EXACT RELATIONS AND LINKS FOR FIBER-REINFORCED ELASTIC COMPOSITES

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by
Meredith Hegg
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Examining Committee Members:

Yury Grabovsky, Mathematics
Shiferaw Berhanu, Mathematics
Benjamin Seibold, Mathematics
Shari Moskow, External Member, Drexel University, Mathematics

ABSTRACT

Exact Relations and Links for Fiber-Reinforced Elastic Composites

Meredith Hegg
DOCTOR OF PHILOSOPHY

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Professor Yury Grabovsky, Chair

Predicting the effective elastic properties of a composite material based on the elastic properties of the constituent materials is extremely difficult, even when the microstructure is known. However, there are cases where certain properties in constituents always carry over to a composite, regardless of the microstructure of the composite. We call such instances *exact relations*. The general theory of exact relations allows us to find all of these instances in a wide variety of contexts including elasticity, conductivity, and piezoelectricity. We combine this theory with ideas from representation theory to find all exact relations for fiber-reinforced polycrystalline composites.

We further extend these ideas to the concept of links. When two composites have the same microstructure but different constituent materials, their effective tensors may be related. We use the theory of exact relations to find such relations, which we call links. In this work we describe a special set of links between elasticity tensors of fiber-reinforced polycrystalline composites. These links allow us to generalize certain results from specific examples to generate new information about this widely-used class of composites. In particular, we apply the link to obtain information about composites made from two transversely isotropic materials and polycrystals made from one orthotropic material.

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CHAPTER 1

INTRODUCTION

The greatest advantage of composite materials lies in the fact that some properties of the constituent materials may remain present in the composite while others may not. Here we use the term composite to describe any material whose structure is heterogeneous on some micro-scale but behaves like a homogeneous material on a macro-scale. Whether by layering, laminating, injecting, or encasing different constituent materials, engineers have developed new composites with effective properties that cannot be found in nature. For example, there now exist materials that are both lightweight and strong and therefore useful in applications ranging from orthopedic casting to aerospace technology. Modern ski construction uses composites to obtain a unique combination of flexibility and torsional rigidity. Yet exactly when and how a composite will retain the properties of its constituent materials and when and how it will not remains a largely unanswered question. Certainly the microstructure of the composite material plays a significant role. However, there are also relations that hold regardless of microstructure.

An interesting example of such a special relation comes from elastically *isotropic* materials. A material is said to be isotropic if its behavior is unaffected by its orientation in space. An elastically isotropic material can be characterized by two numbers: the *bulk* modulus, κ, and the *shear* modulus, μ. The bulk modulus describes how the material responds to uniform compression while the shear modulus tells us how it responds to shearing stresses. In general, a composite material made of two elastically isotropic materials may be anisotropic. However, Hill found [17] that if the two constituent materials are isotropic *and* have the same shear modulus, μ, then the composite will be isotropic with shear modulus μ, regardless of the microstructure of the material. Furthermore, the bulk modulus κ^* will be given by

$$\frac{1}{3\kappa^* + 4\mu} = \frac{\theta_1}{3\kappa_1 + 4\mu} + \frac{\theta_2}{3\kappa_2 + 4\mu} \tag{1.1}$$

where θ_1 and θ_2 represent the volume fractions and κ_1 and κ_2 represent the bulk moduli of the two constituents.

A specific material property that is maintained in the construction of composites regardless of microstructure is called an *exact relation*. In this example by Hill, the set of isotropic materials with a given shear modulus forms an exact relation. Given the high cost of producing composite materials, the ability to obtain results such as these without conducting expensive tests is valuable. Being aware of exact relations allows us to both take advantage of them when we hope to maintain a characteristic and avoid them when we want to change a given property.

We can find exact relations for a number of material properties including elasticity, conductivity, and piezoelectricity. In each of these cases we have two field quantities, say σ and ε, that are related linearly in what is called a constitutive law:

$$C(x)\varepsilon(x) = \sigma(x).$$

The tensor $C(x)$ represents the material properties of the composite at each point x. Much of the early work on exact relations was on *uniform field relations*. These arise whenever there exist constant fields σ and ε such that

$$C(x)\varepsilon = \sigma \quad \forall x.$$

In this case the uniform fields σ and ε also satisfy $C^*\varepsilon = \sigma$, where C^* is the constant *effective tensor*, which describes the composite as a homogeneous material. (We will define effective tensors rigorously in chapter 2.) Therefore the set of materials satisfying a certain uniform field relation form an exact relation. The exact relation in (1.1) stems from this idea.

Hill's work in elasticity was followed by results from Lurie, Cherkaev, and Fedorov [19, 20, 21] and Francfort and Tartar [9]. Cribb [6], Rosen [29], Hashin [14], Schulgasser [31], and Dvorak [8] found exact relations in the context of thermoelasticity. Dvorak also specifically applied uniform field arguments to fiber-reinforced elastic composites [7]. Exact results for piezoelectric composites were discovered by Benveniste and Dvorak [1, 3, 4, 5] while Milgrom and Shtrikman studied thermoelectricity [23, 24, 25]. Benveniste also found exact relations specifically for polycrystalline composites in the context of thermopiezoelectricity [2]. An excellent summary of exact relations can be found in [28]. Finally in [10, 12] and [13], the elegant mathematical theory of exact relations and links was developed, allowing us to find all exact relations in a wide range of physical contexts, including all of the above. In [33], To used this theory to find all exact relations for three-dimensional conductors exhibiting the Hall effect.

In this work we use the theory of exact relations to obtain information about fiber-reinforced elastic composites. Fiber-reinforced composites are those whose microstructure is independent of the longitudinal coordinate and can therefore be described by a single transverse cross-section. Furthermore we focus our attention on *polycrystalline* exact relations. In other words, we require that if a material with elastic tensor C is an admissible constituent, then so is the rotated material $\mathcal{R} \cdot C$ where $\mathcal{R} \in SO_k(2) \subset SO(3)$. Here $SO_k(2)$ is the subgroup of $SO(3)$ consisting of all rotations around the fiber axis $k \in \mathbb{R}^3$. As our notation suggests, $SO_k(2)$ is isomorphic to the group $SO(2)$. The image we have in mind is of a composite created by injecting circular anisotropic fibers into an isotropic matrix and in which the fibers may be rotated arbitrarily around their longitudinal axes. The exact relations we seek hold regardless of fiber position and orientation.

The general theory of exact relations takes a geometric point of view, seeing exact relations as surfaces in the space of elasticity tensors. It utilizes an explicit diffeomorphism that maps all such surfaces to $SO(2)$-invariant subspaces with special algebraic properties. Tools from representation theory then help us to identify all subspaces that satisfy these properties, and hence all exact relations.

We extend our work further by applying the theory of exact relations to compute *links* between tensors. When two composites have the same microstructure but different constituent materials, their effective tensors may be related. In this case we say that a link exists between the two composites. For example, Mendelson [22] found that for any two-dimensional local conductivity tensor $\sigma(x)$, if a second tensor is defined by

$$\sigma'(x) = \frac{\sigma(x)}{\det\, \sigma(x)},$$

then the same relation holds between the effective tensors of the two composites:

$$\sigma'^{*} = \frac{\sigma^*}{\det\, \sigma^*}. \tag{1.2}$$

We are interested in links because in general they give us more information than exact relations. If we apply a single link to different sets of tensors with particular properties, we may obtain multiple exact relations. By re-characterizing links as exact relations in a higher dimensional space, we can use the theory of exact relations to find and describe links.

We begin in chapter 2 with a brief background on the mathematical theory of elastic composites. Then, in chapters 3 and 4, we give an overview of the general theory of exact relations and links. The tools from representation theory which will help us find these exact relations are described in chapter 5 while in chapter 6 we define notation and compute the algebraic structures needed in the case of fiber-reinforced elastic composites. A special set of automorphisms will help us to simplify our list of exact relations. These automorphisms are given in chapter 7. The calculations that lead us to a complete set of exact relations in algebraic variables are described in chapter 8 while the list itself is in appendix A. We then use the theory to compute a special set of links that, in particular, establishes equivalence between exact relation surfaces passing through different points. Chapter 9 shows the calculations that derive this link. Finally, in chapter 10, we apply our link to specific cases to obtain information about fiber-reinforced composites.

CHAPTER 2

ELASTICITY AND COMPOSITES

Let $\Omega \subset \mathbb{R}^3$ represent an elastic body undergoing an infinitesimal deformation given by $y(x) = x + \epsilon u(x)$ for each $x \in \Omega$, where $\epsilon > 0$ is small. Let

$$\varepsilon = e(u) = \frac{1}{2}(\nabla u + \nabla u^T), \tag{2.1}$$

which represents the linear strain, and let $\sigma(x)$ be the Cauchy stress tensor. That is, given any surface $S \subset \Omega$ with unit normal $\mathbf{n}(x)$, the force at $x \in S$ is given by $\sigma(x)\mathbf{n}(x)$. Note that, by construction,

$$\nabla^c \times \nabla^r \times \varepsilon = 0 \tag{2.2}$$

in Ω, where $\nabla^c\times$ and $\nabla^r\times$ denote curl taken by column and by row, respectively. Furthermore, by the balance of forces and the divergence theorem, given any $V \subset \Omega$,

$$0 = \int_{\partial V} \sigma(x)\mathbf{n}(x)dS = \int_V \nabla \cdot \sigma(x)dx$$

which implies

$$\nabla \cdot \sigma = 0 \tag{2.3}$$

in Ω. Considering ε, which by definition is symmetric, the balance of torque tells us

$$0 = \int_{\partial V} (\sigma(x)\mathbf{n}(x) \times x)dS = \int_{\partial V} \pi(x)\sigma(x)\mathbf{n}(x)dS = \int_V \nabla \cdot (\pi(x)\sigma(x))dx$$

where $\pi(x)$ is the skew-symmetric matrix such that $\pi(x)a = a \times x$ for all $a \in \mathbb{R}$. So $\nabla \cdot (\pi(x)\sigma(x)) = 0$ for all $x \in \Omega$, from which it follows that σ is symmetric.

For any vector space U we will use $\mathrm{Sym}(U)$ to denote the set of symmetric linear maps from U to itself. In particular, $\mathrm{Sym}(\mathbb{R}^3)$ represents the Euclidean space of 3×3 symmetric matrices with the inner product

$$\langle E_1, E_2 \rangle = \frac{1}{2}\mathrm{Tr}(E_1 E_2) \tag{2.4}$$

for all E_1, E_2. Note that $\sigma(x), \varepsilon(x) \in \mathrm{Sym}(\mathbb{R}^3)$ for all $x \in \Omega$. By Hooke's law there exists a symmetric fourth-order tensor field C such that $C(x) \in \mathcal{S} := \mathrm{Sym}(\mathrm{Sym}(\mathbb{R}^3))$ for each $x \in \Omega$ and σ and ε satisfy the constitutive relation

$$\sigma(x) = C(x)\varepsilon(x)$$

for all $x \in \Omega$. Furthermore, $C(x)$ is positive definite for all $x \in \Omega$. Let $\mathcal{T} \subset \mathcal{S}$ denote the subset of positive definite, symmetric elasticity tensors. As is customary, we will sometimes use the word *tensor* in place of *tensor field* and simply assume that the context will make our meaning clear.

Now we turn our attention to composite materials. Suppose we have a composite made with two constituent materials whose (constant) elasticity tensors are C_1 and C_2. Then we can write the local elasticity tensor of the composite as

$$C(x) = \chi_1(x)C_1 + \chi_2(x)C_2$$

where χ_1 and χ_2 are the characteristic functions of the regions occupied by the two materials. We would like to be able to solve the elliptic boundary value problem

$$\begin{cases} \nabla \cdot (C(x)e(u(x))) = 0 \quad \text{for} \quad x \in \Omega \\ e(u(x)) = g(x) \quad \text{for} \quad x \in \partial\Omega \end{cases} \tag{2.5}$$

for various $g \in \mathcal{H}^1(\partial\Omega)$. However, the complexity of χ_1 and χ_2 may make this excessively computationally expensive, especially since the values of $C(x)$ oscillate wildly as we move from material C_1 to material C_2. Furthermore, in some cases the precise microstructure is not known.

Instead we focus on periodic composites and the homogenization problem. Let χ_1 and χ_2 now be periodic charactaristic functions with period cell $Q \subset \mathbb{R}^3$. We define

$$C_\epsilon(x) = C(x/\epsilon) = \chi_1(x/\epsilon)C_1 + \chi_2(x/\epsilon)C_2$$

so that as $\epsilon \to 0$, $C_\epsilon(x)$ describes the local elasticity tensor of a composite with increasingly fine microstructure. Suppose we have the sequence of solutions u_ϵ to the elliptic boundary value problem

$$\begin{cases} \nabla \cdot (C_\epsilon(x)e(u_\epsilon(x))) = 0 \quad \text{for} \quad x \in \Omega \\ u_\epsilon(x) = g(x) \quad \text{for} \quad x \in \partial\Omega \end{cases} . \tag{2.6}$$

We will say the constant tensor C^* is the homogenized tensor or *effective tensor* for $C(x)$ if the sequence u_ϵ of solutions to (2.6) converges weakly in $\mathcal{H}^1(Q)$ to u_0 which solves

$$\begin{cases} \nabla \cdot (C^* e(u_0(x))) = 0 \quad \text{for} \quad x \in \Omega \\ u_0(x) = g(x) \quad \text{for} \quad x \in \partial\Omega \end{cases}$$

and $C_\epsilon e(u_\epsilon)$ converges weakly in $L^2(Q)$ to $C^* e(u_0)$.

For periodic composites we can describe C^* in terms of the cell problem. Let $\langle\cdot\rangle$ denote the average of a field over the period cell. Taking any $\xi \in \mathrm{Sym}(\mathbb{R}^3)$, we can find periodic strain $e(u)$ satisfying $\langle e(u)\rangle = \xi$ and

$$\nabla \cdot (C(x)e(u)) = 0.$$

Note that $e(u)$ depends linearly on ξ. If we define C^* as the constant tensor satisfying

$$C^*\xi = C^*\langle e(u)\rangle = \langle C(x)e(u)\rangle,$$

for all pairs $(\xi, e(u))$, then C^* is the effective tensor for $C(x)$ [30, 32]. In general, C^* depends heavily on the microstructure of the composite. We are interested here, however, in special cases where C^* is microstructure-independent.

CHAPTER 3

THEORY OF EXACT RELATIONS

Let $\mathcal{M}$ be a set of elasticity tensors representing a set of materials used to make composites. The *G-closure* of $\mathcal{M}$ is the set of effective tensors of periodic composites made using any combination of materials from $\mathcal{M}$ in any configuration. A smooth surface of codimension greater than zero that is G-closed is called an exact relation. Thus exact relations tell us which properties of tensors are invariant under construction of composites.

As discussed in the introduction, many of the first exact relations to be discovered followed from uniform field relations. For example, we may consider the exact relation identified in [16] and described in [27] regarding materials that exhibit cubic symmetry. If C is the elasticity tensor of such a material and I represents the 3×3 identity matrix, then there exists $\kappa > 0$ such that $CI = \kappa I$. That is, we can think of κ as representing the bulk modulus of the material even though the material is not fully isotropic. A simple uniform field argument tells us that the effective bulk modulus of a statistically isotropic polycrystal made with this material is the same as the bulk modulus of the pure crystal. This result was generalized by He [15] to all materials that respond isotropically to isotropic stress or strain.

Our approach here is based on the general theory of exact relations developed by Milton, Grabovsky, and Sage [10, 12, 13]. While hereafter for simplicity we will use the language and notation of linear elasticity, the general theory allows us to find all exact relations in a wide range of physical contexts including conductivity, piezoelectricity, and thermoelasticity. In each case we have a linear relationship between two field quantities, taking values in a space $\mathfrak{F}$, that satisfy certain differential constraints. In the case of elasticity, the fields $\varepsilon(x)$ and $\sigma(x)$ take values in $\mathfrak{F} = \mathrm{Sym}(\mathbb{R}^3)$.

Let $\mathcal{D} \subset \mathbb{R}^3$ be a set of unit vectors representing some set of admissible directions in space. For each $\mathbf{n} \in \mathcal{D}$, we need subspaces $\mathcal{E}_{\mathbf{n}}$ and $\mathcal{J}_{\mathbf{n}}$ such that $\mathcal{E}_{\mathbf{n}} \oplus \mathcal{J}_{\mathbf{n}} = \mathfrak{F}$ and $\hat{\varepsilon}(\mathbf{n}) \in \mathcal{E}_{\mathbf{n}}$ and $\hat{\sigma}(\mathbf{n}) \in \mathcal{J}_{\mathbf{n}}$ for arbitrary fields ε and σ satisfying the differential constraints. That such subspaces exist in the case of elasticity follows from the differential constraints (2.2) and (2.3). Now fix an arbitrary reference tensor $C_0 \in \mathcal{T}$. Let $\Gamma(\mathbf{n})$ be the orthogonal projection onto $C_0^{1/2}\mathcal{E}_{\mathbf{n}}$. A key property of Γ is the fact that it is local in Fourier space. That is, for any f, an L^2 periodic vector field taking values in $\mathfrak{F}$,

$$\widehat{\Gamma f}(\mathbf{n}) = \Gamma(\mathbf{n})\hat{f}(\mathbf{n})$$

for all $\mathbf{n} \in \mathcal{D}$.

We take advantage of this, following Milton [26], and define:

$$W_{\mathbf{n}}(C) = [I - (I - C_0^{-1/2} C C_0^{-1/2})\Gamma(\mathbf{n})]^{-1}(I - C_0^{-1/2} C C_0^{-1/2}). \tag{3.1}$$

This map has a special property in the case of laminates. These are composites whose microstructures are essentially one-dimensional and which are constructed by layering slabs of two or more materials in some normal direction. Laminates represent an important subset of composites and are included in the set of all fiber-reinforced composites. If $C(x)$ represents a material made via lamination in the direction n, then

$$W_{\mathbf{n}}(C^*) = \langle W_{\mathbf{n}}(C(x)) \rangle.$$

In particular, this tells us that if we laminate materials C_1 and C_2 with volume fractions θ_1 and θ_2, then

$$W_{\mathbf{n}}(C^*) = \theta_1 W_{\mathbf{n}}(C_1) + \theta_2 W_{\mathbf{n}}(C_2).$$

If $\mathcal{M}$ is an exact relation then certainly $\mathcal{M}$ is closed under lamination and so the above implies that $W_{\mathbf{n}}(\mathcal{M})$ is a convex subset of an affine space, $\Pi_{\mathbf{n}}$, of the same dimension as $\mathcal{M}$. If we can pick $C_0 \in \mathcal{M}$, then $\Pi_{\mathbf{n}}$ is in fact a vector space. Furthermore, from [12] we know that $W_{\mathbf{n}}$ is a diffeomorphism on $\mathcal{T}$, the subspaces $\Pi_{\mathbf{n}}$ do not depend on $\mathbf{n}$, and $\Pi := \Pi_{\mathbf{n}}$ (for any $\mathbf{n}$) has an algebraic structure which we will now describe.

Fix $\mathbf{n}_0 \in \mathcal{D}$ and define

$$\mathcal{A} = \mathrm{Span}\{\Gamma(\mathbf{n}) - \Gamma(\mathbf{n}_0) : \mathbf{n} \in \mathcal{X}\}.$$

Then $\mathcal{A}$ encodes the fiber-reinforced structure of the composite and does not depend on our choice of $\mathbf{n}_0$. For each $A \in \mathcal{A}$, define a product $*^A$ on $\mathcal{S}$ by:

$$K_1 *^A K_2 = \frac{1}{2}(K_1 A K_2 + K_2 A K_1) \tag{3.2}$$

for all $K_1, K_2 \in \mathcal{S}$. Then if $\mathcal{M}$ is an exact relation and Π is the vector space containing $W_{\mathbf{n}}(\mathcal{M})$ as described above, from [12] we have:

$$K_1 *^A K_2 \in \Pi \;\; \forall\, A \in \mathcal{A}, \;\; \forall\, K_1, K_2 \in \Pi. \tag{3.3}$$

The product $*^A$ is commutative and non-associative. It is called a *Jordan product* [18] [1]. We say Π is a *Jordan multi-algebra* since Π is closed with respect to a whole family of Jordan multiplications. Note that (3.3) holds if and only if

$$KAK \in \Pi \;\; \forall\, K \in \Pi, \;\; \forall\, A \in \mathcal{A}. \tag{3.4}$$

Also, we will write $K^{*^A 2}$ to mean $K *^A K$.

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[1] Jordan products are commutative but not associative. Instead they satisfy the Jordan identity $(xy)(xx) = x(y(xx))$. Given any associative product, a Jordan product may be defined as in (3.2).

That Π satisfies (3.3) follows from the stability of $\mathcal{M}$ with respect to making laminates and is certainly necessary for $\mathcal{M}$ to be an exact relation. In addition to this necessary condition, we have a related sufficient condition. We say Π satisfies the *j-chain property* if for all $K_1, K_2, ..., K_j \in \Pi$ and $A_1, A_2, ..., A_{j-1} \in \mathcal{A}$,

$$K_1 A_1 K_2 A_2 ... A_{j-1} K_j + K_j A_{j-1} ... A_2 K_2 A_1 K_1 \in \Pi.$$

In [12] it was shown that if Π satisfies the j-chain property for $j = 2, 3,$ and 4, then it represents the image of an exact relation. However, so far every subspace found to satisfy the 2-chain property has satisfied 3 and 4-chain properties.

The general theory also provides us with a simple way to calculate volume fraction relations, i.e., relations that depend only on the volume fractions of the two materials. For any algebraic subspace, Π, define

$$\Pi^{*^A 2} = \mathrm{Span}\{KAK : K \in \Pi, A \in \mathcal{A}\}.$$

Volume fraction relations correspond to subspaces where $\Pi^{*^A 2} \neq \Pi$ since, if $\mathcal{P}_{(\Pi^{*^A 2})^\perp}$ represents the projection onto $(\Pi^{*^A 2})^\perp$, from [11] we have that

$$\mathcal{P}_{(\Pi^{*^A 2})^\perp}(W_{\mathbf{n}}(C^*)) = \mathcal{P}_{(\Pi^{*^A 2})^\perp}\langle W_{\mathbf{n}}(C(x))\rangle.$$

In our complete list of algebraic subspaces in appendix A we will make note of cases where $\Pi^{*^A 2} \neq \Pi$.

To make calculations easier, it is sometimes desirable to put $\mathcal{A}$ in a simpler form. Suppose B_1 is an arbitrary isotropic tensor, not necessarily symmetric. Let $\Pi' = B_1^{-T} \Pi B_1^{-1}$ and let $\mathcal{A}' = B_1 \mathcal{A} B_1^T$. Then

$$K_1 *^A K_2 \in \Pi \Longleftrightarrow K_1' *^{A'} K_2' \in \Pi' \tag{3.5}$$

for all $A' = B_1 A B_1^T \in \mathcal{A}'$ and for all $K_1' = B_1^{-T} K_1 B_1^{-1}, K_2' = B_1^{-T} K_2 B_1^{-1} \in \Pi'$. Thus, choosing an appropriate B_1, we can simplify $\mathcal{A}'$.

Finding subspaces satisfying the 2-chain condition (3.3) is not easy. This, in part, is why we focus on polycrystalline exact relations. An exact relation, $\mathcal{M}$, is polycrystalline if for all $C \in \mathcal{M}$, we have that $\mathcal{R} \cdot C \in \mathcal{M}$ for all $\mathcal{R} \in SO(2)$. In physical terms, $\mathcal{M}$ contains the effective tensors of all composites that can be made by rotating the constituent materials. If our reference tensor $C_0 \in \mathcal{M}$ is isotropic, i.e. $\mathcal{R} \cdot C = C$ for all rotations $\mathcal{R}$, then the general theory tells us that

$$\mathcal{R} \cdot W_{\mathbf{n}}(C) = W_{\mathcal{R}\mathbf{n}}(\mathcal{R} \cdot C)$$

for all $C \in \mathcal{M}$. But since Π does not depend on $\mathbf{n}$, this implies that Π is rotation-invariant if and only if $\mathcal{M}$ is polycrystalline. Hereafter when we say exact relation we will mean polycrystalline exact relation. Thanks to the representation theory of two-dimensional rotations, we can develop a systematic strategy to find all rotation-invariant subspaces of $\mathcal{S}$. We then find which of these subspaces satisfy the algebraic condition (3.3). Finally, we need to invert $W_{\mathbf{n}}$ to describe the polycrystalline exact relations in physical variables.

Inverting $W_{\mathbf{n}}$ is not straightforward. While we could use matrix notation to write elasticity tensors as 6×6 matrices and ask a computer algebra program to calculate the inverse, we would then struggle mightily to interpret the formula. Instead we solve a simpler problem. We take advantage of the following statement proved in [12]. If Π' is an algebraic subspace and $S_0 \in \mathcal{S}$ is such that

$$K'(B_1 \bar{\Gamma} B_1^T - S_0)K' \in \Pi' \tag{3.6}$$

for all $K' \in \Pi'$, then

$$\mathcal{M} = \{C = C_0 - C_0^{1/2} B_1^T (I + K'S_0)^{-1} K' B_1 C_0^{1/2} : K' \in \Pi'\} \cap \mathcal{T} \tag{3.7}$$

is an exact relation. Certainly $S_0 = B_1\bar{\Gamma}B_1^T$ will work in every case, however simpler choices of S_0 are often possible. In our complete list of algebraic subspaces in appendix A we will make note of simple values for S_0.

CHAPTER 4

THEORY OF LINKS

The theory of exact relations becomes even more powerful when extended to *links*. Given local elasticity tensors of two composites $C(x)$ and $C'(x)$, a function $\mathcal{G}$ on $\mathcal{T}\times\mathcal{T}$ is called a link if $\mathcal{G}(C^*, C'^*) = 0$ whenever $\mathcal{G}(C(x), C'(x)) = 0$. In the example from Mendelson above, we have a link given by $\mathcal{G}(\sigma, \sigma') = \frac{\sigma}{\det\ \sigma} - \sigma'$. It is easy to see that links are simply exact relations in the space $\mathcal{T} \times \mathcal{T}$ when we define

$$\hat{\mathcal{M}} = \{(C, C') : \mathcal{G}(C, C') = 0\}$$

and note that $\hat{\mathcal{M}}$ represents a link if $(C, C') \in \hat{\mathcal{M}}$ implies $(C^*, C'^*) \in \hat{\mathcal{M}}$. We can now use the theory described in chapter 3 to find links. If we have a pair of fixed transversely isotropic tensors $(C_1, C_2) \in \hat{\mathcal{M}}$ we can map this link to the set

$$\hat{W}_{\mathbf{n}}(\hat{\mathcal{M}}) = \{(W^1_{\mathbf{n}}(C), W^2_{\mathbf{n}}(C')) : (C, C') \in \hat{\mathcal{M}}\}$$

where $W^i_{\mathbf{n}}$ is defined as in (3.1) using $C_0 = C_i$ for $i = 1, 2$. Then once again, $\hat{W}_{\mathbf{n}}(\hat{\mathcal{M}})$ is a convex subset of some subspace $\hat{\Pi}$ which has the same dimension as $\mathcal{M}$ and does not depend on $\mathbf{n}$. Also, $\hat{\Pi}$ is a Jordan multi-algebra.

Let $\Gamma_i(\mathbf{n})$ be the orthogonal projection onto $C_i^{1/2}\mathcal{E}_{\mathbf{n}}$. For any fixed $\mathbf{n}_0 \in \mathcal{D}$, define

$$\hat{\mathcal{A}} = \mathrm{Span}\{(\Gamma_1(\mathbf{n}) - \Gamma_1(\mathbf{n}_0), \Gamma_2(\mathbf{n}) - \Gamma_2(\mathbf{n}_0) : \mathbf{n} \in \mathcal{D}\}.$$

Rewriting $\hat{A} \in \hat{\mathcal{A}}$ and $\hat{K} \in \hat{\Pi}$ as diagonal matrices, the general theory of exact relations tells us that if $\hat{\mathcal{M}}$ represents a link, then as in (3.4), $\hat{\Pi}$ is a subspace satisfying

$$\begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} \in \hat{\Pi}$$

for all $[A_1, A_2] = \hat{A} \in \hat{\mathcal{A}}$ and for all $[K_1, K_2] = \hat{K} \in \hat{\Pi}$. Furthermore, since the sets

$$\mathcal{M}_1 = \{C : [C, C'] \in \hat{\mathcal{M}} \text{ for some } C'\}$$

$$\mathcal{M}_2 = \{C' : [C, C'] \in \hat{\mathcal{M}} \text{ for some } C\}$$

are exact relations in $\mathcal{T}$, if we define

$$\Pi_1 = \{K_1 : [K_1, K] \in \hat{\Pi} \text{ for some } K\}$$

$$\Pi_2 = \{K_2 : [K, K_2] \in \hat{\Pi} \text{ for some } K\}$$

$$I_1 = \{K \in \Pi_1 : [K, 0] \in \hat{\Pi}\}$$

$$I_2 = \{K \in \Pi_2 : [0, K] \in \hat{\Pi}\}$$

then Π_i is a Jordan multi-algebra and I_i is an ideal in Π_i for $i = 1, 2$. Consider the map $\Psi : \Pi_1/I_1 \longrightarrow \Pi_2/I_2$ defined

$$\Psi(\bar{K_1}) = \bar{K_2}$$

where $\bar{K_i}$ represents the equivalence class of K_i for $i = 1, 2$ and $[K_1, K_2] \in \hat{\Pi}$. We note Ψ is well defined since if $[K_1, K_2], [K_1, K_3] \in \hat{\Pi}$, then $[0, K_2 - K_3] \in \hat{\Pi}$ and so $K_2 - K_3 \in I_2$. That Ψ is one to one and onto is straightforward. Now we can reformulate our search for links as a search for these bijections and ideals.

We will focus here on a special set of links corresponding to bijections $\mathcal{F} : \mathcal{T} \to \mathcal{T}$ so that each link can be written $\mathcal{G}(C, C') = \mathcal{F}(C) - C'$ or

$$\hat{\mathcal{M}} = \{(C, \mathcal{F}(C)) : C \in \mathcal{T}\}.$$

So $\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{T}$, $\Pi_1 = \Pi_2 = \mathcal{S}$, and $I_1 = I_2 = \{0\}$. We must now consider what these functions $\mathcal{F}$ look like in the algebraic variables. Suppose $\hat{\Pi}$ is the subspace containing $\hat{W}_{\mathbf{n}}(\hat{\mathcal{M}})$ as above. Then, since each $W_{\mathbf{n}}^i$ is a diffeomorphism and $\mathcal{F}$ is a bijection, we know there exists a bijection $\Phi : \mathcal{S} \to \mathcal{S}$ such that

$$\hat{\Pi} = \{(K, \Phi(K)) : K \in \mathcal{S}\}.$$

That is, $\Phi = W_{\mathbf{n}}^2 \circ \mathcal{F} \circ (W_{\mathbf{n}}^1)^{-1}$. Since $\hat{\Pi}$ is a subspace, Φ must be linear. Furthermore, since $\hat{\mathcal{M}}$ is an exact relation, we have from (3.3) that

$$\begin{bmatrix} K & 0 \\ 0 & \Phi(K) \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} \tilde{K} & 0 \\ 0 & \Phi(\tilde{K}) \end{bmatrix} + \begin{bmatrix} \tilde{K} & 0 \\ 0 & \Phi(\tilde{K}) \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} K & 0 \\ 0 & \Phi(K) \end{bmatrix} \in \hat{\Pi}$$

for all $K, \tilde{K} \in \mathcal{S}$ and for all $(A_1, A_2) \in \hat{\mathcal{A}}$. But then

$$\Phi(K *^{A_1} \tilde{K}) = \Phi(K) *^{A_2} \Phi(\tilde{K}) \ \ \forall\, (A_1, A_2) \in \hat{\mathcal{A}}, \ \ \forall\, K, \tilde{K} \in \mathcal{S}. \tag{4.1}$$

Rather than searching for Φ that satisfy (4.1), we recall how we modified $\mathcal{A}$ using (3.5), and note that we may also modify $\hat{\mathcal{A}}$ via

$$\hat{B}\hat{A}\hat{B}^T = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} B_1^T & 0 \\ 0 & B_2^T \end{bmatrix}$$

with the hope that we can set

$$B_1 A_1 B_1^T = B_2 A_2 B_2^T = A \in \mathcal{A}. \tag{4.2}$$

We then look for Φ satisfying

$$\Phi(K *^A \tilde{K}) = \Phi(K) *^A \Phi(\tilde{K}) \ \ \forall\ A \in \mathcal{A}, \ \ \forall\ K, \tilde{K} \in \mathcal{S}.$$

Because we are focusing on polycrystalline exact relations and links, we would also like

$$\mathcal{R} \cdot \begin{bmatrix} K & 0 \\ 0 & \Phi(K) \end{bmatrix} = \begin{bmatrix} \mathcal{R} \cdot K & 0 \\ 0 & \mathcal{R} \cdot \Phi(K) \end{bmatrix} \in \hat{\Pi} \tag{4.3}$$

for all $\mathcal{R} \in SO_k(2) \subset SO(3)$, and for all $K \in \mathcal{S}$. Of course this holds if and only if

$$\mathcal{R} \cdot \Phi(K) = \Phi(\mathcal{R} \cdot K). \tag{4.4}$$

We can find all Φ satisfying (4.3) and (4.4) using representation theory. Then our link $\mathcal{F} = (W_n^2)^{-1} \circ \Phi \circ W_n^1$ will map an exact relation in $\mathcal{T}$ passing through C_1 to another exact relation of the same dimension in $\mathcal{T}$ passing through C_2.

CHAPTER 5

REPRESENTATION THEORY AND NOTATION

Because we are focused on fiber-reinforced composites, the set of admissible rotations will be the group of rotations in the transverse plane, $SO_k(2) \cong SO(2)$. Since the transformation $W_{\mathbf{n}}$ maps exact relations in $\mathcal{T}$ to $SO(2)$-invariant subspaces of $\mathcal{S}$, we can take advantage of representation theory of $SO(2)$. By the Peter-Weyl theorem, since $SO(2)$ is a compact Lie group, we can write any $SO(2)$-invariant subspace as an orthogonal direct sum of finite dimensional irreducible representations or *irreps*. We say that a representation is irreducible if its only invariant subspaces are the trivial ones. Over $\mathbb{C}$ irreps must be one-dimensional since $SO(2)$ is commutative. Therefore we can parametrize each irrep by one complex or one real number. Let us also parametrize elements of $SO_k(2)$ by $\theta \in \mathbb{R}$. Then for each irrep $V = \{V(z) : z \in \mathbb{C}\}$, we know each $\mathcal{R}_\theta \in SO_K(2)$ acts on $V(z)$ by complex multiplication of $e^{im\theta}$ by z for some integer m. This integer is called the *weight* of V. Note that V is a weight zero irrep if and only if it is parameterized by a real number.

In order to describe these irreps, we will need to introduce some notation. Without loss of generality, we will assume that the longitudinal fibers are oriented vertically, i.e., we choose the longitudinal axis $k = (0, 0, 1)$. Then we can represent each $\mathcal{R}_\theta \in SO_k(2)$ as a 3×3 matrix of the form:

$$\mathcal{R}_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The action of this element on an arbitrary $E \in \mathrm{Sym}(\mathbb{R}^3)$ is then defined by

$$\mathcal{R}_\theta \cdot E = \mathcal{R}_\theta E \mathcal{R}_\theta^T$$

while the action of $\mathcal{R}_\theta$ on $C \in \mathcal{T}$ is given by

$$(\mathcal{R}_\theta \cdot C)(E) = \mathcal{R}_\theta(C({\mathcal{R}_\theta}^T E \mathcal{R}_\theta))\mathcal{R}_\theta^T \tag{5.1}$$

so that $\mathcal{R}_\theta \cdot C$ describes the elasticity tensor of the rotated material.

If we decompose $\mathbb{R}^3$ as a sum of its irreps then we have

$$\mathrm{Sym}(\mathbb{R}^3) = Y_0' \oplus Y_2 \oplus Y_1 \oplus Y_0 = \mathrm{Sym}(\mathbb{R}^2 \oplus \mathbb{R})$$

where

$$Y_0' = \left\{ \begin{bmatrix} \omega\mathbf{I} & 0 \\ 0 & 0 \end{bmatrix} : \omega \in \mathbb{R} \right\} \text{ where } \mathbf{I} \text{ is the identity in } \mathrm{Sym}(\mathbb{R}^2) \tag{5.2}$$

$$Y_2 = \left\{ \begin{bmatrix} \mathbf{E} & 0 \\ 0 & 0 \end{bmatrix} : \mathbf{E} \in \mathrm{Sym}(\mathbb{R}^2),\ \text{trace-free} \right\} \tag{5.3}$$

$$Y_1 = \left\{ \begin{bmatrix} 0 & e \\ e^T & 0 \end{bmatrix} : e \in \mathbb{R}^2 \right\} \tag{5.4}$$

$$Y_0 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & \varepsilon \end{bmatrix} : \varepsilon \in \mathbb{R} \right\} \tag{5.5}$$

In order to simplify our description of fourth order tensors, we would like to express elements of $\mathrm{Sym}(\mathbb{R}^3)$ as vectors rather than matrices. We take advantage of the following complex formalism. If $z \in \mathbb{C}$ is written $z = \alpha + i\beta$ where $\alpha, \beta \in \mathbb{R}$, let

$$\phi(z) = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \text{ and } \psi(z) = \begin{bmatrix} \alpha & \beta \\ \beta & -\alpha \end{bmatrix}.$$

Then every 2×2 matrix $\mathbf{E}$ can be written $\mathbf{E} = \phi(w) + \psi(z)$ for some $w, z \in \mathbb{C}$. Furthermore, if $\mathbf{E}$ is symmetric, then $w = \omega \in \mathbb{R}$. If we slightly abuse notation and write e for both the vector $\langle e_1, e_2 \rangle \in \mathbb{R}^2$ and the complex number $e_1 + ie_2$, then we can parameterize each 3×3 symmetric matrix, E as

$$E = \begin{bmatrix} \phi(\omega) + \psi(z) & e \\ e^T & \sqrt{2}\varepsilon \end{bmatrix} = \begin{bmatrix} \omega \\ z \\ e \\ \varepsilon \end{bmatrix} \tag{5.6}$$

where $\omega \in \mathbb{R} \cong Y_0'$, $z \in \mathbb{C} \cong Y_2$, $e \in \mathbb{C} \cong Y_1$, and $\varepsilon \in \mathbb{R} \cong Y_0$. The coefficient of $\sqrt{2}$ on ε is convenient so that the invariant inner product $\frac{1}{2}\mathrm{Tr}(EE')$ coincides with the standard dot product $\omega\omega' + \mathrm{Re}(z\bar{z}') + \mathrm{Re}(e\bar{e}') + \varepsilon\varepsilon'$ on $\mathbb{R}^6$. We may now write the action of $\mathcal{R}_\theta$ on $E \in \mathrm{Sym}(\mathbb{R}^3)$ very simply as

$$\mathcal{R}_\theta \cdot E = \begin{bmatrix} \omega \\ e^{2i\theta} z \\ e^{i\theta} e \\ \varepsilon \end{bmatrix}$$

and verify that the subscripts on the subspaces Y_0', Y_2, Y_1, and Y_0 correspond to the weights of these irreps.

It will be useful to observe that

$$\psi(w)a = w\bar{a} \qquad\qquad \phi(z)a = za$$

for any a where on each left hand side we think of a as a vector and on each right hand side we think of it as a complex number. We should also note that

$$\phi(z)\psi(w) = \psi(zw) \qquad\qquad \psi(w)\phi(z) = \psi(w\bar{z})$$

$$\phi(z)\phi(y) = \phi(zy) \qquad\qquad \psi(w)\psi(v) = \phi(w\bar{v})$$

for all $v, w, y, z \in \mathbb{C}$. Then the action of an arbitrary $K \in \mathcal{S}$ on E can be represented as multiplication of the column vector in (5.6) by:

$$K = \begin{bmatrix} \lambda & u^T & b^T & \alpha \\ u & \phi(\mu) + \psi(v) & \phi(c) + \psi(d) & g \\ b & \phi(c)^T + \psi(d) & \phi(\rho) + \psi(f) & h \\ \alpha & g^T & h^T & \gamma \end{bmatrix} \tag{5.7}$$

where $\lambda, \alpha, \mu, \rho, \gamma \in \mathbb{R}$ and $u, b, v, c, d, g, f, h \in \mathbb{C} \cong \mathbb{R}^2$. Therefore, using (5.1), the action of $\mathcal{R}_\theta$ on $K \in \mathcal{S}$ is

$$\mathcal{R}_\theta \cdot K = \begin{bmatrix} \lambda & (e^{2i\theta}u)^T & (e^{i\theta}b)^T & \alpha \\ e^{2i\theta}u & \phi(\mu) + \psi(e^{4i\theta}v) & \phi(e^{i\theta}c) + \psi(e^{3i\theta}d) & e^{2i\theta}g \\ e^{i\theta}b & \phi(e^{i\theta}c)^T + \psi(e^{3i\theta}d) & \phi(\rho) + \psi(e^{2i\theta}f) & e^{i\theta}h \\ \alpha & (e^{2i\theta}g)^T & (e^{i\theta}h)^T & \gamma \end{bmatrix}.$$

We now see that our notation in (5.7) is particularly convenient since each parameter in $\mathbb{R}$ or $\mathbb{C}$ represents a separate irrep. That is, we can decompose K as

$$K = \mathsf{K}_0'(\lambda) + \mathsf{K}_2(u) + \mathsf{K}_0(\mu) + \mathsf{K}_4(v) + \mathcal{L}_1'(b) + \mathcal{L}_1(c) + \mathcal{L}_3(d)$$

$$+ \mathbf{N}_0(\rho) + \mathbf{N}_2(f) + \mathbf{M}_0(\alpha) + \mathbf{M}_2(g) + \mathbf{p}_1(h) + j_0(\gamma)$$

where the weight of each irrep is indicated by the appropriate subscript. We will denote each of the subspaces as, for example,

$$\mathsf{K}_0' = \{\mathsf{K}_0'(\lambda) : \lambda \in \mathbb{R}\} \quad \text{and} \quad \mathsf{K}_2 = \{\mathsf{K}_2(u) : u \in \mathbb{C}\}.$$

It will also be convenient to combine the irreps into the following larger blocks:

- $\mathbb{K} = \mathsf{K}_0' \oplus \mathsf{K}_2 \oplus \mathsf{K}_0 \oplus \mathsf{K}_4 \cong \mathrm{Sym}(\mathrm{Sym}(\mathbb{R}^2)) \cong \mathrm{Sym}(Y_0' \oplus Y_2)$
- $\mathbb{L} = \mathcal{L}_1' \oplus \mathcal{L}_1 \oplus \mathcal{L}_3 \cong \mathrm{Hom}(\mathrm{Sym}(\mathbb{R}^2), \mathbb{R}^2) \cong \mathrm{Hom}((Y_0' \oplus Y_2), Y_1)$
- $\mathbb{M} = \mathbf{M}_0 \oplus \mathbf{M}_2 \cong \mathrm{Hom}(\mathrm{Sym}(\mathbb{R}^2), \mathbb{R}) \cong \mathrm{Hom}((Y_0' \oplus Y_2), Y_0)$
- $\mathbb{N} = \mathbf{N}_0 \oplus \mathbf{N}_2 \cong \mathrm{Sym}(\mathbb{R}^2) \cong \mathrm{Sym}(Y_1)$
- $\mathbb{P} = \mathbf{p}_1 \cong \mathrm{Hom}(\mathbb{R}^2, \mathbb{R}) \cong \mathrm{Hom}(Y_1, Y_0)$
- $\mathbb{J} = j_0 \cong \mathrm{Sym}(\mathbb{R}) \cong \mathrm{Sym}(Y_0)$

where $\mathrm{Hom}(V, W)$ represents the set of symmetric linear maps between V and W. With an abuse of notation wherein we identify each subspace of $\mathcal{S}$ with the components that it represents, we can illustrate the block structure by writing

$$\mathcal{S} = \begin{bmatrix} \mathsf{K}_0' & \mathsf{K}_2 & \mathcal{L}_1' & \mathbf{M}_0 \\ \mathsf{K}_2 & \mathsf{K}_0 \oplus \mathsf{K}_4 & \mathcal{L}_1 \oplus \mathcal{L}_3 & \mathbf{M}_2 \\ \mathcal{L}_1' & \mathcal{L}_1 \oplus \mathcal{L}_3 & \mathbf{N}_0 \oplus \mathbf{N}_2 & \mathbf{p}_1 \\ \mathbf{M}_0 & \mathbf{M}_2 & \mathbf{p}_1 & j_0 \end{bmatrix} = \begin{bmatrix} \mathbb{K} & \mathbb{L} & \mathbb{M} \\ \mathbb{L} & \mathbb{N} & \mathbb{P} \\ \mathbb{M} & \mathbb{P} & \mathbb{J} \end{bmatrix}.$$

CHAPTER 6

ALGEBRAIC STRUCTURE FOR FIBER-REINFORCED COMPOSITES

For fiber-reinforced composites, the set of admissible directions for lamination is

$$\mathcal{D} := \{\mathbf{n} \in \mathbb{R}^3 : \|\mathbf{n}\| = 1, \mathbf{n} = (n, 0) = (n_1, n_2, 0)\}$$

which represents varying angles of rotation in the transverse plane. For each $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$, define the matrix $\mathbf{a} \otimes \mathbf{b}$ component-wise as $(\mathbf{a} \otimes \mathbf{b})_{ij} = a_i b_j$. Then for each $\mathbf{n} \in \mathcal{D}$, we have the subspaces

$$\mathcal{J}_{\mathbf{n}} = \{J \in \mathrm{Sym}(\mathbb{R}^3) : J\mathbf{n} = 0\}$$

$$\mathcal{E}_{\mathbf{n}} = \{\mathbf{a} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{a} : \mathbf{a} \in \mathbb{R}^3\}.$$

Now let us describe the fixed tensor C_0 in (3.1). We say C is *transversely isotropic* if $\mathcal{R}_\theta \cdot C = C$ for all $\mathcal{R}_\theta \in SO_k(2)$. For fiber-reinforced composites, we will require that C_0 be transversely isotropic. An arbitrary transversely isotropic tensor $G_0 \in \mathcal{T}$ can be written

$$G_0 = \begin{bmatrix} \lambda_0 & 0 & 0 & \alpha_0 \\ 0 & \phi(\mu_0) & 0 & 0 \\ 0 & 0 & \phi(\rho_0) & 0 \\ \alpha_0 & 0 & 0 & \gamma_0 \end{bmatrix} = \mathsf{K}_0'(\lambda_0) + \mathsf{K}_0(\mu_0) + \mathbf{N}_0(\rho_0) + \mathbf{M}_0(\alpha_0) + j_0(\gamma_0)$$

for some $\lambda_0, \mu_0, \rho_0 > 0$ and $\lambda_0\gamma_0 - \alpha_0^2 > 0$. We will let $C_0 = G_0^2$ in order to avoid calculating square roots.

Lemma 6.1: The projector $\Gamma(\mathbf{n})$ onto the subspace $C_0^{1/2}\mathcal{E}_{\mathbf{n}}$ is of the form $\Gamma(\mathbf{n}) = (\Gamma(\mathbf{n}) - \bar{\Gamma}(\mathbf{n})) + \bar{\Gamma}(\mathbf{n})$ where $\bar{\Gamma}(\mathbf{n})$ represents the transversely isotropic part and

$$\Gamma(\mathbf{n}) - \bar{\Gamma}(\mathbf{n}) = \frac{1}{\vartheta_0}\begin{bmatrix} 0 & \lambda_0\mu_0 v^T & 0 & 0 \\ \lambda_0\mu_0 v & \psi(-\frac{1}{2}(\lambda_0^2+\alpha_0^2)v^2) & 0 & \alpha_0\mu_0 v \\ 0 & 0 & \psi(\frac{1}{2}\vartheta_0 v) & 0 \\ 0 & \alpha_0\mu_0 v^T & 0 & 0 \end{bmatrix}$$

$$\bar{\Gamma}(\mathbf{n}) = \frac{1}{\vartheta_0}\begin{bmatrix} \lambda_0^2 & 0 & 0 & \alpha_0\lambda_0 \\ 0 & \phi(\frac{1}{2}(\lambda_0^2+\alpha_0^2+2\mu_0^2)) & 0 & 0 \\ 0 & 0 & \phi(\frac{1}{2}\vartheta_0) & 0 \\ \alpha_0\lambda_0 & 0 & 0 & \alpha_0^2 \end{bmatrix}$$

where $\vartheta_0 = \lambda_0^2 + \alpha_0^2 + \mu_0^2$ and $v = n^2$ as complex numbers.

Proof: Fix $\mathbf{n} \in \mathcal{D}$. Then each $\mathbf{b} = (b, \beta) = (b_1, b_2, \beta) \in \mathbb{R}^3$ corresponds to an element $(\mathbf{b}\otimes\mathbf{n}+\mathbf{n}\otimes\mathbf{b}) \in \mathcal{E}_{\mathbf{n}}$. Observing that the 2×2 matrix $(b\otimes n+n\otimes b) = \phi(b\cdot n)+\psi(bn)$, we can write $(\mathbf{b}\otimes\mathbf{n}+\mathbf{n}\otimes\mathbf{b})$ using the form in (5.6):

$$(\mathbf{b}\otimes\mathbf{n}+\mathbf{n}\otimes\mathbf{b}) = \begin{bmatrix} (b\cdot n) \\ bn \\ \beta n \\ 0 \end{bmatrix}.$$

Given an arbitrary $E \in \mathrm{Sym}(\mathbb{R}^3)$, we would like to find the projection of E onto $C_0^{1/2}\mathcal{E}_{\mathbf{n}}$. Let us write E as the sum of its projections onto $C_0^{-1/2}\mathcal{J}_{\mathbf{n}}$ and $C_0^{1/2}\mathcal{E}_{\mathbf{n}}$, say as

$$E = C_0^{-1/2}J + C_0^{1/2}(\mathbf{b}\otimes\mathbf{n}+\mathbf{n}\otimes\mathbf{b})$$

where $J\mathbf{n} = 0$. Then in order to find the projection $\Gamma(\mathbf{n})E = C_0^{1/2}(\mathbf{b} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{b})$, we need to find $\mathbf{b}$.

We use the fact that

$$(C_0^{1/2}E)\mathbf{n} = J\mathbf{n} + C_0(\mathbf{b} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{b})\mathbf{n} = C_0(\mathbf{b} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{b})\mathbf{n}.$$

Letting

$$E = \begin{bmatrix} \omega \\ z \\ e \\ \varepsilon \end{bmatrix}$$

we can now see that

$$\begin{aligned}(C_0^{1/2}E)\mathbf{n} &= \begin{bmatrix} (\lambda_0\omega + \alpha_0\varepsilon)\mathbf{I} + \mu_0\psi(z) & \rho_0 e \\ \rho_0 e^T & \sqrt{2}(\alpha_0\omega + \gamma_0\varepsilon) \end{bmatrix} \begin{bmatrix} n \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} (\lambda_0\omega + \alpha_0\varepsilon)n + \mu_0\psi(z)n \\ \rho_0 e \cdot n \end{bmatrix} \end{aligned} \tag{6.1}$$

while

$$\begin{aligned} &C_0(\mathbf{b} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{b})\mathbf{n} \\ &= \begin{bmatrix} (\lambda_0^2 + \alpha_0^2 - \mu_0^2)(b \cdot n)\mathbf{I} + \mu_0^2(b \otimes n + n \otimes b) & \rho_0^2\beta n \\ \rho_0^2\beta n^T & \sqrt{2}(\lambda_0 + \gamma_0)\alpha_0(b \cdot n) \end{bmatrix} \begin{bmatrix} n \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} (\lambda_0^2 + \alpha_0^2)(b \cdot n)n + \mu_0^2 b \\ \rho_0^2\beta \end{bmatrix}. \end{aligned} \tag{6.2}$$

Equating (6.1) and (6.2), immediately we see that $\beta = \frac{e \cdot n}{\rho_0}$. Applying the inner product with n to both sides of

$$(\lambda_0^2 + \alpha_0^2)(b \cdot n)n + \mu_0^2 b = (\lambda_0\omega + \alpha_0\varepsilon)n + \mu_0\psi(z)n \tag{6.3}$$

we find that

$$b \cdot n = \frac{\lambda_0 \omega + \alpha_0 \varepsilon + \mu_0(\psi(z)n \cdot n)}{\lambda_0^2 + \alpha_0^2 + \mu_0^2}. \tag{6.4}$$

We then substitute (6.4) into (6.3) and solve for

$$b = \frac{\mu_0(\lambda_0 \omega + \alpha_0 \varepsilon) - (\lambda_0^2 + \alpha_0^2)(\psi(z)n \cdot n)}{\mu_0(\lambda_0^2 + \alpha_0^2 + \mu_0^2)} n + \frac{1}{\mu_0}\psi(z)n. \tag{6.5}$$

Now we have

$$\Gamma(\mathbf{n})E = C_0^{1/2} \begin{bmatrix} b \cdot n \\ bn \\ \beta n \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda_0(b \cdot n) \\ \mu_0 bn \\ (n \otimes n)e \\ \alpha_0(b \cdot n) \end{bmatrix}.$$

Using (6.4) and (6.5), we can write

$$\Gamma(\mathbf{n}) = \frac{1}{2\vartheta_0} \begin{bmatrix} 2\lambda_0^2 & 2\lambda_0\mu_0 v^T & 0 & 2\alpha_0\lambda_0 \\ 2\lambda_0\mu_0 v & \phi(\vartheta_0 + \mu_0^2) - \psi((\vartheta_0 - \mu_0^2)v^2) & 0 & 2\alpha_0\mu_0 v \\ 0 & 0 & \phi(\vartheta_0) + \psi(\vartheta_0 v) & 0 \\ 2\alpha_0\lambda_0 & 2\alpha_0\mu_0 v^T & 0 & 2\alpha_0^2 \end{bmatrix}$$

where $\vartheta_0 = \lambda_0^2 + \alpha_0^2 + \mu_0^2$ and $v = n^2$ as complex numbers. We can then easily separate $\Gamma(\mathbf{n})$ into the transversely isotropic part, $\bar{\Gamma}$, and the rest as in the statement of the lemma. $\square$

Note that the transversely isotropic part, $\bar{\Gamma}$ does not depend on $\mathbf{n}$. From [12] we know that

$$\mathcal{A} = \mathrm{Span}\{\Gamma(\mathbf{n}) - \bar{\Gamma} : \mathbf{n} \in \mathcal{D}\}.$$

so we are almost ready to describe $\mathcal{A}$. First we prove a short lemma.

Lemma 6.2: $\mathrm{Span}_{\mathbb{R}}\{(e^{i\theta}, e^{2i\theta}) : \theta \in \mathbb{R}\} = \mathbb{C}^2$.

Proof: Let $\mathcal{B} = \{(e^{i\theta}, e^{2i\theta}) : \theta \in \mathbb{R}\}$. By judicious choice of θ we see that $(1,1)$, $(-1,1)$, $(e^{\frac{\pi i}{4}}, i)$, $(-e^{\frac{\pi i}{4}}, i)$, and $(i,-1) \in \mathcal{B}$. From here we can easily see that $(0,1), (1,0), (0,i)$, and $(i,0) \in \mathrm{Span}_{\mathbb{R}}\{\mathcal{B}\}$ which gives us the lemma. $\square$

From this lemma it follows that

$$\mathcal{A} = \left\{ \begin{bmatrix} 0 & 2\lambda_0\mu_0 z^T & 0 & 0 \\ 2\lambda_0\mu_0 z & \psi(w) & 0 & 2\alpha_0\mu_0 z \\ 0 & 0 & \psi(\vartheta_0 z) & 0 \\ 0 & 2\alpha_0\mu_0 z^T & 0 & 0 \end{bmatrix} \ : \ w, z \in \mathbb{C} \right\}.$$

We now simplify $\mathcal{A}$ by taking advantage of (3.5). If

$$B_1 = \begin{bmatrix} \lambda_1 & 0 & 0 & \alpha_1 \\ 0 & \phi(\mu_1) & 0 & 0 \\ 0 & 0 & \phi(\rho_1) & 0 \\ \beta_1 & 0 & 0 & \gamma_1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0 & 2\lambda_0\mu_0 z^T & 0 & 0 \\ 2\lambda_0\mu_0 z & \psi(w) & 0 & 2\alpha_0\mu_0 z \\ 0 & 0 & \psi(\vartheta_0 z) & 0 \\ 0 & 2\alpha_0\mu_0 z^T & 0 & 0 \end{bmatrix}$$

then

$$B_1 A B_1^T = 2\mu_0\mu_1 \begin{bmatrix} 0 & (\lambda_0\lambda_1 + \alpha_0\alpha_1) z^T & 0 & 0 \\ (\lambda_0\lambda_1 + \alpha_0\alpha_1) z & \psi(\frac{\mu_1^2}{2\mu_0\mu_1} w) & 0 & (\lambda_0\beta_1 + \alpha_0\gamma_1) z \\ 0 & 0 & \psi(\frac{\vartheta_0\rho_1^2}{2\mu_0\mu_1} z) & 0 \\ 0 & (\lambda_0\beta_1 + \alpha_0\gamma_1) z^T & 0 & 0 \end{bmatrix}.$$

Letting

$$\beta_1 = -\frac{\alpha_0\gamma_1}{\lambda_0} \tag{6.6}$$

eliminates the $\mathbf{M}_2$ component of each $A' \in \mathcal{A}'$. While it is tempting to allow $\lambda_1 = -\frac{\alpha_0\alpha_1}{\lambda_0}$ to get rid of the K_2 component, combining this with (6.6) implies that B_1 is singular. Instead, we let

$$\rho_1 = \sqrt{\frac{2\mu_1\mu_0}{\vartheta_0}(\lambda_1\lambda_0 + \alpha_1\alpha_0)}$$

so that the ratio between K_2 and $\mathbf{N}_2$ components is simply one. Therefore, in the case of fiber-reinforced periodic composites, we have

$$\mathcal{A}' = \{\mathsf{K}_2(z) + \mathsf{K}_4(w) + \mathbf{N}_2(z) : z, w \in \mathbb{C}\}.$$

To simplify notation we will now rename our sets so that $\mathcal{A}$ denotes the simplified form.

We are now able to make certain observations about S_0 from (3.7). Since

$$B_1\bar{\Gamma}B_1^T = \begin{bmatrix} \frac{(\lambda_1\lambda_0+\alpha_1\alpha_0)^2}{\vartheta_0} & 0 & 0 & 0 \\ 0 & \phi(\frac{\mu_1^2(\lambda_0^2+\alpha_0^2+2\mu_0^2)}{2\vartheta_0}) & 0 & 0 \\ 0 & 0 & \phi(\frac{1}{2}\rho_1^2) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{6.7}$$

we see that we may in general make S_0 diagonal. Which of the diagonal entries we may set equal to zero depends on the specific algebraic subspace we choose.

The free parameters in B_1 also allow us simplify $C_0^{1/2}B_1^T$ and thus the inversion formula (3.7). Observe that

$$C_0^{1/2}B_1^T = \begin{bmatrix} \lambda_1\lambda_0 + \alpha_1\alpha_0 & 0 & 0 & 0 \\ 0 & \phi\left(\mu_1\mu_0\right) & 0 & 0 \\ 0 & 0 & \phi\left(\rho_0\sqrt{\frac{2\mu_1\mu_0(\lambda_1\lambda_0+\alpha_1\alpha_0)}{\vartheta_0}}\right) & 0 \\ \alpha_0\lambda_1 + \gamma_0\alpha_1 & 0 & 0 & \left(\frac{\lambda_0\gamma_0-\alpha_0^2}{\lambda_0}\right)\gamma_1 \end{bmatrix}.$$

Therefore, assuming $\alpha_0^2 \neq \gamma_0^2$, by setting

$$\alpha_1 = \frac{-\alpha_0 \lambda_1}{\gamma_0}$$

we can ensure that B_1 is invertible and that $C_0^{1/2} B_1^T$ will be diagonal of the form

$$C_0^{1/2} B_1^T = \mathsf{K}_0'(\delta_1) + \mathsf{K}_0(\delta_2) + \mathbf{N}_0\Big(\sqrt{\frac{2\delta_1\delta_2\rho_0^2}{\vartheta_0}}\Big) + j_0(\delta_3)$$

where $\delta_1, \delta_2, \delta_3 \in \mathbb{R}$ are arbitrary.

CHAPTER 7

AUTOMORPHISMS

Before we can identify the subspaces that satisfy our algebraic conditions, we must find a way to describe this infinite list in terms of a finite number of sets. We use equivalence classes of automorphisms of $\mathcal{S}$ to accomplish this goal. Suppose $\Phi : \mathcal{S} \to \mathcal{S}$ is a bijection such that

$$\mathcal{R} \cdot \Phi(K_1) = \Phi(\mathcal{R} \cdot K_1) \tag{7.1}$$

and

$$\Phi(K_1 *^A K_2) = \Phi(K_1) *^A \Phi(K_2) \tag{7.2}$$

hold for all $\mathcal{R} \in SO(2)$, $A \in \mathcal{A}$, and $K_1, K_2 \in \mathcal{S}$. Then Π is a rotationally invariant algebraic subspace if and only if $\Phi(\Pi)$ is as well. Let Υ denote this set of automorphisms. Then we can restrict our list of algebraic subspaces to include just one representative of each Υ-orbit. Conveniently, this set of automorphisms is exactly that which corresponds to the set of links we seek.

As we saw in chapter 4, in general a link corresponds to a bijection on a pair of algebras with certain ideals. The links we discuss in this work correspond to bijections from the entire algebraic space $\mathcal{S}$ to itself, i.e., where both ideals are $\{0\}$. Due to the block structure of $\mathcal{A}$, we can immediately see that $\mathbb{M} \oplus \mathbb{P} \oplus \mathbb{J}$ and $\mathbb{J}$ are also ideals on $\mathcal{S}$. However, as we see in the following theorem, computing the sets of admissible automorphisms for the quotient groups of $\mathcal{S}$ corresponding to these ideals does not result in any additional automorphisms beyond those that are admissible for all of $\mathcal{S}$. We will give the arguments for all three groups in parallel.

Theorem 7.1: All automorphisms satisfying (7.1) and (7.2) are of the form

$$\Phi^{\pm}_{\nu,\tau}(K) = X^{\pm}(\nu,\tau)K(X^{\pm}(\nu,\tau))^T$$

where

$$X^{\pm}(\nu,\tau) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \phi(1) & 0 & 0 \\ 0 & 0 & \phi(\pm 1) & 0 \\ \nu & 0 & 0 & \tau \end{bmatrix} \tag{7.3}$$

for some $\nu, \tau \in \mathbb{R}$. Furthermore, restricting these automorphisms to either $\mathcal{S}\backslash\mathbb{J} \cong \mathcal{S}/\mathbb{J}$ or $\mathcal{S} \setminus (\mathbb{M} \oplus \mathbb{P} \oplus \mathbb{J}) \cong \mathcal{S}/(\mathbb{M} \oplus \mathbb{P} \oplus \mathbb{J})$ gives us precisely the groups of automorphisms satisfying (7.1) and (7.2) on these quotient groups.

Proof: Let Φ be any automorphism satisfying (7.1) and (7.2) for all K_1, K_2 in some algebraic subspace, Π. First we observe the following.

Lemma 7.2: If Φ satisfies (7.1), then Φ respects weight class. That is, for each irrep V of weight $m \in \mathbb{Z}$, $\Phi(V)$ is also an irrep of weight m.

Proof: That Φ satisfies (7.1) implies that for any irrep V, the image $\Phi(V)$ is also an irrep. Since Φ is a bijection, $\mathcal{R}_\theta \cdot \Phi(K) = \Phi(K)$ if and only if $\mathcal{R}_\theta \cdot K = K$. Now note that the following are equivalent for an irrep V:

- V is of weight m
- $\mathcal{R}_\theta \cdot K = K$ holds for all $K \in V$ if and only if $m\theta \equiv 0 \bmod 2\pi$

Therefore $\Phi(V)$ is an irrep of the same weight class as V. $\square$

The remainder of the proof will involve repeated applications of (7.2). We will assume in general that $\Pi = \mathcal{S}$. Note that $\mathcal{S}/(\mathbb{M} \oplus \mathbb{P} \oplus \mathbb{J}) \cong \mathbb{K} \oplus \mathbb{L} \oplus \mathbb{N}$ and that we can immediately see from the block structure of $\mathcal{A}$ that $\mathbb{K} \oplus \mathbb{L} \oplus \mathbb{N}$ is an algebraic subspace. Therefore, by simply ignoring $\mathbb{M} \oplus \mathbb{P} \oplus \mathbb{J}$, we find that these arguments carry directly over to $\mathcal{S}/(\mathbb{M}\oplus\mathbb{P}\oplus\mathbb{J})$. For the case $\Pi = \mathcal{S}/\mathbb{J}$, we will make amendments where appropriate to complete the argument. For reference the entire multiplication table for arbitrary $A \in \mathcal{A}$ is included in appendix B, though we will repeat many of the results in the text for the reader's convenience. First let us define two important subspaces:

$$\Pi_{\mathsf{A}} = \mathsf{K}_0 \oplus \mathsf{K}_4 \oplus \mathbb{N} \oplus \mathcal{L}_1 \oplus \mathcal{L}_3 \tag{7.4}$$

and

$$\Pi_{\mathsf{B}} = \mathsf{K}_0' \oplus \mathsf{K}_2 \oplus \mathcal{L}_1' \oplus \mathbb{M} \oplus \mathbb{P} \oplus \mathbb{J} \tag{7.5}$$

and note that $\Pi_{\mathsf{A}} \oplus \Pi_{\mathsf{B}} = \mathcal{S}$.

Lemma 7.3: Φ is invariant on each irrep $V \in \{\mathsf{K}_0, \mathsf{K}_4, \mathbf{N}_0, \mathbf{N}_2, \mathcal{L}_1, \mathcal{L}_3, \mathbb{J}\}$ as well as on the subspaces Π_{B} and $\mathbb{M} \oplus \mathbb{P} \oplus \mathbb{J}$.

Proof: That Φ is invariant on K_4 and $\mathcal{L}_3$ follows from Lemma 7.2. That Φ is invariant on $\mathbb{J}$ follows from the fact that $\mathbb{J}$ represents the set of annihilators in $\mathcal{S}$, i. e.

$$\mathbb{J} = \mathrm{Ann}_{\mathcal{S}}(\mathcal{S}) = \{J \in \mathcal{S} : J{*^A}K = 0 \ \forall\ K \in \mathcal{S}\}.$$

Since Π_{B} represents the set of elements that are nilpotent for all $A \in \mathcal{A}$, Φ is invariant on Π_{B}. Defining $\mathrm{Ann}_{\mathcal{S}}(V) = \{S \in \mathcal{S} : S{*^A}K = 0 \ \forall\ K \in V\}$, since for any subspace V,

$$\Phi(\mathrm{Ann}_{\mathcal{S}}(V)) \subset \mathrm{Ann}_{\mathcal{S}}(\Phi(V)), \tag{7.6}$$

the facts that $\mathrm{Ann}_{\mathcal{S}}(\mathsf{K}_4) = \mathbb{N} \oplus \mathbb{P} \oplus \mathbb{J}$ and $\Phi(\mathsf{K}_4) = \mathsf{K}_4$ tell us that Φ is invariant on $\mathbb{N} \oplus \mathbb{P} \oplus \mathbb{J}$. In particular, by the Lemma 7.2, Φ is invariant on $\mathbb{P}$, which we will use momentarily. In the case that $\Pi = \mathcal{S}/\mathbb{J}$, we must modify (7.6) to

$$\Phi(\mathrm{Ann}_{\mathcal{S}}(V))*^A\Phi(V) \subset \mathbb{J}.$$

However, since $\Phi(\mathsf{K}_4) = \mathsf{K}_4$, and $(\mathsf{K}_4*^A U) \cap \mathbb{J} = \{0\}$ for all $U \subset \mathcal{S}$, the conclusion that Φ is invariant on (in this case) $\mathbf{N}_0, \mathbf{N}_2$, and $\mathbf{p}_1$ holds.

Observe that (7.2) implies that Φ sends ideals to ideals. We can easily see that $\mathbb{M}\oplus\mathbb{P}\oplus\mathbb{J}$ is an ideal and that since $\Phi(\mathbb{M}\oplus\mathbb{P}\oplus\mathbb{J}) \subset (\Pi_{\mathsf{B}} \setminus \mathcal{L}_1') = (\mathsf{K}_0'\oplus\mathsf{K}_2\oplus\mathbb{M}\oplus\mathbb{P}\oplus\mathbb{J})$ while $\mathsf{K}_0'*^A\mathbb{L} = \mathcal{L}_1'$ and $\mathsf{K}_2*^A\mathbb{L} = \mathbb{L}$ for all $A \in \mathcal{A}$, it is also the only ideal within its potential image under Φ. Therefore Φ must also be invariant on $\mathbb{M} \oplus \mathbb{P} \oplus \mathbb{J}$. Furthermore, $\Phi(\mathbf{p}_1) = \mathbf{p}_1$ and $\mathrm{Ann}_{\mathcal{S}}(\mathbf{p}_1) = \mathbb{K} \oplus \mathbb{M} \oplus \mathbb{J}$ imply that Φ is invariant on $\mathsf{K}_0' \oplus \mathsf{K}_0 \oplus \mathbf{M}_0 \oplus \mathbb{J}$. Since $\mathbf{N}_0*^A\mathbf{N}_2 = \mathbf{N}_0$ for all A and $\mathbb{J}$ is an annihilator, we know $\Phi(\mathbf{N}_0) = \mathbf{N}_0$. Similarly, since $\mathsf{K}_0*^A\mathsf{K}_4 = \mathsf{K}_0$, $\mathsf{K}_0'*^A\mathsf{K}_4 = \mathsf{K}_2$, and $\mathbf{M}_0*^A\mathsf{K}_4 = \mathbf{M}_2$ for all A, we see $\Phi(\mathsf{K}_0) = \mathsf{K}_0$. Finally, since for all A, $\mathcal{L}_1^{*A2} \subset \mathsf{K}_4 \oplus \mathbf{N}_2$ while $(\mathcal{L}_1')^{*A2} = \mathsf{K}_0'$, $\mathbf{p}_1^{*A2} = \mathbb{J}$, $\mathcal{L}_1'*^A\mathbf{p}_1 = \mathbf{M}_0$, $\mathcal{L}_1'*^A\mathcal{L}_1 \subset \mathsf{K}_2 \oplus \mathbb{N}$, and $\mathcal{L}_1*^A\mathbf{p}_1 = \mathbf{M}_2$, we have that $\Phi(\mathcal{L}_1) = \mathcal{L}_1$. $\square$

Before we calculate explicit values for Φ, we would like to have a better sense of how Φ acts on each irrep. Certainly for any weight zero irrep $V = \{V(\beta) : \beta \in \mathbb{R}\}$ there exists $\delta \in \mathbb{R}$ and another weight zero irrep, $W = \{W(\beta) : \beta \in \mathbb{R}\}$, such that for every $\beta \in \mathbb{R}$, $\Phi(V(\beta)) = W(\delta\beta)$. Now suppose $V = \{V(z) : z \in \mathbb{C}\}$ is an irrep of weight m where $m \geq 1$ and $\Phi(V) = V$. Then, by Schur's Lemma, there exists $a \in \mathbb{C}$ such that $\Phi(V(z)) = V(az)$ for all $z \in \mathbb{C}$. So Φ is simply scalar multiplication by a on V. Furthermore, since all irreps of a certain weight are isomorphic, we can extend this result to all irreps V. That is, if $\Phi(V) = W$ and for some fixed $v \in \mathbb{C}$, we know $\Phi(V(v)) = W(w)$, then for any $a \in \mathbb{C}$, $\Phi(V(av)) = W(aw)$. We will use this fact throughout the remainder of the proof.

Lemma 7.4: Define $\Phi_{\mathbb{K}} = \pi_{\mathbb{K}} \circ \Phi$ where $\pi_{\mathbb{K}}$ represents the projection onto the subspace $\mathbb{K}$. Then $\Phi_{\mathbb{K}}$ is the identity map on $\mathbb{K}$. Furthermore, by lemma 7.3,

$$\Phi(\mathsf{K}_0(\mu) + \mathsf{K}_4(v)) = \Phi_{\mathbb{K}}(\mathsf{K}_0(\mu) + \mathsf{K}_4(v)) = \mathsf{K}_0(\mu) + \mathsf{K}_4(v).$$

Proof: For all $A \in \mathcal{A}$, $\mathbb{K}^{*^{A}2} = \mathbb{K}$, $\mathbb{K} *^{A} \mathbb{M} = \mathbb{M}$, $\mathbb{M}^{*^{A}2} = \mathbb{J}$, and all elements of $\mathbb{J}$ annihilate all other elements. When we combine this information with the fact that $\Phi(\mathbb{K}) \subset \mathbb{K} \oplus \mathbb{M} \oplus \mathbb{J}$ we can show that

$$\Phi_{\mathbb{K}}(K_1 *^{A} K_2) = \Phi_{\mathbb{K}}(K_1) *^{A} \Phi_{\mathbb{K}}(K_2)$$

for all $K_1, K_2 \in \mathbb{K}$ and $A \in \mathcal{A}$. Fix $A = \mathsf{K}_2(z) + \mathsf{K}_4(w) + \mathbf{N}_2(z) \in \mathcal{A}$. Then since $\mathsf{K}_4(v)^{*^{A}2} = \mathsf{K}_4(\bar{w}v^2)$ and $\Phi(\mathsf{K}_4) = \mathsf{K}_4$, we have that $\Phi(\mathsf{K}_4(v)) = \mathsf{K}_4(v)$. Similarly, since $\mathsf{K}_0(\mu)^{*^{A}2} = \mathsf{K}_4(w\mu^2)$ and $\Phi(\mathsf{K}_0) = \mathsf{K}_0$, $\Phi(\mathsf{K}_0(\mu)) = \mathsf{K}_0(\beta\mu)$ where $\beta = \pm 1$. But then the fact that

$$\mathsf{K}_2(u) *^{A} \mathsf{K}_0(\mu) = \frac{1}{2}[\mathsf{K}_2(\mu\bar{u}w) + \mathsf{K}_0(\mu \mathrm{Re}(u\bar{z})) + \mathsf{K}_4(\mu u z)]$$

implies that $\beta = 1$ and $\Phi_{\mathbb{K}}(\mathsf{K}_2(u)) = \mathsf{K}_2(u)$. Finally, that $\mathsf{K}_2^{*^{A}2} \subset \mathsf{K}_0' \oplus \mathsf{K}_2$ tells us that $\Phi_{\mathbb{K}}(\mathsf{K}_0'(\lambda)) = \mathsf{K}_0'(\lambda)$. □

Lemma 7.5:

$$\Phi(\mathcal{L}_1(c) + \mathcal{L}_3(d) + \mathbf{N}_0(\rho) + \mathbf{N}_2(f)) =$$

$$\mathcal{L}_1(\beta c) + \mathcal{L}_3(\beta d) + \mathbf{N}_0(\rho) + \mathbf{N}_2(f)$$

where $\beta = \pm 1$

Proof: Let us fix an arbitrary $A = \mathsf{K}_2(z) + \mathsf{K}_4(w) + \mathbf{N}_2(z) \in \mathcal{A}$. As in the case of K_4 above, the facts that $\mathbf{N}_2(f)^{*^{A}2} = \mathbf{N}_2(\bar{z}f^2)$ and $\Phi(\mathbf{N}_2) = \mathbf{N}_2$ imply that $\Phi(\mathbf{N}_2(f)) = \mathbf{N}_2(f)$. Also, $\mathbf{N}_0(\rho)^{*^{A}2} = \mathbf{N}_2(z\rho^2)$ implies $\Phi(\mathbf{N}_0(\rho)) = \mathbf{N}_0(\pm\rho)$. Next the products

$\mathcal{L}_3(d)^{*^A 2} = \mathsf{K}_4(\bar{z}d^2) + \mathbf{N}_2(\bar{w}d^2)$, $\mathcal{L}_1(c)^{*^A 2} = \mathsf{K}_4(zc^2) + \mathbf{N}_2(w\bar{c}^2)$, and $\mathcal{L}_1(c) *^A \mathcal{L}_3(d) = \mathsf{K}_0(\mathrm{Re}(\bar{z}cd)) + \mathbf{N}_0(\mathrm{Re}(w\bar{c}\bar{d}))$ imply

$$\Phi(\mathbf{N}_0(\rho) + \mathcal{L}_1(c) + \mathcal{L}_3(d)) = \mathbf{N}_0(\rho) + \mathcal{L}_1(\beta c) + \mathcal{L}_3(\beta d)$$

where $\beta = \pm 1$. □

Lemma 7.6:

$$\Phi(\mathsf{K}_0'(\lambda) + \mathsf{K}_2(u)) = \mathsf{K}_0'(\lambda) + \mathbf{M}_0(\nu\lambda) + j_0(\nu^2\lambda) + \mathsf{K}_2(u) + \mathbf{M}_2(\nu u)$$

$$\Phi(\mathcal{L}_1'(b)) = \mathcal{L}_1'(\delta b) + \mathbf{p}_1(\delta\nu b)$$

$$\Phi(\mathbf{M}_0(\alpha) + \mathbf{M}_2(g)) = \mathbf{M}_0(\tau\alpha) + j_0(2\tau\nu\alpha) + \mathbf{M}_2(\tau g)$$

$$\Phi(\mathbf{p}_1(h)) = \mathbf{p}_1(\delta\tau h)$$

$$\Phi(j_0(\gamma)) = j_0(\tau^2\gamma)$$

for some $\tau, \nu \in \mathbb{R}$ and $\delta = \pm 1$.

Proof: Fix an arbitrary $A = \mathsf{K}_2(z) + \mathsf{K}_4(w) + \mathbf{N}_2(z) \in \mathcal{A}$. Observe that $\mathbf{M}_2(g)^{*^A 2} = j_0(\mathrm{Re}(w\bar{g}^2))$ and $\mathbf{p}_1(h)^{*^A 2} = j_0(\mathrm{Re}(z\bar{h}^2))$, which imply

$$\Phi(\mathbf{M}_2(g) + \mathbf{p}_1(h) + j_0(\gamma)) = \mathbf{M}_2(\delta_1 g) + \mathbf{p}_1(\delta_1\delta_2 h) + j_0(\delta_1^2\gamma)$$

where $\delta_1^2 \in \mathbb{R}$ and $\delta_2 = \pm 1$. However, when $\Pi = \mathcal{S}/\mathbb{J}$, this argument will not work. Instead we may simply observe that $\mathcal{L}_3(d) *^A \mathbf{p}_1(h) = \mathbf{M}_2(\frac{1}{2}dh\bar{z})$ implies the same, where we disregard the values for j_0. If we let

$$\Phi(\mathsf{K}_0'(\lambda)) = \mathsf{K}_0'(\lambda) + \mathbf{M}_0(\delta_3\lambda) + j_0(\delta_4\lambda)$$

$$\Phi(\mathbf{M}_0(\alpha)) = \mathbf{M}_0(\delta_5\alpha) + j_0(\delta_6\alpha)$$

where $\delta_3, \delta_4, \delta_5, \delta_6 \in \mathbb{R}$, then $\mathsf{K}_0'(\lambda) *^A \mathbf{M}_2(g) = \mathbf{M}_0(\frac{1}{2}\lambda \mathrm{Re}(\bar{z}g))$ and $\mathbf{M}_0(\alpha) *^A \mathbf{M}_2(g) = j_0(\alpha \mathrm{Re}(\bar{z}g))$ imply $\delta_1 = \delta_5$ and $2\delta_1\delta_3 = \delta_6$. Write

$$\Phi(\mathcal{L}_1'(b)) = \mathcal{L}_1'(\delta_7 b) + \mathbf{p}_1(\delta_8 b)$$

for some $\delta_7, \delta_8 \in \mathbb{C}$. Since $\mathcal{L}_1'(b)^{*^A 2} = \mathsf{K}_0'(\mathrm{Re}(z\bar{b}^2))$ and $\mathcal{L}_1'(b) *^A \mathbf{p}_1(h) = \mathbf{M}_0(\frac{1}{2}\mathrm{Re}(z\bar{h}\bar{b}))$, we see that $\delta_7 = \delta_2$, $\delta_2\delta_8 = \delta_3$ and $\delta_8^2 = \delta_3^2 = \delta_4$. Finally, let

$$\Phi(\mathsf{K}_2(u)) = \mathsf{K}_2(u) + \mathbf{M}_2(\delta_9 u)$$

where $\delta_9 \in \mathbb{C}$. Since $\mathsf{K}_2(u) *^A \mathbf{M}_0(\alpha) = \mathbf{M}_0(\frac{1}{2}\mathrm{Re}(\alpha u \bar{z}))$, and using that $\mathbf{M}_0 *^A \mathbf{M}_2 = \mathbb{J}$, we have that $\delta_1\delta_9 = \frac{1}{2}\delta_6 = \delta_1\delta_3$ and therefore that $\delta_9 = \delta_3$. Here, for the case of $\Pi = \mathcal{S}/\mathbb{J}$, we must also use that $\mathsf{K}_0'(\lambda) *^A \mathsf{K}_2(u) = \mathsf{K}_0'(\lambda \mathrm{Re}(u\bar{z})$ and $\mathsf{K}_0'(\lambda) *^A \mathbf{M}_2(g) = \mathbf{M}_0(\frac{1}{2}\lambda \mathrm{Re}(\bar{z}g))$ to show that $\delta_3 = \delta_9$. Letting $\delta_1 = \tau$, $\delta_2 = \delta$, and $\delta_3 = \nu$ yields Φ as described in the statement of the lemma. $\square$

From the observation $\mathsf{K}_0(\mu) *^A \mathcal{L}_1'(b) = \mathcal{L}_1(\frac{1}{4}\mu\bar{b}z) + \mathcal{L}_3(\frac{1}{4}\mu b z)$ we see that the values β and δ in Lemma 7.5 and Lemma 7.6 are equal. Therefore Φ must be of the form in the statement of the theorem.

Conversely, that Φ of the form $\Phi^{\pm}_{\nu,\tau}(K) = X^{\pm}(\nu,\tau)K(X^{\pm}(\nu,\tau))^T$ satisfy (7.1) follows from the fact they preserve weight class. That such Φ satisfy (7.2) follows from the fact that for any $A \in \mathcal{A}$

$$(X^{\pm}(\nu,\tau))^T A X^{\pm}(\nu,\tau) = A. \qquad \square$$

CHAPTER 8

EXACT RELATIONS CALCULATIONS

We can now produce a complete list of subspaces $\Pi \subset \mathcal{S}$ that satisfy the algebraic condition (3.3) and are $SO(2)$-invariant. Hereafter such subspaces will be called algebraic subspaces or simply algebras. They include all images of exact relations. As discussed in chapter 5, by the Peter-Weyl theorem, we can write any Π as a direct sum of complex one-dimensional irreps. Let us fix an arbitrary $A = \mathsf{K}_2(z) + \mathsf{K}_4(w) + \mathbf{N}_2(z) \in \mathcal{A}$ to use throughout this chapter. We begin in section 8.1 by considering minimal algebraic subspaces. We describe in section 8.2 how we identify all subspaces that correspond to uniform field relations. Then we explain in section 8.3 how we may organize the complete list of all algebraic subspaces, while the list itself is contained in appendix A. In section 8.4 we describe how these algebraic subspaces may be converted back into exact relations in physical variables and in section 8.5 we give an example of one such conversion.

8.1 Minimal Algebraic Subspaces

Lemma 8.1: If Z is an irrep within some algebraic subspace and the minimal algebraic subspace containing Z is Π, then the pairing $Z \subset \Pi$ is one of the following:

$$\mathsf{K}_0 \subset (\mathsf{K}_0 \oplus \mathsf{K}_4) \tag{8.1}$$

$$\mathbf{N}_0 \subset (\mathbf{N}_0 \oplus \mathbf{N}_2) = \mathbb{N} \tag{8.2}$$

$$\mathbf{N}_2 \subset \mathbf{N}_2 \tag{8.3}$$

$$\mathcal{L}_1 \subset (\mathcal{L}_1 \oplus \mathsf{K}_4 \oplus \mathbf{N}_2) \tag{8.4}$$

$$\mathcal{L}_3 \subset (\mathcal{L}_3 \oplus \mathsf{K}_4 \oplus \mathbf{N}_2) \tag{8.5}$$

$$\mathsf{K}_4 \subset \mathsf{K}_4 \tag{8.6}$$

$$\Pi_0^{\lambda,\alpha,\gamma} \subset \Pi_0^{\lambda,\alpha,\gamma} \tag{8.7}$$

where we define

$$\Pi_0^{\lambda,\alpha,\gamma} = \{\mathsf{K}_0'(\lambda\delta) + \mathbf{M}_0(\alpha\delta) + j_0(\gamma\delta) : \delta \in \mathbb{R}\} \tag{8.7a}$$

for any $(\lambda, \alpha, \gamma) \in \mathbb{R}^3$;

$$\{\mathcal{L}_1'(\eta_1 d) + \mathbf{p}_1((\eta_2 + i\eta_3)d) : d \in \mathbb{C}\} \subset \ \Pi_1^{\eta_1,\eta_2,\eta_3} \tag{8.8}$$

where we define

$$\Pi_1^{\eta_1,\eta_2,\eta_3} = \{\mathcal{L}_1'(\eta_1 d) + \mathbf{p}_1((\eta_2 + i\eta_3)d) : d \in \mathbb{C}\} \oplus \Pi_0^{\eta_1^2,\eta_1\eta_2,\eta_2^2-\eta_3^3} \oplus \Pi_0^{0,\eta_1\eta_3,2\eta_2\eta_3} \tag{8.8a}$$

for any $(\eta_1, \eta_2, \eta_3) \in \mathbb{R}^3$; and

$$\{\mathsf{K}_2(\eta_1 d) + \mathbf{M}_2(\eta_2 d) : d \in \mathbb{C}\} \subset \ \Pi_2^{\eta_1,\eta_2} \tag{8.9}$$

where we define

$$\Pi_2^{\eta_1,\eta_2} = \{\mathsf{K}_2(\eta_1 d) + \mathbf{M}_2(\eta_2 d) : d \in \mathbb{C}\} \oplus \Pi_0^{\eta_1^2,\eta_1\eta_2,\eta_2^2} \tag{8.9a}$$

for any $(\eta_1, \eta_2) \in \mathbb{R}^2$.

Proof: First observe that for any irrep, the minimal algebraic subspace containing it is unique since the intersection of two algebras is an algebra. We can verify that the Π in (8.1) - (8.6) are algebraic subspaces and that for each Z the given Π is in fact the minimal algebraic subspace containing Z by fixing the parameters appropriately and examining the following calculation:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \phi(\mu)+\psi(v) & \phi(c)+\psi(d) & 0 \\ 0 & \phi(c)^T+\psi(d) & \phi(\rho)+\psi(f) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}^{*_{A}2} =$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \phi(\mathrm{Re}(2\mu\bar{v}w + c\bar{d}z)) & \phi(\mu\bar{d}w + vc\bar{w} + c\bar{f}z + \rho d\bar{z}) & 0 \\ 0 & \phi(\mu\bar{d}w + vc\bar{w} + c\bar{f}z + \rho d\bar{z})^T & \phi(\mathrm{Re}(2\rho f\bar{z} + 2cd\bar{w})) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{8.10}$$

$$+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \psi(\mu^2 w + v^2\bar{w} + c^2 z + d^2\bar{z}) & \psi(\mu\bar{c}w + vd\bar{w} + \rho cz + df\bar{z}) & 0 \\ 0 & \psi(\mu\bar{c}w + vd\bar{w} + \rho cz + df\bar{z}) & \psi(\rho^2 z + f^2\bar{z} + \bar{c}^2 w + d^2\bar{w}) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Similarly, we can verify the (Z, Π) pairs in (8.7) - (8.9) with the calculation

$$\begin{bmatrix} \lambda & u^T & b^T & \alpha \\ u & 0 & 0 & g \\ b & 0 & 0 & h \\ \alpha & g^T & h^T & \gamma \end{bmatrix}^{*_{A}2} =$$

$$\begin{bmatrix} \mathrm{Re}(2\lambda z\bar{u} + w\bar{u}^2 + z\bar{b}^2) & \mathrm{Re}(z\bar{u})u^T & \mathrm{Re}(z\bar{u})b^T & \mathrm{Re}(\alpha z\bar{u} + \lambda z\bar{g} + w\bar{g}\bar{u} + z\bar{b}\bar{h}) \\ \mathrm{Re}(z\bar{u})u & 0 & 0 & \mathrm{Re}(z\bar{g})u \\ \mathrm{Re}(z\bar{u})b & 0 & 0 & \mathrm{Re}(z\bar{g})b \\ \mathrm{Re}(\alpha z\bar{u} + \lambda z\bar{g} + w\bar{g}\bar{u} + z\bar{b}\bar{h}) & \mathrm{Re}(z\bar{g})u^T & \mathrm{Re}(z\bar{g})b^T & \mathrm{Re}(2\alpha z\bar{g} + w\bar{g}^2 + z\bar{h}^2) \end{bmatrix}. \tag{8.11}$$

Now we must show that these are the only candidates for Z. Clearly for weights three and four we can only have $Z = \mathcal{L}_3$ and $Z = \mathsf{K}_4$. We will consider weights zero, one, and two in three separate cases.

Case 1: Weight zero. Let Z be an arbitrary an irrep of weight zero contained within its minimal algebraic subspace Π. Then we can write

$$Z := \{Z(\delta) : \delta \in \mathbb{R}\} := \{\mathsf{K}_0'(\lambda\delta) + \mathsf{K}_0(\mu\delta) + \mathbf{N}_0(\rho\delta) + \mathbf{M}_0(\alpha\delta) + j_0(\gamma\delta) : \delta \in \mathbb{R}\}$$

for some $(\lambda, \mu, \rho, \alpha, \gamma) \in \mathbb{R}^5$ and observe that for each $\delta \in \mathbb{R}$,

$$\begin{aligned} Z(\delta)^{*^A 2} &= 2\mathsf{K}_0'(\lambda\delta) *^A \mathsf{K}_0(\mu\delta) + \mathsf{K}_0(\mu\delta)^{*^A 2} + 2\mathsf{K}_0(\mu\delta) *^A \mathbf{M}_0(\alpha\delta) + \mathbf{N}_0(\rho\delta)^{*^A 2} \\ &= \mathsf{K}_2(\lambda\mu\delta^2 z) + \mathsf{K}_4(\mu^2\delta^2 w) + \mathbf{M}_2(\mu\alpha\delta^2 z) + \mathbf{N}_2(\rho^2\delta^2 z). \end{aligned} \tag{8.12}$$

Letting $z = 0$, we see that if $\mu \neq 0$, then $\mathsf{K}_4 \subset \Pi$. But then

$$\begin{aligned} \mathsf{K}_4(v) *^A Z(\delta) &= \mathsf{K}_4(v) *^A \mathsf{K}_0'(\lambda\delta) + \mathsf{K}_4(v) *^A \mathsf{K}_0(\mu\delta) + \mathsf{K}_4(v) *^A \mathbf{M}_0(\alpha\delta) \\ &= \mathsf{K}_2(\frac{1}{2}\lambda\delta v\bar{z}) + \mathsf{K}_0(\mu\delta \mathrm{Re}(v\bar{w})) + \mathbf{M}_2(\frac{1}{2}\alpha\delta v\bar{z}) \end{aligned}$$

which implies that $\mathsf{K}_0 \subset \Pi$. Since by assumption Z is irreducible, this implies $Z = \mathsf{K}_0$ and $\Pi = \mathsf{K}_0 \oplus \mathsf{K}_4$. Next if $\mu = 0$ and $\rho \neq 0$, then (8.12) implies that $\mathbf{N}_2 \subset \Pi$. Since

$$\mathbf{N}_2(f) *^A Z(\delta) = \mathbf{N}_2(f) *^A \mathbf{N}_0(\rho\delta) = \mathbf{N}_0(\rho\delta \mathrm{Re}(f\bar{z}))$$

we see that $\mathbf{N}_0 \subset \Pi$. Again, since Z was assumed to be irreducible, this implies $Z = \mathbf{N}_0$ and $\Pi = \mathbb{N}$. The remaining cases correspond to varying values for $(\lambda, \alpha, \gamma) \in \mathbb{R}^3$ in (8.7).

Case 2: Weight one. Let Z be an arbitrary irrep of weight one and write

$$Z := \{Z(a) : a \in \mathbb{C}\} := \{\mathcal{L}_1'(ab) + \mathcal{L}_1(ac) + \mathbf{p}_1(ah) : a \in \mathbb{C}\}.$$

Then computing

$$Z(a)^{*^A 2} = \mathcal{L}_1'(ab)^{*^A 2} + \mathcal{L}_1(ac)^{*^A 2} + \mathbf{p}_1(ah)^{*^A 2}$$

$$+ 2\mathcal{L}_1'(ab)*^A\mathcal{L}_1(ac) + 2\mathcal{L}_1(ac)*^A\mathbf{p}_1(ah) + 2\mathcal{L}_1'(ab)*^A\mathbf{p}_1(ah)$$

$$= \mathsf{K}_0'(\mathrm{Re}(a^2b^2\bar{z})) + \mathbf{N}_2(a^2c^2\bar{w}) + \mathsf{K}_4(a^2c^2\bar{z}) + j_0(\mathrm{Re}(a^2h^2\bar{z}))$$

$$+ \mathsf{K}_2(\bar{b}c\|a\|^2z) + \mathbf{N}_0(\mathrm{Re}(\bar{a}^2\bar{b}\bar{c}z)) + \mathbf{N}_2(ab\mathrm{Re}(ac\bar{z}))$$

$$+ \mathbf{M}_2(c\bar{h}\|a\|^2z) + \mathbf{M}_0(\mathrm{Re}(a^2bh\bar{z}))$$

we see that if $c \neq 0$ then setting $z = 0$ shows us that $\mathbf{N}_2 \subset \Pi$. But since

$$\mathbf{N}_2(f)*^A Z(a) = \mathcal{L}_1'\left(\frac{1}{2}afb\bar{z}\right) + \mathcal{L}_1\left(\frac{1}{2}ac\bar{f}z\right) + \mathbf{p}_1\left(\frac{1}{2}afh\bar{z}\right)$$

we see that varying z allow us to isolate $\mathcal{L}_1$. Since Z is irreducible, this implies $Z = \mathcal{L}_1$ and $\Pi = \mathcal{L}_1 \oplus \mathsf{K}_4 \oplus \mathbf{N}_2$. If $c = 0$, then we are left with the irreps of the type in (8.8).

Case 3: Weight two. Let Z be an arbitrary irrep of weight two and write

$$Z := \{Z(a) : a \in \mathbb{C}\} := \{\mathsf{K}_2(au) + \mathbf{N}_2(af) + \mathbf{M}_2(ag) : a \in \mathbb{C}\}.$$

Then

$$Z(a)^{*^A 2} = \mathsf{K}_2(au)^{*^A 2} + \mathbf{N}_2(af)^{*^A 2} + \mathbf{M}_2(ag)^{*^A 2} + 2\mathsf{K}_2(au)*^A\mathbf{M}_2(ag)$$

$$= \mathsf{K}_0'(\mathrm{Re}(a^2u^2\bar{w})) + \mathsf{K}_2(au\mathrm{Re}(au\bar{z})) + \mathbf{N}_2(a^2f^2\bar{z})$$

$$+ j_0(2\mathrm{Re}(a^2g^2\bar{w})) + \mathbf{M}_0(\mathrm{Re}(a^2gu\bar{w})) + \mathbf{M}_2(au\mathrm{Re}(ag\bar{z})).$$

From this we see that if $u/g \notin \mathbb{R}$, then, by judicious choices of a and z, we may isolate $\mathsf{K}_2, \mathbf{N}_2$, and $\mathbf{M}_2$. Of course if $u/g \in \mathbb{R}$, we have an algebraic subspace of the form in (8.9). $\square$

8.2 Identifying Uniform Field Relations

We would like to identify those algebraic subspaces that correspond to uniform field relations. To that end, we repeat the following lemma from [12] for completeness.

Lemma 8.2: The set of rotationally invariant uniform field relations passing through a fixed transversely isotropic tensor C_0 are in one to one correspondence with annihilators of invariant subspaces of $\mathrm{Sym}(\mathbb{R}^3)$.

Proof: Let $\mathcal{M}$ be a rotationally invariant uniform field relation. Then for some index set $\mathcal{N}$, we can write $\mathcal{M}$ as

$$\mathcal{M} = \bigcap_{\beta \in \mathcal{N}} \{C \in \mathcal{T} : C\varepsilon_\beta = \sigma_\beta\}.$$

Let $C_0 \in \mathcal{M}$ be transversely isotropic and define

$$V = \mathrm{Span}(\{\varepsilon_\beta : \beta \in \mathcal{N}\})$$

and

$$\mathcal{M}' = (C_0 + \mathrm{Ann}(V)) \cap \mathcal{T} \tag{8.13}$$

For any $C \in \mathcal{M}$ and any $\beta \in \mathcal{N}$, we have that $C\varepsilon_\beta = C_0\varepsilon_\beta$ and so $\mathcal{M} \subset \mathcal{M}'$. Similarly, given $C \in \mathcal{M}'$, $C\varepsilon_\beta = C_0\varepsilon_\beta = \sigma_\beta$ and so $\mathcal{M}' \subset \mathcal{M}$. Now we must simply show that V is rotationally invariant. By the rotational invariance of $\mathcal{M}$, given any $C \in \mathcal{M}$, $\mathcal{R} \in SO(2)$, and $\varepsilon \in V$,

$$(\mathcal{R}^{-1} \cdot C - C_0)\varepsilon = 0. \tag{8.14}$$

But then, applying $\mathcal{R}$ to (8.14) and using (5.1) and the transverse isotropy of C_0, we see

$$(C - C_0)\varepsilon = (C - C_0)(\mathcal{R} \cdot \varepsilon) = 0.$$

Thus $(\mathcal{R} \cdot \varepsilon) \in V$. $\square$

We now write the set of all nontrivial invariant subspaces, $V \subset \mathrm{Sym}(\mathbb{R}^3)$ using the notation defined in (5.2) - (5.5). Individual elements of the subspaces Y_0' and Y_0 will be written $Y_0'(\eta)$ and $Y_0(\eta)$, respectively.

- $\{Y_0'(\eta_1\eta) \oplus Y_0(\eta_2\eta) : \eta \in \mathbb{R}\}$ for some $\eta_1, \eta_2 \in \mathbb{R}$
- $Y_0' \oplus Y_0$
- Y_1
- Y_2
- $\{Y_0'(\eta_1\eta) \oplus Y_0(\eta_2\eta) : \eta \in \mathbb{R}\} \oplus Y_1$ for some $\eta_1, \eta_2 \in \mathbb{R}$
- $\{Y_0'(\eta_1\eta) \oplus Y_0(\eta_2\eta) : \eta \in \mathbb{R}\} \oplus Y_2$ for some $\eta_1, \eta_2 \in \mathbb{R}$
- $Y_2 \oplus Y_1$
- $\{Y_0'(\eta_1\eta) \oplus Y_0(\eta_2\eta) : \eta \in \mathbb{R}\} \oplus Y_2 \oplus Y_1$ for some $\eta_1, \eta_2 \in \mathbb{R}$
- $Y_0' \oplus Y_2 \oplus Y_0$
- $Y_0' \oplus Y_1 \oplus Y_0$

The annihilators of these subspaces are labeled as such in the list in appendix A.

8.3 List of Algebraic Subspaces

The complete list of algebraic subspaces is divided into three sets on the basis of the subspaces defined in (7.4) and (7.5): those contained in Π_{A}, those contained in Π_{B}, and those that intersect both Π_{A} and Π_{B}. Within each set we begin with the minimal subspaces and then build up. Equations (8.10) and (8.11) immediately tell us that Π_{A} and Π_{B} are disjoint algebraic subspaces. In the following lemma, we see that these subspaces provide a convenient structure for describing all algebraic subspaces of $\mathcal{S}$.

Lemma 8.3: Any algebraic subspace, $\Pi \subset \mathcal{S}$ can be written

$$\Pi = (\Pi \cap \Pi_{\mathsf{A}}) \oplus (\Pi \cap \Pi_{\mathsf{B}})$$

where $(\Pi \cap \Pi_{\mathsf{A}})$ and $(\Pi \cap \Pi_{\mathsf{B}})$ are also algebraic subspaces.

Proof: Observe that the intersection of two algebras is also an algebra. Given $\Pi \subset \mathcal{S}$, an arbitrary algebra, we know $\Pi \cap \Pi_{\mathsf{A}}$ and $\Pi \cap \Pi_{\mathsf{B}}$ are algebras since (8.10) and (8.11) tell us that Π_{A} and Π_{B} are algebras. By the Peter-Weyl theorem and Lemma 8.1, we can decompose Π into an orthogonal direct sum of irreps from the list (8.1) - (8.9). Since this list is divided into those contained in Π_{A} and those contained in Π_{B}, we can then write Π as the direct sum of these two groups:

$$\Pi = \pi_{\mathsf{A}}(\Pi) \oplus \pi_{\mathsf{B}}(\Pi) \tag{8.15}$$

where π_{A} and π_{B} are the projection maps onto Π_{A} and Π_{B}, respectively. However, this implies

$$\Pi = \pi_{\mathsf{A}}(\Pi) \oplus \pi_{\mathsf{B}}(\Pi) \subset (\Pi \cap \Pi_{\mathsf{A}}) \oplus (\Pi \cap \Pi_{\mathsf{B}}) \subset \Pi$$

from which the lemma follows. $\square$

Therefore after identifying algebraic subspaces contained within Π_{A} or within Π_{B}, we may simply take direct sums of these to find all remaining algebras. The complete list is located in appendix A.

8.4 Inversion Formula for Exact Relations

Recall from chapter 6 that the factor $C_0^{1/2}B_1^T$ in the inversion formula (3.7) may be constructed so that it is diagonal of the form $\mathsf{K}_0'(\delta_1) + \mathsf{K}_0(\delta_2) + \mathbf{N}_0\big(\sqrt{\frac{2\delta_1\delta_2\rho_0^2}{\vartheta_0}}\big) + j_0(\delta_3)$ for arbitrary $\delta_1, \delta_2, \delta_3 \in \mathbb{R}$. If we let $C_0^{1/2}B_1^T = \mathsf{K}_0'(1) + \mathsf{K}_0(1) + \mathbf{N}_0\big(\sqrt{\frac{2\rho_0^2}{\vartheta_0}}\big) + j_0(1)$, we can quickly verify that $C_0^{1/2}B_1^T \Pi B_1 C_0^{1/2} = \Pi$ for all algebras Π since this holds for all algebras contained in Π_{A} as well as those in Π_{B}. (The only algebra that requires more than a cursory examination is $\Pi_1^{1,0,1}$. The verification in this case is also straightforward.) Thus whenever we may let $S_0 = 0$, the inversion (3.7) has an especially nice form:

$$\mathcal{M} = \{C = C_0 + K : K \in \Pi\} \cap \mathcal{T}.$$

In particular, this holds for every annihilator of an invariant subspace of $\mathrm{Sym}(\mathbb{R}^3)$, which verifies (8.13).

For Π such that we cannot allow $S_0 = 0$, we will still frequently be able to choose $\delta_1, \delta_2, \delta_3$ such that

$$C_0^{1/2} B_1^T (I + KS_0)^{-1} K B_1 C_0^{1/2} = (I + KS_0)^{-1} K \tag{8.16}$$

for all $K \in \Pi$. In this case

$$\mathcal{M} = \{C = C_0 + (I + KS_0)^{-1} K : K \in \Pi\} \cap \mathcal{T}.$$

8.5 Example of an Exact Relation in Physical Variables

We now focus on the algebraic subspace

$$\Pi_1 = \{\mathcal{L}_1'(b) + \mathbf{p}_1(ib) : b \in \mathbb{C}\} \oplus \mathsf{K}_0' \oplus \mathbf{M}_0 \oplus \mathbb{J}$$

which provides us with a nontrivial example. For this subspace, a straightforward calculation shows us that $S_0 = \mathsf{K}_0'(\vartheta_0^{-1})$ satisfies (3.6). Write an arbitrary $K \in \Pi_1$ as

$$K = \begin{bmatrix} \hat{\lambda} & 0 & \hat{b}^T & \hat{\alpha} \\ 0 & 0 & 0 & 0 \\ \hat{b} & 0 & 0 & i\hat{b} \\ \hat{\alpha} & 0 & (i\hat{b})^T & \hat{\gamma} \end{bmatrix}.$$

Note that for this case, if we fix the parameters of B_1 such that

$$C_0^{1/2}B_1^T = \mathsf{K}_0'(1) + \mathsf{K}_0\left(\frac{\vartheta_0}{2\rho_0^2}\right) + \mathbf{N}_0(1) + j_0(1),$$

then $C_0^{1/2}B_1^T$ satisfies (8.16) for all $K \in \Pi_1$. If the element corresponding to K in the exact relation in physical variables passing through C_0 is

$$C = C_0 - (I + KS_0)^{-1}K \tag{8.17}$$

then we can write the exact relation as consisting of tensors of the form

$$C = \frac{1}{\omega_0\lambda + \zeta_0}\begin{bmatrix} \lambda & 0 & b^T & \alpha \\ 0 & \eta_0\mathbf{I} & 0 & 0 \\ b & 0 & \beta_0\mathbf{I} + b \otimes b & \upsilon_0 b(\alpha - i\lambda + z_0) \\ \alpha & 0 & (\upsilon_0 b(\alpha - i\lambda + z_0))^T & \gamma \end{bmatrix} \tag{8.18}$$

where

$$\lambda = \lambda_0^2 + \alpha_0^2 - \hat{\lambda}$$

$$b = -\hat{b}$$

$$\alpha = (\lambda_0 + \gamma_0)\alpha_0 - \hat{\alpha}$$

$$\gamma = \gamma_0^2 + \alpha_0^2 - \hat{\gamma}(1 + \hat{\lambda}\vartheta_0^{-1}) + \hat{\alpha}^2\vartheta_0^{-1}$$

and all values with subscript 0 are constants depending only on C_0.

We obtain further information for this example by calculating the volume fraction relation. Since $\Pi_1^{*A2} = \mathsf{K}_0' \oplus \mathbf{M}_0 \oplus \mathbb{J}$, we see that

$$(\Pi^{*A2})^{\perp} = \{\mathcal{L}_1'(b) + \mathbf{p}_1(ib) : b \in \mathbb{C}\}$$

and therefore we can augment (8.18) with the fact that $b^* = \langle b \rangle$, where b^* represents the $\mathcal{L}_1'$ component of C^*.

CHAPTER 9

LINK CALCULATIONS

9.1 Simplification of $\hat{\mathcal{A}}$

We begin our discussion of links by computing the simplification in (4.2). Let

$$C_i^{1/2} = \begin{bmatrix} \lambda_{0,i} & 0 & 0 & \alpha_{0,i} \\ 0 & \phi(\mu_{0,i}) & 0 & 0 \\ 0 & 0 & \phi(\rho_{0,i}) & 0 \\ \beta_{0,i} & 0 & 0 & \gamma_{0,i} \end{bmatrix}, \quad B_i = \begin{bmatrix} \lambda_i & 0 & 0 & \alpha_i \\ 0 & \phi(\mu_i) & 0 & 0 \\ 0 & 0 & \phi(\rho_i) & 0 \\ \beta_i & 0 & 0 & \gamma_i \end{bmatrix},$$

and

$$A_i = \frac{1}{\vartheta_{0,i}} \begin{bmatrix} 0 & \lambda_{0,i}\mu_{0,i}v & 0 & 0 \\ \lambda_{0,i}\mu_{0,i}v & \psi(-\frac{1}{2}(\lambda_{0,i}^2 + \alpha_{0,i}^2)v^2) & 0 & \alpha_{0,i}\mu_{0,i}v \\ 0 & 0 & \psi(\frac{1}{2}\vartheta_{0,i}v) & 0 \\ 0 & \alpha_{0,i}\mu_{0,i}v & 0 & 0 \end{bmatrix}$$

for $i = 1, 2$. Our goal is to make the two copies of $\mathcal{A}$ identical. That is, we would like to make $B_1 A_1 B_1^T = B_2 A_2 B_2^T = A$. First we let

$$\beta_i = -\frac{\alpha_{0,i}\gamma_i}{\lambda_{0,i}} \tag{9.1}$$

and

$$\rho_i = \sqrt{\frac{2\mu_i\mu_{0,i}}{\vartheta_{0,i}}(\lambda_i\lambda_{0,i} + \alpha_i\alpha_{0,i})} \tag{9.2}$$

for $i = 1, 2$ so that each copy of $\mathcal{A}$ is simplified as before. That is,

$$B_i A_i B_i^T = \left(\frac{\mu_{0,i}\mu_i}{\vartheta_{0,i}} (\lambda_{0,i}\lambda_i + \alpha_{0,i}\alpha_i) \right) [\mathsf{K}_2(v) + \mathbf{N}_2(v)] - \mathsf{K}_4\big(\frac{\mu_i^2}{2\vartheta_{0,i}} (\lambda_{0,i}^2 + \alpha_{0,i}^2) v^2 \big)$$

for $i = 1, 2$. Now we will also require

$$\mu_2^2 = \mu_1^2 \frac{\vartheta_{0,2}(\lambda_{0,1}^2 + \alpha_{0,1}^2)}{\vartheta_{0,1}(\lambda_{0,2}^2 + \alpha_{0,2}^2)} \tag{9.3}$$

so that the K_4 components in A_1' and A_2' are equal. Similarly we will need to fix α_2 and λ_2 such that

$$\lambda_2\lambda_{0,2} + \alpha_2\alpha_{0,2} = \frac{\mu_{0,1}\sqrt{\vartheta_{0,2}(\lambda_{0,2}^2 + \alpha_{0,2}^2)}}{\mu_{0,2}\sqrt{\vartheta_{0,1}(\lambda_{0,1}^2 + \alpha_{0,1}^2)}} (\lambda_1\lambda_{0,1} + \alpha_1\alpha_{0.1}) \neq 0 \tag{9.4}$$

so that the K_2 and $\mathbf{N}_2$ components are equal. So now our new $\hat{\mathcal{A}}$ is simply

$$\hat{\mathcal{A}} = \{[A, A] : A \in \mathcal{A}\}.$$

Recall that this allows us to view the Φ described in chapter 7 as the algebraic representations of the links we seek. Now we must convert these links to physical variables.

9.2 Inversion of Links

What is the analog of the inversion formula for links? We need M_1, M_2 such that

$$\begin{bmatrix} K' & 0 \\ 0 & \Phi(K') \end{bmatrix} \begin{bmatrix} B_1\bar{\Gamma}_1 B_1^T - M_1 & 0 \\ 0 & B_2\bar{\Gamma}_2 B_2^T - M_2 \end{bmatrix} \begin{bmatrix} K' & 0 \\ 0 & \Phi(K') \end{bmatrix} \in \hat{\Pi}$$

for all $K' \in \Pi_1' = B_1^{-T}\Pi_1 B_1^{-1}$, where $\bar{\Gamma}_i$ is the isotropic part of $\Gamma_i(\mathbf{n})$ (recall that $\bar{\Gamma}_i$ does not depend on $\mathbf{n}$). That is, we need

$$\Phi(K'(B_1\bar{\Gamma}_1 B_1^T - M_1)K') = \Phi(K')(B_2\bar{\Gamma}_2 B_2' - M_2)\Phi(K') \tag{9.5}$$

for all $K' \in \Pi_1'$. But we can write $\Phi(K') = XK'X^T$. Thus we need

$$X^T M_2 X - M_1 = X^T B_2 \bar{\Gamma}_2 B_2^T X - B_1 \bar{\Gamma}_1 B_1^T. \tag{9.6}$$

Now, using (6.7) and the fact that $X^T B_2 \bar{\Gamma}_2 B_2^T X = B_2 \bar{\Gamma}_2 B_2^T$, we see that in general we cannot set B_1, B_2 so that $B_1 \bar{\Gamma}_1 B_1^T = X^T B_2 \bar{\Gamma}_2 B_2^T X$. Therefore we cannot set $M_1 = M_2 = 0$ in (9.6). Furthermore, the difference $B_1 \bar{\Gamma}_1 B_1^T - B_2 \bar{\Gamma}_2 B_2^T$ is fixed. We will return to this fact shortly.

Using a slightly altered version of (3.7), let

$$C = C_1 - C_1^{1/2} B_1^T (K_1^{-1} + M_1)^{-1} B_1 C_1^{1/2} \tag{9.7}$$

$$C' = C_2 - C_2^{1/2} B_2^T (K_2^{-1} + M_2)^{-1} B_2 C_2^{1/2}. \tag{9.8}$$

Then solving (9.7) for K_1^{-1}, using $K_2^{-1} = X^{-T} K_1^{-1} X^{-1}$ and substituting this into (9.8), we see

$$C' = C_2 - C_2^{1/2} B_2^T (X^{-T} [B_1 C_1^{1/2} (C_1 - C)^{-1} C_1^{1/2} B_1^T - M_1] X^{-1} + M_2)^{-1} B_2 C_2^{1/2}. \tag{9.9}$$

Unlike the case of exact relations, for links we want to keep as many free parameters from B_1 and B_2 as possible since they provide us with degrees of freedom to link known results to new results. We will now use the difference $B_1 \bar{\Gamma}_1 B_1^T - B_2 \bar{\Gamma}_2 B_2^T$ to define the fixed tensor

$$H = C_2^{-1/2} B_2^{-1} (-B_1 \bar{\Gamma}_1 B_1^T + B_2 \bar{\Gamma}_2 B_2^T) B_2^{-T} C_2^{-1/2}.$$

Then using (9.6), (9.9), and the fact that $X^{-T} B_i \bar{\Gamma}_i B_i^T X^{-1} = B_i \bar{\Gamma}_i B_i^T$ we can write

$$C' = C_2 - [I + \Delta(C) H]^{-1} \Delta(C) \tag{9.10}$$

where

$$\Delta(C) = S - QCQ^T \tag{9.11}$$

and where S and Q are transversely isotropic tensors defined

$$S = C_2^{1/2} B_2^T X B_1^{-T} B_1^{-1} X^T B_2 C_2^{1/2} \tag{9.12}$$

and

$$Q = C_2^{1/2} B_2^T X B_1^{-T} C_1^{-1/2}. \tag{9.13}$$

9.2.1 New Block Construction

It will now be convenient to write elasticity tensors as 3×3 block matrices by combining the blocks that act on ω and z described in chapter 5. Then we can write the elasticity tensor C effecting the constitutive relation $C\varepsilon = \sigma$ in the form

$$C = \begin{bmatrix} \mathsf{C} & \mathcal{C} & \mathbf{c} \\ \mathcal{C}^T & \mathbf{C} & c \\ \mathbf{c}^T & c^T & \gamma \end{bmatrix}$$

where

$$\mathsf{C} : \mathrm{Sym}(\mathbb{R}^2) \to \mathrm{Sym}(\mathbb{R}^2), \quad \mathcal{C} : \mathbb{R}^2 \to \mathrm{Sym}(\mathbb{R}^2), \quad \mathbf{c} : \mathbb{R} \to \mathrm{Sym}(\mathbb{R}^2),$$

$$\mathbf{C} : \mathbb{R}^2 \to \mathbb{R}^2, \quad c : \mathbb{R} \to \mathbb{R}^2, \text{ and } \gamma : \mathbb{R} \to \mathbb{R}.$$

We may think of C as having the same form as a two-dimensional elasticity tensor. Also, we can think of $\mathbf{C}$ and $\mathbf{c}$ as symmetric 2×2 matrices and c as a vector in $\mathbb{R}^2$.

This form allows us to write H quite simply. Using (9.2), (9.3), and (9.4), we see that $\rho_1^2 = \rho_2^2$. This, together with (9.1) and (6.7), tells us that $H = \mathsf{K}_0'(h_1) + \mathsf{K}_0(h_2)$ where h_1 and h_2 are determined completely by C_1 and C_2:

$$h_1 = \frac{1}{\vartheta_{0,2}} \left(1 - \frac{\mu_{0,2}^2(\lambda_{0,1}^2 + \alpha_{0,1}^2)}{\mu_{0,1}^2(\lambda_{0,2}^2 + \alpha_{0,2}^2)} \right)$$

$$h_2 = \frac{1}{\vartheta_{0,2}} \left(1 - \frac{\mu_{0,1}^2(\lambda_{0,2}^2 + \alpha_{0,2}^2)}{\mu_{0,2}^2(\lambda_{0,1}^2 + \alpha_{0,1}^2)} \right).$$

So we can write

$$H = \begin{bmatrix} \mathsf{H} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \Delta(C) = \begin{bmatrix} \mathsf{D} & \mathcal{D} & \mathbf{d} \\ \mathcal{D}^T & \mathbf{D} & d \\ \mathbf{d}^T & d^T & \delta \end{bmatrix}$$

which implies

$$[I + \Delta(C)H]^{-1} = \begin{bmatrix} \Lambda(\mathsf{D}) & 0 & 0 \\ -\mathcal{D}^T\mathsf{H}\Lambda(\mathsf{D}) & \phi(1) & 0 \\ -\mathbf{d}^T\mathsf{H}\Lambda(\mathsf{D}) & 0 & 1 \end{bmatrix}$$

where $\Lambda(\mathsf{D}) = (\mathsf{I} + \mathsf{DH})^{-1}$ and thus that

$$[I + \Delta(C)H]^{-1}\Delta(C) = \begin{bmatrix} \Lambda(\mathsf{D})\mathsf{D} & \Lambda(\mathsf{D})\mathcal{D} & \Lambda(\mathsf{D})\mathbf{d} \\ (\Lambda(\mathsf{D})\mathcal{D})^T & \mathbf{D} - \mathcal{D}^T\mathsf{H}\Lambda(\mathsf{D})\mathcal{D} & d - \mathcal{D}^T\mathsf{H}\Lambda(\mathsf{D})\mathbf{d} \\ (\Lambda(\mathsf{D})\mathbf{d})^T & (d - \mathcal{D}^T\mathsf{H}\Lambda(\mathsf{D})\mathbf{d})^T & \delta - \mathbf{d}^T\mathsf{H}\Lambda(\mathsf{D})\mathbf{d} \end{bmatrix}. \tag{9.14}$$

Now write the transversely isotropic tensors C_2, S, and Q as

$$C_2 = \begin{bmatrix} \mathsf{C}_2 & 0 & \mathbf{c}_2 \\ 0 & \mathbf{C}_2 & 0 \\ \mathbf{c}_2^T & 0 & \gamma_2 \end{bmatrix} \qquad S = \begin{bmatrix} \mathsf{S} & 0 & \mathbf{s} \\ 0 & \mathbf{S} & 0 \\ \mathbf{s}^T & 0 & \varsigma \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} \mathsf{Q} & 0 & 0 \\ 0 & \mathbf{Q} & 0 \\ \mathbf{q}^T & 0 & \xi \end{bmatrix}$$

where C_2, S, and Q can be thought of as isotropic two-dimensional elasticity tensors and $\mathbf{C}_2, \mathbf{c}_2, \mathbf{S}, \mathbf{s}, \mathbf{Q}$, and $\mathbf{q}$ are scalar 2×2 matrices. That the upper right block of Q is zero follows from (9.1). Let us also write

$$C = \begin{bmatrix} \mathsf{C} & \mathcal{C} & \mathbf{c} \\ \mathcal{C}^T & \mathbf{C} & c \\ \mathbf{c}^T & c^T & \gamma \end{bmatrix} \quad \text{and} \quad C' = \begin{bmatrix} \mathsf{C}' & \mathcal{C}' & \mathbf{c}' \\ (\mathcal{C}')^T & \mathbf{C}' & c' \\ (\mathbf{c}')^T & (c')^T & \gamma' \end{bmatrix}.$$

We will focus first on the upper left block of C'. From (9.11) we have

$$\mathsf{D} = \mathsf{S} - \mathsf{QCQ}$$

and so (9.10) and (9.14) lead us to

$$\mathsf{C}' = \mathsf{C}_2 - (\mathsf{I} + (\mathsf{S} - \mathsf{QCQ})\mathsf{H})^{-1}(\mathsf{S} - \mathsf{QCQ}).$$

Since $\mathsf{H}, \mathsf{S}, \mathsf{Q}$, and C_2 all commute, we can rewrite this in the form

$$\mathsf{C}' = \mathsf{C}_2 - \mathsf{H}^{-1} - \mathsf{H}^{-1}\mathsf{Q}^{-1}(\mathsf{C} - \mathsf{Q}^{-2}(\mathsf{H}^{-1} + \mathsf{S}))^{-1}\mathsf{Q}^{-1}\mathsf{H}^{-1}.$$

Letting $\mathsf{G}_1 = \mathsf{C}_2 - \mathsf{H}^{-1}, \mathsf{G}_2 = \mathsf{H}^{-1}\mathsf{Q}^{-1}$, and $\mathsf{G}_3 = \mathsf{Q}^{-2}(\mathsf{H}^{-1} + \mathsf{S})$, we have

$$\mathsf{C}' = \mathsf{G}_1 - \mathsf{G}_2(\mathsf{C} - \mathsf{G}_3)^{-1}\mathsf{G}_2 \tag{9.15}$$

where $\mathsf{G}_1, \mathsf{G}_2$, and G_3 are uniquely defined (up to ± 1 in the case of G_2).

Let us define

$$\mathsf{G}_0 = \begin{bmatrix} 1 & 0 \\ 0 & \phi(-1) \end{bmatrix}.$$

Note that the map $\mathsf{G}_0 : \mathrm{Sym}(\mathbb{R}^2) \to \mathrm{Sym}(\mathbb{R}^2)$ is defined by its action on $\mathbf{E} = \phi(\omega) + \psi(z) \in \mathrm{Sym}(\mathbb{R}^2)$:

$$\mathsf{G}_0\mathbf{E} = \phi(\omega) - \psi(z).$$

So G_0 representss a two-dimensional isotropic elasticity tensor with bulk modulus $\frac{1}{2}$ and shear modulus $-\frac{1}{2}$ and $\mathsf{G}_0^2 = \mathsf{I}$. In other words, if we define the inner product of two symmetric 2×2 matrices as in (2.4) then

$$\langle \mathsf{G}_0\mathbf{E}, \mathbf{E} \rangle = \det\mathbf{E}.$$

Then, by definition,

$$\mathsf{G}_1 = \frac{\vartheta_{0,1}\mu_{0,2}^2(\lambda_{0,2}^2 + \alpha_{0,2}^2)}{\mu_{0,2}^2(\lambda_{0,1}^2 + \alpha_{0,1}^2) - \mu_{0,1}^2(\lambda_{0,2}^2 + \alpha_{0,2}^2)}\mathsf{G}_0 = g_1\mathsf{G}_0$$

$$\mathsf{G}_2 = \pm\frac{\sqrt{\vartheta_{0,1}\vartheta_{0,2}\mu_{0,1}^2\mu_{0,2}^2(\lambda_{0,1}^2 + \alpha_{0,1}^2)(\lambda_{0,2}^2 + \alpha_{0,2}^2)}}{\mu_{0,2}^2(\lambda_{0,1}^2 + \alpha_{0,1}^2) - \mu_{0,1}^2(\lambda_{0,2}^2 + \alpha_{0,2}^2)}\mathsf{G}_0 = g_2\mathsf{G}_0$$

$$\mathsf{G}_3 = \frac{-\vartheta_{0,2}\mu_{0,1}^2(\lambda_{0,1}^2 + \alpha_{0,1}^2)}{\mu_{0,2}^2(\lambda_{0,1}^2 + \alpha_{0,1}^2) - \mu_{0,1}^2(\lambda_{0,2}^2 + \alpha_{0,2}^2)}\mathsf{G}_0 = g_3\mathsf{G}_0$$

that is, $\mathsf{G}_1, \mathsf{G}_2$, and G_3 are scalar multiples of each other with $g_2^2 = -g_1g_3$. Now we can write the link for two-dimensional elasticity (9.15) as

$$\mathsf{C}' = g_2^2(\mathsf{G}_0(g_3\mathsf{G}_0 - \mathsf{C})^{-1}\mathsf{G}_0 - g_3^{-1}\mathsf{G}_0) = \frac{g_2^2}{g_3^2}(\mathsf{C}^{-1} - g_3^{-1}\mathsf{G}_0)^{-1}$$

where $g_3 \in \mathbb{R}$ must be chosen such that $(\mathsf{C}^{-1} - g_3^{-1}\mathsf{G}_0)$ is positive definite. Written as a relation between compliance tensors C^{-1} and $(\mathsf{C}')^{-1}$, we see that the link is affine. For the remaining computations it will be useful to note that

$$(\mathsf{I} + \mathsf{DH})^{-1} = g_2\mathsf{G}_0(g_3\mathsf{G}_0 - \mathsf{C})^{-1}\mathsf{Q}^{-1}. \tag{9.16}$$

Now we compute the remaining block components of C' using (9.10) and (9.14). We will begin by considering the $\mathbb{L}$ block. Initially we have

$$\mathcal{C}' = -(\mathsf{I} + \mathsf{DH})^{-1}\mathsf{Q}\mathcal{C}\mathbf{Q}.$$

Using (9.16) and the fact that $\mathbf{Q}$ is a scalar operator with scalar value $\pm\frac{\rho_{0,2}}{\rho_{0,1}}$, this simplifies to

$$\mathcal{C}' = \pm\frac{g_2\rho_{0,2}}{\rho_{0,1}}\mathsf{G}_0(g_3\mathsf{G}_0 - \mathsf{C})^{-1}\mathcal{C}.$$

Next, we look at the $\mathbb{M}$ block. This block maps $\mathbb{R} \cong Y_0$ to $\mathrm{Sym}(\mathbb{R}^2)$. It is perhaps easiest to represent this map as the matrix to which it maps $1 \in \mathbb{R}$. Letting $\mathbf{I}_2$ represent the two by two identity matrix, we can write

$$\begin{aligned}\mathbf{c}' &= \mathbf{c}_2 - (\mathsf{I} + \mathsf{DH})^{-1}\mathbf{d}\\ &= (\alpha_{0,2}(\lambda_{0,2} + \gamma_{0,2}) - g_2q_5)\mathbf{I}_2 + g_2\mathsf{G}_0(g_3\mathsf{G}_0 - \mathsf{C})^{-1}\left[\left(g_3q_5 - \frac{s_5}{q_1}\right)\mathbf{I}_2 + \xi\mathbf{c}\right]\end{aligned}$$

where q_1 is the K_0' component of Q and q_5 and s_5 are the $\mathbf{M}_0$ components of Q and S, respectively. Looking at the $\mathbb{N}$ block gives us

$$\begin{aligned}\mathbf{C}' &= \mathbf{C}_2 - \mathbf{D} + \mathcal{D}^T \mathsf{H}(\mathsf{I} + \mathsf{DH})^{-1}\mathcal{D} \\ &= \frac{\rho_{0,2}^2}{\rho_{0,1}^2}(\mathbf{C} + \mathcal{C}^T(g_3\mathsf{G}_0 - \mathsf{C})^{-1}\mathcal{C}).\end{aligned}$$

For the $\mathbb{P}$ block, thinking of this as the vector to which $1 \in \mathbb{R}$ is mapped,

$$\begin{aligned}c' &= \mathcal{D}^T \mathsf{H}(\mathsf{I} + \mathsf{DH})^{-1}\mathbf{d} - d \\ &= \pm\frac{\rho_{0,2}}{\rho_{0,1}}\xi c \pm \frac{\rho_{0,2}}{\rho_{0,1}}\mathcal{C}^T(g_3\mathsf{G}_0 - \mathsf{C})^{-1}\left[\left(g_3q_5 - \frac{s_5}{q_1}\right)\mathbf{I}_2 + \xi\mathbf{c}\right].\end{aligned}$$

Finally, for the $\mathbb{J}$ block we have

$$\gamma' = \gamma_2 - \delta + \mathbf{d}^T\mathsf{H}(\mathsf{I} + \mathsf{DH})^{-1}\mathbf{d}$$

and so

$$\begin{aligned}\gamma' = {} & (\alpha_{0,2}^2 + \gamma_{0,2}^2 - \varsigma + 2q_5\frac{s_5}{q_1} - g_3q_5^2) + \xi^2\gamma \\ & + \frac{1}{2}\mathrm{Tr}\left[\left(\left(g_3q_5 - \frac{s_5}{q_1}\right)\mathbf{I}_2 + \xi\mathbf{c}\right)(g_3\mathsf{G}_0 - \mathsf{C})^{-1}\left(\left(g_3q_5 - \frac{s_5}{q_1}\right)\mathbf{I}_2 + \xi\mathbf{c}\right)\right].\end{aligned}$$

This completes the computations for each of the six blocks of C'. We are now ready to put them together in a simpler final form.

9.2.2 Final General Link Construction

Let

$$a_0 = g_3^{-1}, \quad a_1 = g_2, \quad a_3 = \xi, \quad a_4 = (\alpha_{0,2}(\lambda_{0,2} + \gamma_{0,2}) - g_2q_5)$$
$$a_5 = \xi^{-1}\left(g_3q_5 - \frac{s_5}{q_1}\right), \quad a_6 = \left(\alpha_{0,2}^2 + \gamma_{0,2}^2 - \varsigma + \frac{2q_5s_5}{q_1} - g_3q_5^2\right)$$

and define

$$\Theta(\mathsf{C}) = (a_0^{-1}\mathsf{G}_0 - \mathsf{C})^{-1} \quad \text{and} \quad \Xi(\mathbf{c}) = (\mathbf{c} + a_5\mathbf{I}_2).$$

Then we can write the link as

$$C' = \begin{bmatrix} a_1^2(\mathsf{G}_0\Theta(\mathsf{C})\mathsf{G}_0 - a_0\mathsf{G}_0) & a_1a_2\mathsf{G}_0\Theta(\mathsf{C})\mathcal{C} & a_4\mathbf{I}_2 + a_1a_3\mathsf{G}_0\Theta(\mathsf{C})\Xi(\mathbf{c}) \\ (C'_{1,2})^T & a_2^2(\mathcal{C}^T\Theta(\mathsf{C})\mathcal{C} + \mathbf{C}) & a_2a_3(c + \mathcal{C}^T\Theta(\mathsf{C})\Xi(\mathbf{c})) \\ (C'_{1,3})^T & (C'_{2,3})^T & a_6 + a_3^2(\gamma + \langle\Theta(\mathsf{C})\Xi(\mathbf{c}), \Xi(\mathbf{c})\rangle) \end{bmatrix}. \quad (9.17)$$

In general the values $a_i \in \mathbb{R}$ are independent for $i = 0, ..., 6$. Hereafter they will serve as our parameters for the general link, in place of the parameters from C_1, C_2, B_1, B_2, and X.

9.2.3 Linear Link

We may also consider when $H = 0$, that is, when $B_1\bar{\Gamma}_1B_1^T = B_2\bar{\Gamma}_2B_2^T$. Using (9.3), (9.4), and (6.7), we see that this holds if and only if

$$\frac{\mu_{0,1}^2}{\mu_{0,2}^2} = \frac{\lambda_{0,1}^2 + \alpha_{0,1}^2}{\lambda_{0,2}^2 + \alpha_{0,2}^2}.$$

In this case the inversion (9.10) becomes linear and has the form

$$C' = C_2 - S + QCQ^T$$

where S and Q are as defined in (9.12) and (9.13). Let us write

$$F = C_2 - S = \begin{bmatrix} 0 & 0 & f_1\mathbf{I} \\ 0 & 0 & 0 \\ f_1\mathbf{I} & 0 & f_2 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} d_1\mathsf{I} & 0 & 0 \\ 0 & d_2\mathbf{I} & 0 \\ d_4\mathbf{I} & 0 & d_3 \end{bmatrix}$$

where $f_1, f_2, d_1, d_2, d_3, d_4 \in \mathbb{R}$. The link is then

$$C' = \begin{bmatrix} d_1^2\mathsf{C} & d_1d_2\mathcal{C} & f_1\mathbf{I} + d_1d_3\mathbf{c} + d_1d_4\mathsf{C}\mathbf{I} \\ (C'_{1,2})^T & d_2^2\mathbf{C} & d_2d_3c + d_2d_4\mathcal{C}^T\mathbf{I} \\ (C'_{1,3})^T & (C'_{2,3})^T & f_2 + d_3^2\gamma + 2d_3d_4\langle\mathbf{c}, \mathbf{I}\rangle + d_4^2\langle\mathsf{C}\mathbf{I}, \mathbf{I}\rangle \end{bmatrix}. \quad (9.18)$$

This special case corresponds to the limit of the general link when certain constants go to zero or infinity. Using the Neumann series expansion,

$$\Theta(\mathsf{C}) = a_0\mathsf{G}_0 + a_0^2\mathsf{G}_0\mathsf{C}\mathsf{G}_0 + O(a_0^3).$$

Fix $a_2 = d_2$ and $a_3 = d_3$ and let $a_0 \to 0$ and $a_1, a_4, a_5, a_6 \to \infty$ such that

$$a_0a_1 \to d_1, \quad a_0a_3a_5 \to d_4, \quad a_4 + a_0a_1a_3a_5 \to f_1, \quad \text{and} \quad a_6 + a_0a_3^2a_5^2 \to f_2.$$

Then limit of the general case converges to the linear link $C' = F + QCQ^T$.

We make one further observation to clarify the relationship between the general case and the linear case: the essential nonlinearity in the general case (9.17) is manifested entirely in a_0. That is, we can rewrite any element of the general case as the composition of an element of the linear case and a special element of the general case involving only a_0. More explicitly, let $\mathcal{F}$ represent an arbitrary element of the general case (9.17) with parameters a_i. Then fix $\mathcal{F}_0$ to be a special case of (9.17) with parameters a_i^0 where $a_1^0 = a_2^0 = a_3^0 = 1$, $a_4^0 = a_5^0 = a_6^0 = 0$, and $a_0^0 = a_0$. That is,

$$\mathcal{F}_0(C) = \begin{bmatrix} a_0\mathsf{G}_0\Theta(\mathsf{C})\mathsf{C} & \mathsf{G}_0\Theta(\mathsf{C})\mathcal{C} & \mathsf{G}_0\Theta(\mathsf{C})\mathbf{c} \\ \mathcal{C}^T\Theta(\mathsf{C})\mathsf{G}_0 & \mathcal{C}^T\Theta(\mathsf{C})\mathcal{C} + \mathbf{C} & \mathcal{C}^T\Theta(\mathsf{C})\mathbf{c} + c \\ \mathbf{c}\Theta(\mathsf{C})\mathsf{G}_0 & \mathbf{c}\Theta(\mathsf{C})\mathcal{C} + c^T & \langle\Theta(\mathsf{C})\mathbf{c}, \mathbf{c}\rangle + \gamma \end{bmatrix}.$$

If we let $\mathcal{F}_1$ represents the linear link (9.18) with

$$d_1 = a_1, \quad d_2 = a_2, \quad d_3 = a_3, \quad d_4 = a_0a_3a_5,$$

$$f_1 = a_4 + a_0a_1a_3a_5, \quad \text{and} \quad f_2 = a_6 + a_0a_3^2a_5^2,$$

then we have

$$\mathcal{F}_1(\mathcal{F}_0(C)) = \mathcal{F}(C)$$

for all tensors $C \in \mathcal{T}$.

CHAPTER 10

APPLICATIONS

Now that we can describe our links using the parameters a_i or f_i and d_i, we will reset our notation and reuse the variables $\lambda_i, \mu_i, \alpha_i, \rho_i$, and γ_i to define new elasticity tensors, not necessarily the fixed tensors C_0, C_1, C_2, B_1, or B_2.

10.1 Composite Made from Two Isotropic Materials

Let us apply the link to Hill's exact relation regarding two isotropic materials with the same shear modulus. Suppose we make a composite with two transversely isotropic materials, C_1 and C_2. We can write these as

$$C_i = \begin{bmatrix} \mathsf{C}_i(\lambda_i, \mu_i) & 0 & \alpha_i \mathbf{I}_2 \\ 0 & \rho_i \mathbf{I}_2 & 0 \\ \alpha_i \mathbf{I}_2 & 0 & \gamma_i \end{bmatrix} \tag{10.1}$$

where $\mathsf{C}_i(\lambda_i, \mu_i)$ represents the elasticity tensor of a two-dimensional isotropic material with shear modulus $\frac{1}{2}\mu_i$ and bulk modulus $\frac{1}{2}\lambda_i$ for $i = 1, 2$. The parameters above relate to the standard engineering constants in Voigt notation in the following ways:

$$\lambda_i = C_{11}^i + C_{12}^i, \quad \mu_i = 2C_{66}^i, \quad \rho_i = 2C_{44}^i, \quad \gamma_i = C_{33}^i, \quad \text{and} \quad \alpha_i = \sqrt{2}C_{13}^i.$$

Alternately, [7] uses the following six constants and one relation to describe transversely isotropic materials. The Young's moduli in the longitudinal and transverse directions are given by

$$E_L^i = \frac{1}{\lambda_i}(\gamma_i \lambda_i - \alpha_i^2)$$

and

$$E_T^i = \frac{2\mu_i(\lambda_i \gamma_i - \alpha_i^2)}{(\lambda_i + \mu_i)\gamma_i - \alpha_i^2}.$$

The Poisson ratio for loading on the transversal axis is

$$\nu_L^i = \frac{\alpha_i}{\sqrt{2}\lambda_i}$$

while the Poisson ratio describing the orthogonal contraction within the transversal plane due to tension applied in the transversal plane is

$$\nu_T^i = \frac{(\lambda_i - \mu_i)\gamma_i - \alpha_i^2}{(\lambda_i + \mu_i)\gamma_i - \alpha_i^2}.$$

Finally, the shear moduli in the longitudinal and transverse directions are simply

$$G_L^i = \frac{1}{2}\rho_i \quad \text{and} \quad G_T^i = \frac{1}{2}\mu_i.$$

The relation indicating the dependence between these six constants is

$$G_T^i = \frac{E_T^i}{2(1 + \nu_T^i)}.$$

Our linear link maps these transversely isotropic materials to two new transversely isotropic materials given by

$$
\begin{aligned}
C_i' &= \begin{bmatrix} \mathsf{C}_i'(\lambda_i', \mu_i') & 0 & \alpha_i' \mathbf{I}_2 \\ 0 & \rho_i' \mathbf{I}_2 & 0 \\ \alpha_i' \mathbf{I}_2 & 0 & \gamma_i' \end{bmatrix} \\
&= \begin{bmatrix} d_1^2 \mathsf{C}_i(\lambda_i, \mu_i) & 0 & (f_1 + d_1 d_3 \alpha_i + d_1 d_4 \lambda_i) \mathbf{I}_2 \\ 0 & d_2^2 \rho_i \mathbf{I}_2 & 0 \\ (f_1 + d_1 d_3 \alpha_i + d_1 d_4 \lambda_i) \mathbf{I}_2 & 0 & f_2 + d_3^2 \gamma_i + 2 d_3 d_4 \alpha_i + d_4^2 \lambda_i \end{bmatrix}.
\end{aligned}
$$

We can always use the free parameters to set $\alpha_i' = 0$ so that the C_i' are block diagonal, which allows us to take advantage of the following lemma.

Lemma 10.1: If C represents the elasticity tensor of a fiber-reinforced composite and is block diagonal of the form

$$
C = \begin{bmatrix} \mathsf{C} & 0 & 0 \\ 0 & \mathbf{C} & 0 \\ 0 & 0 & \gamma \end{bmatrix}
$$

then its effective tensor is of the form

$$
C^* = \begin{bmatrix} \mathsf{C}^* & 0 & 0 \\ 0 & \mathbf{C}^* & 0 \\ 0 & 0 & \langle \gamma \rangle \end{bmatrix}
$$

where C^* and $\mathbf{C}^*$ represent the effective elasticity and conductivity tensors of a two-dimensional composite with local elasticity tensor C and local conductivity tensor $\mathbf{C}$ and the same microstructure as the original fiber-reinforced composite's transversal cross-section.

Proof: Let $\mathbf{u} = (u_1, u_2, u_3) = (u', u_3)$ represent a deformation. Then we can write

$$e(\mathbf{u}) = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}) & \frac{1}{2}(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}) \\ \frac{1}{2}(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}) \\ \frac{1}{2}(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}) & \frac{1}{2}(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}) & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

$$= \begin{bmatrix} e(u') & \frac{1}{2}(\frac{\partial u'}{\partial x_3} + \nabla' u_3) \\ \frac{1}{2}(\frac{\partial u'}{\partial x_3} + \nabla' u_3)^T & \frac{\partial u_3}{\partial x_3} \end{bmatrix} \in \mathrm{Sym}(\mathbb{R}^3)$$

and an arbitrary fixed matrix

$$\zeta = \begin{bmatrix} \zeta' & \bar{\zeta} \\ \bar{\zeta}^T & \sqrt{2}\zeta_{33} \end{bmatrix} \in \mathrm{Sym}(\mathbb{R}^3).$$

We assume for fiber-reinforced composites that C is independent of x_3. Let us now suppose that a solution $\mathbf{u}$ exists to

$$\nabla \cdot C(e(\mathbf{u}) + \zeta) = \mathbf{0} \tag{10.2}$$

which is independent of x_3 as well. Then (10.2) becomes

$$\nabla' \cdot (\mathsf{C}(e(u') + \zeta')) = \mathbf{0} \tag{10.3}$$

$$\nabla' \cdot \left(\mathbf{C} \left(\frac{1}{2} \nabla' u_3 + \bar{\zeta} \right) \right) = 0 \tag{10.4}$$

where the top line is a vector equation while the bottom is a scalar equation. But we know solutions u' and u_3 to (10.3) and (10.4) exist and are unique. Therefore the unique solution $\mathbf{u} = (u', u_3)$ to (10.2) is x_3-independent.

Since the effective tensor is defined

$$C^* \zeta = \langle C(e(u) + \zeta) \rangle$$

for all $\zeta \in \mathrm{Sym}(\mathbb{R}^3)$ and C is block diagonal, we have

$$\begin{bmatrix} \mathsf{C}^*\zeta' + \mathcal{C}^*\bar{\zeta} + \mathbf{c}^*\zeta_{33} \\ \mathcal{C}^{*T}\zeta' + \mathbf{C}^*\bar{\zeta} + c^*\zeta_{33} \\ \mathbf{c}^{*T}\zeta' + c^{*T}\bar{\zeta} + \gamma^*\zeta_{33} \end{bmatrix} = \begin{bmatrix} \langle \mathsf{C}(e(u') + \zeta')\rangle \\ \langle \mathbf{C}(\frac{1}{2}\nabla' u_3 + \bar{\zeta})\rangle \\ \langle \gamma\zeta_{33}\rangle \end{bmatrix}$$

which implies $\mathcal{C}^*, \mathbf{c}^*$, and c^* are all zero. Furthermore, the effective tensors C^* and $\mathbf{C}^*$ are defined by the formulas

$$\mathsf{C}^*\zeta' = \langle \mathsf{C}(e(u') + \zeta')\rangle$$

and

$$\mathbf{C}^*\bar{\zeta} = \left\langle \mathbf{C}\left(\frac{1}{2}\nabla' u_3 + \bar{\zeta}\right)\right\rangle$$

for all $\zeta' \in \mathrm{Sym}(\mathbb{R}^2)$ and all $\bar{\zeta} \in \mathbb{R}^2$, while $\gamma^* = \langle\gamma\rangle$. $\square$

If $\mu_1 = \mu_2 = \mu$, then $\mu_1' = \mu_2'$ and we can apply Hill's exact relation to the two-dimensional elasticity block. Since we do not need to change this block or the two-dimensional conductivity block, we may assume $d_1 = d_2 = 1$. Let us assume C_1 and C_2 are ordered such that $\lambda_1 > \lambda_2$. Then we can set

$$d_4 = -\frac{\alpha_1 - \alpha_2}{\lambda_1 - \lambda_2} \quad \text{and} \quad f_1 = -\alpha_1 + \frac{\alpha_1 - \alpha_2}{\lambda_1 - \lambda_2}\lambda_1$$

to ensure $\alpha_1' = \alpha_2' = 0$ and

$$f_2 = 2\frac{\alpha_1 - \alpha_2}{\lambda_1 - \lambda_2}\alpha_2$$

to ensure that $\lambda_i' > 0$ for $i = 1, 2$. Note that we can rewrite (1.1) as

$$\frac{1}{\lambda^* + \mu} = \left\langle \frac{1}{\lambda + \mu}\right\rangle.$$

Combining Lemma 10.1 with the link and Hill's exact relation on the two-dimensional elasticity block we have effective tensors

$$(C')^* = \begin{bmatrix} \mathsf{C}(\lambda^*, \mu) & 0 & 0 \\ 0 & \mathbf{C}^* & 0 \\ 0 & 0 & \langle \gamma' \rangle \end{bmatrix}$$

and

$$C^* = \begin{bmatrix} \mathsf{C}(\lambda^*, \mu) & 0 & \alpha^* \mathbf{I}_2 \\ 0 & \mathbf{C}^* & 0 \\ \alpha^* \mathbf{I}_2 & 0 & \gamma^* \end{bmatrix}$$

where

$$\lambda^* = \left\langle \frac{1}{\lambda + \mu} \right\rangle^{-1} - \mu,$$

$$\alpha^* = \langle \alpha \rangle + \frac{\alpha_1 - \alpha_2}{\lambda_1 - \lambda_2} (\lambda^* - \langle \lambda \rangle),$$

$$\gamma^* = \langle \gamma \rangle + \left(\frac{\alpha_1 - \alpha_2}{\lambda_1 - \lambda_2} \right)^2 (\lambda^* - \langle \lambda \rangle),$$

and $\mathbf{C}^*$ is the effective two-dimensional conductivity tensor of the composite whose local conductivity tensor is given by $\mathbf{C}$ and whose microstructure is the same as the microstructure of the transversal plane of the original composite. This result was found by Rosen [29] in the context of two-dimensional thermoelasticity where $\mathsf{C}(\lambda^*, \mu)$ is the effective two-dimensional elasticity tensor, $\alpha^* \mathbf{I}_2$ is the effective thermal expansion tensor, and γ^* is the coefficient of specific heat.

10.2 Polycrystal Made from an Orthotropic Monocrystal

We can also apply the link to the case of a polycrystal made from an *orthotropic* material. We define an orthotropic material to be one that can be rotated into an orientation in which the material is invariant to 180 degree flips about each of the three coordinate axes. The tensor of such a material (rotated into the appropriate position) is therefore invariant with respect to rotations in the group

$$\mathcal{Q} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

A straightforward calculation shows us that $\mathcal{Q}$-invariant tensors can be written in the form

$$C = \begin{bmatrix} \mathsf{C} & 0 & \mathbf{c} \\ 0 & \mathbf{C} & 0 \\ \mathbf{c} & 0 & \gamma \end{bmatrix}$$

where C sends diagonal matrices to diagonal matrices and $\mathbf{C}$ and $\mathbf{c}$ are diagonal.

We would like to set the $\mathbf{c}'$-block equal to zero so that we once again map C to a block diagonal C'. We may let $d_1 = d_2 = d_3 = 1$ and fix d_4 and f_1 so that

$$\mathbf{c}' = \mathbf{c} + f_1\mathbf{I} + d_4\mathsf{C}\mathbf{I} = 0. \tag{10.5}$$

This is possible if and only if $\mathbf{c}$ is a scalar multiple of the identity *or* $\mathbf{I}$ is *not* an eigenvector of C. Since the linked tensor C' is now block diagonal, we may again apply Lemma 10.1 to see that for a polycrystal made using C', the effective tensor $(C')^*$ is block diagonal. For the polycrystal made using C we may then write

$$C^* = \begin{bmatrix} \mathsf{C}^* & 0 & \mathbf{c}^* \\ 0 & \mathbf{C}^* & 0 \\ \mathbf{c}^* & 0 & \gamma^* \end{bmatrix}$$

where if

$$\mathsf{C}\mathbf{I} = \begin{bmatrix} \lambda_0 + \upsilon_0 & 0 \\ 0 & \lambda_0 - \upsilon_0 \end{bmatrix} \text{ and } \mathbf{c} = \begin{bmatrix} \alpha_0 + \zeta_0 & 0 \\ 0 & \alpha_0 - \zeta_0 \end{bmatrix}$$

then

$$\mathbf{c}^* = \alpha_0\mathbf{I} + \frac{\zeta_0}{\upsilon_0}(\mathsf{C}^*\mathbf{I} - \lambda_0\mathbf{I}) \quad \text{and} \quad \gamma_0^* = \gamma_0 + \frac{\zeta_0^2}{\upsilon_0^2}(\langle \mathsf{C}^*\mathbf{I}, \mathbf{I}\rangle - \lambda_0). \tag{10.6}$$

This result shown was shown by Hashin [14] for the case where C is isotropic and generalized by Schulgasser [31], both in context of two-dimensional thermoelasticity.

In particular, if C sends scalar matrices to scalar matrices, i.e. if $\upsilon_0 = 0$, then [15] tells us that C^* does as well and $\lambda^* = \lambda_0$. In order to establish (10.5) we will need $\mathbf{c} = \alpha_0\mathbf{I}$. In this case, the equations in (10.6) become

$$\mathbf{c}^* = \alpha_0\mathbf{I} \quad \text{and} \quad \gamma_0^* = \gamma_0. \tag{10.7}$$

Furthermore, if the texture of the polycrystal is statistically isotropic, then, taking advantage of (1.2) on the $\mathbf{C}$-block, we have (10.7) and

$$\mathbf{C}^* = \mathbf{I}\sqrt{\det \mathbf{C}}.$$

Of course this also holds for the special case when $\mathbf{C}$ itself is scalar. Tensors satisfying all of these conditions, i.e. tensors such that

$$\upsilon_0 = \zeta_0 = 0 \text{ and } \mathbf{C} = \rho_0\mathbf{I}$$

represent materials that are *tetragonal*. Such materials have a fourfold rotational symmetry about the transverse axis. The above then tells us that the effective tensor of a fiber-reinforced polycrystalline composite made with one tetragonal material will itself be tetragonal with

$$\lambda^* = \lambda_0, \alpha^* = \alpha_0, \rho^* = \rho_0, \text{ and } \gamma^* = \gamma_0.$$

CHAPTER 11

CONCLUSION

The general theory of exact relations gives us necessary algebraic conditions for exact relations in many different contexts. However, finding all subspaces that satisfy these conditions is a highly nontrivial task. Furthermore, fiber-reinforced elasticity represents an especially challenging context since three-dimensional elasticity provides us with a relatively large space of tensors while the restriction of fiber-reinforced geometry further increases the anticipated number of relations.

We are able to address this challenge by focusing on polycrystalline exact relations and taking advantage of representation theory of $SO(2)$. Since polycrystalline exact relations correspond to $SO(2)$-invariant algebraic subspaces, we are able to describe these subspaces in terms of their decompositions into irreps. We are then able to systematically find all $SO(2)$-invariant algebraic subspaces and hence all polycrystalline exact relations. We also use certain automorphisms Φ to describe the infinite list of all polycrystalline exact relations in terms of finitely many equivalence classes mod Φ.

These same automorphisms, Φ, correspond to the algebraic representations of links, $\mathcal{F}$, that map one exact relation to another. These links are parameterized by seven real parameters, one of which represents the nonlinear part of the link while the remaining six comprise the linear portion. In particular we apply these links to obtain information about composites made from two transversely isotropic materials and polycrystalline composites made from one orthotropic material. Finally, we observe that we can use this link to map two general elasticity tensors to a uniform field relation.

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APPENDIX A

COMPLETE LIST OF ALGEBRAIC SUBSPACES

A.1 Overview of the List

As discussed in chapter 3, for each algebra we will make note of simple choices for S_0 satisfying (3.6). If we write $B_1\bar{\Gamma}B_1^T = \mathsf{K}_0'(\tilde{\lambda}) + \mathsf{K}_0(\tilde{\mu}) + \mathbf{N}_0(\tilde{\rho})$, then each choice of S_0 can be described in terms of this block structure. We will also make note of cases where volume fractions exist, i.e. where $\Pi^{*^A 2} \neq \Pi$, and cases that correspond to uniform field relations, i.e., when an algebra is an annihilator of an invariant subspace of $\mathrm{Sym}(\mathbb{R}^3)$.

A.2 Algebraic Subspaces within Π_{A}

First note that for all algebras $\Pi \subset \Pi_{\mathsf{A}}$, we have that $\Pi^{*^A 2} = \Pi$. Intersecting with $\mathbb{K}$ (clearly an algebra) we have

$$\mathsf{K}_4 \qquad S_0 = \mathsf{K}_0(\tilde{\mu})$$

$$\mathsf{K}_0 \oplus \mathsf{K}_4 = \mathrm{Ann}(Y_0' \oplus Y_0 \oplus Y_1) \qquad S_0 = 0$$

Intersecting with $\mathbb{N}$ (also an algebra) we have

$$\mathbf{N}_2 \qquad S_0 = \mathbf{N}_0(\tilde{\rho})$$

$$\mathbb{N} = \mathbf{N}_0 \oplus \mathbf{N}_2 = \mathrm{Ann}(Y_0' \oplus Y_2 \oplus Y_0) \qquad S_0 = 0$$

Since $\mathbb{K} *^A \mathbb{N} = \{0\}$, we see that given $\Pi_1 \subset (\mathsf{K}_0 \oplus \mathsf{K}_4)$ and $\Pi_2 \subset (\mathbf{N}_0 \oplus \mathbf{N}_2)$, $\Pi_1 \oplus \Pi_2$ is an algebra. Therefore, combining the above we have:

$$\mathsf{K}_4 \oplus \mathbf{N}_2 \qquad S_0 = \mathsf{K}_0(\tilde{\mu}) + \mathbf{N}_0(\tilde{\rho})$$

$$\mathsf{K}_0 \oplus \mathsf{K}_4 \oplus \mathbf{N}_2 \qquad S_0 = \mathbf{N}_0(\tilde{\rho})$$

$$\mathsf{K}_4 \oplus \mathbf{N}_0 \oplus \mathbf{N}_2 \qquad S_0 = \mathsf{K}_0(\tilde{\mu})$$

$$\mathsf{K}_0 \oplus \mathsf{K}_4 \oplus \mathbf{N}_0 \oplus \mathbf{N}_2 \qquad S_0 = 0$$

If we have either of the irreps from $\mathbb{L}$, we know that we must add irreps from $\mathbb{K}$ and $\mathbb{N}$:

$$\mathsf{K}_4 \oplus \mathbf{N}_2 \oplus \mathcal{L}_1 \qquad\qquad S_0 = \mathsf{K}_0(\tilde{\mu}) + \mathbf{N}_0(\tilde{\rho})$$

$$\mathsf{K}_4 \oplus \mathbf{N}_2 \oplus \mathcal{L}_3 \qquad\qquad S_0 = \mathsf{K}_0(\tilde{\mu}) + \mathbf{N}_0(\tilde{\rho})$$

If we have $\mathcal{L}_1 \oplus \mathcal{L}_3$ or if we add K_0 or $\mathbf{N}_0$ to either of the above, we immediately have

$$\Pi_{\mathsf{A}} = \mathrm{Ann}(Y_0' \oplus Y_0) \qquad S_0 = 0$$

These are the only algebraic subspaces possible within Π_{A}.

A.3 Algebraic Subspaces within Π_{B}

There are, of course, infinitely many algebraic subspaces within Π_{B} since we have three infinite families of minimal irreps given in (8.7), (8.8), and (8.9). However, we find that if we group these algebras by equivalence class with respect to the family of automorphisms of the form $\Phi^{\pm}_{\nu,\tau}(K) = X^{\pm}(\nu,\tau)K(X^{\pm}(\nu,\tau))^T$ where $X^{\pm}(\nu,\tau)$ is as in (7.3), then we can represent each infinite family by a finite number of representatives of these equivalence classes.

Lemma A.1: The equivalence classes of irreps of the form

$$Z = \{Z(\delta) : \delta \in \mathbb{R}\} = \Pi_0^{\lambda,\alpha,\gamma} = \{\mathsf{K}_0'(\lambda\delta) + \mathbf{M}_0(\alpha\delta) + j_0(\gamma\delta) : \delta \in \mathbb{R}\}$$

are

$$\mathsf{K}_0' = \mathrm{Ann}(Y_2 \oplus Y_1 \oplus Y_0) \qquad S_0 = 0 \qquad \Pi^{*^{A}2} = 0$$

$$\mathbf{M}_0 \qquad S_0 = \mathsf{K}_0'(\tilde{\lambda}) \qquad \Pi^{*^{A}2} = 0$$

$$j_0 = \mathrm{Ann}(Y_0' \oplus Y_2 \oplus Y_1) \qquad S_0 = 0 \qquad \Pi^{*^{A}2} = 0$$

$$\{\mathsf{K}_0'(\lambda) + j_0(\lambda) : \lambda \in \mathbb{R}\} \qquad S_0 = 0 \qquad \Pi^{*^{A}2} = 0$$

$$\{\mathsf{K}_0'(\lambda) + j_0(-\lambda) : \lambda \in \mathbb{R}\} \qquad S_0 = 0 \qquad \Pi^{*^{A}2} = 0$$

Proof: First observe that

$$\Phi^{\pm}_{\nu,\tau}(Z(\delta)) = \mathsf{K}_0'(\lambda\delta) + \mathbf{M}_0((\lambda\nu + \alpha\tau)\delta) + j_0((\lambda\nu^2 + 2\alpha\nu\tau + \gamma\tau^2)\delta). \tag{A.1}$$

Therefore if $\lambda = \alpha = 0$, i.e., if $Z = j_0$, then Z forms its own equivalence class. Similarly, if $\lambda = 0$ but $\alpha \neq 0$, then Z is a member of the equivalence class of $\mathbf{M}_0$. Now assume $\lambda \neq 0$. Let the following two coefficients from (A.1) be named

$$\alpha' = \lambda\nu + \alpha\tau$$

$$\gamma' = \lambda\nu^2 + 2\alpha\nu\tau + \gamma\tau^2.$$

Solving for τ yields

$$\tau^2 = \frac{(\alpha')^2 - \lambda\gamma'}{\alpha^2 - \lambda\gamma}.$$

Since we need $\tau^2 \geq 0$, we see we have three more equivalence classes which include all the remaining algebras of this type:

- $\{\Pi_0^{\lambda,\alpha,\gamma} : \lambda\gamma = \alpha^2\} = \{\Pi_0^{\lambda,\alpha,\gamma} : \Pi_0^{\lambda,\alpha,\gamma} \cong \mathsf{K}_0\}$
- $\{\Pi_0^{\lambda,\alpha,\gamma} : \lambda\gamma > \alpha^2\} = \{\Pi_0^{\lambda,\alpha,\gamma} : \Pi_0^{\lambda,\alpha,\gamma} \cong \{\mathsf{K}_0'(\lambda) + j_0(\lambda) : \lambda \in \mathbb{R}\}\}$
- $\{\Pi_0^{\lambda,\alpha,\gamma} : \lambda\gamma < \alpha^2\} = \{\Pi_0^{\lambda,\alpha,\gamma} : \Pi_0^{\lambda,\alpha,\gamma} \cong \{\mathsf{K}_0'(\lambda) + j_0(-\lambda) : \lambda \in \mathbb{R}\}\}$ $\square$

Note also that

$$\Phi^{\pm}_{\beta,\tau}(\mathsf{K}_0') = \mathrm{Ann}(Y_2 \oplus Y_1 \oplus \{Y_0'(-\beta\omega) + Y_0(\omega) : \omega \in \mathbb{R}\}$$

for each $\beta \in \mathbb{R}$.

Since $(\mathsf{K}_0' \oplus \mathbf{M}_0 \oplus \mathbb{J})^{*^{A}2} = \{0\}$, the remaining algebras composed entirely of weight zero irreps are all possible two-dimensional combinations as well as the three dimensional algebra of all three irreps:

$\mathbf{M}_0 \oplus \{\mathsf{K}_0'(\lambda) + j_0(-\lambda) : \lambda \in \mathbb{R}\}$	$S_0 = \mathsf{K}_0'(\tilde{\lambda})$	$\Pi^{*^{A}2} = 0$
$\mathbf{M}_0 \oplus \{\mathsf{K}_0'(\lambda) + j_0(\lambda) : \lambda \in \mathbb{R}\}$	$S_0 = \mathsf{K}_0'(\tilde{\lambda})$	$\Pi^{*^{A}2} = 0$
$\mathbf{M}_0 \oplus \mathbb{J}$	$S_0 = 0$	$\Pi^{*^{A}2} = 0$
$\mathsf{K}_0' \oplus \mathbf{M}_0$	$S_0 = \mathsf{K}_0'(\tilde{\lambda})$	$\Pi^{*^{A}2} = 0$
$\mathsf{K}_0' \oplus \mathbb{J}$	$S_0 = 0$	$\Pi^{*^{A}2} = 0$
$\mathsf{K}_0' \oplus \mathbf{M}_0 \oplus \mathbb{J} = \mathrm{Ann}(Y_2 \oplus Y_1)$	$S_0 = 0$	$\Pi^{*^{A}2} = 0$

Lemma A.2: Algebras of the form

$$\Pi_1^{\eta_1,\eta_2,\eta_3} = \{\mathcal{L}_1'(\eta_1 d) + \mathbf{p}_1((\eta_2 + i\eta_3)d) : d \in \mathbb{C}\} \oplus \Pi_0^{\eta_1^2,\eta_1\eta_2,\eta_2^2-\eta_3^3} \oplus \Pi_0^{0,\eta_1\eta_3,2\eta_2\eta_3}$$

for some $\eta_1, \eta_2, \eta_3 \in \mathbb{R}$ where $\Pi_0^{\lambda,\alpha,\gamma}$ is as defined in (8.7a) are in one of the following equivalence classes:

$\mathsf{K}_0' \oplus \mathcal{L}_1'$	$S_0 = \mathsf{K}_0'(\tilde{\lambda})$	$\Pi^{*^{A}2} = \mathsf{K}_0'$
$\mathbb{P} \oplus \mathbb{J}$	$S_0 = 0$	$\Pi^{*^{A}2} = \mathbb{J}$
$\Pi_1^{1,0,1}$	$S_0 = \mathsf{K}_0'(\tilde{\lambda}) + \mathbf{N}_0(\tilde{\rho})$	$\Pi^{*^{A}2} = \Pi_0^{1,0,-1}$

Proof: Since

$$\Phi^{\pm}_{\nu,\tau}(\mathsf{K}_0'(\lambda)+\mathcal{L}_1'(b)) = \mathsf{K}_0'(\lambda)+\mathbf{M}_0(\nu\lambda)+j_0(\nu^2\lambda)+\mathcal{L}_1'(\pm b)+\mathbf{p}_1(\pm\nu b)$$

and

$$\Phi^{\pm}_{\nu,\tau}(\mathbf{p}_1(h)+j_0(\gamma)) = \mathbf{p}_1(\pm\tau h)+j_0(\tau^2\gamma)$$

we see that all algebras of this type that have either $\eta_1=0$ or $\eta_3=0$ are equivalent to one of the above two. Now suppose $\eta_1,\eta_3\neq 0$. Then the algebra contains a two dimensional weight zero subspace and therefore is clearly not equivalent to either of the two above. Since $\eta_1\neq 0$ we may fix $\eta_1=1$, i.e., we are absorbing it into $b\in\mathbb{C}$. Then, since:

$$\Phi^{\pm}_{\nu,\tau}(\mathcal{L}_1'(b)+\mathbf{p}_1(ib)+\mathbf{M}_0(\alpha)+\mathsf{K}_0'(\lambda)+j_0(-\lambda)) =$$

$$\mathcal{L}_1'(\pm b)+\mathbf{p}_1(\pm\nu b\pm\tau ib)+\mathsf{K}_0'(\lambda)+\mathbf{M}_0(\nu\lambda+\tau\alpha)+j_0(\nu^2\lambda+2\nu\tau\alpha-\tau^2\gamma)$$

we see that if $\nu=\eta_2$ and $\tau=\eta_3$, we can map this algebra to any of the remaining algebras of this type. $\square$

Since $(\mathcal{L}_1'+\mathbf{p}_1)*^A(\mathsf{K}_0'+\mathbf{M}_0+\mathbb{J})=0$, the remaining algebras containing only weight zero and weight one irreps are again simply all distinct combinations of those we have already seen:

<u>Weight zero 2D $\oplus$ weight one 1D</u>

$\mathsf{K}_0'\oplus\mathcal{L}_1'\oplus\mathbf{M}_0$	$S_0=\mathsf{K}_0'(\tilde{\lambda})$	$\Pi^{*^A2}=\mathsf{K}_0'$
$\mathsf{K}_0'\oplus\mathcal{L}_1'\oplus\mathbb{J}$	$S_0=\mathsf{K}_0'(\tilde{\lambda})$	$\Pi^{*^A2}=\mathsf{K}_0'$
$\mathbb{P}\oplus\mathbf{M}_0\oplus\mathbb{J}$	$S_0=0$	$\Pi^{*^A2}=\mathbb{J}$
$\mathsf{K}_0'\oplus\mathbb{P}\oplus\mathbb{J}$	$S_0=0$	$\Pi^{*^A2}=\mathbb{J}$

Weight zero 3D $\oplus$ weight one 1D

$\mathsf{K}_0' \oplus \mathcal{L}_1' \oplus \mathbf{M}_0 \oplus \mathbb{J}$	$S_0 = \mathsf{K}_0'(\tilde{\lambda})$	$\Pi^{*^A 2} = \mathsf{K}_0'$
$\mathsf{K}_0' \oplus \mathbf{M}_0 \oplus \mathbb{P} \oplus \mathbb{J}$	$S_0 = 0$	$\Pi^{*^A 2} = \mathbb{J}$
$\mathsf{K}_0' \oplus \{\mathcal{L}_1'(b) + \mathbf{p}_1(ib) : b \in \mathbb{C}\} \oplus \mathbf{M}_0 \oplus \mathbb{J}$	$S_0 = \mathsf{K}_0'(\tilde{\lambda})$	$\Pi^{*^A 2} = \mathbf{M}_0 \oplus \Pi_0^{1,0,-1}$

Weight zero 3D $\oplus$ weight one 2D

$\mathsf{K}_0' \oplus \mathcal{L}_1' \oplus \mathbf{M}_0 \oplus \mathbb{P} \oplus \mathbb{J}$	$S_0 = \mathsf{K}_0'(\tilde{\lambda})$	$\Pi^{*^A 2} = \mathsf{K}_0' \oplus \mathbf{M}_0 \oplus \mathbb{J}$

Lemma A.3: Algebras of the form

$$\{\mathsf{K}_2(\eta_1 g) + \mathbf{M}_2(\eta_2 g) : g \in \mathbb{C}\} \oplus \{\mathsf{K}_0'(\eta_1^2 \delta) + \mathbf{M}_0(\eta_1 \eta_2 \delta) + j_0(\eta_2^2 \delta) : \delta \in \mathbb{R}\}$$

for some $(\eta_1, \eta_2) \in \mathbb{R}^2$ are equivalent to one of the following:

$\mathsf{K}_0' \oplus \mathsf{K}_2$	$S_0 = \mathsf{K}_0'(\tilde{\lambda})$	
$\mathbf{M}_2 \oplus \mathbb{J}$	$S_0 = 0$	$\Pi^{*^A 2} = \mathbb{J}$

Proof: We know that

$$\Phi^{\pm}_{\nu,\tau}(\mathsf{K}_0'(\lambda) + \mathsf{K}_2(u)) = \mathsf{K}_0'(\lambda) + \mathbf{M}_0(\lambda\nu) + j_0(\lambda\nu^2) + \mathsf{K}_2(u) + \mathbf{M}_2(\nu u)$$

which shows that the first representative, $\mathsf{K}_0' \oplus \mathsf{K}_2$, is equivalent to all algebras of the general type except for those where $\eta_1 = 0$, while

$$\Phi^{\pm}_{\nu,\tau}(\mathbf{M}_2(g) + j_0(\gamma)) = \mathbf{M}_2(\nu g) + j_0(\nu^2\gamma)$$

shows that $\mathbf{M}_2 \oplus \mathbb{J}$ is clearly its own equivalence class. $\square$

We can now list the remaining algebras containing only weight zero and weight two irreps. Since $\mathbb{J}$ is the set of annihilators in $\mathcal{S}$ while $\mathsf{K}_2 *^A \mathbf{M}_0 = \mathbf{M}_0$, $\mathbf{M}_2 *^A \mathsf{K}'_0 = \mathbf{M}_0$, and $\mathbf{M}_2 *^A \mathbf{M}_0 = \mathbb{J}$, the algebras containing only weight zero and weight two irreps are:

$\mathsf{K}'_0 \oplus \mathsf{K}_2 \oplus \mathbb{J}$	$\mathcal{S}_0 = \mathsf{K}'_0(\tilde{\lambda})$	$\Pi^{*^A 2} = \mathsf{K}'_0 \oplus \mathsf{K}_2$
$\mathsf{K}'_0 \oplus \mathsf{K}_2 \oplus \mathbf{M}_0$	$S_0 = \mathsf{K}'_0(\tilde{\lambda})$	
$\mathbb{M} \oplus \mathbb{J}$	$S_0 = 0$	$\Pi^{*^A 2} = \mathbb{J}$
$\mathsf{K}'_0 \oplus \mathsf{K}_2 \oplus \mathbf{M}_0 \oplus \mathbb{J}$	$\mathcal{S}_0 = \mathsf{K}'_0(\tilde{\lambda})$	$\Pi^{*^A 2} = \mathsf{K}'_0 \oplus \mathsf{K}_2 \oplus \mathbf{M}_0$
$\mathsf{K}'_0 \oplus \mathbb{M} \oplus \mathbb{J}$	$S_0 = 0$	$\Pi^{*^A 2} = \mathbb{M} \oplus \mathbb{J}$
$\mathsf{K}'_0 \oplus \mathsf{K}_2 \oplus \mathbb{M} \oplus \mathbb{J}$	$S_0 = 0$	

Now we combine weight one and weight two irreps. Since $\mathsf{K}_2 *^A \mathcal{L}'_1 = \mathcal{L}'_1$, $\mathbf{M}_2 *^A \mathbf{p}_1 = \{0\}$, $\mathbf{M}_2 *^A \mathcal{L}'_1 = \mathbf{p}_1$, and $\mathsf{K}_2 *^A \mathbf{p}_1 = \{0\}$ we have the following algebras:

<u>Weight zero 1D $\oplus$ weight one 1D $\oplus$ weight two 1D</u>

$\mathsf{K}'_0 \oplus \mathsf{K}_2 \oplus \mathcal{L}'_1$	$S_0 = \mathsf{K}'_0(\tilde{\lambda})$	
$\mathbf{M}_2 \oplus \mathbb{P} \oplus \mathbb{J}$	$S_0 = 0$	$\Pi^{*^A 2} = \mathbb{J}$

Weight zero 2D $\oplus$ weight one 1D $\oplus$ weight two 1D

$$\mathsf{K}_0' \oplus \mathsf{K}_2 \oplus \mathcal{L}_1' \oplus \mathbf{M}_0 \qquad S_0 = \mathsf{K}_0'(\tilde{\lambda})$$

$$\mathsf{K}_0' \oplus \mathsf{K}_2 \oplus \mathcal{L}_1' \oplus \mathbb{J} \qquad S_0 = \mathsf{K}_0'(\tilde{\lambda}) \qquad \Pi^{*^A 2} = \mathsf{K}_0' \oplus \mathsf{K}_2 \oplus \mathcal{L}_1'$$

$$\mathsf{K}_0' \oplus \mathsf{K}_2 \oplus \mathbb{P} \oplus \mathbb{J} \qquad S_0 = \mathsf{K}_0'(\tilde{\lambda}) \qquad \Pi^{*^A 2} = \mathsf{K}_0' \oplus \mathsf{K}_2 \oplus \mathbb{J}$$

$$\mathbb{M} \oplus \mathbb{P} \oplus \mathbb{J} \qquad S_0 = 0 \qquad \Pi^{*^A 2} = \mathbb{J}$$

Weight zero 3D $\oplus$ weight one 1D $\oplus$ weight two 1D

$$\mathsf{K}_0' \oplus \mathsf{K}_2 \oplus \mathcal{L}_1' \oplus \mathbf{M}_0 \oplus \mathbb{J} \qquad S_0 = \mathsf{K}_0'(\tilde{\lambda}) \qquad \Pi^{*^A 2} = \mathsf{K}_0' \oplus \mathsf{K}_2 \oplus \mathcal{L}_1' \oplus \mathbf{M}_0$$

$$\mathsf{K}_0' \oplus \mathsf{K}_2 \oplus \mathbf{M}_0 \oplus \mathbb{P} \oplus \mathbb{J} \qquad S_0 = \mathsf{K}_0'(\tilde{\lambda}) \qquad \Pi^{*^A 2} = \mathsf{K}_0' \oplus \mathsf{K}_2 \oplus \mathbf{M}_0 \oplus \mathbb{J}$$

$$\mathsf{K}_0' \oplus \mathbb{M} \oplus \mathbb{P} \oplus \mathbb{J} \qquad S_0 = 0 \qquad \Pi^{*^A 2} = \mathbf{M}_0 \oplus \mathbb{J}$$

Weight zero 3D $\oplus$ weight one 1D $\oplus$ weight two 2D

$$\mathsf{K}_0' \oplus \mathsf{K}_2 \oplus \mathbb{M} \oplus \mathbb{P} \oplus \mathbb{J} \qquad S_0 = \mathsf{K}_0'(\tilde{\lambda}) \qquad \Pi^{*^A 2} = \mathsf{K}_0' \oplus \mathsf{K}_2 \oplus \mathbb{M} \oplus \mathbb{J}$$

Weight zero 3D $\oplus$ weight one 2D $\oplus$ weight two 1D

$$\mathsf{K}_2 \oplus \mathsf{K}_0' \oplus \mathcal{L}_1' \oplus \mathbf{M}_0 \oplus \mathbb{P} \oplus \mathbb{J} \qquad S_0 = \mathsf{K}_0'(\tilde{\lambda}) \qquad \Pi^{*^A 2} = \mathsf{K}_2 \oplus \mathsf{K}_0' \oplus \mathcal{L}_1' \oplus \mathbf{M}_0 \oplus \mathbb{J}$$

$$\mathsf{K}_0' \oplus \mathcal{L}_1' \oplus \mathbb{M} \oplus \mathbb{P} \oplus \mathbb{J} \qquad S_0 = \mathsf{K}_0'(\tilde{\lambda}) \qquad \Pi^{*^A 2} = \mathsf{K}_0' \oplus \mathbf{M}_0 \oplus \mathbb{J}$$

Weight zero 3D $\oplus$ weight one 2D $\oplus$ weight two 2D

$$\Pi_{\mathsf{B}} \qquad S_0 = \mathsf{K}_0'(\tilde{\lambda})$$

A.4 Algebraic Subspaces Intersecting both Π_{A} and Π_{B}

In this subsection we will list all algebras, Π such that $\Pi \cap \Pi_{\mathsf{A}} \neq \{0\}$ and $\Pi \cap \Pi_{\mathsf{B}} \neq \{0\}$. This process is simplified by the fact that in many cases when we add an algebraic subspace of Π_{A} to an algebraic subspace of Π_{B}, we find that we must include all of Π_{A} to obtain a new algebraically closed subspace. Let us begin by adding K_4 to Π_{B}. Since $\mathsf{K}_0' *^A \mathsf{K}_4 = \mathsf{K}_2$ and $\mathsf{K}_2 *^A \mathsf{K}_4 \subset \mathsf{K}_0 \oplus \mathsf{K}_2 \oplus \mathsf{K}_4$, there are no additional algebras contained within $\mathbb{K}$ beyond

$$\mathbb{K} = \mathrm{Ann}(Y_1 \oplus Y_0) \qquad S_0 = 0$$

Furthermore, for each $\beta \in \mathbb{R}$,

$$\Phi^{\pm}_{\beta,\tau}(\mathbb{K}) = \mathrm{Ann}(Y_1 \oplus \{Y_0'(-\beta\omega) + Y_0(\omega) : \omega \in \mathbb{R}\}).$$

Since $\mathsf{K}_4 *^A \mathcal{L}_1' \subset \mathcal{L}_1 \oplus \mathcal{L}_3$, there are no algebras containing K_4 and $\mathcal{L}_1'$ that do not contain all of Π_{A}. We will return to these later. Observing that $(\mathsf{K}_0 \oplus \mathsf{K}_4) *^A \mathbb{M} = \mathbf{M}_2$, the algebras left in $\mathbb{K} \oplus \mathbb{M} \oplus \mathbb{J}$ are

$\mathsf{K}_4 \oplus \mathbf{M}_2 \oplus \mathbb{J}$	$S_0 = \mathsf{K}_0(\tilde{\mu})$	
$\mathsf{K}_4 \oplus \mathbb{M} \oplus \mathbb{J}$	$S_0 = \mathsf{K}_0(\tilde{\mu})$	$\Pi^{*^A 2} = \mathsf{K}_4 \oplus \mathbf{M}_2 \oplus \mathbb{J}$
$\mathsf{K}_0 \oplus \mathsf{K}_4 \oplus \mathbf{M}_2 \oplus \mathbb{J} = \mathrm{Ann}(Y_0' \oplus Y_1)$	$S_0 = 0$	
$\mathsf{K}_0 \oplus \mathsf{K}_4 \oplus \mathbb{M} \oplus \mathbb{J}$	$S_0 = 0$	$\Pi^{*^A 2} = \mathsf{K}_0 \oplus \mathsf{K}_4 \oplus \mathbf{M}_2 \oplus \mathbb{J}$
$\mathbb{K} \oplus \mathbb{M} \oplus \mathbb{J} = \mathrm{Ann}(Y_1)$	$S_0 = 0$	

Because $(\mathbb{N} \oplus \mathbb{P}) *^A (\mathbb{K} \oplus \mathbb{M} \oplus \mathbb{J}) = \{0\}$ while $\mathbb{N} *^A \mathbb{P} = \mathbb{P}$ and $\mathbb{N} *^A \mathcal{L}_1' = \mathcal{L}_1'$, it is convenient to group together the remaining algebras in $\mathbb{K} \oplus \mathcal{L}_1' \oplus \mathbb{N} \oplus \mathbb{M} \oplus \mathbb{P} \oplus \mathbb{J}$ as follows.

First, with two exceptions stated below, given $\Pi_1 \subset \Pi_{\mathsf{B}}$ and $\Pi_2 \subset \mathbb{N}$, we see that $\Pi = \Pi_1 \oplus \Pi_2$ is an algebra. For $\Pi_1 \subset \mathbb{K} \oplus \mathbb{M} \oplus \mathbb{J}$, the fact that $(\mathbb{K} \oplus \mathbb{M} \oplus \mathbb{J}) *^A \mathbb{N} = 0$ makes this immediate. If $\mathbf{p}_1 \subset \Pi_1$ or $\mathcal{L}'_1 \subset \Pi_1$, we are also fine thanks to the following products:

$$\mathbf{N}_2(f) *^A \mathcal{L}'_1(b) = \mathcal{L}'_1(\frac{1}{2} b f \bar{z}) \tag{A.2}$$

$$\mathbf{N}_2(f) *^A \mathbf{p}_1(h) = \mathbf{p}_1(\frac{1}{2} f h \bar{z}) \tag{A.3}$$

$$\mathbf{N}_0(\rho) *^A \mathcal{L}'_1(b) = \mathcal{L}'_1(\frac{1}{2} \rho \bar{b} z) \tag{A.4}$$

$$\mathbf{N}_0(\rho) *^A \mathbf{p}_1(h) = \mathbf{p}_1(\frac{1}{2} \rho \bar{h} z) \tag{A.5}$$

Our only difficulty comes when Π_1 is either $\Pi_1^{1,0,1}$ or $(\Pi_1^{1,0,1} \oplus \mathsf{K}'_0 \oplus \mathbf{M}_0 \oplus \mathbb{J}) = (\Pi_1^{1,0,1} \oplus \mathbb{J})$ *and* $\Pi_2 = \mathbb{N}$. In this case, due to (A.4) and (A.5), we find that $\Pi_1 \oplus \Pi_2$ is not algebraically closed and we must instead choose a larger subspace of Π_{B}, i.e., one containing $\mathcal{L}'_1 \oplus \mathbf{p}_1$.

Having already written the complete lists for both Π_{B} and $\mathbb{N}$, we will not list all possible combinations here, but we will make a few observations. Whether or not $\Pi^{*^A 2} = \Pi$ depends on whether this holds for Π_1, further keeping in mind that $\mathcal{L}'_1 *^A \mathbb{N} = \mathcal{L}'_1$ and $\mathbf{p}_1 *^A \mathbb{N} = \mathbf{p}_1$. Furthermore, the value of S_0 depends independently on Π_1 and Π_2. That is, $S_0 = \mathsf{K}'_0(\tilde{\lambda}) + \mathbf{N}_0(\tilde{\mu})$ will always work, but if $S_0 = 0$ is admissible for Π_1, we may let the K'_0 component of S_0 be zero for Π. Similarly, if $S_0 = 0$ is admissible for Π_2, then we may let the $\mathbf{N}_0$ component be zero. Finally, the algebras intersecting Π_{B} and $\mathbb{N}$ that correspond to uniform field relations are

$$\mathsf{K}'_0 \oplus \mathcal{L}'_1 \oplus \mathbb{N} \oplus \mathbf{M}_0 \oplus \mathbb{P} \oplus \mathbb{J} = \mathrm{Ann}(Y_2) \qquad S_0 = 0$$

$$\mathbb{N} \oplus \mathbb{P} \oplus \mathbb{J} = \mathrm{Ann}(Y'_0 \oplus Y_2) \qquad S_0 = 0$$

$$\mathsf{K}'_0 \oplus \mathcal{L}'_1 \oplus \mathbb{N} = \mathrm{Ann}(Y_2 \oplus Y_0) \qquad S_0 = 0$$

and, for each $\beta \in \mathbb{R}$,

$$\Phi^{\pm}_{\beta,\tau}(\mathsf{K}'_0 \oplus \mathcal{L}'_1 \oplus \mathbb{N}) = \mathrm{Ann}(Y_2 \oplus \{Y'_0(-\beta\omega) + Y_0(\omega) : \omega \in \mathbb{R}\}).$$

Second, given $\Pi_1 \subset (\mathbb{K} \oplus \mathbb{M} \oplus \mathbb{J})$ and $\Pi_2 \subset (\mathbb{N} \oplus \mathbb{P} \oplus \mathbb{J})$, $\Pi = \Pi_1 \oplus \Pi_2$ is an algebra because $(\mathbb{K} \oplus \mathbb{M} \oplus \mathbb{J}) *^A (\mathbb{N} \oplus \mathbb{P} \oplus \mathbb{J}) = \{0\}$. We assume that $\mathsf{K}_4 \subset \Pi_1$ since otherwise we are covered by the previous case. The important observation here is that not every algebra in $\mathbb{M} \oplus \mathbb{J}$ may be combined in a direct sum with K_4. We can only use the algebras identified in $\mathbb{K} \oplus \mathbb{M} \oplus \mathbb{J}$ above. Again, we will not list all such combinations $\Pi = \Pi_1 \oplus \Pi_2$ but will make relevant comments regarding $\Pi^{*^A 2}$ and S_0. Since $(\mathbb{K} \oplus \mathbb{M} \oplus \mathbb{J}) *^A (\mathbb{N} \oplus \mathbb{P} \oplus \mathbb{J}) = \{0\}$, whether or not $\Pi^{*^A 2} = \Pi$ depends independently on whether this holds for Π_1 and Π_2. Also, the value of S_0 again depends independently on the values needed for Π_1 and Π_2. None of these algebras correspond to annihilators of invariant subspaces of $\mathrm{Sym}(\mathbb{R}^3)$.

Now let us consider separately adding $\mathcal{L}_1$ and $\mathcal{L}_3$ to algebraic subspaces of Π_{B}. Since $\mathsf{K}'_0 *^A (\mathcal{L}_1 \oplus \mathcal{L}_3) = \mathcal{L}'_1$ and $\mathcal{L}'_1 *^A (\mathcal{L}_1 \oplus \mathcal{L}_3) \subset \mathsf{K}_2 \oplus \mathbb{N}$ we quickly realize that adding K'_0 or $\mathcal{L}'_1$ to $\mathcal{L}_1$ or $\mathcal{L}_3$ leads us to include all of Π_{A}. So instead we consider first $\mathbb{M} \oplus \mathbb{P} \oplus \mathbb{J}$. Observe that $\mathbb{M} *^A (\mathcal{L}_1 \oplus \mathcal{L}_3) = \mathbb{P}$ and $\mathbb{P} *^A (\mathcal{L}_1 \oplus \mathcal{L}_3) = \mathbf{M}_2$. Therefore we are left with the following algebras:

$$\mathsf{K}_4 \oplus \mathbf{N}_2 \oplus \mathcal{L}_1 \oplus \mathbb{J} \qquad S_0 = \mathsf{K}_0(\tilde{\mu}) + \mathbf{N}_0(\tilde{\rho}) \qquad \Pi^{*^A 2} = \Pi \setminus \mathbb{J}$$

$$\mathsf{K}_4 \oplus \mathbf{N}_2 \oplus \mathcal{L}_3 \oplus \mathbb{J} \qquad S_0 = \mathsf{K}_0(\tilde{\mu}) + \mathbf{N}_0(\tilde{\rho}) \qquad \Pi^{*^A 2} = \Pi \setminus \mathbb{J}$$

$$\mathsf{K}_4 \oplus \mathbf{N}_2 \oplus \mathcal{L}_1 \oplus \mathbf{M}_2 \oplus \mathbb{P} \oplus \mathbb{J} \qquad S_0 = \mathsf{K}_0(\tilde{\mu}) + \mathbf{N}_0(\tilde{\rho})$$

$$\mathsf{K}_4 \oplus \mathbf{N}_2 \oplus \mathcal{L}_3 \oplus \mathbf{M}_2 \oplus \mathbb{P} \oplus \mathbb{J} \qquad S_0 = \mathsf{K}_0(\tilde{\mu}) + \mathbf{N}_0(\tilde{\rho})$$

$$\mathsf{K}_4 \oplus \mathbf{N}_2 \oplus \mathcal{L}_1 \oplus \mathbb{M} \oplus \mathbb{P} \oplus \mathbb{J} \qquad S_0 = \mathsf{K}_0(\tilde{\mu}) + \mathbf{N}_0(\tilde{\rho}) \qquad \Pi^{*^A 2} = \Pi \setminus \mathbf{M}_0$$

$$\mathsf{K}_4 \oplus \mathbf{N}_2 \oplus \mathcal{L}_3 \oplus \mathbb{M} \oplus \mathbb{P} \oplus \mathbb{J} \qquad S_0 = \mathsf{K}_0(\tilde{\mu}) + \mathbf{N}_0(\tilde{\rho}) \qquad \Pi^{*^A 2} = \Pi \setminus \mathbf{M}_0$$

What remains are algebras including all of Π_{A} and some algebraic subspace of Π_{B}. Noting that $\mathsf{K}_0' *^A \mathbf{M}_2 = \mathbf{M}_0$ we have

$$\Pi_{\mathsf{A}} \oplus \mathbb{J} \qquad S_0 = 0 \qquad \Pi^{*^A 2} = \Pi_{\mathsf{A}}$$

$$\Pi_{\mathsf{A}} \oplus \mathsf{K}_0' \oplus \mathsf{K}_2 \oplus \mathcal{L}_1' = \mathbb{K} \oplus \mathbb{L} \oplus \mathbb{N} = \mathrm{Ann}(Y_0) \qquad S_0 = 0$$

Furthermore, for each $\beta \in \mathbb{R}$,

$$\Phi^{\pm}_{\beta,\tau}(\mathbb{K} \oplus \mathbb{L} \oplus \mathbb{N}) = \mathrm{Ann}(\{Y_0'(-\beta\omega) + Y_0(\omega) : \omega \in \mathbb{R}\}).$$

Finally, we have

$$\Pi_{\mathsf{A}} \oplus \mathbf{M}_2 \oplus \mathbb{P} \oplus \mathbb{J} = \mathrm{Ann}(Y_0') \qquad S_0 = 0$$

$$\Pi_{\mathsf{A}} \oplus \mathsf{K}_0' \oplus \mathsf{K}_2 \oplus \mathcal{L}_1' \oplus \mathbb{J} = \mathbb{K} \oplus \mathbb{L} \oplus \mathbb{N} \oplus \mathbb{J} \qquad S_0 = 0 \qquad \Pi^{*^A 2} = \Pi \setminus \mathbb{J}$$

$$\Pi_{\mathsf{A}} \oplus \mathbb{M} \oplus \mathbb{P} \oplus \mathbb{J} \qquad S_0 = 0 \qquad \Pi^{*^A 2} = \Pi \setminus \mathbf{M}_0$$

$$\mathcal{S} \qquad S_0 = 0$$

APPENDIX B

MULTIPLICATION TABLE

B.1 Multiplication Table Summary

	K_0'	K_2	K_0	K_4	$\mathcal{L}_1'$	$\mathcal{L}_1$	$\mathcal{L}_3$	$\mathbf{N}_0$	$\mathbf{N}_2$	$\mathbf{M}_0$	$\mathbf{M}_2$	$\mathbf{p}_1$	j_0
K_0'	0	K_0'	K_2	K_2	0	$\mathcal{L}_1'$	$\mathcal{L}_1'$	0	0	0	$\mathbf{M}_0$	0	0
K_2		$\mathsf{K}_0' \oplus \mathsf{K}_2$	$\mathsf{K}_0 \oplus \mathsf{K}_2 \oplus \mathsf{K}_4$	$\mathsf{K}_0 \oplus \mathsf{K}_2 \oplus \mathsf{K}_4$	$\mathcal{L}_1'$	$\mathcal{L}_1' \oplus \mathcal{L}_1 \oplus \mathcal{L}_3$	$\mathcal{L}_1' \oplus \mathcal{L}_1 \oplus \mathcal{L}_3$	0	0	$\mathbf{M}_0$	$\mathbf{M}_0 \oplus \mathbf{M}_2$	0	0
K_0			K_4	K_0	$\mathcal{L}_1 \oplus \mathcal{L}_3$	$\mathcal{L}_3$	$\mathcal{L}_1$	0	0	$\mathbf{M}_2$	$\mathbf{M}_2$	0	0
K_4				K_4	$\mathcal{L}_1 \oplus \mathcal{L}_3$	$\mathcal{L}_1$	$\mathcal{L}_3$	0	0	$\mathbf{M}_2$	$\mathbf{M}_2$	0	0
$\mathcal{L}_1'$					K_0'	$\mathsf{K}_2 \oplus \mathbf{N}_0 \oplus \mathbf{N}_2$	$\mathsf{K}_2 \oplus \mathbf{N}_0 \oplus \mathbf{N}_2$	$\mathcal{L}_1'$	$\mathcal{L}_1'$	0	$\mathbf{p}_1$	$\mathbf{M}_0$	0
$\mathcal{L}_1$						$\mathsf{K}_4 \oplus \mathbf{N}_2$	$\mathsf{K}_0 \oplus \mathbf{N}_0$	$\mathcal{L}_3$	$\mathcal{L}_1$	$\mathbf{p}_1$	$\mathbf{p}_1$	$\mathbf{M}_2$	0
$\mathcal{L}_3$							$\mathsf{K}_4 \oplus \mathbf{N}_2$	$\mathcal{L}_1$	$\mathcal{L}_3$	$\mathbf{p}_1$	$\mathbf{p}_1$	$\mathbf{M}_2$	0
$\mathbf{N}_0$								$\mathbf{N}_2$	$\mathbf{N}_0$	0	0	$\mathbf{p}_1$	0
$\mathbf{N}_2$									$\mathbf{N}_2$	0	0	$\mathbf{p}_1$	0
$\mathbf{M}_0$										0	j_0	0	0
$\mathbf{M}_2$											j_0	0	0
$\mathbf{p}_1$												j_0	0
j_0													0

B.2 Complete Multiplication Table

First products within each block are given. Products between blocks follow.

$\mathbb{J}$ block

$$\mathbb{J} *^A \mathcal{S} = 0 \tag{B.1}$$

$\mathbb{K}$ block

$$\mathsf{K}_0'(\lambda)^{*^A 2} = 0 \tag{B.2}$$

$$\mathsf{K}_2(u)^{*^A 2} = \mathsf{K}_0'(\mathrm{Re}(u^2\bar{w})) + \mathsf{K}_2(u\mathrm{Re}(u\bar{z})) \tag{B.3}$$

$$\mathsf{K}_0(\mu)^{*^A 2} = \mathsf{K}_4(w\mu^2) \tag{B.4}$$

$$\mathsf{K}_4(v)^{*^A 2} = \mathsf{K}_4(\bar{w}v^2) \tag{B.5}$$

$$\mathsf{K}_0'(\lambda) *^A \mathsf{K}_2(u) = \mathsf{K}_0'(\lambda\mathrm{Re}(\bar{z}u)) \tag{B.6}$$

$$\mathsf{K}_0'(\lambda) *^A \mathsf{K}_0(\mu) = \mathsf{K}_2(\frac{1}{2}\lambda\mu z) \tag{B.7}$$

$$\mathsf{K}_0'(\lambda) *^A \mathsf{K}_4(v) = \mathsf{K}_2(\frac{1}{2}\lambda v\bar{z}) \tag{B.8}$$

$$\mathsf{K}_2(u) *^A \mathsf{K}_0(\mu) = \frac{1}{2}[\mathsf{K}_2(\mu\bar{u}w) + \mathsf{K}_0(\mu\mathrm{Re}(u\bar{z})) + \mathsf{K}_4(\mu uz) \tag{B.9}$$

$$\mathsf{K}_2(u) *^A \mathsf{K}_4(v) = \frac{1}{2}[\mathsf{K}_2(uv\bar{w}) + \mathsf{K}_0(\mathrm{Re}(u\bar{v}z)) + \mathsf{K}_4(uv\bar{z})] \tag{B.10}$$

$$\mathsf{K}_0(\mu) *^A \mathsf{K}_4(v) = \mathsf{K}_0(\mathrm{Re}(\mu v\bar{w})) \tag{B.11}$$

$\mathbb{L}$ block

$$\mathcal{L}_1'(b)^{*^A 2} = \mathsf{K}_0'(\mathrm{Re}(z\bar{b}^2)) \tag{B.12}$$

$$\mathcal{L}_1(c)^{*^A 2} = \mathsf{K}_4(zc^2) + \mathbf{N}_2(w\bar{c}^2) \tag{B.13}$$

$$\mathcal{L}_3(d)^{*^A 2} = \mathsf{K}_4(\bar{z}d^2) + \mathbf{N}_2(\bar{w}d^2) \tag{B.14}$$

$$\mathcal{L}_1'(b) *^A \mathcal{L}_1(c) = \frac{1}{2}[\mathsf{K}_2(\bar{b}cz) + \mathbf{N}_0(\mathrm{Re}(\bar{b}\bar{c}z)) + \mathbf{N}_2(b\mathrm{Re}(c\bar{z}))] \tag{B.15}$$

$$\mathcal{L}_1'(b) *^A \mathcal{L}_3(d) = \frac{1}{2}[\mathsf{K}_2(bd\bar{z}) + \mathbf{N}_0(\mathrm{Re}(b\bar{d}z) + \mathbf{N}_2(bd\bar{z})] \tag{B.16}$$

$$\mathcal{L}_1(c) *^A \mathcal{L}_3(d) = \mathsf{K}_0(\mathrm{Re}(\bar{z}cd)) + \mathbf{N}_0(\mathrm{Re}(w\bar{c}\bar{d})) \tag{B.17}$$

$\mathbb{M}$ block

$$\mathbf{M}_0(\alpha)^{*^A 2} = 0 \tag{B.18}$$

$$\mathbf{M}_2(g)^{*^A 2} = j_0(\mathrm{Re}(w\bar{g}^2)) \tag{B.19}$$

$$\mathbf{M}_0(\alpha) *^A \mathbf{M}_2(g) = j_0(\alpha \mathrm{Re}(\bar{z}g)) \tag{B.20}$$

$\mathbb{N}$ block

$$\mathbf{N}_0(\rho)^{*^A 2} = \mathbf{N}_2(\rho^2 z) \tag{B.21}$$

$$\mathbf{N}_2(f)^{*^A 2} = \mathbf{N}_2(f^2 \bar{z}) \tag{B.22}$$

$$\mathbf{N}_0(\rho) *^A \mathbf{N}_2(f) = \mathbf{N}_0(\mathrm{Re}(\bar{z}\rho f)) \tag{B.23}$$

$\mathbb{P}$ block

$$\mathbf{p}_1(h)^{*^A 2} = j_0(\mathrm{Re}(z\bar{h}^2)) \tag{B.24}$$

Products with $\mathbb{K}$

$$\mathbb{K} *^A \mathbb{N} = \mathbb{K} *^A \mathbb{P} = 0 \tag{B.25}$$

$$\mathsf{K}_0'(\lambda) *^A \mathcal{L}_1'(b) = 0 \tag{B.26}$$

$$\mathsf{K}_0'(\lambda) *^A \mathcal{L}_1(c) = \mathcal{L}_1'(\frac{1}{2}\lambda \bar{c} z) \tag{B.27}$$

$$\mathsf{K}_0'(\lambda) *^A \mathcal{L}_3(d) = \mathcal{L}_1'(\frac{1}{2}\lambda d \bar{z}) \tag{B.28}$$

$$\mathsf{K}_0'(\lambda) *^A \mathbf{M}_0(\alpha) = 0 \tag{B.29}$$

$$\mathsf{K}_0'(\lambda) *^A \mathbf{M}_2(g) = \mathbf{M}_0(\frac{1}{2}\lambda \mathrm{Re}(\bar{z}g)) \tag{B.30}$$

$$\mathsf{K}_2(u) *^A \mathcal{L}_1'(b) = \mathcal{L}_1'(\frac{1}{2} b \mathrm{Re}(\bar{z}u)) \tag{B.31}$$

$$\mathsf{K}_2(u) *^A \mathcal{L}_1(c) = \mathcal{L}_1'(\frac{1}{2}\bar{c}\bar{u}w) + \mathcal{L}_1(\frac{1}{4}cu\bar{z}) + \mathcal{L}_3(\frac{1}{4}\bar{c}uz) \tag{B.32}$$

$$\mathsf{K}_2(u) *^A \mathcal{L}_3(d) = \mathcal{L}_1'(\frac{1}{2}ud\bar{w}) + \mathcal{L}_1(\frac{1}{4}u\bar{d}z) + \mathcal{L}_3(\frac{1}{4}ud\bar{z}) \tag{B.33}$$

$$\mathsf{K}_2(u) *^A \mathbf{M}_0(\alpha) = \mathbf{M}_0(\frac{1}{2}\mathrm{Re}(\alpha u \bar{z})) \tag{B.34}$$

$$\mathsf{K}_2(u) *^A \mathbf{M}_2(g) = \frac{1}{2}[\mathbf{M}_0(\mathrm{Re}(ug\bar{w})) + \mathbf{M}_2(u\mathrm{Re}(g\bar{z}))] \tag{B.35}$$

$$\mathsf{K}_0(\mu)*^A\mathcal{L}_1'(b) = \mathcal{L}_1(\frac{1}{4}\mu\bar{b}z) + \mathcal{L}_3(\frac{1}{4}\mu bz) \tag{B.36}$$

$$\mathsf{K}_0(\mu)*^A\mathcal{L}_1(c) = \mathcal{L}_3(\frac{1}{2}\mu\bar{c}w) \tag{B.37}$$

$$\mathsf{K}_0(\mu)*^A\mathcal{L}_3(d) = \mathcal{L}_1(\frac{1}{2}\mu\bar{d}w) \tag{B.38}$$

$$\mathsf{K}_0(\mu) *^A \mathbf{M}_0(\alpha) = \mathbf{M}_2(\frac{1}{2}\alpha\mu z) \tag{B.39}$$

$$\mathsf{K}_0(\mu)*^A\mathbf{M}_2(g) = \mathbf{M}_2(\frac{1}{2}\mu\bar{g}w) \tag{B.40}$$

$$\mathsf{K}_4(v)*^A\mathcal{L}_1'(b) = \mathcal{L}_1(\frac{1}{4}\bar{b}v\bar{z}) + \mathcal{L}_3(\frac{1}{4}bv\bar{z}) \tag{B.41}$$

$$\mathsf{K}_4(v)*^A\mathcal{L}_1(c) = \mathcal{L}_1(\frac{1}{2}cv\bar{w}) \tag{B.42}$$

$$\mathsf{K}_4(v)*^A\mathcal{L}_3(d) = \mathcal{L}_3(\frac{1}{2}dv\bar{w}) \tag{B.43}$$

$$\mathsf{K}_4(v)*^A\mathbf{M}_0(\alpha) = \mathbf{M}_2(\frac{1}{2}\alpha v\bar{z}) \tag{B.44}$$

$$\mathsf{K}_4(v)*^A\mathbf{M}_2(g) = \mathbf{M}_2(\frac{1}{2}gv\bar{w}) \tag{B.45}$$

Products with $\mathbb{L}$

$$\mathcal{L}_1'(b)*^A\mathbf{N}_0(\rho) = \mathcal{L}_1'(\frac{1}{2}\rho\bar{b}z) \tag{B.46}$$

$$\mathcal{L}_1'(b)*^A\mathbf{N}_2(f) = \mathcal{L}_1'(\frac{1}{2}bf\bar{z}) \tag{B.47}$$

$$\mathcal{L}_1'(b)*^A\mathbf{M}_0(\alpha) = 0 \tag{B.48}$$

$$\mathcal{L}_1'(b) *^A \mathbf{M}_2(g) = \mathbf{p}_1(\frac{1}{2}b\mathrm{Re}(\bar{z}g)) \tag{B.49}$$

$$\mathcal{L}_1'(b) *^A \mathbf{p}_1(h) = \mathbf{M}_0(\frac{1}{2}\mathrm{Re}(bh\bar{z})) \tag{B.50}$$

$$\mathcal{L}_1(c)*^A\mathbf{N}_0(\rho) = \mathcal{L}_3(\frac{1}{2}\rho cz) \tag{B.51}$$

$$\mathcal{L}_1(c)*^A\mathbf{N}_2(f) = \mathcal{L}_1(\frac{1}{2}c\bar{f}z) \tag{B.52}$$

$$\mathcal{L}_1(c)*^A\mathbf{M}_0(\alpha) = \mathbf{p}_1(\frac{1}{2}\alpha\bar{c}z) \tag{B.53}$$

$$\mathcal{L}_1(d)*^A\mathbf{M}_2(g) = \mathbf{p}_1(\frac{1}{2}\bar{c}\bar{g}w) \tag{B.54}$$

$$\mathcal{L}_1(c)*^A\mathbf{p}_1(h) = \mathbf{M}_2(\frac{1}{2}c\bar{h}z) \tag{B.55}$$

$$\mathcal{L}_3(d){*^A}\mathbf{N}_0(\rho) = \mathcal{L}_1(\frac{1}{2}\rho d\bar{z}) \tag{B.56}$$

$$\mathcal{L}_3(d){*^A}\mathbf{N}_2(f) = \mathcal{L}_3(\frac{1}{2}df\bar{z}) \tag{B.57}$$

$$\mathcal{L}_3(d){*^A}\mathbf{M}_0(\alpha) = \mathbf{p}_1(\frac{1}{2}\alpha d\bar{z}) \tag{B.58}$$

$$\mathcal{L}_3(d){*^A}\mathbf{M}_2(g) = \mathbf{p}_1(\frac{1}{2}dg\bar{w}) \tag{B.59}$$

$$\mathcal{L}_3(d) *^A \mathbf{p}_1(h) = \mathbf{M}_2(\frac{1}{2}\bar{z}dh) \tag{B.60}$$

Products with $\mathbb{M}$

$$\mathbb{M}{*^A}\mathbb{N} = \mathbb{M}{*^A}\mathbb{P} = 0 \tag{B.61}$$

Products with $\mathbb{N}$

$$\mathbf{N}_0(\rho){*^A}\mathbf{p}_1(h) = \mathbf{p}_1(\frac{1}{2}\rho\bar{h}z) \tag{B.62}$$

$$\mathbf{N}_2(f){*^A}\mathbf{p}_1(h) = \mathbf{p}_1(\frac{1}{2}fh\bar{z}) \tag{B.63}$$