf = f,\ \|f\| \leq 1,\ \|f\|_{L^2(\Gamma)} \leq \epsilon\right\},$$

for $\Gamma = [-1,1] + ih$, where recall that $Sf(\zeta) = \overline{f(-\bar{\zeta})}$. Let us in fact consider the more general situation and assume that $\Gamma \Subset \mathbb{H}_+$ is a rectifiable curve of finite length symmetric with respect to the imaginary axis (i.e. $\tau \in \Gamma$ iff $-\bar{\tau} \in \Gamma$) and let $z \in \mathbb{H}_+ \backslash cl(\Gamma)$. Let us introduce the analogous quantity without the symmetry constraint:

$$A_z(\epsilon) := \sup\left\{|f(z)| : f \in H^2 \text{ and } \|f\| \leq 1,\ \|f\|_{L^2(\Gamma)} \leq \epsilon\right\}. \tag{5.33}$$

Theorem 5.5. *With the notation introduced above for any $\epsilon > 0$*

$$\tfrac{1}{2}A_z(\epsilon) \leq A_z^S(\epsilon) \leq A_z(\epsilon). \tag{5.34}$$

Remark 5.2. *Note that the above theorem shows that the symmetry constraint $Sf = f$ has no effect on the asymptotic behavior of $A_z^S(\epsilon)$ as $\epsilon \to 0$. In particular, if $A_z(\epsilon) \simeq \epsilon^{\gamma(z)}$, then (5.34) implies that also $A_z^S(\epsilon) \simeq \epsilon^{\gamma(z)}$ with the same exponent $\gamma(z)$.*

Immediately from the above theorem and Theorem 5.4 we obtain

Corollary 5.2. *Let A_h be defined by (5.7), then for any $h' \in (0,h)$*

$$A_h(\epsilon) \lesssim \Delta_h(\epsilon) \lesssim A_{h'}(\epsilon),$$

as $\epsilon \to 0$, where the implicit constants depend only on h and h'.

Proof of Theorem 5.5. In analyzing the problem without the symmetry constraint we reduced it to a maximization of a linear target functional, under convex quadratic constraints. Indeed, this was possible since the two constraints $\|f\| \leq 1$ and $\|f\|_{L^2(\Gamma)} \leq \epsilon$ are invariant under multiplying f by a constant phase factor, therefore equivalently we can maximize $\mathfrak{Re} f(z)$ instead of $|f(z)|$. This reduction does not work in presence of the symmetry constraint $Sf = f$, since multiplication by nonreal factors breaks the symmetry. Nevertheless, linearization can still be achieved by observing that

$$|f(z)| = \max_{|\lambda|=1} \mathfrak{Re}(\lambda f(z))$$

and interchanging the order of maximums with respect to λ and f. That is, for a fixed λ we first maximize the linear functional $\mathfrak{Re}(\lambda f(z))$ subject to the three constraints of A_z^S and then maximize the result in λ.

Let us next show that we can completely eliminate the symmetry constraint. Recall that $(\cdot,\cdot)$ denotes the inner product of H^2, or equivalently of $L^2(\mathbb{R})$. It is easy to check that for $f \in H^2$, satisfying the symmetry constraint we have

$$\mathfrak{Re}(\overline{\lambda} f(z)) = \mathfrak{Re}(f, \lambda p_z) = \mathfrak{Re}(f, q_{z,\lambda}), \qquad q_{z,\lambda} = \frac{\lambda p_z + S(\lambda p_z)}{2}.$$

We can now discard the symmetry constraint. We claim that the maximizer function of the problem

$$A_{\lambda,z}(\epsilon) = \sup\left\{\mathfrak{Re}(f, q_{z,\lambda}) : f \in H^2 \text{ with } \|f\| \leq 1,\ \|f\|_{L^2(\Gamma)} \leq \epsilon\right\} \tag{5.35}$$

automatically has the required symmetry. Indeed, if $f \in H^2$ solves (5.35), we can decompose it into its symmetric and antisymmetric parts $f = f_s + f_a$, which are mutually orthogonal both in H^2 and $L^2(\Gamma)$. Thus,

$$\|f\|^2 = \|f_s\|^2 + \|f_a\|^2 \geq \|f_s\|^2, \quad \|f\|^2_{L^2(\Gamma)} = \|f_s\|^2_{L^2(\Gamma)} + \|f_a\|^2_{L^2(\Gamma)} \geq \|f_s\|^2_{L^2(\Gamma)},$$

which implies that

$$\kappa = \max\left\{\|f_s\|, \frac{\|f_s\|_{L^2(\Gamma)}}{\epsilon}\right\} \leq 1.$$

Also, by the symmetry of $q_{z,\lambda}$ we find that

$$\mathfrak{Re}(f, q_{z,\lambda}) = \mathfrak{Re}(f_s, q_{z,\lambda}).$$

But then the function f_s/κ satisfies the constraints of (5.35) and strictly increases the value of target functional unless $f_a = 0$. Thus, if f is the maximizer, then it has to be symmetric.

According to Theorem 2.1 part (i), the maximizer function $f_\epsilon^*(\zeta)$ for (5.33) has the property that $f_\epsilon^*(z) = A_z(\epsilon) > 0$. Since removing the symmetry constraint increases the set of admissible functions we have an obvious inequality

$$A_z^S(\epsilon) \le f_\epsilon^*(z) = A_z(\epsilon).$$

Our foregoing discussion suggests that the function $v_{\lambda,\epsilon} = \lambda f_\epsilon^*$ must be a good candidate for the maximizer in $A_{\lambda,z}(\epsilon)$. Using it as a test function we get the inequality

$$A_{\lambda,z}(\epsilon) \ge \mathfrak{Re}(\lambda f_\epsilon^*, q_{z,\lambda}) = \frac{f_\epsilon^*(z)}{2} + \frac{1}{2}\mathfrak{Re}(\lambda^2(f_\epsilon^*, Sp_z)).$$

We conclude that

$$A_z^S(\epsilon) = \max_{|\lambda|=1} A_{\lambda,z}(\epsilon) \ge \frac{f_\epsilon^*(z)}{2} + \frac{1}{2}|(f_\epsilon^*, Sp_z)| \ge \frac{f_\epsilon^*(z)}{2} = \frac{1}{2}A_z(\epsilon).$$

□

5.6.1 Applications

• Let $\Gamma = C(i, r)$ be the circle centered at i with radius $r < 1$, Theorem 5.5 implies that adding the symmetry constraint $Sf = f$ does not affect the power law (4.7).

• Let $\Gamma = [-1, 1]$, we can show that adding the symmetry constraint $Sf = f$ does not affect the power law (4.15) directly, without referring to Theorem 5.5. Indeed, consider the slight modification of the maximizer function (4.14)

$$M^S(\zeta) := \frac{i\epsilon}{\zeta + i} e^{\frac{i}{\pi} \ln \epsilon \ln \frac{1+\zeta}{1-\zeta}}, \qquad \zeta \in \mathbb{H}_+.$$

Note that $M^S \in H^2(\mathbb{H}_+)$ satisfies the symmetry constraint, and

$$|M^S(\zeta)| = \frac{\epsilon^{\gamma(\zeta)}}{|\zeta + i|},$$

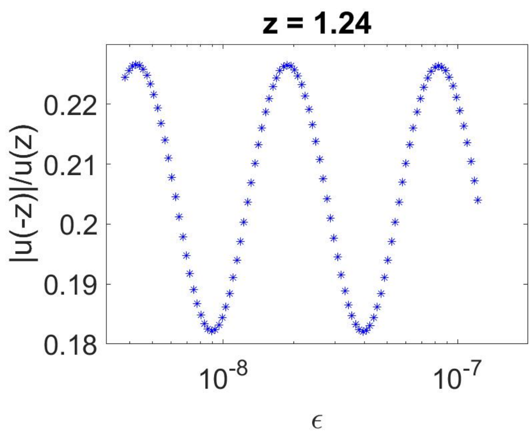
where γ is given by (4.13). In particular $\|M^S\|_{H^2} \lesssim 1$, $\|M^S\|_{L^2(-1,1)} \lesssim \epsilon$, i.e. M^S satisfies all the constraints and attains the optimal bound $|M^S(z)| \simeq \epsilon^{\gamma(z)}$.

• Let $\Gamma = [-1, 1] + ih$ with $h > 0$ and consider extrapolation points $z + ih$ with $z > 1$. The equality of the exponents for problems with and without symmetry shown in Figure 5.3b can be explained by the "quantitative asymmetry" of the solution u_ϵ of the integral equation $\mathscr{K}u + \epsilon^2 u = p_z$ (where $\mathscr{K}, p_z$ are given in (4.3), (4.2) respectively):

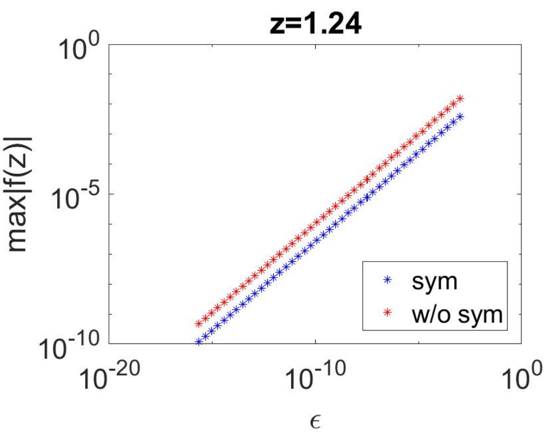
$$\overline{\lim_{\epsilon\to 0}} \frac{|u_\epsilon(z)|}{|u_\epsilon(-z)|} < 1. \tag{5.36}$$

Indeed, the symmetrized solution $v_\epsilon(\zeta) = u_\epsilon(\zeta) + \overline{u_\epsilon(-\bar{\zeta})}$ has the same order of magnitude at $\zeta = z$ as $u_\epsilon(z)$, as $\epsilon \to 0$. While numerically (5.36) is seen to hold, we do not have a mathematical proof of this inequality. Nonetheless, the equality of the exponents for problems with and without symmetry follows from Theorem 5.5. Figure 5.3a (corresponding to $h = 1$) verifies (5.36) and shows that the ratio in (5.36) does not converge to a limit as $\epsilon \to 0$, but exhibits an oscillatory behavior instead. We show below (cf. (5.38)) that this oscillatory pattern is a consequence of the exponential decay of eigenvalues λ_n of $\mathscr{K}$. Specifically, if $\lambda_n \simeq e^{-\alpha n}$, then, as $\epsilon \to 0$ the ratio (5.36) is well-approximated by a smooth function of $\{2\alpha^{-1} \ln \epsilon^{-1}\}$, where $\{\cdot\}$ denotes the fractional part of a number. In particular, this means that the ratio (5.36) must look like a $\alpha/2$-periodic function of $\ln(1/\epsilon)$, when ϵ is sufficiently small. This explains the periodic pattern in Figure 5.3a.

Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis of H^2 with $\mathscr{K}e_n = \lambda_n e_n$. The eigenfunctions e_n can be chosen to satisfy the symmetry constraint $Se_n = e_n$. Indeed, since S commutes with $\mathscr{K}$, then e_n and Se_n belong to the same eigenspace, which is one dimensional (cf. Section 5.7). Therefore, one is a complex multiple of the other, but since both have the same norms the multiplicative constant must be on the unit circle, i.e. $\exists \theta_n \in [0, 2\pi]$ such that



(a)



(b)

Figure 5.3: Adding symmetry has no effect on $\gamma(z)$

$Se_n = e^{i\theta_n} e_n$. Now $e^{i\theta_n/2} e_n$ has the required symmetry.

Using the eigenbasis expansion of u in (2.17) and the symmetry of e_n we can write

$$\overline{u(-\overline{z})} = \sum_n \frac{e_n(z)^2}{\epsilon^2 + \lambda_n}, \qquad u(z) = \sum_n \frac{|e_n(z)|^2}{\epsilon^2 + \lambda_n}. \tag{5.37}$$

Now, following Conjecture 5.1 assume

$$\lambda_n \simeq e^{-\alpha n}, \qquad e_n(z)^2 \simeq e^{(-\beta + i\theta)n}, \qquad 0 < \beta < \alpha.$$

Replacing these in (5.37) we obtain, that as $j \to \infty$

$$\frac{|u_{\epsilon_j}(-\overline{z})|}{u_{\epsilon_j}(z)} \sim \frac{\left| \sum_n \frac{e^{(-\beta+i\theta)n}}{\epsilon_j^2 + e^{-\alpha n}} \right|}{\sum_n \frac{e^{-\beta n}}{\epsilon_j^2 + e^{-\alpha n}}} \sim L_0(t), \tag{5.38}$$

where, using Lemma 2.4, $t = \lim_{j\to\infty} \left\{ 2\alpha^{-1} \ln \epsilon_j^{-1} \right\}$ depends on how ϵ_j approaches to zero and

$$L_0(t) = \left| \sum_{k\in\mathbb{Z}} \frac{e^{(-\beta+i\theta)k}}{e^{-\alpha t} + e^{-\alpha k}} \right| \Big/ \sum_{k\in\mathbb{Z}} \frac{e^{-\beta k}}{e^{-\alpha t} + e^{-\alpha k}}.$$

Clearly $L_0(t) < 1$ and depending on how ϵ_j approaches to zero, several limiting values t are possible, creating the oscillatory behavior of the ratio.

5.7 Analytic continuation in $\mathbb{H}_+$ from $[-1,1]+ih$

Assume that $\Gamma = [-1,1] + ih$ with $h > 0$. Corollary 5.2 shows that the analysis of $\Delta(\epsilon)$ is reduced to that of

$$A_z(\epsilon) = \sup\left\{|f(z)| : f \in H^2(\mathbb{H}_+) \text{ and } \|f\|_{H^2(\mathbb{H}_+)} \le 1,\ \|f\|_{L^2(\Gamma)} \le \epsilon\right\},$$

to which Theorems 2.1 and 2.2 apply. In particular, $A_z(\epsilon) \simeq M_{\epsilon,z}(z)$, where $M_{\epsilon,z}$ is given by (4.5). One of the key ingredients in the asymptotic behavior of $M_{\epsilon,z}(z)$, as shown by Theorem 2.2, is the exponential decay rate of eigenvalues λ_n of the operator $\mathscr{K}$ given by (4.3). The exponential upper bound on λ_n is a consequence of the displacement rank 1 structure:

$$(L\mathscr{K} - \mathscr{K}L^*)u = \frac{i}{2\pi}\int_\Gamma u(\tau)|d\tau| =: Ru, \tag{5.39}$$

where $L : L^2(\Gamma) \to L^2(\Gamma)$ is the operator of multiplication by $\tau \in \Gamma$: $(Lu)(\tau) = \tau u(\tau)$. The operator R on the right-hand side of (5.39) is a rank-one operator, since its range consists of constant functions.

Then, according to [4],

$$\lambda_{n+1} \le \rho_1\lambda_n, \qquad \rho_1 = \inf_{r\in\mathscr{M}} \frac{\max_{\tau\in\Gamma}|r(\tau)|}{\min_{\tau\in\Gamma}|r(\overline{\tau})|}, \tag{5.40}$$

for all $n \ge 1$, where $\mathscr{M}$ is the set of all Möbius transformations

$$r(\tau) = \frac{a\tau + b}{c\tau + d}.$$

It is easy to see that $\rho_1 < 1$ by considering Möbius transformations that map upper half-plane into the unit disk. Then Γ will be mapped to a curve inside the unit disk, so that $m = \max_{\tau\in\Gamma}|r(\tau)| < 1$. By the symmetry property of Möbius transformations the image of $\overline{\Gamma}$ will be symmetric to the image of Γ

with respect to the inversion in the unit circle. Thus, $\min_{\tau\in\Gamma}|r(\overline{\tau})| = 1/m$, so that $\rho_1 \leq m^2 < 1$. In particular this implies that all eigenvalues have multiplicity 1.

The implied exponential upper bound $\lambda_{n+1} \leq \rho_1^n\lambda_1$ is not the best that one can derive from the rank-1 displacement structure (5.39). According to a theorem of Beckermann and Townsend [4], $\lambda_n \leq Z_n(\Gamma,\overline{\Gamma})\lambda_1$, where Z_n is the nth Zolotarev number [74]. When n is large, the Zolotarev numbers decay exponentially $\ln Z_n(\Gamma,\overline{\Gamma}) \sim -n\ln\rho_\Gamma$, where ρ_Γ is the Riemann invariant , whereby the annulus $\{1 < |\omega| < \rho_\Gamma\}$ is conformally equivalent to the Riemann sphere with Γ and $\overline{\Gamma}$ removed [37]. Hence,

$$\lambda_n \lesssim \rho_\Gamma^{-n}. \tag{5.41}$$

Conjecture 5.1. *The eigenvalues λ_n of $\mathscr{K}$ and the magnitudes of eigenfunctions $|e_n(z)|$ have exponential decay asymptotics* (2.21). *Moreover, we also conjecture that the asymptotic upper bound (5.41) captures the rate of exponential decay of λ_n, i.e. $\alpha = \ln\rho_\Gamma$.*

There is substantial evidence supporting this conjecture, including the explicit formula for $\gamma(z)$ in the limiting case when $\Gamma \subset \partial\mathbb{H}_+$, given in Theorem 4.2. Also, if the L^2 norm of $f \in H^2$ were of order ϵ on a compact subdomain $G \subset \mathbb{H}_+$, instead of the curve Γ, then the conjectured asymptotics of λ_n would hold, as shown in [58], provided the boundary of G is sufficiently smooth. The curve Γ could also be regarded as a limiting case of a domain. However, its boundary would not be smooth and the analysis in [58] would not apply.

The operator $\mathscr{K}+\epsilon^2$ in the integral equation $(\mathscr{K}+\epsilon^2)u = p_z$ is almost singular when ϵ is small, since $\mathscr{K}$ is compact and has no bounded inverse. It was the idea of Leslie Greengard to solve the integral equation directly numerically using quadruple precision floating point arithmetic available in FORTRAN. He has written the code and shared the FORTRAN libraries for Gauss quadrature, linear systems solver and eigenvalues and eigenvectors routines for Hermitian matrices. For the numerical computations we took extrapolation points $z+ih$, $z \geq 1$. Quadruple precision allowed us to compute all eigenvalues of $\mathscr{K}$ that are larger than 10^{-33} and solve the integral equation for values of ϵ as low as 10^{-16}. Note that for the particular choice $\Gamma = [-1,1] + ih$ the operator $\mathscr{K}$ is a finite convolution type operator with kernel $k(t) = \frac{i(2\pi)^{-1}}{t+2ih}$. Asymptotics of eigenvalues of positive self-adjoint finite convolution operators with real-valued kernels (i.e. even real functions $k(t)$) were obtained by Widom in [71]. To apply these results we note that $\widehat{k}(\xi) = e^{-2h\xi}\chi_{(0,+\infty)}(\xi)$, which has exact exponential decay when $\xi \to +\infty$. The operator $\mathscr{K}_0$ with the even real kernel $k_0(t) = 2\mathfrak{Re}\,k(t)$ has symbol $\widehat{k}_0(\xi) = e^{-2h|\xi|}$ to which Widom's theory applies. Widom's formula gives

$$\ln \lambda_n(\mathscr{K}_0) \sim -Wn, \qquad \text{as } n \to \infty, \qquad W = -\pi \frac{K\left(\operatorname{sech}\left(\frac{\pi}{2h}\right)\right)}{K\left(\tanh\left(\frac{\pi}{2h}\right)\right)},$$

where $K(k)$ is the complete elliptic integral of the first kind. We therefore obtain an upper bound for $\lambda_n = \lambda_n(\mathscr{K})$

$$\ln \lambda_n(\mathscr{K}) \leq \ln \lambda_n(\mathscr{K}_0) \sim -Wn. \tag{5.42}$$

The lower bound can be obtained from the same formula using an inequality

$$\lambda_n(\mathscr{K}_0) \leq \lambda_{n/2}(\mathscr{K}) + \lambda_{n/2}(\overline{\mathscr{K}}) = 2\lambda_{n/2}(\mathscr{K}),$$

so that

$$\ln \lambda_n(\mathscr{K}) \geq \ln \tfrac{1}{2} + \ln \lambda_{2n}(\mathscr{K}_0) \sim -2Wn. \tag{5.43}$$

Figure 5.4a, where $h = 1$ supports the exponential decay conjecture of λ_n in Conjecture 5.1 and shows that estimates (5.42), (5.43) are not asymptotically sharp. By contrast, Figure 5.4a shows that the Beckermann-Townsend upper bound (5.41) matches the asymptotics of λ_n very well. The explicit transformation Ψ of the extended complex plane with $[-1,1] \pm ih$ removed onto the annulus $\{\omega \in \mathbb{C} : \rho_\Gamma^{-1/2} < |\omega| < \rho_\Gamma^{1/2}\}$ has been derived in [2] in terms of the elliptic functions and integrals

$$\Psi^{-1}(\omega) = \frac{h}{\pi}\left(\zeta\left(\frac{\ln\omega}{2\pi i}\middle|\,\tau\right) - \zeta\left(\frac{1}{2}\middle|\,\tau\right)\frac{\ln\omega}{\pi i}\right), \qquad \tau = \frac{K(1-m)}{K(m)}, \quad (5.44)$$

where $\zeta(z|\tau)$ is the Weierstrass zeta function with quasi-periods 1 and $i\tau$. The Riemann invariant $\rho_\Gamma = e^{2\pi\tau}$ is computed after finding the unique solution $m \in (0,1)$ of

$$K(m)E(x(m)|m) - E(m)F(x(m)|m) = \frac{\pi}{2h}, \quad x(m) = \sqrt{\frac{K(m)-E(m)}{mK(m)}}.$$

Figure 5.4b shows the logarithmic plot of values of $M_{\epsilon,z}(z)$ as a function of ϵ, supporting the power law principle

$$M_{\epsilon,z}(z) \simeq \epsilon^{\gamma(z)}. \quad (5.45)$$

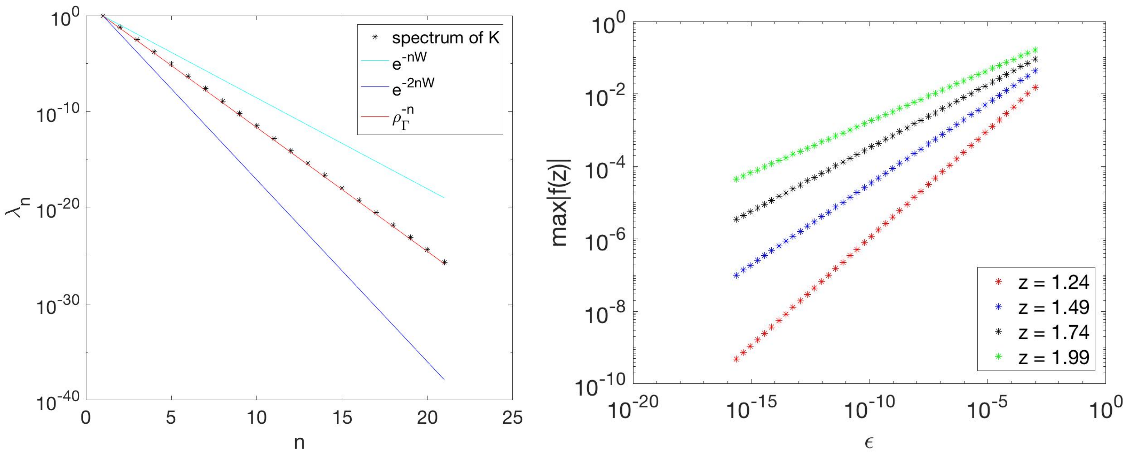
We remark that under the exponential decay assumptions of Conjecture 5.1, by Theorem 2.2 the two quantities inside minimum in the definition of $M_{\epsilon,z}$ (4.5) are comparable and hence the maximizer $M_{\epsilon,z}$ can be taken to be $\epsilon u_{\epsilon,z}/\|u_{\epsilon,z}\|_{L^2(\Gamma)}$. Further, again by Theorem 2.2 $u_{\epsilon,z} \simeq \|u_{\epsilon,z}\|^2_{L^2(\Gamma)}$ and so $M_{\epsilon,z}(z) \simeq \epsilon\|u_{\epsilon,z}\|_{L^2(\Gamma)}$, which then implies

$$\gamma(z) = 1 - \lim_{\epsilon\to 0^+} \frac{\ln\|u_{\epsilon,z}\|_{L^2(\Gamma)}}{\ln(1/\epsilon)}. \quad (5.46)$$

5.7.1 Upper bound on the power law exponent

A lower bound on $M_{\epsilon,z}(z)$ (or equivalently, an upper bound on the exponent $\gamma(z)$) can be obtained by constructing a test function that is small on the data curve, but is not as small at the point z.

The explicit formula (4.9) of the maximizer function when the data curve is a circle, suggests such a construction. Recall that in (4.9) m maps $\mathbb{H}_+\backslash\Gamma$ onto the annulus $\{\rho < |\omega| < 1\}$, in our case Ψ maps $\mathbb{H}_+\backslash\Gamma$ onto the annulus



(a) (b)

Figure 5.4: Numerical support for the power law transition principle.

$\{\rho_\Gamma^{-1/2} < |\omega| < 1\}$. So, let us replace m with Ψ, and ρ with $\rho_\Gamma^{-1/2}$ and consider the function

$$M_1(\zeta) = \frac{\epsilon^{2-\gamma_1(z)}}{\zeta + ih} \sum_{n=1}^{\infty} \frac{\left(\overline{\Psi(z)}\Psi(\zeta)\right)^n}{\epsilon^2 + \rho_\Gamma^{-n}}, \tag{5.47}$$

where $z \in \mathbb{H}_+ \backslash \Gamma$ is the extrapolation point and (in analogy to γ in (4.8))

$$\gamma_1(z) = \frac{\ln|\Psi(z)|}{\ln \rho_\Gamma^{-1/2}} \in (0, 1). \tag{5.48}$$

Then, in view of Corollary 2.2 and the facts that Ψ maps Γ onto the circle $\{|\omega| = \rho_\Gamma^{-1/2}\}$ and $\mathbb{R}$ onto the unit circle, we have

$$M_1(z) \simeq \epsilon^{\gamma_1(z)}, \qquad \|M_1\|_{L^2(\Gamma)} \lesssim \epsilon, \qquad \|M_1\|_{H^2(\mathbb{H}_+)} \lesssim 1.$$

Thus, M_1 is an admissible function for (5.7) and we have proved the following bound:

Corollary 5.3. $M_{\epsilon,z}(z) \gtrsim \epsilon^{\gamma_1(z)}$ *with the implicit constant independent of* ϵ.

In view of (5.45), the above corollary implies the following upper bound on the optimal exponent γ:

$$\gamma(z) \leq \gamma_1(z).$$

We compare the computed exponents $\gamma(z)$ with the estimate (5.48) for $\Gamma = [-1, 1] + ih$ and extrapolation points $z + ih$, $z > 1$. Figure 5.5 shows $\gamma(z)$ (obtained by least squares linear fit of the data for various values of z, four of which are shown in Figure 5.4b) and the upper bound $\gamma_1(z)$ given by (5.48). We remark that by virtue of transplanting the actual maximizer of $|f(z)|$ from one geometry to the other, the structure of the test function (5.47) resembles the optimal one. In fact, for values of $h > 0.6$ the bound $\gamma_1(z)$ is virtually indistinguishable from $\gamma(z)$.

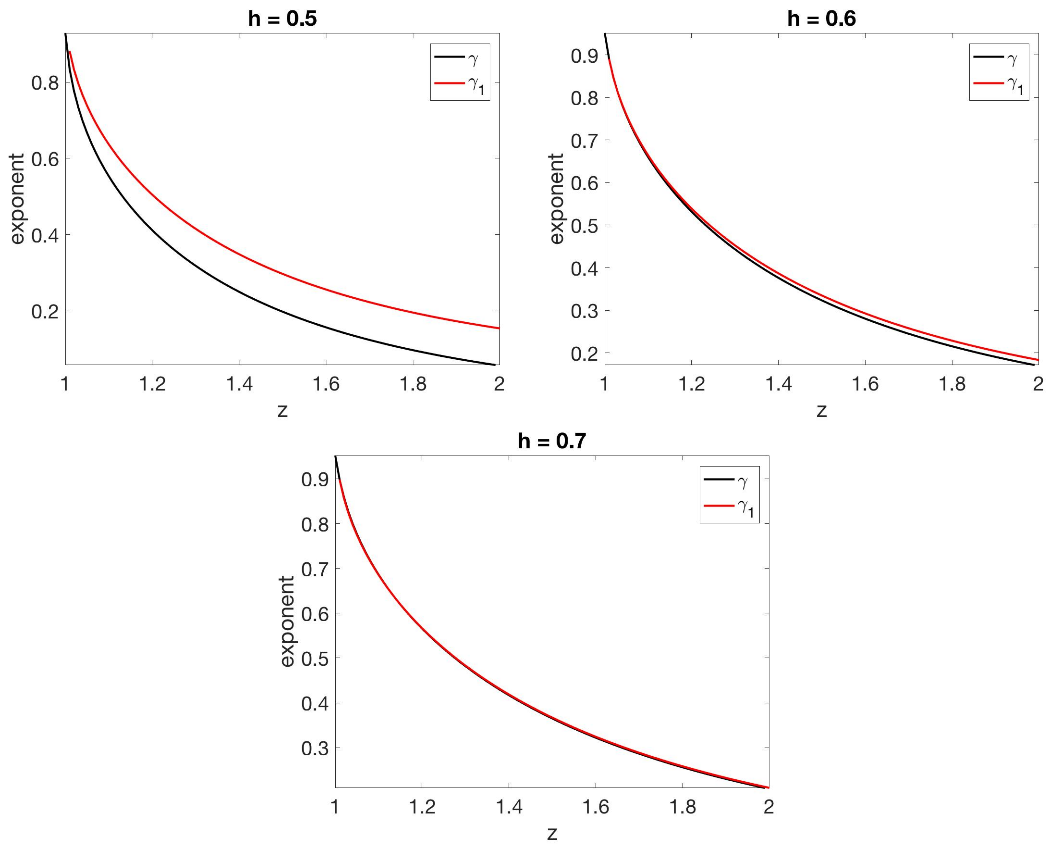


Figure 5.5: Comparison of γ and γ_1

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APPENDIX

Here we present some auxiliary results used in Chapter 5.

A.1 Extension of positivity

Proposition A.1. *Let f be analytic in $\mathbb{H}_h$ with $Sf = f$ (cf.* (5.24)*) and $f(\zeta) \sim -A\zeta^{-2}$ as $\zeta \to \infty$ for some $A > 0$. In addition assume $f'(0) \neq 0$, then the following are equivalent:*

(i) $\mathfrak{Im}\, f(x) > 0$ for all $x > 0$;

(ii) $\exists h' \in (0, h)$ s.t. $\mathfrak{Im}\, f(x - ih') > 0$ for all $x > 0$.

Proof. The second item immediately implies the first one. Indeed, the symmetry $Sf = f$ implies that $\mathfrak{Im} f = 0$ on the imaginary axis. Let $\Omega = \{\zeta : \mathfrak{Im}\zeta > -h', \ \mathfrak{Re}\zeta > 0\}$, note that $\mathfrak{Im} f \geq 0$ on $\partial\Omega$ and in fact $\min_{\partial\Omega} \mathfrak{Im} f = 0$, since $\mathfrak{Im} f$ approaches to zero at infinity (because of $f(\zeta) \sim -A\zeta^{-2}$) applying the strong maximum principle we conclude that $\mathfrak{Im} f > 0$ in Ω. (Note that the assumption $f'(0) \neq 0$ was not used here).

Let us now turn to the converse implication. Let $h_0 \in (0, h)$, then f is analytic in the closure $\overline{\mathbb{H}}_{h_0}$ and in particular is bounded inside the semidisc $D = \{\zeta : |\zeta + ih_0| \leq M\}$, where $M > 0$ is a large number that can be chosen such that $|f(\zeta)| \leq 2A/|\zeta|^2$ for all $\zeta \notin D$. With these two inequalities, it is straightforward to show that $\int_{\mathbb{R}} |f(x + iy)|^2 dx$ is bounded uniformly for $y > -h_0$. Thus, $f \in H^2(\mathbb{H}_{h_0})$ and following the proof of Lemma 5.3 we obtain the representation

$$f(\zeta) = \int_0^\infty \frac{d\sigma(\lambda)}{\lambda - (\zeta + ih_0)^2}, \qquad \zeta \in \mathbb{H}_{h_0},$$

where $d\sigma(\lambda) = \frac{1}{\pi}\mathfrak{Im} f(\sqrt{\lambda} - ih_0)d\lambda$. Using this, it is easy to find that f must have the more precise asymptotics, as $\zeta \to \infty$ in $\mathbb{H}_{h_0}$:

$$f(\zeta) \sim A\left(-\frac{1}{\zeta^2} + \frac{2ih_0}{\zeta^3}\right), \qquad A = \int_0^\infty d\sigma(\lambda).$$

But then for any $t \in (0, h_0)$

$$\mathfrak{Im}\, f(x - it) \sim \frac{2A(h_0 - t)}{x^3} > 0, \qquad x \to +\infty. \tag{A.1}$$

Assume, for the sake of contradiction that for each $t \in (0, h_0)$ there exists $x_t > 0$, such that $\mathfrak{Im} f(x_t - it) \leq 0$. Clearly, (A.1) implies that x_t remains bounded as $t \to 0^+$. Let us now extract convergent subsequence (without relabeling it) $x_t \to x_0 \geq 0$ as $t \to 0^+$, but then $\mathfrak{Im}\, f(x_0) \leq 0$. Assumption (i) implies that $x_0 = 0$. Let us show that this leads to a contradiction. Since $\mathfrak{Im}\, f(x_t) > 0$ and $\mathfrak{Im} f(x_t - it) \leq 0$, by continuity we conclude that $\exists \theta_t \in (0, 1]$ such that $\mathfrak{Im} f(x_t - i\theta_t t) = 0$. The symmetry $Sf = f$ implies that $\mathfrak{Im} f(-i\theta_t t) = 0$, therefore by the mean value theorem $\mathfrak{Im} f'(\tilde{x}_t - i\theta_t t) = 0$ for some $\tilde{x}_t \in (0, x_t)$. Taking limits as $t \to 0^+$ we obtain $\mathfrak{Im} f'(0) = 0$, but $f'(0) \in i\mathbb{R}$, hence this contradicts to the assumption $f'(0) \neq 0$. □

A.2 Power law bounds

Let $A_{z,h}(\epsilon)$ and $\gamma(z, h)$ be defined by (5.7) and (5.8) respectively, i.e.

$$A_{z,h}(\epsilon) = \sup\left\{|f(z)| \;: f \in H^2(\mathbb{H}_h) \text{ and } \|f\|_{H^2(\mathbb{H}_h)} \leq 1, \; \|f\|_{L^2(-1,1)} \leq \epsilon\right\}$$

and

$$\gamma(z, h) = \varliminf_{\epsilon \to 0} \frac{\ln A_{z,h}(\epsilon)}{\ln \epsilon}.$$

Note that $A_{z,h}(\epsilon)$ is non-increasing in h. Indeed, $\mathbb{H}_{h_1} \subset \mathbb{H}_{h_2}$ for $h_1 \leq h_2$ and so admissible functions for $A_{z,h_2}(\epsilon)$ are also admissible for $A_{z,h_1}(\epsilon)$, showing that $A_{z,h_2}(\epsilon) \leq A_{z,h_1}(\epsilon)$. Now dividing by $\ln \epsilon < 0$ and taking liminf in ϵ we conclude that $\gamma(z,h)$ is non-decreasing in h.

Let us turn to deriving power law upper and lower bounds on $A_{z,h}(\epsilon)$. We are going to use the following two results. The first one is analytic continuation from a boundary interval (cf. Section 4.2): for any $s \in \mathbb{H}_+$

$$\sup\{|f(s)| : f \in H^2(\mathbb{H}_+) \text{ and } \|f\|_{H^2(\mathbb{H}_+)} \leq 1,\ \|f\|_{L^2(-1,1)} \leq \delta\} \leq C(s)\delta^{\alpha(s)}, \tag{A.2}$$

where $C(s)^{-2} = \frac{s_i}{9}\left(\arctan\frac{s_r+1}{s_i} - \arctan\frac{s_r-1}{s_i}\right)$ with $s = s_r + is_i$ and $\alpha(s) = -\frac{1}{\pi}\arg\frac{s+1}{s-1} \in (0,1)$ is the angular size of $[-1,1]$ as seen from s, measured in the units of π radians. Moreover, the bound is optimal in δ and maximizer function attaining the bound (up to a constant independent of δ) in (A.2) is given by

$$G(\omega) = \frac{\delta}{\omega - \overline{s}} e^{\frac{i}{\pi}\ln\delta\ln\frac{1+\omega}{1-\omega}}, \qquad \omega \in \mathbb{H}_+ \tag{A.3}$$

where ln denotes the principal branch of logarithm.

The second one is analytic continuation from a circle (cf. Section 4.1). Namely let $\Gamma \subset \mathbb{H}_+$ be a circle and $s \in \mathbb{H}_+$ a point lying outside of Γ, then

$$\sup\{|f(s)| : f \in H^2(\mathbb{H}_+) \text{ and } \|f\|_{H^2(\mathbb{H}_+)} \leq 1,\ \|f\|_{L^2(\Gamma)} \leq \epsilon\} \simeq \epsilon^{\beta(s)}, \tag{A.4}$$

with implicit constants independent of ϵ and $\beta(s) = \frac{\ln|m(s)|}{\ln\rho}$, where m is the Möbius map transforming the upper half-plane into the unit disc and the circle Γ into a concentric circle of radius $\rho < 1$.

Lemma A.1. *There exist $r_0, r_1 \in (0,1)$ (depending on z,h) such that*

$$\epsilon^{r_1} \lesssim A_{z,h}(\epsilon) \lesssim \epsilon^{r_0}, \tag{A.5}$$

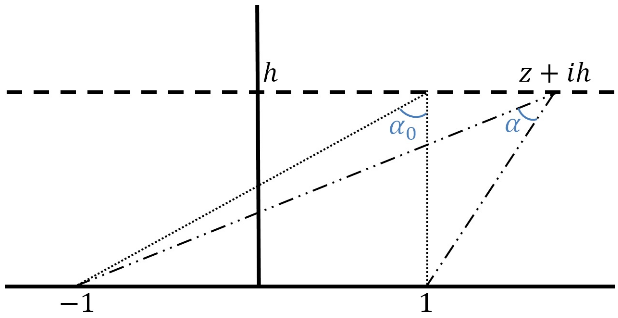

Figure A.1: Comparison of angles.

where the implicit constants depend only on h *and* z*. Moreover,* $r_1(h) \to 0$ *as* $h \to 0^+$*.*

Proof. The lower bound is obtained by introducing an ansatz function admissible for $A_{z,h}(\epsilon)$. Consider the function G in (A.3) with $s = ih$, then the ansatz function is going to be $f(\zeta) = G(\zeta + ih)$. Note that we can rewrite

$$G(\omega) = \frac{\delta^{\alpha(\omega)} e^{i\theta_\delta(\omega)}}{\omega + ih}, \qquad \theta_\delta(\omega) = \frac{1}{\pi} \ln \delta \ln \left| \frac{1+\omega}{1-\omega} \right|.$$

It is now clear that

$$\|G\|_{L^2((-1,1)+ih)} \lesssim \delta^{\alpha_0}, \qquad \alpha_0 = \min_{x\in[-1,1]} \alpha(x + ih) = \frac{1}{\pi} \arctan \frac{2}{h} \in (0, 1)$$

and $|G(z + ih)| \gtrsim \delta^\alpha$, where $\alpha = \alpha(z + ih) < \alpha_0$ (see Figure A.1). Thus,

$$\|f\|_{H^2(\mathbb{H}_h)} \lesssim 1, \qquad \|f\|_{L^2(-1,1)} \lesssim \delta^{\alpha_0}, \qquad |f(\omega_0)| \gtrsim \delta^\alpha. \tag{A.6}$$

Letting $\epsilon = \delta^{\alpha_0}$ we see that cf is an admissible function for $A_{z,h}(\epsilon)$, for some constant $c > 0$ independent of δ, hence

$$A_{z,h}(\epsilon) \geq c|f(z)| \gtrsim \delta^\alpha = \epsilon^{r_1},$$

where $r_1 = r_1(z, h) = \alpha/\alpha_0 \in (0, 1)$. It remains to notice that $r_1(z, h) \to 0$ as $h \to 0^+$.

Let us now turn to the upper bound. Let f be an admissible function for $A_{z,h}(\epsilon)$, it is clear that f is also admissible for (A.2) with $\delta = \epsilon$. However, applying the estimate in (A.2) at the point $z > 1$ doesn't give a useful bound, since $\alpha(z) = 0$. Instead let us apply (A.2) at the points s lying on the circle $\mathcal{C} = \{s \in \mathbb{H}_+ : |s - i| = \frac{1}{2}\}$. It is clear that the angle $\alpha(s)$ is the smallest at the top point of the circle, i.e. at $s_0 = \frac{3}{2}i$. Moreover, obviously the constant $C(s)$ in (A.2) is uniformly bounded for all $s \in \mathcal{C}$. Thus,

$$|f(s)| \lesssim \epsilon^{\beta_0}, \qquad \forall s \in \mathcal{C}, \quad \text{where} \quad \beta_0 = \alpha(s_0) = \frac{1}{\pi} \arctan \frac{12}{5}$$

and the implicit constant is independent of s and ϵ. In particular, $\|f\|_{L^2(\mathcal{C})} \lesssim \epsilon^{\beta_0}$. Now we can apply (A.4) to the function $f(\cdot - ih)$ at the point $s = z + ih$ and obtain

$$|f(z)| \lesssim \epsilon^{r_0}, \qquad r_0 = \beta_0 \cdot \beta(z + ih) = \beta_0 \frac{\ln |m(z + ih)|}{\ln \rho}, \tag{A.7}$$

where $m(z) = \frac{z - z_0}{z + z_0}$ with $z_0 = \frac{i}{2}\sqrt{4h^2 + 8h + 3}$ and $\rho = 2h + 2 - \sqrt{4h^2 + 8h + 3}$. Taking supremum over f in (A.7) we conclude the proof of the upper bound. □

As an immediate corollary from Lemma A.1 we see that for any $h > 0$

$$\gamma(z, h) \in [r_0(z, h), r_1(z, h)] \subset (0, 1)$$

and also $\gamma(z, h) \to 0$ as $h \to 0^+$.

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