ON SOME DEGENERATE BOUNDARY VALUE PROBLEMS

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Cristian G. Guriţă
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ABSTRACT

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Cristian G. Guriţă
DOCTOR OF PHILOSOPHY

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Professor Gerardo Mendoza, Chair

This thesis studies a boundary value problem for an operator which is elliptic inside the domain but fails to be elliptic on the boundary. The classical (elliptic) boundary value problems, posed for elliptic operators, are by now well understood. In solving such an elliptic problem it is essential that the operators involved are elliptic on a domain that is larger than the original one. Failure of ellipticity on the boundary causes the degeneracy of the boundary value problem.

The operator studied here is constructed as a sum of squares of vector fields. It fails to be elliptic on the boundary in a relatively mild way, in just one direction of the cotangent bundle and the characteristic set is symplectic. As a result the existence of parametrices for this operator can be studied using some classes of symbols that make a distinction between this direction of vanishing and the transversal directions.

The next step is to rewrite the classical compatibility conditions between the equation inside the domain and the equations on the boundary in terms of these new classes of operators used.

Since the operator used here is just a model for a larger class of operators (the immediate generalization being an operator constructed with the same vector fields but using a positive definite matrix instead of the identity matrix), the method presented in this thesis promises to be applicable, after further development, to a larger class of boundary value problems.

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TO *Patience and Tobacco*

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CHAPTER 1

INTRODUCTION. STATEMENT OF THE PROBLEM

1.1 The Problem

Suppose that in $\mathbb{R}^{2n+2}$, with the variables denoted by $x = (x_0, \ldots, x_{2n+1})$ we are given the family $\mathcal{D}$ of $2n+2$ vector fields:

$$\begin{aligned} L_0 &= x_{2n+1}\partial_{x_0} \\ L_j &= \partial_{x_j} + \frac{1}{2}x_{n+j}\partial_{x_0} \\ L_{n+j} &= \partial_{x_{n+j}} - \frac{1}{2}x_j\partial_{x_0} \\ L_{2n+1} &= \partial_{x_{2n+1}} \end{aligned} \tag{1.1}$$

for $j = 1, \ldots, n$. Let

$$\mathcal{L} = \sum_{j=0}^{2n+1} L_j^2. \tag{1.2}$$

The main purpose of this thesis is to examine a degenerate Boundary Value Problem of the type:

$$\begin{cases} \mathcal{L}u &= f \qquad \text{in } \mathbb{R}_+^{2n+2} = \{x \in \mathbb{R}^{2n+2} : x_{2n+1} > 0\} \\ B\gamma u &= g \qquad \text{on } \mathbb{R}^{2n+1} = \{x_{2n+1} = 0\}) \end{cases} \tag{1.3}$$

namely to find out which boundary conditions allow us to find parametrices for the problem near the boundary. Here $\gamma u = (\gamma_0 u, \gamma_1 u)$ where $\gamma_0 u(x_0, \ldots, x_{2n}) = u(x_0, \ldots, x_{2n}, 0)$ and $\gamma_1 u(x_0, \ldots, x_{2n}) = -i\partial_{x_{2n+1}} u(x_0, \ldots, x_{2n}, 0)$. B is a 1×2 matrix of pseudodifferential operators on which we want to state conditions that will make the boundary value problem solvable.

This problem is well understood when instead of the operator $\mathcal{L}$ we use an elliptic operator P on a bounded domain with smooth boundary. In this elliptic case the conditions that insure that a boundary value problem is Fredholm are known as the *Shapiro-Lopatinski conditions*. We aim at studying similar conditions for our non-elliptic operator.

The span S_x of the vector fields in (1.1) at any x with $x_{2n+1} \neq 0$ is the full tangent space at x, $T_x(\mathbb{R}^{2n+2})$. Since the coefficient of ∂_{x_0} in L_0 vanishes on $\{x_{2n+1}\} = 0$, the vector field ∂_{x_0} is missing over the set $\{x_{2n+1}\} = 0$. However, if we compute the brackets, we get that all of them are zero, except

$$\begin{aligned} [L_0, L_{2n+1}] &= -\partial_{x_0} \\ [L_j, L_{n+j}] &= -\partial_{x_0} \end{aligned} \tag{1.4}$$

so one can obtain ∂_{x_0}, including over $\{x_{2n+1} = 0\}$, by using first order brackets instead of linear combinations of the given vector fields. Hence on the whole space $\mathbb{R}^{2n+2}$ these vector fields satisfy Hörmander's step two bracket condition.

If $x \in \partial\mathbb{R}^{2n+2}_+ = \{x_{2n+1} = 0\}$, then the part of S_x tangent to $\{x_{2n+1} = 0\}$ is spanned by the family $\mathcal{D}'$, consisting of $L_1, \ldots, L_{2n}$:

$$\begin{aligned} L_j &= \partial_{x_j} + \frac{1}{2} x_{n+j} \partial_{x_0} \\ L_{n+j} &= \partial_{x_{n+j}} - \frac{1}{2} x_j \partial_{x_0} \end{aligned} \tag{1.5}$$

for $j = 1, \ldots, n$. Note again that these vector fields satisfy the step 2 condition.

1.2 The Operator $\mathcal{L}$

Using the given $2n+2$ vector fields (1.1) we construct, in $\mathbb{R}^{2n+2}$ the operator

$$\mathcal{L} = \sum_{j=0}^{2n+1} L_j^2 \tag{1.6}$$

while using the $2n$ vector fields defined over the boundary we define a second operator, acting only on the boundary:

$$\mathcal{L}' = \sum_{j=1}^{2n} L_j^2. \tag{1.7}$$

Our purpose will be to use these operators in a similar way the Laplacian operator Δ is used in the classical theory, like defining Sobolev spaces (non-isotropic in this case) in our domain and also over the boundary, obtaining trace theorems, inverting operators, with the final purpose of "solving" our degenerate boundary value problem.

As opposed to the regular Laplacian Δ, our operators $\mathcal{L}$ and $\mathcal{L}'$ are not elliptic. We will investigate the failure of these operators to being elliptic. Due to the nature of the vector fields, in the given coordinates, the total symbol of $\mathcal{L}$ is its principal symbol, namely

$$\sigma(\mathcal{L}) = -(x_{2n+1}\xi_0)^2 - \sum_{j=1}^{n}(\xi_j + \frac{1}{2}x_{n+j}\xi_0)^2 - \sum_{j=1}^{n}(\xi_{n+j} - \frac{1}{2}x_j\xi_0)^2 - \xi_{2n+1}^2. \tag{1.8}$$

The characteristic set of this operator is obtained by solving the equations:

$$\begin{aligned} x_{2n+1}\xi_0 &= 0 \\ \xi_j + \frac{1}{2}x_{n+j}\xi_0 &= 0, \qquad j = 1, \ldots, n \\ \xi_{n+j} - \frac{1}{2}x_j\xi_0 &= 0, \qquad j = 1, \ldots, n \\ \xi_{2n+1} &= 0 \end{aligned} \tag{1.9}$$

We see that over the points outside the boundary, i.e. over $x_{2n+1} \neq 0$, $\mathcal{L}$ has no characteristic set. Over the set $x_{2n+1} = 0$, though, we obtain as characteristic set the one-dimensional sub-bundle of the cotangent bundle

$$\{(x_0, \ldots, x_{2n}, 0; \xi_0, -\frac{1}{2}x_{n+1}\xi_0, \ldots, -\frac{1}{2}x_{2n}\xi_0, \frac{1}{2}x_1\xi_0, \ldots, \frac{1}{2}x_n\xi_0, 0)\} \tag{1.10}$$

over each point $(x_0, \ldots, x_{2n}, 0)$, where ξ_0 is an arbitrary real parameter, i.e. the characteristic set is spanned by the 1-form

$$\Theta = dx_0 + \frac{1}{2} \sum_{j=1}^{n} (x_j dx_{n+j} - x_{n+j} dx_j). \tag{1.11}$$

By using the embedding given by

$$\begin{aligned} \mathbb{R}^{2n+1} &\xrightarrow{i} \mathbb{R}^{2n+2} \\ (x_0, \ldots, x_{2n}) &\xrightarrow{i} (x_0, \ldots, x_{2n}, 0) \end{aligned} \tag{1.12}$$

we get the map

$$i^* : T^*(\mathbb{R}^{2n+2}) \to T^*(\mathbb{R}^{2n+1}) \tag{1.13}$$

which restricted to Σ is bijective onto Σ', the characteristic set of $\mathcal{L}'$.

The natural setup (see Folland and Stein [10] or Beals and Greiner [1]) for problems involving these vector fields, due to their behavior on the boundary, is the Heisenberg group. We will then consider our $\mathbb{R}^{2n+1}$ to be in fact H^n, the Heisenberg group and $\mathbb{R}^{2n+2}$ to be $H^n \times \mathbb{R}$. In this situation we have powerful instruments, given by the non-isotropic version of the Fourier transform.

But before getting there we will first see the symplectic character of the setup. Recall that a *symplectic manifold* is a manifold with a *symplectic form* on it, i.e. a closed nondegenerate 2-form (traditionally denoted by ω). The prime example of a symplectic manifold is the cotangent bundle of an arbitrary manifold M with the canonical symplectic form $\omega = \sum dx_j \wedge d\xi_j$. The characteristic set Σ of $\mathcal{L}$ considered as a $2n+2$ dimensional submanifold of $T^*(\mathbb{R}^{2n+2})$ is a symplectic manifold, the symplectic form being the pullback to Σ of the canonical symplectic form on $T^*(\mathbb{R}^{2n+2})$. Its expression is:

$$i^*\omega = \Theta \wedge d\xi_0 - \xi_0 \sum_{j=1}^{n} dx_j \wedge dx_{n+j}. \tag{1.14}$$

It is easy to see that $i^*\omega$ is closed and nondegenerate, hence that it is a symplectic form on Σ.

Similarly, the characteristic set Σ' of $\mathcal{L}'$ considered as a submanifold of $T^*(\mathbb{R}^{2n+1})$ is a symplectic manifold. The symplectic form in this case has the same expression, this being explained by the identification of the two characteristic sets mentioned in the previous paragraphs.

CHAPTER 2

THE CLASSICAL ELLIPTIC PROBLEM

2.1 The Classical Elliptic Problem

We will start with a description of a classical elliptic boundary value problem, pointing out the differences in our degenerate case and the ideas we use in order to overcome these supplementary difficulties.

The classical boundary value problem is:

$$\begin{cases} P(x, D_x)u & = f \qquad \text{in } \Omega \\ B(x, D_{x'})\gamma u & = g \qquad \text{in } \partial\Omega. \end{cases} \tag{2.1}$$

In this classical problem, P is a globally defined elliptic differential operator, B is a $\mu \times m$ matrix of pseudo-differential operators $(B_{j,k})_{j,k}$, with $j = 0, \ldots, \mu - 1$ and $k = 0, \ldots, m - 1$, defined only over the boundary. γ is the operator of taking traces to the boundary of functions or distributions defined in Ω along a vector field ν transversal to the boundary, defined in a neighborhood of the boundary. It is written as a column matrix $\gamma = (\gamma_j u)_j$, $j = 0, \ldots, m - 1$ and its components are defined by $\gamma_j u = \left[\left(\frac{1}{i}\nu\right)^j u\right] |_{\partial\Omega}$. The orders of the operators $B_{j,k}$ are $d_j - k$.

The difficulty in solving a boundary value problem resides in the fact that the equation in Ω and the equation on $\partial\Omega$ are of essentially different natures (the sets on which they reside are Ω and $\partial\Omega$, respectively) and cannot be paired together naturally, in order to find compatibility conditions between them. One of the methods typically used (see [8]) is to "sweep" the interior equation to the boundary, and pair this new equation with the existing boundary conditions; this pairing now becomes a pairing between equations of the same nature, so it is much more natural.

First, one should construct a (global) parametrix Q of P. This is long but not technically difficult if P is elliptic, and the parametrix thus obtained is a classical pseudo-differential operator. We will be able to construct parametrices in our non-elliptic case by using a special kind of pseudo-differential operators in the class $\Psi^{m,k}(\mathbb{R}^{2n+2}, \Sigma)$, so defined that they take into account the non-ellipticity along the characteristic set $\Sigma \subset T^*(\mathbb{R}^{2n+2})$.

Next, one constructs $\widetilde{P}$, the "trace to the boundary" of the operator P. For a function $u \in C^\infty(\overline{\Omega})$ we define u^0 to be its extension by 0 outside the set Ω. P being defined in a neighborhood of $\bar{\Omega}$ one can define an operator $\widetilde{P}$ acting over the boundary by:

$$\widetilde{P}\gamma u = P(u^0) - (Pu)^0. \tag{2.2}$$

where γu is the trace of u to the boundary of Ω along a vector field ν transversal to the boundary.

Now, if we apply Q (a globally defined classical pseudo-differential operator, the parametrix of P) on the left and restrict to Ω we obtain:

$$u = (Q\widetilde{P}\gamma u)|_\Omega + (Q(f^0))|_\Omega + \text{ smooth.} \tag{2.3}$$

We can apply the same technique in our degenerate case; only the class of operators we will be working with will be different.

By taking traces in this equation, one obtains (disregarding the smoothing part):

$$\gamma u = \gamma[(Q\widetilde{P}\gamma u)|_\Omega] + \gamma[(Q(f^0))|_\Omega]. \tag{2.4}$$

In the classical case one would consider now the *Calderón projector* C, a pseudo-differential operator (defined on the boundary, acting on m-tuples of functions on the boundary, these being essentially the traces to the boundary of functions in Ω), defined by

$$Cv = \gamma[(Q\widetilde{P}v)|_{\Omega}]. \tag{2.5}$$

The initial equation in Ω, $(Pu = f)$ is readily transformed into the equation on the boundary involving v, the trace of u:

$$v = Cv + \gamma[Q(f^0)|_{\Omega}] \tag{2.6}$$

(modulo smooth functions). In fact, this equation on $\partial\Omega$ is equivalent to the initial equation on u in Ω; if v satisfies it, by putting

$$u = Q(f^0)|_{\Omega} + Q\widetilde{P}v|_{\Omega} \tag{2.7}$$

then v is the trace of u and u satisfies the equation $Pu = f$ in Ω.

The advantage now is that since the equation in v resides on the boundary, it is much easier to be paired with the boundary conditions: $B\gamma u = g$ (on $\partial\Omega$) than the initial equation in Ω. One proves in the classical case and using a similar proof one can prove in our degenerate case that the Calderón projector is indeed a projector; the conditions for the compatibility of the two equations on the boundary are absolutely natural conditions to pose. The new system of equations being

$$\begin{cases} (I - C)v & = \gamma((Qf^0)|_{\Omega}) \\ Bv & = g \end{cases} \tag{2.8}$$

and C being a projector, by taking the symbols b of B and c of C, the compatibility condition becomes: b restricted to the range of c is surjective. This condition is known as the Shapiro-Lopatinski condition and we will provide a similar condition in the class of operators we use.

CHAPTER 3

TRACES IN SOBOLEV SPACES

3.1 Classical and Non Isotropic Sobolev Spaces

Classical (isotropic) Sobolev spaces are defined in the literature in essentially two ways; the first one is very transparent, but it only allows us to work with spaces H^m, for m positive integer while the second one is more abstract, but it is suited to deal with any real s instead of the positive integer m.

The definitions are the following:

Definition 3.1. For Ω open subset of $\mathbb{R}^n$ and $m \in \mathbb{N}$ we define the Sobolev spaces $H^m(\Omega)$ by:

$$H^m(\Omega) = \left\{u \in L^2(\Omega) \mid (\partial/\partial x)^\alpha u \in L^2(\Omega)\right\} \tag{3.1}$$

for all $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n \leq m$.

$H^m(\Omega)$ is a Hilbert space with the inner product:

$$\langle u, v\rangle_m = \sum_{|\alpha|\leq m} \int (\partial/\partial x)^\alpha u(x)\overline{(\partial/\partial x)^\alpha v(x)}\, dx \tag{3.2}$$

which also gives the norm:

$$\|u\|_m^2 = \sum_{|\alpha|\leq m} \|(\partial/\partial x)^\alpha u\|_{L^2(\Omega)}^2. \tag{3.3}$$

In other words, these functions are differentiable up to m times in the L^2 sense. Note that this first definition depends intimately on the naturally existing vector fields on $\mathbb{R}^n$, $\partial/\partial_{x_1}, \dots, \partial/\partial_{x_n}$.

Definition 3.2. For $s \in \mathbb{R}$ we define the Sobolev spaces $H^s(\mathbb{R}^n)$ by:

$$H^s(\mathbb{R}^n) = \left\{ u \in \mathcal{S}'(\mathbb{R}^n) \ : \ \int |\widehat{u}(\xi)|^2 (1 + |\xi|^2)^s \, d\xi < \infty \right\} \tag{3.4}$$

which says that $\widehat{u} \in L^2(\mathbb{R}^n, \mu)$, with μ being $(1 + |\xi|^2)^s$ times the Lebesgue measure. $H^s(\mathbb{R}^n)$ is a Hilbert space with the inner product:

$$\langle u, v \rangle_s = \int \widehat{u}(\xi)\overline{\widehat{v}(\xi)}(1 + |\xi|^2)^s \, d\xi \tag{3.5}$$

which also gives the norm:

$$\|u\|_s^2 = \int |\widehat{u}(\xi)|^2 (1 + |\xi|^2)^s \, d\xi. \tag{3.6}$$

Due to the global character of the Fourier transform this second definition cannot be applied to open subsets $\Omega \subsetneq \mathbb{R}^n$, except after truncation with functions in $C_0^\infty(\Omega)$ (this being the way to obtain the spaces $H^s_{loc}(\Omega)$) so we will have to define the Sobolev spaces in an open subset Ω of $\mathbb{R}^n$ differently.

The main feature of Definition 3.2 is that if we use it we can apply the powerful methods of Fourier analysis.

Remark 3.3. Note that when $s = m \in \mathbb{N}$ and $\Omega = \mathbb{R}^n$ the two definitions coincide. Definition 3.1, as well as all other definitions which only deal with integer indices for the Sobolev spaces can be extended to arbitrary real indices by complex interpolation [6]. The spaces obtained this way are the same as the spaces defined by using Definition 3.2 directly (as well as other definitions that deal from the start with arbitrary real indices).

In fact, for our purposes, since we will have to work with traces on hyperplanes, it will pay off to become more abstract and assign a special direction, given by the last variable x_n (this privileged direction will be later the direction of the vector field ν used to define traces at the boundary), so we will reinterpret (redefine) $H^m(\mathbb{R}^n)$

(resp. $H^m(\overline{\mathbb{R}^n_+})$) as a space of functions on the real line $\mathbb{R}$ (resp. on the positive half-line $\mathbb{R}_+$) in the variable x_n, with values in a space of functions of the first $n-1$ variables, $x' = (x_1, \dots, x_{n-1})$:

Definition 3.4. We say that $u \in H^m(\mathbb{R}^n)$ if $u \in L^2(\mathbb{R}, H^m(\mathbb{R}^{n-1}))$ such that

$$D_n^j u \in L^2(\mathbb{R}, H^{m-j}(\mathbb{R}^{n-1})) \text{ for all } j = 0, 1, \dots, m. \tag{3.7}$$

We give this space the norm:

$$\|u\|^2_{H^m(\mathbb{R}^n)} = \sum_{j=0}^{m} \|D_n^j u\|^2_{L^2(\mathbb{R}, H^{m-j}(\mathbb{R}^{n-1}))}. \tag{3.8}$$

This definition is easily seen to be equivalent to Definition 3.1, since what it says is that every time we have to take several derivatives, in order to verify the definition, we should take the ones in the privileged direction (the x_n direction) first.

We define $H^m(\overline{\mathbb{R}^n_+})$ similarly. The important thing is that although defined in different setups and with various methods, all these three definitions mean the same thing: regularity of a certain order of the functions belonging to these spaces; we will feel free to use any of them, according to the needs.

It is natural to try to generalize Definition 3.1 by using an arbitrary family D of vector fields in place of the coordinate vector fields $\partial/\partial x_j$:

Definition 3.5. Let D be a family of smooth vector fields on an open subset Ω of $\mathbb{R}^n$. Denote by $H^m_D(\Omega)$ the space of functions $u \in L^2(\Omega)$ such that $L_{j_1} \dots L_{j_k} u \in L^2(\Omega)$ for all $L_{j_1}, \dots, L_{j_k} \in D$, $k \le m$. The norm in this space is given by:

$$\|u\|^2_{H^m_D(\Omega)} = \sum_{|j| \le m} \|L_{j_1} \dots L_{j_k} u\|^2_{L^2(\Omega)}. \tag{3.9}$$

(See, for example, Berhanu-Pesenson [2].)

First, note that we did not require that the subspace spanned by these vector fields have constant dimension in $T(\Omega)$. Second, it is clear that if the family D spans the whole tangent space at each point of a compact set $\bar{\Omega}$ and the vector fields are smooth up to the boundary, this definition is equivalent to Definition 3.1. On the other hand suppose that the family D does not span the whole tangent space but it is involutive, (i.e. the bracket of each two vector fields in D lays in the span of D, which implies by the Frobenius theorem that locally there is an integral manifold Σ for the family D). In this case, we obtain some information about the regularity along Σ, but we have no information at all about the transversal directions. The interesting cases will lie in between these two extremes.

It is also known that if the vector fields in D satisfy a step k Hörmander condition, meaning that together with their brackets up to order $k-1$ they span the whole tangent space at each point then besides the trivial inclusion $H^m_{loc}(\Omega) \subset H^m_{D,loc}(\Omega)$ we also have $H^m_{D,loc}(\Omega) \subset H^{[m/k]}_{loc}(\Omega)$, where $[m/k]$ is the integer part of m/k. This is essentially in [11].

Remark 3.6. When one of the vector fields in D is $\partial/\partial x_n$ and the other vector fields are tangent to the $(n-1)$-dimensional planes $\{x_n = \text{ constant }\}$, another definition can be given, analogous to Definition 3.4, by using (3.9) for the Sobolev norm in the lower-dimensional planes.

We can use these equivalent definitions to construct the nonisotropic Sobolev spaces $H^m_{\mathcal{D}}(\mathbb{R}^{2n+2})$ and $H^m_{\mathcal{D}'}(\mathbb{R}^{2n+1})$ for m a positive integer, where $\mathcal{D}$ and $\mathcal{D}'$ are the families of vector fields given in (1.1) and (1.5).

Recall that the vector fields in (1.1) (respectively (1.5)) satisfy a step 2 Hörmander condition, meaning that together with their brackets span the whole space $T_x(\mathbb{R}^{2n+2})$ (respectively $T_x(\mathbb{R}^{2n+1})$) at every x (since $[L_j, L_{j+n}] = \partial_{x_0}$, the only missing direction).

A simple relation between these spaces and the classical Sobolev spaces is given by $H^m_{loc}(\mathbb{R}^{2n+1}) \subset H^m_{\mathcal{D}',loc}(\mathbb{R}^{2n+1})$ (since the linear span of $\{\partial_{x_0}, \ldots, \partial_{x_{2n}}\}$ includes the span of $\{L_1, \ldots, L_{2n}\}$) while in the other direction we obtain that $H^m_{\mathcal{D}',loc}(\mathbb{R}^{2n+1}) \subset H^{[m/2]}_{loc}(\mathbb{R}^{2n+1})$, where $[m/2]$ is the integer part of $k/2$.

3.2 Trace Theorems on Regular and Nonisotropic Sobolev Spaces

We will first show a known proof, taken from Treves [20], page 242, of the fact that the trace operator, taking the trace of a function $u \in H^m(\mathbb{R}^n_+)$ on the hyperplane

$x_n = 0$ is a continuous operator between $H^m(\mathbb{R}^n_+)$ and $H^{m-1/2}(\mathbb{R}^{n-1})$. Using the density of C_0^∞ in H^m it is enough to prove that the operator is continuous when applied to functions $u \in C_0^\infty$. The remarkable fact about this proof is that it can be extended ad literam to the nonisotropic case.

3.2.1 Classical Case

Let u be the restriction to $\mathbb{R}^n_+$ of a function in $C_0^\infty(\mathbb{R}^n)$ and denote by $\widehat{u}(\xi', t)$ its Fourier transform in the first $n-1$ variables. As a function of $t \geq 0$ it is differentiable and we have:

$$\begin{aligned} \frac{\partial}{\partial t}\left(|\widehat{u}(\xi',t)|^2\right) &= \frac{\partial}{\partial t}\left(\widehat{u}(\xi',t)\overline{\widehat{u}(\xi',t)}\right) \\ &= \frac{\partial \widehat{u}}{\partial t}(\xi',t)\overline{\widehat{u}(\xi',t)} + \widehat{u}(\xi',t)\overline{\frac{\partial \widehat{u}}{\partial t}(\xi',t)} \\ &= 2\Re\left(\frac{\partial \widehat{u}}{\partial t}(\xi',t)\overline{\widehat{u}(\xi',t)}\right). \end{aligned} \tag{3.10}$$

Integrating in t over $[0, \infty)$ we get:

$$|\widehat{u}(\xi',0)|^2 = -2\Re \int_0^\infty \frac{\partial \widehat{u}}{\partial t}(\xi',t)\overline{\widehat{u}(\xi',t)}\, dt. \tag{3.11}$$

Multiply on both sides by $(1+|\xi'|^2)^{m-1/2}$ and integrate in ξ' over $\mathbb{R}^{n-1}$:

$$\begin{aligned} &\int_{\mathbb{R}^{n-1}} |\widehat{u}(\xi',0)|^2(1+|\xi'|^2)|^{m-1/2}\, d\xi' \\ &= -2\Re \iint_{\mathbb{R}^{n-1}\times[0,\infty)} \frac{\partial \widehat{u}}{\partial t}(\xi',t)\overline{\widehat{u}(\xi',t)}(1+|\xi'|^2)^{m-1/2}\, d\xi'\, dt. \end{aligned} \tag{3.12}$$

Now use the inequality $\Re(z) \leq |z|$ and the Cauchy-Schwartz inequality in the integral on the right hand side after splitting $m - 1/2 = (m/2 - 1/2) + (m/2)$:

$$\begin{aligned} &\int_{\mathbb{R}^{n-1}} |\widehat{u}(\xi',0)|^2(1+|\xi'|^2)|^{m-1/2}\, d\xi' \\ &\leq C\left|\iint_{\mathbb{R}^{n-1}\times[0,\infty)} \frac{\partial \widehat{u}}{\partial t}(\xi',t)\overline{\widehat{u}(\xi',t)}(1+|\xi'|^2)|^{m-1/2}\, d\xi'\, dt\right| \\ &\leq C\left(\iint_{\mathbb{R}^{n-1}\times[0,\infty)} \left|\frac{\partial \widehat{u}}{\partial t}(\xi',t)\right|^2 (1+|\xi'|^2)|^{m-1}\, d\xi'\, dt\right)^{1/2} \times \\ &\times \left(\iint_{\mathbb{R}^{n-1}\times[0,\infty)} |\widehat{u}(\xi',t)|^2 (1+|\xi'|^2)|^{m}\, d\xi'\, dt\right)^{1/2} \end{aligned} \tag{3.13}$$

Now the left hand side is $\|\gamma u\|^2_{H^{m-1/2}(\mathbb{R}^{n-1})}$ while each of the factors on the right is bounded by $\|u\|_{H^m(\mathbb{R}^n_+)}$. This gives the desired continuity.

3.2.2 Nonisotropic Case

A similar proof, using the Heisenberg group Fourier transform instead of the regular Fourier transform and the corresponding Plancherel theorem for passing from the $L^2(\mathbb{R}^{2n+1})$ norm to the Hilbert-Schmidt norm in the Fourier transform space (as in [22]) will show that by taking traces in the class of nonisotropic Sobolev spaces we have the same expected loss of 1/2 class of regularity.

Take $\varphi \in C_0^\infty(\mathbb{R}^{2n+2})$. We have first:

$$\begin{aligned}
&\|\varphi(\cdot,0)\|^2_{H^{m-1/2}_{\mathcal{D}'}(\mathbb{R}^{2n+1})} \\
&= C_n \|\mathcal{L}_0^{\frac{m-1/2}{2}}\varphi(\cdot,0)\|^2_{L^2(\mathbb{R}^{2n+1})} \\
&= C_n \int_{-\infty}^{\infty} \|\pi_\lambda(\mathcal{L}_0^{\frac{m-1/2}{2}}\varphi(\cdot,0))\|^2_{\text{HS}} \, |\lambda^n| \, d\lambda \\
&\le C_n \int_{-\infty}^{\infty} \left[\int_0^\infty \left| \langle \frac{\partial}{\partial t}\pi_\lambda(\mathcal{L}_0^{\frac{m-1/2}{2}}\varphi(\cdot,t)), \pi_\lambda(\mathcal{L}_0^{\frac{m-1/2}{2}}\varphi(\cdot,t))\rangle_{\text{HS}} \right| dt\right] |\lambda^n| \, d\lambda
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
\le C \int_{-\infty}^{\infty} \Bigg[\int_0^\infty \Bigg| & \langle \pi_\lambda(\frac{\partial \varphi}{\partial t}(\cdot,t)) \circ (-\Delta + |x|^2)^{\frac{m-1/2}{2}}, \\
& (-\Delta + |x|^2)^{\frac{m-1/2}{2}} \circ \pi_\lambda \varphi(\cdot,t)\rangle_{\text{HS}} \Bigg| \, dt\Bigg] \, |\lambda^n| \, d\lambda.
\end{aligned} \tag{3.15}$$

We split the two exponents each equal to $(m-1/2)/2$ into an $(m-1)/2$ and an $m/2$. Then by Cauchy-Schwartz we have:

$$\begin{aligned}
\le C \int_{-\infty}^{\infty} \Bigg[\int_0^\infty & \|\pi_\lambda(\frac{\partial \varphi}{\partial t}(\cdot,t)) \circ (-\Delta + |x|^2)^{\frac{m-1}{2}}\|_{\text{HS}} \times \\
& \|(-\Delta + |x|^2)^{\frac{m}{2}} \circ \pi_\lambda(\varphi(\cdot,t))\|_{\text{HS}} \, dt\Bigg]^{1/2} \, |\lambda^n| \, d\lambda
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
\le C & \left\{ \int_{-\infty}^{\infty} \left[\int_0^\infty \|\pi_\lambda(\frac{\partial \varphi}{\partial t}(\cdot,t)) \circ (-\Delta + |x|^2)^{\frac{m-1}{2}}\|_{\text{HS}} \, dt\right]^2 |\lambda^n| \, d\lambda \right\}^{1/2} \times \\
& \left\{ \int_{-\infty}^{\infty} \left[\int_0^\infty \|(-\Delta + |x|^2)^{\frac{m}{2}} \circ \pi_\lambda(\varphi(\cdot,t))\|_{\text{HS}} \, dt\right]^2 |\lambda^n| \, d\lambda \right\}^{1/2}
\end{aligned} \tag{3.17}$$

But as in the classical case, these are

$$\leq C \left(\int_0^\infty \| \frac{\partial \varphi}{\partial t}(\cdot, t) \|^2_{H^{m-1}_{\mathcal{D}'}(\mathbb{R}^{2n+1})} \, dt \right)^{1/2} \times \left(\int_0^\infty \| \varphi(\cdot, t) \|^2_{H^m_{\mathcal{D}'}(\mathbb{R}^{2n+1})} \, dt \right)^{1/2} \tag{3.18}$$

and now each of the two is $\leq C\|\varphi\|_{H^m_{\mathcal{D}}(\mathbb{R}^{2n+2})}$, giving the desired continuity in the non-isotropic case.

CHAPTER 4

PARAMETRIX OF $\mathcal{L}$ IN THE CLASS $\Psi^{m,k}$

4.1 The Nonisotropic Setup

We saw that the first obstacle in solving our degenerate boundary value problem was that, as opposed to the classical elliptic case when it was easy to find a (global) parametrix of the operator P, we cannot find a parametrix (in the usual sense) of the operator $\mathcal{L} \in \Psi^2(\mathbb{R}^{2n+2})$ since it has a (conic) characteristic set Σ over the boundary $\{x_{2n+1} = 0\}$. The solution is to consider more general classes of operators (see Sjöstrand [19], Boutet de Monvel [4], Beals and Greiner [1] and Cancelier, Chemin, Xu [7]). This section describes the approach we use, taken from Mendoza [18]. The classes of operators defined there, denoted $\Psi^{m,k}(\mathbb{R}^{2n+2}, \Sigma)$, take into account the behavior of the symbol in different directions of the cotangent space (along Σ and transversal to it) and allow us to find a parametrix of an operator in $\Psi^{m,k}(\mathbb{R}^{2n+2}, \Sigma)$ as an operator in $\Psi^{-m,-k}(\mathbb{R}^{2n+2}, \Sigma)$.

Definition 4.1. Let U be an open subset of $\mathbb{R}^n$, $\Gamma \subset U \times \mathbb{R}^N$ an open cone. For $m, k \in \mathbb{R}$, $S^{m,k}_{1,0}(\Gamma, \mathbb{R}^d)$ is the space of smooth maps $a : \Gamma \times \mathbb{R}^d \to \mathbb{C}$ such that for any $K \subset\subset U$ and any α, β, γ multiindices there exists C such that

$$\left| \partial_x^\alpha \partial_\theta^\beta \partial_v^\gamma a(x, \theta; v) \right| \leq C(1 + |\theta|)^{m-|\beta|}(1 + |v|)^{k-|\gamma|}.$$

A symbol $a \in S^{m,k}_{1,0}(\Gamma, \mathbb{R}^d)$ is said to be semiclassical if there are functions $a_{m-i/2} : \Gamma \times \mathbb{R}^d \to \mathbb{C}$ such that $a_{m-i/2}(x, \theta; v)$ is homogeneous of order $m - i/2$ in θ for θ large and $a \sim \sum_i a_{m-i/2}$ in the sense that for any M

$$a - \sum_{i=0}^{M} a_{m-i/2} \in S^{m-(M+1)/2,k}_{1,0}(\Gamma, \mathbb{R}^d)$$

and furthermore, for each i there are smooth functions $a_{m-i/2,k-j} : \Gamma \times \mathbb{R}^d \setminus 0 \to \mathbb{C}$, homogeneous of order $k - j$ in the last variable, such that $a_{m-i/2} \sim \sum_j a_{m-i/2,k-j}$ in the sense that for any M

$$a_{m-i/2} - \chi \sum_{j=0}^{M} a_{m-i/2,k-j} \in S^{m-i/2,k-M-1}_{1,0}(\Gamma, \mathbb{R}^d)$$

if $\chi \in C^\infty(\mathbb{R}^d)$ vanishes near 0 and equals 1 outside some neighborhood of 0. We denote the space of semiclassical symbols in $S^{m,k}_{1,0}$ by $S^{m,k}$.

Following the typical procedure of the theory of classical pseudodifferential operators (where after defining symbols $a(x,\xi) \in S^m(U \times \mathbb{R}^n)$, symbols $a(x,y,\xi) \in S^m(U \times U \times \mathbb{R}^n)$ are used instead in defining the class of pseudodifferential operators $\Psi^m(U)$), for U open subset of $\mathbb{R}^n$ we will consider an open cone $\Gamma = U \times U \times \mathbb{R}^n$. We will also consider a set of functions $\lambda = (\lambda_1, \ldots, \lambda_{2l}) : U \times \mathbb{R}^n \to \mathbb{R}^{2l}$, smooth and homogeneous of order 1/2 in the second variable, $\theta \in \mathbb{R}^n$. We assume these functions λ_j to be defining functions of a symplectic submanifold Σ of $U \times \mathbb{R}^n \setminus 0$.

Then, we define the operators of class $\Psi^{m,k}(U, \Sigma)$ to be operators written in the form:

$$Au(x) = \left\{ \frac{1}{(2\pi)^n} \int e^{i(x-x')\cdot\xi} a(x, x', \xi; \lambda(\frac{x+x'}{2}, \xi)) u_0(x')\, dx'\, d\xi \right\} |dx|^{1/2} \qquad (4.1)$$

when they act on half-densities $u = u_0 |dx'|^{1/2}$, with the symbol $a \in S^{m,k}(U \times U \times \mathbb{R}^n, \mathbb{R}^{2l})$. The λ^j are introduced in combination with the special behavior of a in v to allow for a different behavior of the symbol in the directions normal to Σ. The Weyl convention is used to simplify the calculus.

By relaxing the condition on the symbols to $a \in S^{m,k}_{1,0}(U \times U \times \mathbb{R}^n, \mathbb{R}^{2l})$, we will also define the less restrictive class of operators $\Psi^{m,k}_{1,0}(U, \Sigma)$. The symbols of these operators will only have to satisfy the bounds in Definition 4.1, but will not be required to have the double expansion in ξ and v.

Proposition 4.2. *The sequence*

$$0 \to \Psi^{m-1/2,k}(M,\Sigma) \xrightarrow{\imath} \Psi^{m,k}(M,\Sigma) \xrightarrow{\sigma} S_h^{m,k}(M,\Sigma) \to 0$$

is exact.

$S_h^{m,k}(M,\Sigma)$ is the space of principal symbols of operators in $\Psi^{m,k}(M,\Sigma)$. Such a principal symbol of an operator A consists of two parts: a classical (standard) part, $\sigma^s(A)$, which is the regular symbol at points outside Σ, and the Σ part, a function defined on $T\Sigma^\perp$ which is a classical symbol of order k on every fiber of $T\Sigma^\perp$ and homogeneous of order m with respect to a nonisotropic action of $\mathbb{R}_+$ on T^*M, which separates the directions along Σ and the directions transversal to Σ. In addition, the classical and the Σ symbols have to satisfy a compatibility condition in order to define an operator in $\Psi^{m,k}(M,\Sigma)$.

The algebra structure of $S_h^{m,k}(M,\Sigma)$ is the following: If $(a^s,a^\Sigma) \in S_h^{m,k}$ and $(b^s,b^\Sigma) \in S_h^{m',k'}$, then $(a^s,a^\Sigma)(b^s,b^\Sigma) = (c^s,c^\Sigma)$ where $c^s = a^s \cdot b^s$ (product of classical symbols) and $c^\Sigma = a^\Sigma \sharp b^\Sigma$ (composition of symbols in Weyl calculus), meaning: pick a symplectic basis $e_1,\dots,e_l,f_1,\dots,f_l$ for $T_\rho\Sigma^\perp$, define an operator on the span of the e_j by

$$\mathrm{op}(a^\Sigma)(\varphi)(\sum_{j=1}^{l} u_j e_j) = \frac{1}{(2\pi)^l}\int e^{i(u-u')\cdot w} a(\rho; \frac{u+u'}{2}\cdot e + w\cdot f)\varphi(u')du'dw. \tag{4.2}$$

This is a pseudodifferential operator of order k. Let

$$C = \mathrm{op}(a^\Sigma)\circ \mathrm{op}(b^\Sigma), \tag{4.3}$$

let $k_C = k(u,u')$ the Schwartz kernel of C. We then define

$$(a^\Sigma \sharp b^\Sigma)(u\cdot e + v\cdot f) = \int k(u+\frac{1}{2}v, u-\frac{1}{2}v)e^{-i\theta\cdot v}\,d\theta. \tag{4.4}$$

Working in this setup, it turns out that our operator $\mathcal{L} \in \Psi^2(\mathbb{R}^{2n+2})$ can be considered an operator in the space $\Psi^{1,2}(\mathbb{R}^{2n+2},\Sigma)$ and as such we can construct a parametrix Q, which will be in $\Psi^{-1,-2}(\mathbb{R}^{2n+2},\Sigma)$.

4.2 The Symbol of $\mathcal{L}\circ\mathbf{op}(b)$

In order to study the existence of a parametrix $Q \in \Psi^{-1,-2}(\mathbb{R}^{2n+2}, \Sigma)$ of $\mathcal{L}$ we will compute the symbol of the composition of $\mathcal{L}$ with op (b), with $b \in S^{m,k}(\mathbb{R}^{2n+2}, \Sigma)$. For this, we will compute first the way a single operator L_j acts on a symbol

$$b \in S^{m,k}(\mathbb{R}^{2n+2}, \Sigma)$$

according to the rules of formal calculus in this space.

Let $b\left(\frac{x+y}{2}, \xi, \lambda(\frac{x+y}{2}, \xi)\right)$ be such a symbol. The corresponding Schwartz kernel will then be

$$\int e^{i(x-y)\cdot\xi} b\left(\frac{x+y}{2}, \xi, \lambda(\frac{x+y}{2}, \xi)\right) d\xi. \tag{4.5}$$

For $j = 1, \ldots, n$ the action of L_j on this kernel is:

$$\begin{aligned} &L_j(x)\left(\int e^{i(x-y)\cdot\xi} b\left(\frac{x+y}{2}, \xi, \lambda(\frac{x+y}{2}, \xi)\right) d\xi\right) \\ &= \int e^{i(x-y)\cdot\xi} i\sigma_j(x,\xi) b\left(\frac{x+y}{2}, \xi, \lambda(\frac{x+y}{2}, \xi)\right) d\xi \\ &\quad + \int e^{i(x-y)\cdot\xi} L_j(x)\left(b\left(\frac{x+y}{2}, \xi, \lambda(\frac{x+y}{2}, \xi)\right)\right) d\xi \\ &= I + II. \end{aligned} \tag{4.6}$$

In order to simplify the computations, we will introduce a notation for the indices. Suppose $j \in \{0, 1, \ldots, 2n+1\}$. We denote by $\not{j}$ its "conjugate", i. e. that index to which j is naturally paired:

$$\not{j} = \begin{cases} 2n+1, & \text{if } j = 0 \\ j+n, & \text{if } j \in \{1, \ldots, n\} \\ j-n, & \text{if } j \in \{n+1, \ldots, 2n\} \\ 0, & \text{if } j = 2n+1. \end{cases} \tag{4.7}$$

Using this notation, since $i\sigma_j(x,\xi) = i\sigma_j\left(\frac{x+y}{2}, \xi\right) + \frac{i}{4}(x_{\not{j}} - y_{\not{j}})\xi_0$, we have to integrate by parts in I, in order to eliminate $(x_{\not{j}} - y_{\not{j}})$, conforming hence to the formal calculus of the $S^{m,k}$ spaces (the Weyl convention). We obtain:

$$\begin{aligned}
I = &\int e^{i(x-y)\cdot\xi} i\sigma_j(\frac{x+y}{2},\xi) b(\frac{x+y}{2},\xi,\lambda(\frac{x+y}{2},\xi))\, d\xi - \\
&\quad - \frac{1}{4}\int e^{i(x-y)\cdot\xi}\xi_0 \partial_{\xi_{j'}}\left(b(\frac{x+y}{2},\xi,\lambda(\frac{x+y}{2},\xi))\right) d\xi \\
= &\int e^{i(x-y)\cdot\xi} i\sigma_j(\frac{x+y}{2},\xi) b(\frac{x+y}{2},\xi,\lambda(\frac{x+y}{2},\xi))\, d\xi - \\
&\quad - \frac{1}{4}\int e^{i(x-y)\cdot\xi}\xi_0 \frac{\partial b}{\partial \xi_{j'}}(\frac{x+y}{2},\xi,\lambda(\frac{x+y}{2},\xi))\, d\xi \\
&\quad - \frac{1}{4}\int e^{i(x-y)\cdot\xi}\xi_0 \sum_{l=0}^{2n+1} \frac{\partial b}{\partial v_l}(\frac{x+y}{2},\xi,\lambda(\frac{x+y}{2},\xi)) \frac{\partial \lambda_l}{\partial \xi_{j'}}(\frac{x+y}{2},\xi)\, d\xi
\end{aligned} \tag{4.8}$$

The first of these three symbols in I is of order $(m+1/2, k+1)$, the second one is of order (m,k) and the third one is of order $(m+1/2, k-1)$.

Remark 4.3. We note that the first symbol in I, $i\sigma_j(\frac{x+y}{2},\xi)b(\frac{x+y}{2},\xi,\lambda(\frac{x+y}{2},\xi))$ can also be considered a symbol of order $(m+1,k)$, but since $\sigma_j = \xi_0^{1/2}\lambda_j$ we will prefer to look at it as of class $(m+1/2, k+1)$ since this puts more weight on the behavior in the special v_j directions over the behavior in the ξ directions.

Since $L_j(x) = L_j(\frac{x+y}{2}) + \frac{1}{4}(x_{j'} - y_{j'})\partial_{x_0}$, we have to also integrate by parts in II, in order to eliminate $(x_{j'} - y_{j'})$ (and thus bring it to Weyl form). II becomes:

$$\begin{aligned}
II &= \int e^{i(x-y)\cdot\xi}\left(L_j(\frac{x+y}{2})\right)\left(b(\frac{x+y}{2},\xi,\lambda(\frac{x+y}{2},\xi))\right)d\xi \\
&\quad + \frac{1}{4}\int e^{i(x-y)\cdot\xi}(x_{j'} - y_{j'})\partial_{x_0}\left(b(\frac{x+y}{2},\xi,\lambda(\frac{x+y}{2},\xi))\right)d\xi \\
&= \frac{1}{2}\int e^{i(x-y)\cdot\xi}(L_j b)(\frac{x+y}{2},\xi,\lambda(\frac{x+y}{2},\xi))\,d\xi \\
&\quad + \frac{1}{2}\int e^{i(x-y)\cdot\xi}\sum_{l=0}^{2n+1}\frac{\partial b}{\partial v_l}(\frac{x+y}{2},\xi,\lambda(\frac{x+y}{2},\xi))(L_j\lambda_l)(\frac{x+y}{2},\xi)\,d\xi \\
&\quad + \frac{1}{4}\int e^{i(x-y)\cdot\xi}(x_{j'} - y_{j'})\partial_{x_0}\left(b(\frac{x+y}{2},\xi,\lambda(\frac{x+y}{2},\xi))\right)d\xi \\
&= \frac{1}{2}\int e^{i(x-y)\cdot\xi}(L_j b)(\frac{x+y}{2},\xi,\lambda(\frac{x+y}{2},\xi))\,d\xi \\
&\quad + \frac{1}{2}\int e^{i(x-y)\cdot\xi}\sum_{l=0}^{2n+1}\frac{\partial b}{\partial v_l}(\frac{x+y}{2},\xi,\lambda(\frac{x+y}{2},\xi))(L_j\lambda_l)(\frac{x+y}{2},\xi)\,d\xi \\
&\quad - \frac{1}{4i}\int e^{i(x-y)\cdot\xi}\partial_{\xi_{j'}}\partial_{x_0}\left(b(\frac{x+y}{2},\xi,\lambda(\frac{x+y}{2},\xi))\right)d\xi
\end{aligned} \tag{4.9}$$

In II, then, the first symbol is of order (m, k) and the second one is of order $(m + 1/2, k - 1)$. The third term, taken separately, gives:

$$\begin{aligned}
&- \frac{1}{4i}\int e^{i(x-y)\cdot\xi}\partial_{\xi_{j'}}\partial_{x_0}\left(b(\frac{x+y}{2},\xi,\lambda(\frac{x+y}{2},\xi))\right)d\xi \\
&= -\frac{1}{8i}\int e^{i(x-y)\cdot\xi}\partial_{\xi_{j'}}\left[\frac{\partial b}{\partial x_0}(\frac{x+y}{2},\xi,\lambda(\frac{x+y}{2},\xi))\right]d\xi \\
&= -\frac{1}{8i}\int e^{i(x-y)\cdot\xi}\frac{\partial^2 b}{\partial x_0\partial\xi_{j'}}(\frac{x+y}{2},\xi,\lambda(\frac{x+y}{2},\xi))\,d\xi \\
&\quad - \frac{1}{8i}\int e^{i(x-y)\cdot\xi}\sum_{l=0}^{2n+1}\frac{\partial^2 b}{\partial x_0\partial v_l}(\frac{x+y}{2},\xi,\lambda(\frac{x+y}{2},\xi))\frac{\partial\lambda_l}{\partial\xi_{j'}}(\frac{x+y}{2},\xi)\,d\xi
\end{aligned} \tag{4.10}$$

(all the terms containing $\partial\lambda_l/\partial\xi_0$ are zero), hence it gives a part of order $(m - 1, k)$ and a part of order $(m - 1/2, k - 1)$.

By putting now all these computations together and observing that $L_j - \frac{1}{2}\xi_0\partial_{\xi_{j'}} = H_{\sigma_j}$ (the Hamiltonian vector field of σ_j) we finally obtain that when L_j (for $j = 1, \ldots, n$) is composed to the left with an operator with symbol $b \in S^{m,k}$, the composition is an operator whose "principal part" in ξ, of order $m + 1/2$ in ξ, has two terms,

$$|\xi_0|^{1/2}(iv_j b + \frac{1}{2}\{\lambda_j, \lambda_{j'}\}\frac{\partial b}{\partial v_{j'}}), \tag{4.11}$$

of orders $(m+1/2, k+1)$ and $(m+1/2, k-1)$ respectively. In fact similar computations will show that the same formula gives the principal part for all $j = 0, \ldots, 2n+1$. For $j = 1, \ldots, 2n$ the "lower order parts" are

$$\frac{1}{2}H_{\sigma_j} b - \frac{|\xi_0|^{-1/2}\epsilon_j}{8i}\frac{\partial^2 b}{\partial x_0 \partial v_{j'}} - \frac{\epsilon_j}{8i}\frac{\partial^2 b}{\partial x_0 \partial \xi_{j'}} \tag{4.12}$$

of orders (m, k), $(m - 1/2, k - 1)$ and $(m - 1, k)$ respectively. For $j = 0$ we have $4i$ instead of the $8i$ in the last two denominators, while for $j = 2n + 1$ the lower order part is only the first term: $\frac{1}{2}H_{\sigma_j} b$. In order to have a single formula for all $j = 0, \ldots, 2n+1$ we consider ϵ_j to be 1 if j is "small", i.e. $j = 0, \ldots, n$, and -1 if j is "large", i.e. $j = n+1, \ldots, 2n+1$.

We conclude that when considering b to be an element in $S^{m,k}$ and applying L_j twice, the highest order part obtained (of order $m + 1$ in ξ) will be

$$\begin{aligned} b \to \left[|\xi_0|^{1/2}(iv_j + \frac{1}{2}\{\lambda_j, \lambda_{j'}\}\frac{\partial}{\partial v_{j'}})\right]^2 b = \\ = \left[|\xi_0|^{1/2}(iv_j - \epsilon_j \mathrm{sgn}(\xi_0)\frac{1}{2}\frac{\partial}{\partial v_{j'}})\right]^2 b \\ = |\xi_0| \left[-v_j^2 - \epsilon_j i\, \mathrm{sgn}(\xi_0) v_j \frac{\partial}{\partial v_{j'}} + \frac{1}{4}\frac{\partial^2}{\partial v_{j'}^2}\right] b \end{aligned} \tag{4.13}$$

(since $\{\lambda_j, \lambda_{j'}\}$ is $-\mathrm{sgn}(\xi_0)$ for j "small" and $\mathrm{sgn}(\xi_0)$ for j "large"). The full $L_j^2 b$ will have terms whose orders in ξ range from $m + 1$ to $m - 2$. We sum over j and we separate the operator $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_{1/2} + \cdots + \mathcal{L}_{-2}$ according to the effect on the order of ξ.

In order to find a parametrix for $\mathcal{L} = \sum_{j=0}^{2n+1} L_j^2$ (that is, solve $\mathcal{L}b \equiv 1$ for b modulo a smoothing operator) we need to consider $\mathcal{L}_1$ and invert it (in some reasonable sense), since in that case we will be able to find an asymptotic expansion of the symbol b of the parametrix iteratively in the order of ξ, the usual technique of pseudodifferential operators. We do that next.

4.3 Inverting $\mathcal{L}_1$

Let's consider the equation

$$\mathcal{L}_1 b = |\xi_0| \sum_{j=0}^{2n+1} \left[-v_j^2 - \epsilon_j i \operatorname{sgn}(\xi_0) v_j \frac{\partial}{\partial v_{\jmath'}} + \frac{1}{4} \frac{\partial^2}{\partial v_{\jmath'}^2} \right] b = 1. \tag{4.14}$$

where b is a symbol of class $S^{m,k}$. Noting that the equation only depends on ξ_0 and the v-s and that (with the exception of sgn (ξ_0), which we will disregard since the computations are the same no matter if $\xi_0 > 0$ or $\xi_0 < 0$) they occur in separate factors, we will look for a solution of the form:

$$b(\xi_0, v) = \alpha(\xi_0)\beta(v). \tag{4.15}$$

We remark here that in the case of the full operator $\mathcal{L}$ we are looking for an approximate solution in ξ (a parametrix), but for the equation $\mathcal{L}_1\beta = 1$ we need an exact solution $\beta(v)$, since otherwise for the equation $\mathcal{L}b = 1$ we obtain a sum of errors in the spaces $S^{j,-\infty}$, which are of order j in ξ, so that will not be a parametrix in ξ of $\mathcal{L}$. Hence the "reasonable" sense mentioned earlier, in which $\mathcal{L}_1$ has to be inverted, means here that we have to invert it exactly.

For the ξ_0 part the solution is simple: just take $\alpha(\xi_0) = 1/\xi_0$, defined outside the hyperplane $\{\xi_0 = 0\}$ which only intersects any cone around Σ only at the origin of the fibers.

In order to show existence for the v part of the equation, which is a PDE, we will use an unorthodox method, of showing existence for a more complicated object: an operator equation. Consider the equation in $\beta(v)$

$$\sum_{j=0}^{2n+1} \left[-v_j^2 - \epsilon_j i\, v_j \frac{\partial}{\partial v_{\jmath'}} + \frac{1}{4} \frac{\partial^2}{\partial v_{\jmath'}^2} \right] \beta = 1 \tag{4.16}$$

(as mentioned, we will consider only the case $\xi_0 > 0$ hence $\operatorname{sgn}(\xi_0) = +1$, since if $\xi_0 < 0$ the computations are similar). The remaining equation only depends on v-s:

$$\sum_{j=0}^{2n+1} \left[-v_j^2 - \epsilon_j i\, v_j \frac{\partial}{\partial v_{\jmath'}} + \frac{1}{4} \frac{\partial^2}{\partial v_{\jmath'}^2} \right] \beta = 1 \tag{4.17}$$

and the behavior of β when taking derivatives in the "small" v_j-s (for $j = 0, \dots n$) and the "large" v_j-s (for $j = n+1, \dots, 2n+1$) is similar (i.e. we cannot make a distinction between what variables should be considered the space variables and what variables should be considered the phase variables for β, viewed as a symbol). We arrive hence naturally to considering a new type of symbols, for which the behavior in the space variables and the phase variables is similar.

This is consistent with a remark of Chazarain and Piriou in [8], stating that there is not just one single type of symbols but rather we can say that there are almost as many of them as there are problems, and we need to be able to adapt one to another.

To solve this equation in β let us then consider a new type of symbols, a class $\widetilde{S}^k(\mathbb{R}^{2N})$ of symbols with symmetric behavior in x and ξ:

$$\widetilde{S}^k(\mathbb{R}^{2N}) = \{\beta(v) \in C^\infty(\mathbb{R}^{2N}) : |\partial_v^\alpha \beta(v)| \le C_\alpha (1 + |v|)^{k-|\alpha|}\}. \tag{4.18}$$

We will write u_j for the the "small" v_j-s, (i.e. $u_j = v_j$ for $j = 0, \dots, n$) and w_j for the "large" v_j-s (i.e. $w_j = v_{n+j}$ for $j = 1, \dots, n$, while $w_0 = v_{2n+1}$. We treat v_{2n+1} this way since v_{2n+1} is the conjugate of v_0 and we want to reestablish a "natural" numbering of the variables. We will consider the u-s to be the space variables in this new setup while the w-s will be the phase variables.

If β is a symbol in $\widetilde{S}^k(\mathbb{R}^{2N})$ acting on functions defined in $\mathbb{R}^N$, its action on a function ϕ is (we also use the Weyl convention in this space of symbols):

$$\mathrm{op}(\beta)\phi(u) = \int e^{i(u-\widetilde{u})\cdot w} \beta(\frac{u+\widetilde{u}}{2}, w)\phi(\widetilde{u})\, d\widetilde{u}\, dw \tag{4.19}$$

which means that the kernel of such an operator is

$$\ker\,(\mathrm{op}(\beta))(u, \widetilde{u}) = \int e^{i(u-\widetilde{u})\cdot w} \beta(\frac{u+\widetilde{u}}{2}, w)\, dw. \tag{4.20}$$

If in this space, $\widetilde{S}^k(\mathbb{R}^{2N})$, we compute the kernel of the composition of multiplication by $iu_j, j = 0, \dots, n$ with $\mathrm{op}(\beta)$ we obtain:

$$(iv_j - \frac{1}{2}\partial_{v_{N+j}})\beta \tag{4.21}$$

while for the kernel of the composition of $\partial u_j, j = 0, \dots, n$ with $\mathrm{op}(\beta)$ we get:

$$(iv_{N+j} + \frac{1}{2}\partial_{v_j})\beta. \tag{4.22}$$

In simple words, principal parts of L_j-s with "small" j-s correspond to the operators iu_j in $\widetilde{S}^k$ while principal parts of L_j-s with "large" j-s correspond to ∂u_j. Our equation in β:

$$\begin{aligned}&\sum_{j=0}^{2n+1}\left[-v_j^2-\epsilon_j i\, v_j\frac{\partial}{\partial v_{j'}}+\frac{1}{4}\frac{\partial^2}{\partial v_{j'}^2}\right]\beta\\&=\sum_{j=0}^{n}\left[-v_j^2-i\, v_j\frac{\partial}{\partial v_{j'}}+\frac{1}{4}\frac{\partial^2}{\partial v_{j'}^2}\right]\beta+\sum_{j=n+1}^{2n+1}\left[-v_j^2+i\, v_j\frac{\partial}{\partial v_{j'}}+\frac{1}{4}\frac{\partial^2}{\partial v_{j'}^2}\right]\beta\\&=\sum_{j=0}^{n}(iv_j-\frac{1}{2}\frac{\partial}{\partial v_{j'}})^2\beta+\sum_{j=n+1}^{2n+1}(iv_j+\frac{1}{2}\frac{\partial}{\partial v_{j'}})^2\beta=1\end{aligned}\tag{4.23}$$

is equivalent to the operator equation in $\mathrm{op}(\beta)$

$$\mathcal{H}\circ\mathrm{op}(\beta)=\sum_{j=0}^{n}\left[(\partial_{u_j})^2+(iu_j)^2\right]\mathrm{op}(\beta)=Id.\tag{4.24}$$

But $\mathcal{H}$ is a representation of the Heisenberg Laplacian on the Heisenberg group, called *the harmonic oscillator Hamiltonian* (see e.g. [22]). It arises in the Schrödinger equation and is studied in detail in quantum mechanics texts. There is a lot of information known about it.

The most important piece of information at this point is that $\mathcal{H}$ is invertible, but we also know about it that it is a positive, self adjoint, unbounded operator; $\mathcal{H}$ in $\mathbb{R}^n$ splits very nicely into analogous $\mathcal{H}$-s in one-dimensional $\mathbb{R}$; in this case (for $n=1$) its domain is:

$$\mathcal{D}(\mathcal{H})=\{u\in L^2(\mathbb{R})\, u''\in L^2(\mathbb{R}),\, x^2u\in L^2(\mathbb{R})\}\tag{4.25}$$

and $L^2(\mathbb{R})$ has an orthonormal basis consisting of eigenfunctions for $\mathcal{H}$; each eigenspace is one-dimensional, the eigenvalues are $\{1,3,5,7,\dots\}$, the corresponding eigenfunctions are Hermite polynomials times $e^{-x^2/2}$, appropriately normalized and moreover, there are simple ways to put back together all this information about $\mathcal{H}$ in $\mathbb{R}$ to obtain information about the corresponding $\mathcal{H}$ in $\mathbb{R}^n$.

Hence working within the frame of the calculus on the Heisenberg group, solving the initial equation for the symbol β, (which instead of being regarded as a symbol in $S^{m,k}(\mathbb{R}^{2n+2},\Sigma)$ independent of x and ξ can be thought of as a symbol $\beta\in\widetilde{S}^k(\mathbb{R}^{2n+2})$)

is equivalent to solving

$$\mathcal{H}\cdot\ \text{op}\ (\beta) = (-\Delta + |u|^2)\cdot\ \text{op}\ (\beta) = Id. \tag{4.26}$$

for op(β). But the known theory states $\mathcal{H}$ is invertible so a solution op(β) exists, hence we can also solve the initial equation in β, i.e. invert $\mathcal{L}_1$.

We have obtained:

Theorem 4.4. *$\mathcal{L}_1$, the principal part of $\mathcal{L}$, is invertible.*

We will now examine the full parametrix of $\mathcal{L}$.

4.4 A Parametrix of $\mathcal{L}$

Once we know that this "principal part" $\mathcal{L}_1$ of $\mathcal{L} = \sum_{j=0}^{2n+1} L_j^2$ is invertible, we can apply the standard recursive construction to find an asymptotic expansion in ξ of the symbol b of a parametrix of $\mathcal{L}$. Recall that $\mathcal{L}$ is split into terms $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_{1/2} + \cdots + \mathcal{L}_{-2}$ according to their effect on the order in ξ of the symbol b. Writing this way the equation $\mathcal{L}b = 1$ we obtain:

$$\mathcal{L}_1 b_{-1} + (\mathcal{L}_1 b_{-3/2} + \mathcal{L}_{1/2} b_{-1}) + (\mathcal{L}_1 b_{-2} + \mathcal{L}_{1/2} b_{-3/2} + \mathcal{L}_0 b_{-1}) + \cdots = 1 \tag{4.27}$$

which means, again separating according to the order in ξ and then taking the term of order zero:

$$\mathcal{L}_1 b_{-1} = 1 \tag{4.28}$$

which gives

$$b_{-1} = \mathcal{L}_1^{-1}(1). \tag{4.29}$$

By taking the terms of order 1/2 on both sides we obtain

$$\mathcal{L}_1 b_{-3/2} + \mathcal{L}_{1/2} b_{-1} = 0 \tag{4.30}$$

which gives

$$b_{-3/2} = -\mathcal{L}_1^{-1}(\mathcal{L}_{1/2} b_{-1}). \tag{4.31}$$

Similarly, from

$$\mathcal{L}_1 b_{-2} + \mathcal{L}_{1/2} b_{-3/2} + \mathcal{L}_0 b_{-1} = 0 \tag{4.32}$$

we obtain

$$b_{-2} = -\mathcal{L}_1^{-1}(\mathcal{L}_{1/2} b_{-1/2} + \mathcal{L}_0 b_{-1}) \tag{4.33}$$

so from the obvious recursion formula (we have a recursion of order seven) the result is transparent: in order to find b_{-l} we apply $\mathcal{L}_1^{-1}$ to a symbol of order $-l+1$ previously obtained by acting on the terms already solved for with the parts of the appropriate orders in the expansion of $\mathcal{L}$. Moreover, by induction, all the terms b_{-l} will have expansions in the v variables, making the symbol of the parametrix a symbol in $S^{-1,-2}(\mathbb{R}^{2n+2}, \Sigma)$.

We have obtained:

Theorem 4.5. *For $\mathcal{L} = \sum_{j=0}^{2n+1} L_j^2$, considered as an operator of class $\Psi^{1,2}(\mathbb{R}^{2n+2}, \Sigma)$ there exists a parametrix $Q \in \Psi^{-1,-2}(\mathbb{R}^{2n+2}, \Sigma)$.*

Remark 4.6. It will be also important for our work to notice that the solution we obtain for the equation $\mathcal{L}_1 b(x, \xi, v) = 1$ only depends on ξ_0 and on $v_0, \ldots, v_{2n+1}$, and not on x or the other ξ-s.

Remark 4.7. We only proved existence of an exact solution of the equation (4.17). So let us now solve equation (4.17) asymptotically in enough detail to get more information about the actual solution. At the same time, this will also give us the second expansion (in the v variables) of the terms in the ξ expansion, proving that the symbol of the parametrix is semiclassical.

4.5 Expansion in v of b_{-1}

We showed that b_{-1}, the first term in the ξ expansion of the symbol b of the desired parametrix is obtained by solving the equation

$$\mathcal{L}_1 b_{-1} = 1 \tag{4.34}$$

and that the solution is the product

$$b_{-1}(\xi_0, v) = \alpha(\xi_0)\beta(v) \tag{4.35}$$

with $\beta(v)$ solution of the equation (4.17)

$$\sum_{j=0}^{2n+1}\left[-v_j^2 - \epsilon_j i\, v_j \frac{\partial}{\partial v_{j'}} + \frac{1}{4}\frac{\partial^2}{\partial v_{j'}^2}\right]\beta = 1.$$

This $\beta(v)$ can be considered both an element in $S^{m,k}(\mathbb{R}^{2n+2}, \Sigma)$, independent of x and ξ and as an element in $\widetilde{S}^k(\mathbb{R}^{2n+2})$. Symbols in both these classes have the same behavior; the difference is that the same equation in β will arise from different operator equations in $\mathrm{op}(\beta)$ depending on where we consider β to live. This is though of no relevance at this point, where we are simply solving the given equation in β.

Consider then $\beta \in \widetilde{S}^k(\mathbb{R}^{2n+2})$, $\beta(v) \sim \sum_{j=0}^{\infty} \beta_{k-j}(v)$, with $\beta_{k-j}(v)$ eventually homogeneous of order $k - j$ in v. Call A the operator acting on β and split it according to the effect on the order in v:

$$\begin{aligned} A &= \sum_{j=0}^{2n+1}\left[-v_j^2 - \epsilon_j i\, v_j \frac{\partial}{\partial v_{j'}} + \frac{1}{4}\frac{\partial^2}{\partial v_{j'}^2}\right] \\ &= \sum_{j=0}^{2n+1}(-v_j^2) + \sum_{j=0}^{2n+1}(-\epsilon_j i\, v_j \frac{\partial}{\partial v_{j'}}) + \sum_{j=0}^{2n+1}(\frac{1}{4}\frac{\partial^2}{\partial v_{j'}^2}) \\ &= A_2 + A_0 + A_{-2}. \end{aligned} \tag{4.36}$$

In order to have $A\beta = 1$ (asymptotically), k should be -2 and we obtain:

$$A_2\beta_{-2} = \sum_{j=0}^{2n+1}(-v_j^2)\beta_{-2} = 1, \tag{4.37}$$

hence

$$\beta_{-2}(v) = A_2^{-1}(1) = \frac{-1}{\sum_{j=0}^{2n+1}(v_j^2)} \tag{4.38}$$

outside the origin. We also obtain that $\beta_{-3} = 0$, as well as all other terms with odd indices. From the equation:

$$A_0\beta_{-2} + A_2\beta_{-4} = 0 \tag{4.39}$$

we obtain the next possibly nonzero term which is:

$$\begin{aligned}\beta_{-4}(v) &= -(A_2)^{-1}(A_0\beta_{-2}) \\ &= \frac{1}{\sum_{j=0}^{2n+1}(v_j^2)}\left[\sum_{j=0}^{2n+1} -i\epsilon_j v_j \frac{\partial}{\partial v_{j'}}\right](\beta_{-2}(v)) = 0.\end{aligned} \tag{4.40}$$

We obtain that in fact $\beta_{-4} = 0$; we will retain the computation for reuse in case of lower order terms. The third nonzero term, β_{-6} is obtained from the equation:

$$A_2\beta_{-6} + A_0\beta_{-4} + A_{-2}\beta_{-2} = 0 \tag{4.41}$$

and it is:

$$\begin{aligned}\beta_{-6}(v) &= -(A_2)^{-1}\left(A_0\beta_{-4} + A_{-2}\beta_{-2}\right) \\ &= \frac{1}{\sum_{j=0}^{2n+1}(v_j^2)}\left[\sum_{j=0}^{2n+1} -i\epsilon_j v_j \frac{\partial}{\partial v_{j'}}(\beta_{-4}(v)) + \frac{1}{4}\sum_{j=0}^{2n+1}\frac{\partial^2}{\partial v_{j'}^2}(\beta_{-2}(v))\right].\end{aligned} \tag{4.42}$$

By doing these computations we see that

$$\beta_{-6}(v) = \frac{n-1}{(\sum_{j=0}^{2n+1} v_j^2)^3}. \tag{4.43}$$

From this point on, the general equation (for $l \geq 3$) is

$$A_2\beta_{-2l} + A_0\beta_{-2l+2} + A_{-2}\beta_{-2l+4} = 0 \tag{4.44}$$

which gives us the general formula for the expansion in v:

$$\beta_{-2l}(v) = \frac{1}{\sum_{j=0}^{2n+1}(v_j^2)}\left[\sum_{j=0}^{2n+1}(-i\epsilon_j v_j \frac{\partial}{\partial v_{j'}})(\beta_{-2l+2}) + \frac{1}{4}\sum_{j=0}^{2n+1}\left(\frac{\partial^2}{\partial v_j^2}\right)(\beta_{-2l+4})\right]. \tag{4.45}$$

For later use, we will explore what happens if we assume that in equations (4.37),(4.39), (4.41), (4.44), instead of having 1 or 0 on the right hand side we have more general terms. Namely, we will assume that equation (4.17) becomes

$$A\beta = \beta'. \tag{4.46}$$

The authors of [8] assume in a similar situation a condition on an operator $A \in \Psi_c^\mu(\mathbb{R}^n)$ that they simply call *condition (2.1)*; this condition is:

A is proper and each term of the complete symbol $\sum_{j\geq 0} a_j(x,\xi)$ *of A is a rational fraction in* ξ*.*

Their *condition (2.1)* is slightly stronger than the *transmission property* in [15] and their results can be extended to operators satisfying the *transmission property*. In our case *condition (2.1)* proved not to be sufficient for results regarding symbols of class $\Psi^{m,k}(\mathbb{R}^{2n+2},\Sigma)$, so we will assume here that β' satisfies a similar but stronger condition, which we will call *condition (2.1')*:

The expansion $\beta'(v) = \sum_{k\geq 0} \beta'_{l-2k}$ *in* v *consists of terms which are rational fractions in* v*, of the form:*

$$\beta'_{l-2k}(v) = \frac{P_l(v)}{(\sum_{j=0}^{2n+1}(v_j^2))^k} \tag{4.47}$$

where $P_l(v)$ *is a homogeneous polynomial in* v *of order* l*.*

Note that the initial right hand side terms of our equations, the constants 1 and 0 are of this form. The actions of A_0 and A_{-2} on a term satisfying condition (2.1') are:

$$A_0\left(\frac{P_l(v)}{\left(\sum_{j=0}^{2n+1}(v_j^2)\right)^k}\right) = \frac{1}{\left(\sum_{j=0}^{2n+1}(v_j^2)\right)^k}\sum_{j=0}^{2n+1} -i\epsilon_j v_j \frac{\partial P_l}{\partial v_{j'}} \tag{4.48}$$

and

$$\begin{aligned} A_{-2}\left(\frac{P_l(v)}{(\sum_{j=0}^{2n+1}(v_j^2))^k}\right) = \frac{1}{4}\frac{\Delta P_l}{\left(\sum_{j=0}^{2n+1}(v_j^2)\right)^k} + \\ + \frac{1}{\left(\sum_{j=0}^{2n+1}(v_j^2)\right)^{k+1}}\left[k\sum_{j=0}^{2n+1} v_{j'}\frac{\partial P_l}{\partial v_{j'}} + k(k-n)P_l\right]. \end{aligned} \tag{4.49}$$

We can then state:

Remark 4.8. The expansion in v of the symbol β that we obtain by solving iteratively the equation $A\beta = \beta'$, where β' satisfies condition (2.1') will also satisfy condition (2.1'). The orders of the fractions β_{-2l} as symbols are decreasing by 2 at each step of iteration.

In fact, in the special case we have here ($\beta' = 1$) the derivatives make all the terms in the recursion formula vanish, except the last one, and the remark becomes:

Theorem 4.9. *If the initial term is $\beta' = 1$, the symbol β has an expansion $\beta \sim \sum_{l=0}^{\infty} \beta_{-2-4l}$ where :*

$$\beta_{-2-4l}(v) = \frac{C_{-2-4l}}{\left[\sum_{j=0}^{2n+1}(v_j^2)\right]^{2l+1}} \tag{4.50}$$

and C_{-2-4l} are coefficients obtained by recursion from the formula:

$$C_{-2-4l} = C_{-2-4(l-1)}(4l-2)(4l-2-n). \tag{4.51}$$

Remark 4.10. Note that in this expansion only the terms whose indices are odd multiples of 2 are nonzero; moreover, if n is an odd multiple of 2 this expansion is finite.

4.6 Expansion in v of a General b_{-l}

Similarly to the way b_{-1} is obtained from the equation $\mathcal{L}_1 b_{-1} = 1$, the following terms of the ξ expansion of b are obtained from equations of the form:

$$\mathcal{L}_1 b_{-1-j/2} + \mathcal{L}_{1/2} b_{-1-j/2+1/2} + \cdots + \mathcal{L}_{-2} b_{-1-j/2+3} = 0 \tag{4.52}$$

where the terms of a higher order than $-1-j/2$ have already been obtained (for values of j smaller than 7 only part of these terms are necessary). This equation gives

$$b_{-1-j/2} = -\mathcal{L}_1^{-1}[\mathcal{L}_{1/2} b_{-1-j/2+1/2} + \cdots + \mathcal{L}_{-2} b_{-1-j/2+3}]. \tag{4.53}$$

But we saw that b_{-1} satisfies condition (2.1'). By applying the operators $\mathcal{L}_{1/2}, \ldots \mathcal{L}_{-2}$ and $\mathcal{L}_1^{-1}$ this condition is preserved, so using Remark 4.8 we obtain by iteration:

Theorem 4.11. *The symbol of the parametrix of $\mathcal{L}$, $b(x, \xi, v)$ is a symbol of class $S^{-1,-2}(\mathbb{R}^{2n+2}, \Sigma)$ and has an asymptotic expansion in ξ and v, as in the definition of the classes $S^{m,k}$. Moreover, the symbols $b_{-1-j/2}$ are of the form: the appropriate power of ξ_0 times an asymptotic expansion in v, whose terms are rational functions in v, the denominators being powers of $(\sum_{j=0}^{2n+1}(v_j^2))$ and the numerators being homogeneous polynomials in v, of appropriate degree.*

CHAPTER 5

TRACES AND THE CALDERÓN PROJECTOR

5.1 Traces to the Boundary

In solving an elliptic Boundary Value Problem, one of the methods, used e.g. by Chazarain and Piriou [8] is to construct the Calderón projector associated to the operator P, the domain Ω and the vector field ν transversal to the boundary which is used to define the traces on the boundary. This Calderón projector acts on traces to the boundary of functions in $C^\infty(\bar{\Omega})$. Compatibility between the Calderón projector and the operators giving the boundary conditions is the key in solving a boundary value problem.

The Calderón projector is an operator acting on the boundary, defined by (2.5):

$$Cv = \gamma[(Q\widetilde{P}v)|_\Omega].$$

We will generically call these operators acting on the boundary, obtained from an operator P acting inside the domain, *traces to the boundary* of P. The Calderón projector is the prime example; the operator $\widetilde{P}$ obtained from P is a simpler example. Within the frame of classical pseudodifferential operators it is known that the trace to the boundary of a classical pseudodifferential operator is a classical pseudodifferential operator; moreover, if $P \in \Psi^m(\mathbb{R}^n)$ and v is a function or distribution on $\mathbb{R}^{n-1}$,

$v \to \gamma_0(P(v \otimes \delta(x_n))$ is an operator in $\Psi^{m+1}(\mathbb{R}^{n-1})$. [18] does not provide details in this direction, so our next task is to analyze traces to the boundary of operators in $\Psi^{m,k}(\mathbb{R}^{2n+2}, \Sigma)$.

After a short computation of the *jump formula* for the operator $\mathcal{L}$, the first important step in this direction is to show that if Q is an operator in $\Psi^{m,k}_{1,0}(\mathbb{R}^{2n+2}, \Sigma)$, (i.e. its symbol $q(x, \xi, v)$ satisfies the bounds in Definition 4.1), then the symbol of its trace to the boundary satisfies similar bounds.

The next step is to show that if the symbol of Q has the double asymptotic expansion in ξ and v, the symbol of its trace to the boundary has a similar expansion.

In both these cases the microlocal behavior in directions far away from Σ is similar to the classical theory; we are hence interested only in what happens microlocally in an open cone around Σ.

Finally, we will see how these conclusions apply to our special operator $\mathcal{L}$.

5.2 The Operator $\widetilde{\mathcal{L}}$

In this section we will compute $\widetilde{\mathcal{L}}$, the trace to the boundary of the operator $\mathcal{L} = \sum_{j=0}^{2n+1} L_j^2$.

For a function $u \in C^\infty(\bar{\Omega})$ we define u^0 to be its extension by zero outside $\bar{\Omega}$. Then for a globally defined operator P we define:

$$\widetilde{P}\gamma u = P(u^0) - (Pu)^0 \tag{5.1}$$

which is a distribution supported on $\partial\Omega$, depending only on P and γu.

Since in our case Ω is $\mathbb{R}^{2n+2}_+$, taking u^0 for a function u is equivalent to multiplying it by the Heaviside function H in the last variable:

$$u^0(x) = H(x_{2n+1})u(x) \tag{5.2}$$

while constructing $\widetilde{P}$ for an operator P means in fact computing the commutator $[P, M_H]$ of P and M_H, the operator of multiplication by the Heaviside function in the last variable.

This is essentially the general case, since for a general Ω (bounded, hence with compact boundary) all we have to do is to localize u with a partition of unity supported around $\partial\Omega$ and then use a chart to transport $\partial\Omega$ into $\{x_{2n+1} = 0\}$ and Ω into $\{x_{2n+1} > 0\}$.

In our case, computing first $\widetilde{L}_j$ for $j = 0, \ldots, 2n$ we obtain zero in all cases, which is easy to explain by the fact that the first $2n+1$ vector fields only contain derivatives in the tangential directions to the boundary of Ω, $\{x_{2n+1} = 0\}$ hence they commute with M_H. Obviously, $\widetilde{L_j^2} = 0$ also, for these j.

Computation of $\widetilde{L}_{2n+1}$ gives:

$$\widetilde{L}_{2n+1}\gamma u = \gamma_0 u \otimes \delta(x_{2n+1}) \tag{5.3}$$

using the notation from [8]; the authors call this *the jump formula*. Similarly,

$$\widetilde{L_{2n+1}^2}\gamma u = i(\gamma_0 u) \otimes D_{x_{2n+1}}\delta(x_{2n+1}) + i(\gamma_1 u) \otimes \delta(x_{2n+1}). \tag{5.4}$$

The same formula applies if instead of being a smooth function, u is a distribution which is regular in the last variable, i.e. $u \in C^\infty(\mathbb{R}, \mathcal{D}'(\mathbb{R}^{2n+1}))$. Note that, following [8], we use D, not ∂ when we take traces to the boundary.

Now, $\widetilde{\mathcal{L}} = \widetilde{L_{2n+1}^2}$ is an operator from $C_0^\infty(\mathbb{R}^{2n+1}, \mathbb{C}^2)$ to the set of distributions in $\mathcal{D}'(\mathbb{R}^{2n+2})$ which are supported in $\{x_{2n+1} = 0\}$ and it can be written as the sum of the two operators above. Putting $(\gamma_0 u) = v_0$ and $(\gamma_1 u) = v_1$ we can write:

Proposition 5.1. *The jump formula giving $\widetilde{\mathcal{L}}$ for the operator $\mathcal{L}$ can be written as:*

$$\widetilde{\mathcal{L}}((v_0, v_1)) = (iv_0 \otimes D_{x_{2n+1}}\delta(x_{2n+1})) + (iv_1 \otimes \delta(x_{2n+1})) \tag{5.5}$$

where v_0 and v_1 are functions (or distributions) on $\{x_{2n+1} = 0\}$, (arising in a boundary value problem as traces of orders 0 and 1 of u on $\{x_{2n+1} = 0\}$).

We will analyze next the way an operator $Q \in \Psi^{m,k}(\mathbb{R}^{2n+2}, \Sigma)$ acts on functions (or distributions) on $\mathbb{R}^{2n+2}$ of this form.

5.3 Traces to the Boundary of Operators of Class $\Psi^{m,k}_{1,0}$

Suppose we have a symbol $q \in S^{m,k}_{1,0}(\mathbb{R}^{2n+2}, \Sigma)$ giving the operator Q (by substituting the λ-s, i.e. the defining functions of Σ for the v-s). We will show here that if we take the trace of this operator to the boundary, (i.e., in terms of the symbol, we set $x_{2n+1} = 0$, we substitute the λ_{2n+1} for v_{2n+1} and we integrate in ξ_{2n+1} on the intersection with a cone around Σ), then we get a decent symbol which once we substitute the λ-s for the remaining $2n+1$ v-s will give an operator of the same type on the boundary, i.e. an operator of some class $\Psi^{m',k'}_{1,0}(\partial\Omega, \Sigma)$.

We take a function $q(x, \theta, v)$ satisfying the estimate

$$|q(x, \theta, v)| \leq C(1 + |\theta|^2)^{m/2}(1 + |v|^2)^{k/2} \tag{5.6}$$

and analyze what estimates we can get for the integral of

$$q(x', 0; \xi', \xi_{2n+1}; v', \lambda_{2n+1}(\xi_0, \xi_{2n+1})) \tag{5.7}$$

in ξ_{2n+1} over the intersection with a cone around Σ, defined by $|\xi_{2n+1}|^2 \leq \varepsilon|\xi'|^2$. Note that λ_{2n+1} only depends on ξ_0, ξ_{2n+1} and that we only integrate inside a small cone around Σ. We will separate ξ' and ξ_{2n+1} in the bound above. On one hand if $m \geq 0$ we have

$$\begin{aligned} (1 + |\xi|^2)^{m/2} &= (1 + |\xi'|^2 + |\xi_{2n+1}|^2)^{m/2} \\ \leq (1 + |\xi'|^2 + \varepsilon|\xi'|^2)^{m/2} &\leq (1 + (1 + \varepsilon)|\xi'|^2)^{m/2} \end{aligned} \tag{5.8}$$

while if $m < 0$ we have:

$$\begin{aligned} (1 + |\xi|^2)^{m/2} &= (1 + |\xi'|^2 + |\xi_{2n+1}|^2)^{m/2} \\ &\leq (1 + |\xi'|^2)^{m/2} \end{aligned} \tag{5.9}$$

so in both cases above we get for the first factor the estimate

$$(1 + |\xi|^2)^{m/2} \leq C(1 + |\xi'|^2)^{m/2} \tag{5.10}$$

if ξ is inside the cone.

In order to bound the second factor inside the cone, we substitute $\lambda_{2n+1} = \frac{\xi_{2n+1}}{|\xi_0|^{1/2}} = \frac{\xi_{2n+1}|\xi_0|^{1/2}}{|\xi_0|}$ for v_{2n+1}. If $k \geq 0$ we obtain

$$\begin{aligned}(1+|v|^2)^{k/2} &= (1+|v'|^2+|\lambda_{2n+1}|^2)^{k/2} \\ &\leq (1+|v'|^2)^{k/2}(1+|\lambda_{2n+1}|^2)^{k/2} \\ &= (1+|v'|^2)^{k/2}(1+\frac{\xi_n^2|\xi_0|}{|\xi_0|^2})^{k/2} \\ &\leq (1+|v'|^2)^{k/2}(1+\varepsilon|\xi_0|)^{k/2} \\ &\leq (1+|v'|^2)^{k/2}(1+\varepsilon|\xi|')^{k/2}\end{aligned} \tag{5.11}$$

while for $k < 0$ we obtain

$$\begin{aligned}(1+|v|^2)^{k/2} &= (1+|v'|^2+|v_{2n+1}|^2)^{k/2} \\ &\leq (1+|v'|^2)^{k/2}\end{aligned} \tag{5.12}$$

so we have, no matter what k is,

$$(1+|v|^2)^{k/2} \leq (1+|v'|^2)^{k/2}(1+\varepsilon|\xi'|)^{\max(0,k/2)}. \tag{5.13}$$

Summing up, if we have a function $q(x,\theta,v)$ satisfying

$$|q(x,\theta,v)| \leq C(1+|\theta|^2)^{m/2}(1+|v|^2)^{k/2} \tag{5.14}$$

then inside a cone $|\xi_{2n+1}|^2 \leq \varepsilon|\xi'|^2$ we have the estimate:

$$|q(x,\xi',\xi_{2n+1};v',\lambda_{2n+1}(\xi_0,\xi_{2n+1}))| \leq C(1+|\xi'|^2)^{m/2+\max(0,k/4)}(1+|v'|^2)^{k/2}. \tag{5.15}$$

Integrating now in ξ_{2n+1} between $-\varepsilon^{1/2}|\xi|'$ and $\varepsilon^{1/2}|\xi|'$, the integral will be bounded by:

$$C(1+|\xi'|^2)^{(m+1)/2+\max(0,k/4)}(1+|v'|^2)^{k/2}. \tag{5.16}$$

Let's suppose now that we have a symbol q of class (m,k), meaning that it satisfies the estimates:

$$|\partial_x^\alpha\partial_\theta^\beta\partial_v^\gamma q(x,\theta,v)| \leq C(1+|\theta|)^{m-|\beta|}(1+|v|)^{k-|\gamma|}. \tag{5.17}$$

According to the bound proved above,

$$\left| \int \chi(x,\xi)\partial_x^\alpha \partial_\theta^\beta \partial_v^\gamma q(x',0;\xi',\xi_{2n+1};v',\lambda_{2n+1}(\xi_0,\xi_{2n+1}))\, d\xi_{2n+1} \right| \\ \leq C(1+|\xi'|^2)^{(m+1-|\beta|)/2+\max(0,(k-|\gamma|)/4)}(1+|v'|^2)^{(k-|\gamma|)/2} \tag{5.18}$$

We obtained:

Theorem 5.2. *If the initial symbol $q(x,\xi,v)$ is in a class $S_{1,0}^{m,k}(\mathbb{R}^{2n+2},\Sigma)$, with $k<0$ (in our case $k=-2$) then the symbol we obtained by integrating in the cone is of class $S_{1,0}^{m+1,k}(\mathbb{R}^{2n+1},\Sigma)$ over the boundary $\{x_{2n+1}=0\}$.*

5.4 Computation of the Calderón Projector for $b_{-1,-2}$

We will first compute the Calderón projector corresponding to $\widetilde{\mathcal{L}}$ and $b_{-1,-2}$; One reason is that we will need to know exactly what it is; the other reason is that these computations will be a model for the computation of the Calderón projector corresponding to more general operators. Recall the formula (5.5) we obtained:

$$\widetilde{\mathcal{L}}((v_0,v_1)) = (iv_0 \otimes D_{y_{2n+1}}\delta(y_{2n+1})) + (iv_1 \otimes \delta(y_{2n+1}))$$

The Calderón projector will be a 2 by 2 matrix of pseudo-differential operators, $\gamma \circ \mathrm{op}(b_{-1,-2}) \circ \widetilde{\mathcal{L}}$. Its matrix of principal symbols will be:

$$\begin{pmatrix} c_{0,0} & c_{0,1} \\ c_{1,0} & c_{1,1} \end{pmatrix}. \tag{5.19}$$

We will only consider the case $\xi_0>0$, the case $\xi_0<0$ being similar; $\vec{v}$ will be a short notation for $(v_1,\ldots,v_{2n})$. In order to compute the second column of this matrix (the part depending on v_1) we start with:

$$\begin{aligned}
&\mathrm{op}(b_{-1,-2})(iv_1\otimes\delta(y_{2n+1}))(x',x_{2n+1}) = \\
&= \frac{1}{(2\pi)^{2n+2}}\int e^{i(x-y)\cdot\xi}\frac{-1}{\xi_0}\frac{iv_1(y')\otimes\delta(y_{2n+1})}{\left(\frac{x_{2n+1}+y_{2n+1}}{2}\xi_0^{1/2}\right)^2+|\vec{v}|^2+\left(\frac{\xi_{2n+1}}{\xi_0^{1/2}}\right)^2}\,dy\,d\xi \\
&= \frac{1}{(2\pi)^{2n+1}}\int e^{i(x'-y')\cdot\xi'}\left[\frac{-i}{2\pi\xi_0}\int\frac{e^{ix_{2n+1}\xi_{2n+1}}d\xi_{2n+1}}{\left(\frac{1}{2}x_{2n+1}\xi_0^{1/2}\right)^2+|\vec{v}|^2+\left(\frac{\xi_{2n+1}}{\xi_0^{1/2}}\right)^2}\right]v_1(y')\,dy'\,d\xi'.
\end{aligned} \tag{5.20}$$

The bracket above, taken separately, will give us the symbols of the pseudo-differential operators on the boundary that we are looking for. Let $\beta^2 = \left(\frac{1}{2}x_{2n+1}\xi_0^{1/2}\right)^2+|\vec{v}|^2$, $\beta>0$; change the variable $\frac{\xi_{2n+1}}{\xi_0^{1/2}}=\eta$; split the fraction $\frac{1}{\eta^2+\beta^2}=\frac{1}{2i\beta}(\frac{1}{\eta-i\beta}-\frac{1}{\eta+i\beta})$ and obtain, by shifting to a contour in the half-space $\Im\eta>0$ and using residues:

$$\begin{aligned}
&\frac{-i}{2\pi\xi_0}\int\frac{e^{ix_{2n+1}\xi_{2n+1}}}{\left(\frac{1}{2}x_{2n+1}\xi_0^{1/2}\right)^2+|\vec{v}|^2+\left(\frac{\xi_{2n+1}}{\xi_0^{1/2}}\right)^2}\,d\xi_{2n+1} \\
&\quad= \frac{-i}{2\pi\xi_0^{1/2}}2\pi i\mathrm{Res}_{\eta=i\beta}\frac{e^{ix_{2n+1}\xi_0^{1/2}\eta}}{\eta-i\beta} \\
&\quad= \frac{e^{-x_{2n+1}\xi_0^{1/2}\beta}}{2i\beta\xi_0^{1/2}}.
\end{aligned} \tag{5.21}$$

If in this expression we take the zero-th order trace (that is, we make $x_{2n+1}=0$) we obtain

$$c_{0,1} = \frac{1}{2i|\vec{v}|\xi_0^{1/2}}. \tag{5.22}$$

If we take the first order trace (we take $\mathrm{D}_{x_{2n+1}}$ and then we make $x_{2n+1}=0$) we obtain

$$c_{1,1} = \frac{1}{2}|\vec{v}|. \tag{5.23}$$

To compute the entries of the first column of the matrix (the part depending on

v_0) we start with:

$$\begin{aligned}&\operatorname{op}(b_{-1,-2})(iv_0\otimes D_{y_{2n+1}}\delta(y_{2n+1}))(x',x_{2n+1}) = \\ &= \frac{1}{(2\pi)^{2n+2}}\int e^{i(x-y)\cdot\xi}\frac{-1}{\xi_0}\frac{iv_0(y')\otimes D_{y_{2n+1}}\delta(y_{2n+1})}{\left(\frac{x_{2n+1}+y_{2n+1}}{2}\xi_0^{1/2}\right)^2+|\vec{v}|^2+\left(\frac{\xi_{2n+1}}{\xi_0^{1/2}}\right)^2}\,dy\,d\xi.\end{aligned} \tag{5.24}$$

We compute first the effect of $D_{y_{2n+1}}\delta(y_{2n+1})$:

$$\begin{aligned}&\int e^{-iy_{2n+1}\xi_{2n+1}}\frac{\partial_{y_{2n+1}}\delta(y_{2n+1})}{\left(\frac{x_{2n+1}+y_{2n+1}}{2}\xi_0^{1/2}\right)^2+|\vec{v}|^2+\left(\frac{\xi_{2n+1}}{\xi_0^{1/2}}\right)^2}\,dy_{2n+1} \\ &\quad = \int \partial_{y_{2n+1}}\left[-\frac{e^{-iy_{2n+1}\xi_{2n+1}}}{\left(\frac{x_{2n+1}+y_{2n+1}}{2}\xi_0^{1/2}\right)^2+|\vec{v}|^2+\left(\frac{\xi_{2n+1}}{\xi_0^{1/2}}\right)^2}\right]\delta(y_{2n+1})\,dy_{2n+1}.\end{aligned} \tag{5.25}$$

Computing the derivative of the bracket and setting $y_{2n+1}=0$ gives that we should be looking for:

$$\begin{aligned}&\operatorname{op}(b_{-1,-2})(iv_0\otimes D_{y2n+1}\delta(y_{2n+1}))(x',x_{2n+1}) = \\ &= \frac{1}{(2\pi)^{2n+1}}\int e^{i(x'-y')\cdot\xi'}v_0(y')\times \\ &\times\left\{\frac{1}{2\pi}\int\left[\frac{i\xi_{2n+1}e^{ix_{2n+1}\xi_{2n+1}}\frac{-1}{\xi_0}}{\left(\frac{1}{2}x_{2n+1}\xi_0^{1/2}\right)^2+|\vec{v}|^2+\left(\frac{\xi_{2n+1}}{\xi_0^{1/2}}\right)^2}\right.\right. \\ &\left.\left.+\frac{\frac{-1}{2}x_{2n+1}e^{ix_{2n+1}\xi_{2n+1}}}{\left[\left(\frac{1}{2}x_{2n+1}\xi_0^{1/2}\right)^2+|\vec{v}|^2+\left(\frac{\xi_{2n+1}}{\xi_0^{1/2}}\right)^2\right]^2}\right]d\xi_{2n+1}\right\}dy'\,d\xi'\end{aligned} \tag{5.26}$$

with the zeroth and the first order traces of the integral inside the braces giving us the symbols which are the entries we need for the first column.

By doing the same kind of computations, shifting contours to the upper complex half-plane and using residues, the first term in this integral gives a zeroth order trace of $\frac{1}{2}$ and a first order trace of $\frac{i}{2}\xi_0^{1/2}|\vec{v}|$. The second term's zero-th order trace vanishes, while its first order trace is $\frac{\xi_0^{1/2}}{16\pi|\vec{v}|^3}$, a term of lower order. We can write this result as:

Proposition 5.3. *The Calderón projector corresponding to the top order term $b_{-1,-2}$ of the parametrix Q, and $\widetilde{\mathcal{L}}$ has as matrix of principal symbols:*

$$\begin{pmatrix} \frac{1}{2} & \frac{-i}{2\xi_0^{1/2}|\vec{v}|} \\ \frac{i}{2}\xi_0^{1/2}|\vec{v}| & \frac{1}{2} \end{pmatrix} \tag{5.27}$$

and has a matrix of lower order terms:

$$\begin{pmatrix} 0 & 0 \\ \frac{\xi_0^{1/2}}{16\pi|\vec{v}|^3} & 0 \end{pmatrix}. \tag{5.28}$$

5.5 Computation of the Calderón Projector for the Terms $b_{-1,-2-4l}$

We start from the same formula (5.5) we obtained:

$$\widetilde{\mathcal{L}}((v_0, v_1)) = (iv_0 \otimes D_{y_{2n+1}}\delta(y_{2n+1})) + (iv_1 \otimes \delta(y_{2n+1}))$$

and from the formula for the symbol we obtained in (4.50):

$$b_{-1,-2-4l}(\xi, v) = \frac{1}{\xi_0}\frac{C_{-2-4l}}{\left[\sum_{j=0}^{2n+1}(v_j^2)\right]^{2l+1}}.$$

Just to ease the notation burden in the lengthy computations that follow we will use instead a symbol of the form

$$b_{-1,-2l} = \frac{1}{\xi_0}\frac{C_{-2l}}{\left[\sum_{j=0}^{2n+1}(v_j^2)\right]^{l}}. \tag{5.29}$$

Similarly to the way we proceeded when we computed the Calderón projector for $b_{-1,-2}$, we will compute first the second column of the Calderón projector:

$$\begin{aligned}
&\operatorname{op}(b_{-1,-2l})(iv_1 \otimes \delta(y_{2n+1}))(x', x_{2n+1}) = \\
&= \frac{1}{(2\pi)^{2n+2}} \int e^{i(x-y)\cdot\xi} \frac{1}{\xi_0} \frac{iC_{-2l} v_1(y') \otimes \delta(y_{2n+1})}{\left[\left(\frac{x_{2n+1}+y_{2n+1}}{2}\xi_0^{1/2}\right)^2 + |\vec{v}|^2 + \left(\frac{\xi_{2n+1}}{\xi_0^{1/2}}\right)^2\right]^l} \, dy \, d\xi \\
&= \frac{1}{(2\pi)^{2n+1}} \int e^{i(x'-y')\cdot\xi'} \left[\frac{iC_{-2l}}{2\pi\xi_0} \int \frac{e^{ix_{2n+1}\xi_{2n+1}} d\xi_{2n+1}}{\left[\left(\frac{1}{2}x_{2n+1}\xi_0^{1/2}\right)^2 + |\vec{v}|^2 + \left(\frac{\xi_{2n+1}}{\xi_0^{1/2}}\right)^2\right]^l} \right] v_1(y') \, dy' \, d\xi'
\end{aligned} \tag{5.30}$$

after computing the effect of $\delta(y_{2n+1})$. The zeroth and the first trace of the bracket above will give us the entries of the second column the Calderón projector. We use the same technique we used for $b_{-1,-2}$: let $\beta^2 = \left(\frac{1}{2}x_{2n+1}\xi_0^{1/2}\right)^2 + |\vec{v}|^2$, $\beta > 0$; change the variable of integration, $\frac{\xi_{2n+1}}{\xi_0^{1/2}} = \eta$, $d\xi_{2n+1} = \xi_0^{1/2} d\eta$; shift the contour of integration in the upper complex ξ_{2n+1} plane; use residues. After computing the residues we will just have to multiply by $\frac{iC_{-2l}}{2\pi\xi_0} 2\pi i \xi_0^{1/2} = \frac{-C_{-2l}}{\xi_0^{1/2}}$. We have one pole of order l for the integrand:

$$F(z) = \frac{e^{ix_{2n+1}\xi_0^{1/2} z}}{(\beta^2 + \eta^2)^l} \tag{5.31}$$

at $z = i\beta$. We use the formula for such a residue:

$$\operatorname{Res}_{z=z_0} F(z) = \frac{1}{(l-1)!} \frac{\partial^{l-1}}{\partial z^{l-1}} \left[(z-z_0)^l F(z)\right]\Big|_{z=z_0}. \tag{5.32}$$

If we differentiate $l-1$ times in z in:

$$\frac{1}{(l-1)!}(z-z_0)^l F(z) = \frac{1}{(l-1)!} e^{ix_{2n+1}\xi_0^{1/2} z} (z+i\beta)^{-l} \tag{5.33}$$

we obtain

$$\frac{1}{(l-1)!} \sum_{k=0}^{l-1} \binom{l-1}{k} (ix_{2n+1}\xi_0^{1/2})^k e^{ix_{2n+1}\xi_0^{1/2} z} \frac{\partial^{l-1-k}}{\partial z^{l-1-k}} \left[(z+i\beta)^{-l}\right] \tag{5.34}$$

For the $(0,1)$ entry of the Calderón projector we have to set first $z = i\beta$ and then $x_{2n+1} = 0$ in (5.34). Only the term with $k = 0$ will survive:

$$\left\{ \frac{1}{(l-1)!} \binom{l-1}{0} e^{-x_{2n+1}\xi_0^{1/2}\beta} (-1)^{l-1} \frac{(2l-2)!}{(l-1)!} (2i\beta)^{-2l+1} \right\} \Big|_{x_{2n+1}=0} \tag{5.35}$$

so the (0,1) entry is:

$$c_{0,1} = \frac{-C_{-2l}}{\xi_0^{1/2}} \frac{1}{(l-1)!} (-1)^{l-1} \frac{(2l-2)!}{(l-1)!} (2i|\vec{v}|)^{-2l+1}. \tag{5.36}$$

For the (1,1) entry of the Calderón projector we have to set $z = i\beta$, differentiate in x_{2n+1} (using D, not ∂) and then set $x_{2n+1} = 0$ in (5.34). The only terms that might not vanish when setting $x_{2n+1} = 0$ are those with $k = 0$ and $k = 1$. We analyze them next; their coefficient will be $\frac{-C_{-2l}}{\xi_0^{1/2}}$.

In the term obtained for $k = 0$ in (5.34) we set $z = i\beta$ and we take $D_{x_{2n+1}}$. We obtain:

$$\frac{1}{(l-1)!} \binom{l-1}{0} e^{-x_{2n+1}\xi_0^{1/2}\beta} \left\{ D_{x_{2n+1}} (-x_{2n+1}\xi_0^{1/2}\beta) (-1)^{l-1} \frac{(2l-2)!}{(l-1)!} (2i\beta)^{-2l+1} + \right.$$
$$\left. (-1)^{l-1} \frac{(2l-2)!}{(l-1)!} D_{x_{2n+1}} [(2i\beta)^{-2l+1}] \right\}. \tag{5.37}$$

Since $\partial_{x_{2n+1}}\beta|_{x_{2n+1}=0} = 0$, the second term in the braces vanishes and we obtain for this term coming from $k = 0$:

$$c_{1,1}^0 = -C_{-2l} \frac{1}{(l-1)!} i|\vec{v}| (-1)^{l-1} \frac{(2l-2)!}{(l-1)!} (2i|\vec{v}|)^{-2l+1}. \tag{5.38}$$

In the term obtained for $k = 1$ in (5.34) we also set $z = i\beta$, we apply $D_{x_{2n+1}}$ and then we set $x_{2n+1} = 0$ to obtain:

$$c_{1,1}^1 = -C_{-2l} \frac{1}{(l-1)!} \binom{l-1}{1} (-1)^{l-2} \frac{(2l-3)!}{(l-1)!} (2i|\vec{v}|)^{-2l+2}. \tag{5.39}$$

The sum of these two terms, $c_{1,1}^0 + c_{1,1}^1$ is the $c_{1,1}$ entry of the Calderón projector associated to $b_{-1,-2l}$. It vanishes.

In order to compute now the first column of the Calderón projector associated to

$b_{-1,-2l}$ we write:

$$\begin{aligned}&\mathrm{op}(b_{-1,-2l})(iv_0 \otimes D_{y_{2n+1}}\delta(y_{2n+1}))(x', x_{2n+1}) = \\ &= \frac{1}{(2\pi)^{2n+2}} \int e^{i(x-y)\cdot\xi} \frac{1}{\xi_0} \frac{iC_{-2l} v_0(y') \otimes D_{y_{2n+1}}\delta(y_{2n+1})}{\left[\left(\frac{x_{2n+1}+y_{2n+1}}{2}\xi_0^{1/2}\right)^2 + |\vec{v}|^2 + \left(\frac{\xi_{2n+1}}{\xi_0^{1/2}}\right)^2\right]^l} \, dy \, d\xi. \end{aligned} \tag{5.40}$$

We compute first the action of $D_{y_{2n+1}}\delta(y_{2n+1})$:

$$\begin{aligned}&\int e^{-iy_{2n+1}\xi_{2n+1}} \frac{D_{y_{2n+1}}\delta(y_{2n+1})}{\left[\left(\frac{x_{2n+1}+y_{2n+1}}{2}\xi_0^{1/2}\right)^2 + |\vec{v}|^2 + \left(\frac{\xi_{2n+1}}{\xi_0^{1/2}}\right)^2\right]^l} \, dy_{2n+1} \\ &= \frac{-1}{i} \int \partial_{y_{2n+1}} \left\{ \frac{e^{-iy_{2n+1}\xi_{2n+1}}}{\left[\left(\frac{x_{2n+1}+y_{2n+1}}{2}\xi_0^{1/2}\right)^2 + |\vec{v}|^2 + \left(\frac{\xi_{2n+1}}{\xi_0^{1/2}}\right)^2\right]^l} \right\} \delta(y_{2n+1}) \, dy_{2n+1} \\ &\qquad = \frac{-1}{i} \left\{ -i\xi_{2n+1} \left[\left(\frac{x_{2n+1}}{2}\xi_0^{1/2}\right)^2 + |\vec{v}|^2 + \left(\frac{\xi_{2n+1}}{\xi_0^{1/2}}\right)^2\right]^{-l} \right. \\ &\qquad\qquad \left. - \, l \left[\left(\frac{x_{2n+1}}{2}\xi_0^{1/2}\right)^2 + |\vec{v}|^2 + \left(\frac{\xi_{2n+1}}{\xi_0^{1/2}}\right)^2\right]^{-l-1} \frac{x_{2n+1}}{2}\xi_0 \right\}. \end{aligned} \tag{5.41}$$

Similarly to (5.30) we can write this part of the Calderón projector (the part acting on $v_0(y')$) as:

$$\mathrm{op}(b_{-1,-2l})(iv_0 \otimes D_{y_{2n+1}}\delta(y_{2n+1}))(x', x_{2n+1}) =$$

$$= \frac{1}{(2\pi)^{2n+1}} \int e^{i(x'-y')\cdot\xi'} \left\{ \frac{iC_{-2l}}{2\pi\xi_0} \int \frac{e^{ix_{2n+1}\xi_{2n+1}}\xi_{2n+1}}{\left[\left(\frac{1}{2}x_{2n+1}\xi_0^{1/2}\right)^2 + |\vec{v}|^2 + \left(\frac{\xi_{2n+1}}{\xi_0^{1/2}}\right)^2\right]^l} d\xi_{2n+1} \right.$$

$$\left. + \frac{lC_{-2l}}{4\pi} \int \frac{e^{ix_{2n+1}\xi_{2n+1}}x_{2n+1}}{\left[\left(\frac{1}{2}x_{2n+1}\xi_0^{1/2}\right)^2 + |\vec{v}|^2 + \left(\frac{\xi_{2n+1}}{\xi_0^{1/2}}\right)^2\right]^{l+1}} d\xi_{2n+1} \right\} v_0(y')\,dy'\,d\xi'. \tag{5.42}$$

We will compute now the zeroth and the first order trace of each of the two terms inside the braces.

The first term is:

$$\frac{iC_{-2l}}{2\pi} \int e^{ix_{2n+1}\xi_0^{1/2}\eta} \frac{\eta}{(\beta^2+\eta^2)^l} d\eta \tag{5.43}$$

after introducing the variables β and η and shifting the contour; $i\beta$ is again a pole of order l and we use the same formula for residues:

$$\frac{1}{(l-1)!}\frac{\partial^{l-1}}{\partial z^{l-1}}\left[e^{ix_{2n+1}\xi_0^{1/2}z}\frac{z}{(z+i\beta)^l}\right]\Bigg|_{z=i\beta}. \tag{5.44}$$

We have a first splitting in order to apply the general Leibniz formula and obtain:

$$\frac{1}{(l-1)!}\sum_{k=0}^{l-1}\binom{l-1}{k}(ix_{2n+1}\xi_0^{1/2})^k e^{ix_{2n+1}\xi_0^{1/2}z}\frac{\partial^{l-1-k}}{\partial z^{l-1-k}}\left[\frac{z}{(z+i\beta)^l}\right] \tag{5.45}$$

and for the zeroth trace we will only needed to compute the term with $k = 0$, while for the first trace we will only need to compute the terms with $k = 0$ and $k = 1$, the only ones that don't obviously vanish when setting $x_{2n+1} = 0$.

We compute first the zeroth trace of the integral, so $k = 0$. We use the general Leibniz formula for the $l-1$ derivative of the fraction (the last factor in (5.45) with $k = 0$) and we have:

$$\frac{1}{(l-1)!}e^{ix_{2n+1}\xi_0^{1/2}z}\sum_{j=0}^{l-1}\binom{l-1}{j}\frac{\partial^j}{\partial z^j}(z)\frac{\partial^{l-1-j}}{\partial z^{l-1-j}}\left[(z+i\beta)^{-l}\right]. \tag{5.46}$$

The only derivatives of z which won't make it vanish are the ones for $j = 0$ and $j = 1$. For $j = 0$ we obtain after setting $z = i\beta$

$$\begin{aligned} c'_{0,0} &= \frac{1}{(l-1)!} e^{-x_{2n+1}\xi_0^{1/2}\beta}(i\beta)(-1)^{l-1}\frac{(2l-2)!}{(l-1)!}(2i\beta)^{-2l+1}\Big|_{x_{2n+1}=0} \\ &= \frac{1}{(l-1)!}(i|\vec{v}|)(-1)^{l-1}\frac{(2l-2)!}{(l-1)!}(2i|\vec{v}|)^{-2l+1}. \end{aligned} \tag{5.47}$$

For $j = 1$ we obtain after setting $z = i\beta$

$$\begin{aligned} c''_{0,0} &= \frac{1}{(l-1)!} e^{-x_{2n+1}\xi_0^{1/2}\beta}\binom{l-1}{1}(-1)^{l-2}\frac{(2l-3)!}{(l-1)!}(2i\beta)^{-2l+2}\Big|_{x_{2n+1}=0} \\ &= \frac{1}{(l-1)!}\binom{l-1}{1}(-1)^{l-2}\frac{(2l-3)!}{(l-1)!}(2i|\vec{v}|)^{-2l+2}. \end{aligned} \tag{5.48}$$

The sum of these two last results, $c'_{0,0} + c''_{0,0} = c^0_{0,0} = 0$ is the zeroth trace of the first of two integrals that we had to compute. We will see later that the zeroth trace of the second integral is zero, so we can state that $c_{0,0} = 0$.

For the first trace of the first integral, as noted above, at the first splitting we need to take into account the terms with $k = 0$ and $k = 1$ in (5.45). For $k = 0$ we have again (5.46), and we are only interested in the terms with $j = 0$ and $j = 1$ of the second splitting. For $j = 0$ we obtain the first line in (5.47) but instead of setting $x_{2n+1} = 0$ we compute $D_{x_{2n+1}}$ of that and then set $x_{2n+1} = 0$. We obtain:

$$\frac{1}{(l-1)!}(-\frac{1}{i}\xi_0^{1/2}|\vec{v}|)(i|\vec{v}|)(-1)^{l-1}\frac{(2l-2)!}{(l-1)!}((2i|\vec{v}|)^{-2l+1} \tag{5.49}$$

while for $j = 1$ by doing the same we obtain

$$\frac{1}{(l-1)!}(-\frac{1}{i}\xi_0^{1/2}|\vec{v}|)\binom{l-1}{1}(-1)^{l-2}\frac{(2l-3)!}{(l-1)!}((2i|\vec{v}|)^{-2l+2}. \tag{5.50}$$

These two terms also cancel each other. So for $k = 0$ there is no contribution from the first integral to the first order trace.

Going back to (5.45), splitting and analyzing the case $k = 1$ we obtain:

$$\frac{1}{(l-1)!}\binom{l-1}{1}(ix_{2n+1}\xi_0^{1/2})e^{ix_{2n+1}\xi_0^{1/2}z}\sum_{j=0}^{l-2}\binom{l-2}{j}\frac{\partial^j}{\partial z^j}(z)\frac{\partial^{l-2-j}}{\partial z^{l-2-j}}\left[(z+i\beta)^{-l}\right]. \tag{5.51}$$

We are again interested in the terms obtained for $j = 0$ and $j = 1$, the only ones that could give us nonzero first traces. If $j = 0$ after computing the derivatives in z and setting $z = i\beta$ we obtain:

$$\frac{1}{(l-1)!}\binom{l-1}{1}(ix_{2n+1}\xi_0^{1/2})e^{-x_{2n+1}\xi_0^{1/2}\beta}\binom{l-2}{0}(i\beta)(-1)^{l-2}\frac{(2l-3)!}{(l-1)!}(2i\beta)^{-2l+2}. \tag{5.52}$$

Taking $D_{x_{2n+1}}$ of this and setting $x_{2n+1} = 0$ we obtain

$$\frac{1}{(l-1)!}\binom{l-1}{1}\xi_0^{1/2}(i|\vec{v}|)(-1)^{l-2}\frac{(2l-3)!}{(l-1)!}(2i|\vec{v}|)^{-2l+2}. \tag{5.53}$$

If $j = 1$ we obtain similarly:

$$\frac{1}{(l-1)!}\binom{l-1}{1}\xi_0^{1/2}\binom{l-2}{1}(-1)^{l-3}\frac{(2l-4)!}{(l-1)!}(2i|\vec{v}|)^{-2l+3}. \tag{5.54}$$

Summing these two terms we obtain that the part of the $(1, 0)$ entry of the Calderón projector arising from the first integral is:

$$c^I_{1,0} = -\frac{1}{2}C_{-2l}\frac{1}{(l-1)!}\binom{l-1}{1}\xi_0^{1/2}(-1)^{l-2}\frac{(2l-4)!}{(l-1)!}(2i|v|)^{-2l+3} \tag{5.55}$$

The second integral was, in (5.42),

$$\frac{lC_{-2l}}{4\pi}\int\frac{e^{ix_{2n+1}\xi_{2n+1}}x_{2n+1}}{\left[\left(\frac{1}{2}x_{2n+1}\xi_0^{1/2}\right)^2+|\vec{v}|^2+\left(\frac{\xi_{2n+1}}{\xi_0^{1/2}}\right)^2\right]^{l+1}}d\xi_{2n+1}.$$

We call again $\beta^2 = \left(\frac{1}{2}x_{2n+1}\xi_0^{1/2}\right)^2 + |\vec{v}|^2$, $\beta > 0$; change the variable of integration, $\frac{\xi_{2n+1}}{\xi_0^{1/2}} = \eta$, $d\xi_{2n+1} = \xi_0^{1/2}d\eta$; shift the contour of integration in the upper complex ξ_{2n+1} plane; use residues. This time we have an integrand:

$$F(z) = \frac{e^{ix_{2n+1}\xi_0^{1/2}z}x_{2n+1}}{(\beta^2+z^2)^{l+1}}. \tag{5.56}$$

A factor of $\xi_0^{1/2}2\pi i$ will arise as a result of the change of variables and using residues. The residue itself will be:

$$\frac{x_{2n+1}}{l!}\frac{\partial^l}{\partial z^l}\left[\frac{e^{ix_{2n+1}\xi_0^{1/2}z}}{(z+i\beta)^{l+1}}\right]\Bigg|_{z=i\beta} \tag{5.57}$$

It is clear that the x_{2n+1} factor will make the zeroth trace vanish; for the first trace, the only nonzero term will be:

$$\frac{1}{i}\frac{1}{l!}\frac{\partial^l}{\partial z^l}\left[\frac{e^{ix_{2n+1}\xi_0^{1/2}z}}{(z+i\beta)^{l+1}}\right]\Bigg|_{z=i\beta} \tag{5.58}$$

This is in fact something we computed before (for l instead of $l+1$) when we computed the $(0,1)$ entry of the Calderón projector; using that result we obtain

$$c^{II}_{1,0} = \frac{lC_{-2l}\xi_0^{1/2}}{2}\frac{1}{l!}(-1)^l\frac{(2l)!}{l!}(2i|\vec{v}|)^{-2l-1}. \tag{5.59}$$

After all these computations, we reverse from the temporary notation we introduced in (5.29) and we state:

Proposition 5.4. *The Calderón projector associated to the term $b_{-1,-2-4l}$ of the form (4.50) in the expansion of b_{-1} is a 2×2 matrix with entries:*

$$\begin{aligned} c_{0,0} &= 0 \\ c_{0,1} &= \frac{-C_{-2-4l}}{\xi_0^{1/2}}\frac{1}{(2l)!}\frac{(4l)!}{(2l)!}(2i|\vec{v}|)^{-1-4l} \\ c_{1,0} &= \frac{C_{-2-4l}}{2}\frac{\xi_0^{1/2}}{(2l-1)!}\frac{(4l-2)!}{(2l)!}(2i|\vec{v}|)^{1-4l} \\ &\quad - \frac{C_{-2-4l}}{2}\frac{\xi_0^{1/2}}{(2l)!}\frac{(4l+4)!}{(2l+1)!}(2i|\vec{v}|)^{-3-4l} \\ c_{1,1} &= 0 \end{aligned} \tag{5.60}$$

As we saw in the previous section, there is a lower order entry $c_{1.0}$ being carried over from the top order term of the expansion; for all the other terms of the expansion there also is a $c_{1.0}$ being carried over to the next term of the expansion in the formula above, and there is one coming in from the previous term. Taking this into account, we can state:

Theorem 5.5. *The Calderón projector associated to the term of top order -1 in ξ of the parametrix of $\mathcal{L}$ has an expansion in v with top order term (5.27):*

$$\begin{pmatrix} \frac{1}{2} & \frac{-i}{2\xi_0^{1/2}|\vec{v}|} \\ \frac{i}{2}\xi_0^{1/2}|\vec{v}| & \frac{1}{2} \end{pmatrix}$$

with the following terms of the expansion (for $l \geq 1$) being:

$$c_{-1,-2-4l} = \begin{pmatrix} 0 & C'_{-2-4l}\xi_0^{-1/2}|\vec{v}|^{-1-4l} \\ C''_{-2-4l}\xi_0^{1/2}|\vec{v}|^{1-4l} & 0 \end{pmatrix} \tag{5.61}$$

5.6 Traces to the Boundary of Operators of Class $\Psi^{m,k}$

When we computed, in the last two chapters, the symbol of the parametrix Q of $\mathcal{L}$ and then we showed that its trace to the boundary is an operator of the same type $\Psi^{m,k}(\mathbb{R}^{2n+1}, \Sigma)$, we explicitly used the form of $\mathcal{L}$. However, we emphasized that the symbol of Q satisfies condition (2.1') and that the computations of the Calderón projector corresponding to $\mathcal{L}$ and Q depend essentially on property (2.1'), and not on the specific expressions of the operators.

By using similar computations it is easy to prove a more general theorem:

Theorem 5.6. *If $Q \in \Psi^{m,k}(\mathbb{R}^{2n+2}, \Sigma)$ satisfies condition (2.1') then the operator $v \to \gamma_0\left((Q\widetilde{P}v)|_\Omega\right)$ is in $\Psi^{m+1,k}_{cl}(\mathbb{R}^{2n+1}, \Sigma)$ on the boundary.*

5.7 The Calderón Projector for the Classical Part of the Parametrix Q

We have concentrated so far on the microlocal behavior of the symbol of Q and its trace to the boundary in a conic neighborhood around Σ. However, we also need to know the behavior of the simpler, classical part of the symbol of Q outside such a cone. We do this now.

When analyzing the classical part of the symbol of our parametrix we are situating ourselves outside a cone containing Σ, the characteristic set of $\mathcal{L}$. In such a region $\sigma(\mathcal{L})$ is nonzero, and we can write:

$$\sigma(Q)(x,\xi) = q(x,\xi) = 1/\sigma(\mathcal{L})(x,\xi). \tag{5.62}$$

In order to find the Calderón projector of this classical part we have to do computations similar to those in Section 5.4 in this simpler case. The method and the pseudodifferential operators used here are all classical. Recall again formula (5.5) we obtained:

$$\widetilde{\mathcal{L}}((v_0, v_1)) = (iv_0 \otimes D_{y_{2n+1}}\delta(y_{2n+1})) + (iv_1 \otimes \delta(y_{2n+1}))$$

This Calderón projector will be a 2 × 2 matrix of classical pseudo-differential operators. In order to compute the second column of this matrix (the part depending on v_1) we need:

$$\begin{aligned} &op(q)(iv_1 \otimes \delta(y_{2n+1}))(x', x_{2n+1}) \\ &\qquad = \frac{1}{(2\pi)^{2n+2}} \int e^{i(x-y)\cdot\xi} \frac{-1}{\sum_{j=0}^{2n+1} \sigma_j^2(x,\xi)} iv_1(y')\delta(y_{2n+1})\, dy\, d\xi \\ &\qquad = \frac{1}{(2\pi)^{2n+1}} \int e^{i(x'-y')\cdot\xi'} \left[\frac{-i}{2\pi} \int \frac{e^{ix_{2n+1}\xi_{2n+1}}}{\sum_{j=0}^{2n+1} \sigma_j^2(x,\xi)}\, d\xi_{2n+1}\right] v_1(y')\, dy'\, d\xi'. \end{aligned} \tag{5.63}$$

The zeroth and the first order traces of the bracket above will give us $c_{0,1}$ and $c_{1,1}$ respectively. Call $\sum_{j=0}^{2n} \sigma_j^2(x,\xi) = \beta^2$, $\beta > 0$, split

$$\frac{1}{\beta^2 + \xi_{2n+1}^2} = \frac{1}{2i\beta}\left(\frac{1}{\xi_{2n+1} - i\beta} - \frac{1}{\xi_{2n+1} + i\beta}\right), \tag{5.64}$$

shift the contour to $\Im\xi_{2n+1} > 0$ and compute (with residues) the integral given by the first partial fraction (the second fraction has no poles in $\Im\xi_{2n+1} > 0$).

We obtain that the bracket above is

$$\begin{aligned} &\frac{-1}{4\pi\beta} \int \frac{e^{ix_{2n+1}}}{\xi_{2n+1} - i\beta}\, d\xi_{2n+1} \\ &\qquad = \frac{-i}{2\beta} e^{-x_{2n+1}\beta}. \end{aligned} \tag{5.65}$$

By taking the zero-th trace of this we obtain:

$$c_{0,1} = \frac{-i}{2\left(\sum_{j=0}^{2n} \sigma_j^2(x,\xi)\right)^{1/2}} \tag{5.66}$$

and by taking the first trace:

$$c_{1,1} = \frac{1}{2}. \tag{5.67}$$

Similarly, for the first column of this matrix (the part depending on v_0), we start from:

$$op(q)(iv_0 \otimes D_{y_{2n+1}}\delta(y_{2n+1}))(x', x_{2n+1}) =$$
$$= \frac{1}{(2\pi)^{2n+2}} \int e^{i(x-y)\cdot\xi} \frac{-1}{\sum_{j=0}^{2n+1} \sigma_j^2(x,\xi)} iv_0(y') \otimes \frac{1}{i}\partial_{y_{2n+1}}\delta(y_{2n+1})\, dy\, d\xi$$
$$= \frac{1}{(2\pi)^{2n+1}} \int e^{i(x'-y')\cdot\xi'}$$
$$\left[\frac{-1}{2\pi} \int \frac{e^{i(x_{2n+1}-y_{2n+1})\xi_{2n+1}}\partial_{y_{2n+1}}\delta(y_{2n+1})}{\sum_{j=0}^{2n+1} \sigma_j^2(x,\xi)}\, dy_{2n+1}\, d\xi_{2n+1}\right] v_0(y')\, dy'\, d\xi'. \tag{5.68}$$

The zeroth and the first order traces of the bracket above will give us $c_{0,0}$ and $c_{1,0}$, respectively. First we eliminate the derivative acting on $\delta(y_{2n+1})$:

$$\begin{aligned} &\int e^{-iy_{2n+1}\xi_{2n+1}}\partial_{y_{2n+1}}\delta(y_{2n+1})\, dy_{2n+1} \\ &\quad = -\int \partial_{y_{2n+1}}\left(e^{-iy_{2n+1}\xi_{2n+1}}\right)\delta(y_{2n+1})\, dy_{2n+1} \\ &\quad = i\xi_{2n+1}. \end{aligned} \tag{5.69}$$

Calling again $\sum_{j=0}^{2n} \sigma_j^2(x,\xi) = \beta^2$, with $\beta > 0$, splitting

$$\frac{\xi}{\beta^2 + \xi_{2n+1}^2} = \frac{1}{2}\left(\frac{1}{\xi_{2n+1} + i\beta} + \frac{1}{\xi_{2n+1} - i\beta}\right), \tag{5.70}$$

shifting the contour to $\Im\xi_{2n+1} > 0$ and computing (with residues) the integral given by the second partial fraction (the first fraction has no poles in $\Im\xi_{2n+1} > 0$), we obtain the value of the bracket:

$$\frac{1}{2}e^{-x_{2n+1}\beta} \tag{5.71}$$

and we have to take now the zeroth and the first order traces of this. The zeroth order trace gives

$$c_{0,0} = \frac{1}{2} \tag{5.72}$$

while the first order trace gives

$$c_{1.0} = \frac{i}{2}\left(\sum_{j=0}^{2n} \sigma_j^2(x,\xi)\right)^{1/2}. \tag{5.73}$$

We have obtained:

Proposition 5.7. *The Calderón projector of the classical part of the parametrix Q of $\mathcal{L}$ has principal symbol:*

$$c = \begin{pmatrix} \frac{1}{2} & \frac{-i}{2\left(\sum_{j=0}^{2n} \sigma_j^2(x,\xi)\right)^{1/2}} \\ \frac{i}{2}\left(\sum_{j=0}^{2n} \sigma_j^2(x,\xi)\right)^{1/2} & \frac{1}{2} \end{pmatrix}. \tag{5.74}$$

CHAPTER 6

THE CALDERÓN PROJECTOR OF THE Σ SYMBOL

6.1 The Σ Symbol

After studying the Calderón projector for the classical part of the symbol of $\mathcal{L}$ we will now do the same for its Σ symbol. We saw that for a fixed point $\rho^0 = (x^0, \xi^0) \in \Sigma$, the Σ symbol is a symbol defined on the fiber at ρ^0 of $T_{\rho^0}\Sigma^\perp$ which can also be interpreted as an operator, namely $\mathcal{H}$, the harmonic oscillator Hamiltonian. We restrict ourselves to a given fiber of $T\Sigma^\perp$ at ρ^0 and we study the Calderón projector of this operator. We will use for this another type of Sobolev spaces, which we proceed to define in the next section.

6.2 Sobolev Spaces $H^s_{\mathcal{H}}(\mathbb{R}^n)$

Consider, in $\mathbb{R}^n$, the harmonic oscillator Hamiltonian

$$\mathcal{H} = -\Delta + |x|^2 = -\sum_{j=1}^{n} \partial^2_{x_j} + \sum_{j=1}^{n} x_j^2. \tag{6.1}$$

Similarly to the way classical Sobolev spaces are defined using the operator $1 + \Delta$, we define $\mathcal{H}$-Sobolev spaces using this $\mathcal{H}$, for any $s \in \mathbb{R}$ by

Definition 6.1.

$$H^s_{\mathcal{H}}(\mathbb{R}^n) = \{u \in \mathcal{S}' \mid (-\Delta + |x|^2)^{s/2} u \in L^2(\mathbb{R}^n)\}. \tag{6.2}$$

The norm in this space is:

$$\|u\|_s = \|(-\Delta + |x|^2)^{s/2} u\|_{L^2(\mathbb{R}^n)}. \tag{6.3}$$

We have seen in Section 4.3 that the operator $\mathcal{H}$ is very well known. Taking advantage of the spectral theorem for $\mathcal{H}$ we can write expansions in the eigenfunctions of $\mathcal{H}$ for $u \in \mathcal{S}'$:

$$u = \sum_k \langle u, \varphi_k \rangle \varphi_k \tag{6.4}$$

and give equivalent definitions for the space and the norm.

Definition 6.2. The spaces $H^s_{\mathcal{H}}(\mathbb{R}^n)$ can also be written as:

$$\{u \in \mathcal{S}' \mid \sum_k \lambda_k^{s/2} \langle u, \varphi_k \rangle \varphi_k \in L^2(\mathbb{R}^n). \tag{6.5}$$

or

$$\{u \in \mathcal{S}' \mid \sum_k \lambda_k^s |\langle u, \varphi_k \rangle|^2 < \infty\}. \tag{6.6}$$

The norm in this space can also be written as:

$$\|u\|_s^2 = \sum_k \lambda_k^s |\langle u, \varphi_k \rangle|^2. \tag{6.7}$$

Defining these spaces by using the same operator $\mathcal{H}$ whose Calderón projector we want to study is natural at this point since we are in the Heisenberg group setup and the analog of the classical Laplacian Δ is $\mathcal{H}$; the spectral theorem applied to $\mathcal{H}$ will also prove to be very useful. Otherwise, these spaces are very similar to the classical Sobolev spaces. We can prove, for example, a similar trace theorem as in the classical Sobolev spaces.

Theorem 6.3. *Let $s \in \mathbb{R}, s > \frac{1}{2}$. The trace operator $\gamma : C_0^\infty(\mathbb{R}^{n+1}) \to C_0^\infty(\mathbb{R}^n)$ defined by $\gamma u(x') = u(x', 0)$ extends into a continuous linear operator (again denoted by γ) from $H^s_{\mathcal{H}}(\mathbb{R}^{n+1})$ into $H^{s-1/2}_{\mathcal{H}}(\mathbb{R}^n)$.*

Proof. Take $u \in C_0^\infty(\mathbb{R}^{n+1})$; using the expansion of u in the basis φ_k we can write

$$u(x',0) = \sum_{k',k_{n+1}} \langle u, \varphi_{k'} \otimes \varphi_{k_{n+1}} \rangle \varphi_{k'}(x') \otimes \varphi_{k_{n+1}}(0) \tag{6.8}$$

which means, as in the case of the classical Fourier transform, from which we borrow the notation:

$$(\widehat{\gamma u})(k') = \sum_{k_{n+1}} \langle u, \varphi_{k'} \otimes \varphi_{k_{n+1}} \rangle \varphi_{k_{n+1}}(0). \tag{6.9}$$

The present proof depends essentially on the following relation between the weights used in defining the norms in $H^s_{\mathcal{H}}(\mathbb{R}^{n+1})$ and $H^{s-1/2}_{\mathcal{H}}(\mathbb{R}^n)$ respectively:

$$\sum_{k_{n+1}} \lambda^{-s}_{(k',k_{n+1})} |\varphi_{k_{n+1}}(0)|^2 \leq C_s \lambda^{-s+1/2}_{k'} \qquad \text{if } s > 1/2. \tag{6.10}$$

(Note that this inequality is very similar to

$$\int (1 + |\xi'|^2 + |\xi_{n+1}|^2)^{-s} \, d\xi_{n+1} = C_s (1 + |\xi'|^2)^{-s+1/2} \qquad \text{if } s > 1/2 \tag{6.11}$$

which is used for a very natural and meaningful proof of the similar trace theorem in classical Sobolev spaces, as in [8], page 113.) Assuming

(6.10), we write

$$(\widehat{\gamma u})(k') = \sum_{k_{n+1}} \lambda^{-s/2}_{(k',k_{n+1})} \varphi_{k_{n+1}}(0) \lambda^{s/2}_{(k',k_{n+1})} \langle u, \varphi_{k'} \otimes \varphi_{k_{n+1}} \rangle \tag{6.12}$$

and by using the Cauchy-Schwartz inequality:

$$\begin{aligned} |(\widehat{\gamma u})(k')|^2 &\leq \left(\sum_{k_{n+1}} \lambda^{-s}_{(k',k_{n+1})} |\varphi_{k_{n+1}}(0)|^2 \right) \left(\sum_{k_{n+1}} \lambda^{s}_{(k',k_{n+1})} |\langle u, \varphi_{k'} \otimes \varphi_{k_{n+1}} \rangle|^2 \right) \\ &\leq C_s \lambda^{-s+1/2}_{k'} \left(\sum_{k_{n+1}} \lambda^{s}_{(k',k_{n+1})} |\langle u, \varphi_{k'} \otimes \varphi_{k_{n+1}} \rangle|^2 \right) \end{aligned} \tag{6.13}$$

hence

$$|(\widehat{\gamma u})(k')|^2 \lambda^{s-1/2}_{k'} \leq C_s \left(\sum_{k_{n+1}} \lambda^{s}_{(k',k_{n+1})} \widehat{u}(k', k_{n+1}) \right) \tag{6.14}$$

and now, by summing in k' we obtain

$$\|\gamma u\|^2_{s-1/2} \leq C_s \|u\|^2_s \tag{6.15}$$

which proves the theorem, modulo the assumption.

Now, in order to prove the important relation (6.10), considering that $\lambda_{(k',k_{n+1})} = \lambda_{k'} + \lambda_{k_{n+1}}$, that $\varphi_{k_{n+1}}(0) \neq 0$ only if $k_{n+1} = 2m$ and in that case $\varphi^2_{k_{2m}}(0) = ((2m)!)/(2^{2m}(m!)^2) \sim (\pi m)^{-1/2}$, by using the Stirling approximation, and knowing that $\lambda_{2m} = 4m+1$, it is enough to show:

$$\lambda_{k'}^{-1/2} \sum_m \left(\frac{\lambda_{k'}}{\lambda_{k'} + 4m + 1} \right)^s \frac{1}{m^{1/2}} \leq C_s \tag{6.16}$$

with C_s independent of $\lambda_{k'}$. For this, using the inequality between a lower Darboux sum and the integral and making a change of variable $m = Ax$ in the integral, we obtain

$$\begin{aligned} \sum_m \left(\frac{A}{A + 4m + 1} \right)^s \frac{1}{m^{1/2}} &\leq \int_0^\infty \left(\frac{A}{A + 4m + 1} \right)^s \frac{1}{m^{1/2}}\, dm \\ &= A^{1/2} \int_0^\infty \frac{1}{(1 + 4x + 1/A)^s} \frac{1}{x^{1/2}}\, dx \\ &\leq A^{1/2} \int_0^\infty \frac{1}{(1+4x)^s x^{1/2}}\, dx \end{aligned} \tag{6.17}$$

where the last integral is convergent and its value, C_s, is independent of A. Hence

$$\lambda_{k'}^{-1/2} \sum_m \left(\frac{\lambda_{k'}}{\lambda_{k'} + 4m + 1} \right)^s \frac{1}{m^{1/2}} \leq \lambda_{k'}^{-1/2} \lambda_{k'}^{1/2} \int_0^\infty \frac{1}{(1+4x)^s x^{1/2}}\, dx = C_s \tag{6.18}$$

which ends the proof of the theorem. □

Remark 6.4. The present proof shows that traces are continuous from the $\mathcal{H}$-Sobolev space of the whole $\mathbb{R}^{n+1}$ to the $\mathcal{H}$-Sobolev space of $\mathbb{R}^n$, with loss of half an order of regularity. Its idea is different from the idea in the proofs in Section 3.2. Using the idea of those proofs one can prove that traces are continuous from the $\mathcal{H}$-Sobolev spaces of a half space, $H^s_{\mathcal{H}}(\mathbb{R}^{n+1}_+)$ (defined as restrictions to $\mathbb{R}^{n+1}_+$ of elements of $H^s_{\mathcal{H}}(\mathbb{R}^{n+1})$) to the $\mathcal{H}$-Sobolev spaces of the boundary, $H^{s-1/2}_{\mathcal{H}}(\mathbb{R}^n)$, with the same 1/2 loss of regularity.

CHAPTER 7

APPLICATION TO THE BOUNDARY VALUE PROBLEM

7.1 The Boundary Value Problem

Let Ω be $\mathbb{R}^{2n+2}_+$, an open subset of $\mathbb{R}^{2n+2}$ and Σ the closed embedded symplectic submanifold of $T^*(\partial\Omega)$ of codimension $2n$ already defined. The projection $\Sigma \to \partial\Omega$ is a submersion. Suppose we have

$$\begin{aligned} \mathcal{P} : C^\infty(\bar{\Omega}) &\to C^\infty(\bar{\Omega}) \times C^\infty(\partial\Omega, \mathbb{C}) \\ u \to &\ (\mathcal{L}u, B\gamma u). \end{aligned} \tag{7.1}$$

where $\mathcal{L}$ is defined as in (1.2) and B is a 2 dimensional row vector of operators of type $\Psi^{\cdot,\cdot}(\partial\Omega, \Sigma)$, therefore suited for analyzing a Dirichlet problem. Remember that the Calderón projector associated to the problem is a 2×2 matrix of operators of type $\Psi^{\cdot,\cdot}$ whose matrix of orders is

$$\begin{pmatrix} (0,0) & (-\frac{1}{2}, -1) \\ (\frac{1}{2}, 1) & (0,0) \end{pmatrix}. \tag{7.2}$$

We will denote by b the principal symbol of B, i.e. $(b_j)_{j=1,2}$ is the matrix of principal symbols of B_j, of orders (m_j, k_j), each of them being a pair $(\sigma^s_j, \sigma^\Sigma_j)$ whose first entry is the classical part of the principal symbol of B_j and the second entry is the Σ symbol of B_j. The orders m_j are equal to $d/2, d/2-1/2$ and the orders k_j are $d', d'-1$. For every $(x', \xi') \in T^*(\partial\Omega) \setminus \Sigma$, the matrix $(\sigma_j^s)_j$ given by the classical parts of the principal symbols defines a linear mapping of $\mathbb{C}^2$ into $\mathbb{C}$. The classical Shapiro-Lopatinski conditions require $\sigma^s(B) \cdot \sigma^s(C)$ be surjective.

For every $\rho \in \Sigma$, the entries of the row vector $(\sigma_j^\Sigma)_j$ given by the Σ part of the principal symbol of $(B_j)_j$ are operators in $\Psi^{d'}(\mathbb{R}^n)$ and $\Psi^{d'-1}(\mathbb{R}^n)$, respectively. Considering the Σ part of the matrix of the Calderón projector to be an operator

$$\begin{pmatrix} H^r_{\mathcal{H}}(\mathbb{R}^n) \\ H^{r-1}_{\mathcal{H}}(\mathbb{R}^n) \end{pmatrix} \to \begin{pmatrix} H^r_{\mathcal{H}}(\mathbb{R}^n) \\ H^{r-1}_{\mathcal{H}}(\mathbb{R}^n) \end{pmatrix} \tag{7.3}$$

for some $r \in \mathbb{R}$, the Σ symbols of b will give an operator

$$\begin{pmatrix} H^r_{\mathcal{H}}(\mathbb{R}^n) \\ H^{r-1}_{\mathcal{H}}(\mathbb{R}^n) \end{pmatrix} \longrightarrow H^{r-d'}_{\mathcal{H}}(\mathbb{R}^n). \tag{7.4}$$

To the classical Shapiro-Lopatinski conditions we will add the condition that $\sigma^\Sigma(b) \cdot \sigma^\Sigma(C)$ be also surjective.

Theorem 7.1. *Assume that BC is surjective at the symbol level. Then $\mathcal{P}$ admits a right parametrix $\mathcal{Q}$. The orthogonal of $Im\mathcal{P}$ in $\mathcal{E}'(\bar{\Omega}) \times \mathcal{E}'(\partial\Omega, \mathbb{C})$ is contained in $C^\infty(\bar{\Omega}) \times C^\infty(\partial\Omega, \mathbb{C})$.*

Proof. The hypothesis implies that BC is right-elliptic in the sense of Agmon-Douglis-Nirenberg, in a sense adapted to the symbols of type $S^{\cdot,\cdot}$.

An adaptation of the Agmon-Douglis-Nirenberg theorem is straightforward. The difference is that we have to take into account the two parts of the principal symbols of the entries separately and also consider separately the different meanings of surjectivity of the classical part and of the Σ part. This adjusted theorem shows then that there exists A, a 2×1 matrix of operators with entries

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \in \begin{pmatrix} \Psi^{-d/2,-d'}(\partial\Omega) \\ \Psi^{1/2-d/2,1-d'}(\partial\Omega) \end{pmatrix} \tag{7.5}$$

such that $BC \cdot A = I + r$, $r \in \Psi^{-\infty}(\partial\Omega, \mathbb{C})$. We define $\mathcal{Q}$ by

$$\mathcal{Q}(f,g) = \{Q(f^0) + Q\widetilde{\mathcal{L}}A(g - B\gamma[Q(f^0)]|_\Omega\}|_\Omega. \tag{7.6}$$

The operator $\mathcal{Q}$ is continuous from $C^\infty(\bar{\Omega}) \times C^\infty(\partial\Omega, \mathbb{C})$ into $C^\infty(\bar{\Omega})$. Clearly $\mathcal{P}\mathcal{Q} = I + \mathcal{R}$, where $\mathcal{R}$ has two components, one in $\bar{\Omega}$ and one in $\partial\Omega$:

$$\mathcal{R}(f, g) = \Big(\{R\widetilde{\mathcal{L}}A(g - B\gamma[Q(f^0)]|_\Omega + R(f^0)\}|_\Omega,\ r(g - B\gamma[Q(f^0)]|_\Omega)\Big). \tag{7.7}$$

This is shown as in the classical case; the only difference is that the operators are now of type $\Psi^{\cdot,\cdot}$ instead of $\Psi^{\cdot}$. The operator $\mathcal{R}$ is regularizing since R and r are regularizing.

In order to prove the second part of the theorem, notice that the orthogonal of $\mathrm{Im}\mathcal{P}$ is the kernel of ${}^t\mathcal{P}$ and since ${}^t\mathcal{Q}\,{}^t\mathcal{P} = I + {}^t\mathcal{R}$, with ${}^t\mathcal{R}$ also regularizing, if (f, g) is in $\mathrm{Im}\mathcal{P}^\perp$ then $(f, g) = -\,{}^t\mathcal{R}(f, g)$ are in $C^\infty(\bar{\Omega}) \times C^\infty(\partial\Omega, \mathbb{C})$.

$\square$

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