Some Aspects of the Theory of the Adelic Zeta Function Associated to the Space of Binary Cubic Forms

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ABSTRACT

Some Aspects of the Theory of the Adelic Zeta Function Associated to the Space of Binary Cubic Forms

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This paper gives a classification of the lattices of a four dimensional vector space over a number field K, which are invariant under a certain action of $GL_2(O_K)$.

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This thesis would not have been possible without the efforts of my advisor, Boris Datskovsky, to whom I am rightly thankful.

To my family, Charles Sr., Teresita,

and Philip.

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CHAPTER 1

Introduction

In 2008, Ohno, Taniguchi and Wakatsuki obtained a classification of all $GL_2(\mathbb{Z})$-invariant lattices in $V_{\mathbb{Q}} = \mathbb{Q}^4$. In this paper, we aim to generalize their result by replacing the rational field with an arbitrary algebraic number field, K. We conclude the paper by connecting the lattices described in our main result to a zeta function developed by Datskovsky and Wright, which yields a functional equation for certain Dirichlet series attached to the lattices.

We begin our labors with a discussion of the space of binary cubic forms over K. This is necessary to describe the action of GL_2 on K^4, and to define the zeta function mentioed above. To simplify our exposition, we introduce some notation.

Notation 1.1 *Throughout this thesis, V denotes the four dimensional affine space. Also, we will let G denote the general linear group of order 2. B will represent the subgroup of G consisting of lower triangular matrices. Thus $V_K = K^4$, $G_K = GL_2(K)$, and $B_K = \{A \in G_K : A$ is lower triangular$\}$.*

The space of binary cubic forms over K is the set

$$\{x_1u^3 + x_2u^2v + x_3uv^2 + x_4v^3 : x_i \in K\}.$$

We identify the cubic form $x_1u^3 + x_2u^2v + x_3uv^2 + x_4v^3$ with the point $x = (x_1, x_2, x_3, x_4) \in V_K$, and we will denote the form as either x or $F_x(u, v)$. The group G_K acts on the space of binary cubic forms by linear change of variables. Indeed, we have:

Definition 1.1 *Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G_K$, and let $x \in V_K$. We define the action of g on x by*

$$g \cdot x = F_{g \cdot x}(u, v) = \det(g)^{-1} F_x \left((u\, v) \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \right).$$

The twist by $\det(g)^{-1}$ ensures that if $g = \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$, then $g \cdot x = ax$.

For a form x, we let $P(x)$ denote the discriminant of the polynomial $F_x(u, 1)$: for $x = (x_1, x_2, x_3, x_4)$, we have

$$P(x) = x_2^2 x_3^2 + 18 x_1 x_2 x_3 x_4 - 4 x_2^3 x_4 - 4 x_1 x_3^3 - 27 x_1^2 x_4^2.$$

Observe that for $g \in G_K$, $P(g \cdot x) = \det(g)^2 P(x)$. We call a form x nonsingular if $P(x) \neq 0$.

We will refer to the roots of the polynomial $F_x(u, 1)$ as the roots of the form x, and $K(x)$ will be the splitting field of $F_x(u, 1)$ over K. This is either a cyclic extension of K of degree 3 or less, or a degree 6 extension with Galois group S_3. In this latter case, we may think of $K(x)$ as a conjugacy class of noncyclic cubic extensios of K. We now recall a Proposition from Section 2 of [11], giving the orbits in our action of G_K on V_K.

Proposition 1.1 *The G_K-orbits in V_K are as follows:*
$S_0 = \{0\}$ *(the zero form)*
$S_{1,K} = \{x \in V_K : x$ *has a triple root.* $\}$
$S_{2,K} = \{x \in V_K : x$ *has a double root, as well as a simple root.* $\}$
$V_K(K') = \{x \in V_K : P(x) \neq 0, K(x) = K'\}$
In the fourth class of orbit, K' runs over all conjugacy classes of extensions of K with degree at most 3.

Indeed, the works of Datskovsky and Wright cited here rely on the fact that the map $x \to K(x)$ induces a one to one correspondence between the orbits of nonsingular binary cubic forms over K, and the conjugacy classes of extensions of K of degree not exceeding 3.

[11] also chooses standard representatives for each type of orbit, which we will use in a future calculation. For S_0, $S_{1,K}$, and $S_{2,K}$, these are $(0,0,0,0)$, $(1,0,0,0)$, and $(0,1,0,0)$, respectively. For nonsingular forms x with $K(x) = K$, we choose $(0,1,1,0)$ as our standard representative. For forms such that the degree of $K(x)$ over K is 2, we consider θ such that $K(x) = K(\theta)$, and pick $(0,1,\theta+\theta', \theta\theta')$, where θ' is the Galois conjugate of θ over K. When $K(x)$ is a conjugacy class of cubic extensions, we again choose θ which generates a member of this class over K, and pick $(1, \theta+\theta'+\theta", \theta\theta'+\theta\theta"+\theta'\theta", \theta\theta'\theta")$ to be our standard representative, where again, θ' and $\theta"$ are the conjugates of θ.

The stabilizers of our nonsingular forms are also known. The following Proposition originally appears in [11].

Proposition 1.2 *Let $x \in V_K$, with $P(x) \neq 0$.*
(i) If $[K(x) : K] = 1$, then $|\mathrm{Stab}_{G_K}(x)| = 6$.
(ii) If $[K(x) : K] = 2$, then $|\mathrm{Stab}_{G_K}(x)| = 2$.
(iii) If $[K(x) : K] = 3$, and $K(x)$ is cyclic over K, then $|\mathrm{Stab}_{G_K}(x)| = 3$.
(iv) If $K(x)$ is a conjugacy class of noncyclic extensions of K, then $|\mathrm{Stab}_{G_K}(x)| = 1$.

We let O_K be the ring of integers in the number field K. $M(K)$ will stand for the set of places of K, while $M_\infty(K)$ and $M_0(K)$ will refer to the sets of infinite and finite places of K, respectively. For $\nu \in M(K)$, we let K_ν denote the completion of K at ν, and we let O_ν stand for the ring of integers in K_ν. Moreover, $\mathbb{A}_K$ will stand for the ring of adeles of K.

To define the zeta function, we first need to introduce invariant measures on K_ν and G_{K_ν}. On K_ν, we choose the additive measure dx_ν, normalized so that the measure of O_ν is 1, and the multiplicative measure d^*x_ν, which is scaled so that the measure of O_ν^* is 1. We define our measure on G_{K_ν} in stages. First, we introduce a maximal compact subgroup, U_{K_ν} of G_{K_ν}. When ν is real, we let U_{K_ν} be the group of orthogonal matrices of $\mathbb{R}$. If ν is complex, U_{K_ν} will be the unitary group. When ν is finite, we let $U_{K_\nu} = GL_2(O_\nu)$. By the Iwasawa decomposition, we know that an element $g \in G_{K_\nu}$ can be written in the form

$$g = k \left(\begin{smallmatrix} t & 0 \\ 0 & u \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right),$$

for some $k \in U_{K_\nu}$, $t, u \in K_\nu^*$, $c \in K_\nu$. We define

$$a(t,u) = \left(\begin{smallmatrix} t & 0 \\ 0 & u \end{smallmatrix}\right)$$

and

$$n(c) = \left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right).$$

We have chosen U_{K_ν} to be compact, so we have a measure, dk_ν such that U_{K_ν} has measure 1. Observe that $a(t,u)n(c) \in B_{K_\nu}$, and every element of B_{K_ν} is of this form. We specify our measure on B_{K_ν} by the formula

$$\int_{B_{K_\nu}} f(b)db_\nu = \int_{K_\nu^*}\int_{K_\nu^*}\int_{K_\nu} \left|\frac{u}{t}\right|_\nu f(a(t,u)n(c))dc_\nu d^*t_\nu d^*u_\nu,$$

where the absolute value is the standard choice on K_ν. This enables us to give a measure, dg_ν on G_{K_ν} via

$$\int_{G_{K_\nu}} f(g)dg_\nu = \int_{U_{K_\nu}}\int_{B_{K_\nu}} f(kb)db_\nu dk_\nu.$$

We define an additive measure on $\mathbb{A}_K$ via $\prod_{\nu \in M(K)} dx_\nu$. We similarly define a multiplicative measure on $\mathbb{A}_K$, and a measure on $GL_2(\mathbb{A}_K)$ as products of local measures.

With our measures defined, we may go on to define the adelic zeta function. Let Φ be a function on V_K admitting a product $\Phi = \prod_{\nu \in M(K)} \Phi_\nu$ such that Φ_ν is rapidly decreasing if ν is infinite, or locally constant with compact support if ν is finite, and all but finitely many of the Φ_ν are characteristic functions of $O_{K_\nu}^4$. We will refer to such Φ as Schwartz-Bruhat functions. Let $s \in \mathbb{C}$, and let V_K' denote the set of nonsingular forms in V_K. The adelic zeta function is then defined as

$$Z(s,\Phi) = \int_{G_{\mathbb{A}_K}/G_K} |det(g)|_{\mathbb{A}_K}^s \sum_{x \in V_K'} \Phi(g \cdot x)dg,$$

which converges absolutely for $\mathrm{Re}(s) > 2$. Having defined the function, we proceed to describe some of its basic properties. One has a functional equation for the zeta function, obtained in [11] by means of the Poisson summation formula. We briefly recall the Fourier transform on V_K used in [11]. Let $x = (x_1, x_2, x_3, x_4)$ and $y = (y_1, y_2, y_3, y_4)$ lie in V_K. We introduce the bilinear form

$$[x, y] = x_1 y_4 - \frac{1}{3} x_2 y_3 + \frac{1}{3} x_3 y_2 - x_4 y_1,$$

and let $<>$ be a nontrivial additive character on K. We let dv be the Haar measure on V_K self-dual with respect to the character $< [x, y] >$ on V_K^2. If Φ is a Schwartz-Bruhat function on V_K, we define its Fourier transform by

$$\hat{\Phi}(y) = \int_{V_K} \Phi(x) < [x, y] > dx.$$

After considerable labor, Wright arrives at the functional equation

$$Z(2 - s, \hat{\Phi}) = Z(s, \Phi).$$

CHAPTER 2

Invariant Lattices

2.1 Overview

At this point, we have all the background we need to begin describing our lattices, and hence, our main result. We begin with a definition, following Weil [10].

Definition 2.1 *Let K be an algebraic number field, and let O_K be the ring of (algebraic) integers of K. Let V be a finite-dimensional vector space over K. A lattice of V is a finitely generated O_K-module in V which contains a basis of V over K.*

If K_ν is the completion of K at a finite prime ν, we define lattices in K_ν-vector spaces in the same way. Working over the completions, one finds that the ring of integers is a local ring, which greatly simplifies this case. Moreover, we have the following lemma, whose proof is given by Weil (Chapter 5 of [We3]).

Lemma 2.1 *Let $M_0(K)$ be the set of finite primes of K, and let $L \subseteq K$ be a lattice of K^4. For $\nu \in M_0(K)$, denote by L_ν the closure of L in K_ν^4. Then $L = \bigcap_{\nu \in M_0(K)} (L_\nu \cap K^4)$.*

In light of the preceeding lemma, we are able to carry out most of our work in the local fields, and reconstruct our global classification from these results. Observe that for K_ν, our lattices are free O_ν-modules of rank 4. To explain our classification, we recall the notion of primitivity.

Definition 2.2 *We say a K-lattice, L, is primitive if it is contained in O_K^4, and for every prime ideal, $\mathfrak{p}$, of O_K, we have $\mathfrak{p}^{-1}L \nsubseteq O_K^4$.*

Again, we make an analogous definition for local fields. Note that in the local case, we may replace a prime ideal by any of its uniformizers.

We say that two lattices, L and L' are equivalent if there is a fractional ideal, $\mathfrak{p}$, of the integer ring of the base field such that $\mathfrak{p}L = L'$. Every lattice (of a four dimensional vector space) is equivalent to a primitive lattice. Our goal is to enumerate the primitive, invariant lattices in K^4.

Notation 2.1 *Throughout, E_i, for $i = 1, 2, 3, 4$, will denote the standard basis vectors for K^4. We will write $u(\alpha)$ in place of $\left(\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix}\right)$, and ω for $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Finally, we let $\Psi(x) = (u(1) \cdot x) - x$, for $x \in K^4$. Indeed, we have $\Psi(x) = (x_2 + x_3 + x_4, 2x_3 + 3x_4, 3x_4, 0)$.*

We end our overview by presenting the original result of [7]; we will adhere to their notations for the lattices we introduce.

Theorem 2.1 *The $SL_2(\mathbb{Z})$-invariant lattices in $\mathbb{Q}^4$ are as follows:*

$L_1 = \mathbb{Z}^4$
$L_2 = \{(a, b, c, d) \in \mathbb{Z}^4 : 3|b, c\}$
$L_3 = \{(a, b, c, d) \in \mathbb{Z}^4 : 2|b + c\}$
$L_4 = \{(a, b, c, d) \in \mathbb{Z}^4 : 2|b + c, a, d; 3|b, c\}$
$L_5 = \{(a, b, c, d) \in \mathbb{Z}^4 : 2|b + c, a, d\}$
$L_6 = \{(a, b, c, d) \in \mathbb{Z}^4 : 2|b + c; 3|b, c\}$
$L_7 = \{(a, b, c, d) \in \mathbb{Z}^4 : 2|a + b + c, b + c + d\}$
$L_8 = \{(a, b, c, d) \in \mathbb{Z}^4 : 2|a + b + d, a + c + d; 3|b, c\}$
$L_9 = \{(a, b, c, d) \in \mathbb{Z}^4 : 2|a + b + d, b + c + d\}$
$L_{10} = \{(a, b, c, d) \in \mathbb{Z}^4 : 2|a + b + c, b + c + d; 3|b, c\}$.

2.2 Results and Proofs

Lemma 2.2 *If* $\nu \nmid 2, 3$, *then* $(L)_\nu = (L_1)_\nu = O_\nu^4$.

Proof: Let $x = (x_1, x_2, x_3, x_4)$ be primitive for ν. First, suppose that either x_1 or x_4 is a unit of O_ν. By applying ω if necessary, we may assume, without loss of generality, that $x_4 \in O_\nu^*$, the group of units of O_ν. Let $y_1 = x_4^{-1}u(-3^{-1}x_4^{-1}x_3) \cdot x$. Then the third and fourth coordinates of y_1 are 0 and 1, respectively, so we see that $6^{-1}\Psi(\Psi(y_1)) = (1, 1, 0, 0)$. Now, $E_2 = u(-1) \cdot (1, 1, 0, 0)$, and $E_1 = \Psi(E_2)$, so $E_1, E_2 \in (L)_\nu$, and one sees easily that $E_3, E_4 \in (L)_\nu$ as well. Hence $(L)_\nu = O_\nu^4$. Next, suppose $x_1, x_4 \notin O_\nu^*$. By primitivity, either x_2 or x_3 is a unit, and again, we may assume $x_3 \in O_\nu^*$ by means of ω. Now, consider $u(1) \cdot x + u(-1) \cdot x - 2x$ This element has $2x_3$ as its first coordinate, and $2x_3 \in O_\nu^*$. Thus, we have reduced the problem to the previous case. This proves our lemma. $\square$

Lemma 2.3 *If* $\nu \mid 3$, *then* $(L)_\nu = O_\nu \bigoplus O_\nu^m \bigoplus O_\nu^m \bigoplus O_\nu$ *for some* $m \in 0, 1, 2, ..., ord_\nu(3)$.

Proof: Let x be primitive for ν. We first assume x_2 or x_3 to be a unit. As in the preceeding lemma, we may simply assume that x_3 is a unit. Set $y_1 = (2x_3 + 3x_4)^{-1}\Psi(x) = (x_1^{'}, 1, x_3^{'}, 0)$, and also $y_2 = (2x_3 + 6x_4)^{-1}\Psi(\Psi(x)) = (1, x_2^{'}, 0, 0)$. Because, $x_2^{'} = 6x_4(2x_3+6x_4)^{-1}$ and $x_3^{'} = 3x_4(2x_3+3x_4)^{-1}$, we have $x_2^{'}, x_3^{'} \in 3O_\nu$. Let $y_3 = y_1 - x_1^{'}y_2 = (0, 1 - x_2^{'}x_1^{'}, x_3^{'}, 0)$. Now, $1 - x_2^{'}x_1^{'} \in O_\nu^*$, and we have $y_4 = (1 - x_2^{'}x_1^{'})^{-1}(\omega \cdot y_3) = (0, x_2^{''}, 1, 0)$, where we note that $x_2^{''} \notin O_\nu^*$, since it is divisible by $x_3^{'}$. So let $y_5 = u(-2^{-1}x_2^{''}) \cdot y_4 = (x_1^{''}, 0, 1, 0)$, and $y_6 = \Psi(y_5) = (1, 2, 0, 0)$. Then $E_1 = 2^{-1}\Psi(y_6)$, and $E_2 = 2^{-1}(y_6 - E_1)$, so $E_1, E_2, \in (L)_\nu$, and as before, $(L)_\nu = O_\nu^4$.

Let us now suppose that $x_2, x_3, \notin O_\nu^*$. By primitivity, and possibly using ω, we may assume, without loss of generality, that $x_4 \in O_\nu^*$. Let $y_7 = \Psi(x) = (x_2+x_3+x_4, 2x_3+3x_4, 3x_4, 0)$. Then, we see that $x_2+x_3+x_4 \in O_\nu^*$, $3x_4 \in 3O_\nu^*$, and $2x_3 + 3x_4 \in \pi_\nu O_\nu$. For brevity, we write $u = x_2 + x_3 + x_4$, $a = 2x_3 + 3x_4$, and $b = 3x_4$. Then $y_7 = (u, a, b, 0)$. We have $\frac{1}{2}(y_7 + \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \cdot y_7) = (0, a, 0, 0)$. So $y_8 = (u, 0, b, 0) \in (L)_\nu$. $x_4^{-1}\Psi(y_8) = 3E_1 + 6E_2$, and $2^{-1}\Psi(3E_1 + 6E_2) = 3E_1$. Also, $3E_2 = 2^{-1}((3E_1 + 6E_2) - 3E1)$, and $E_1 = u^{-1}(y_8 - x_4 3E_3)$, so $E_1, 3E_2 \in (L)_\nu$. We already saw that $(0, a, 0, 0) \in (L)_\nu$, and it follows that $\pi_\nu^{ord_\nu(a)} E_2 \in (L)_\nu$. Let m be the smallest integer such that $\pi_\nu^m E_2 \in (L)_\nu$, and note that $m \in 0, 1, 2, ..., ord_\nu(3)$. Suppose $cE_2 + dE_3$ lies in $(L)_\nu$, and that either c or d has order at ν less than m. (We may disregard that E_1 and E_4 coordinates since we know $E_1, E_4 \in (L)_\nu$). Applying $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and halving, we see that either $(0, c, 0, 0)$ or $(0, 0, d, 0)$ lies in $(L)_\nu$, violating minimality of m. So we conclude that $(L)_\nu = O_\nu \bigoplus O_\nu^m \bigoplus O_\nu^m \bigoplus O_\nu$.

Lemma 2.4 *If $(L)_\nu$ contains an element of the form $(\alpha, 1, 1, 0)$, then it contains $2E_i$, for $i = 1, 2, 3, 4$.*

Proof: Let $f = (\alpha, 1, 1, 0) \in (L)_\nu$. Then $\Psi(f) = 2E_1 + 2E_2$. Now, $\Psi(2E_1 + 2E_2) = 2E_1$, and the lemma follows easily.

Lemma 2.5 *If $\nu \mid 2$, and $[O_\nu/\pi_\nu O_\nu : \mathbb{Z}_2/2\mathbb{Z}_2] > 1$, then $(L)_\nu = (L_1)_\nu$.*

Proof: As before, we choose $x = (x_1, x_2, x_3, x_4)$ to be primitive for ν. First, we suppose that either $x_1 \in O_\nu^*$ or $x_4 \in O_\nu^*$. Applying ω if necessary, we may simply assume that $x_4 \in O_\nu^*$. We set $y_1 = u(-\frac{1}{3}x_4^{-1}x_3) \cdot x$, and note that the third and fourth entries of y_1 are 0 and x_4, respectively. We next set $y_2 = (3x_4)^{-1}\Psi(y_1)$, which becomes $(x_1', 1, 1, 0)$. By Lemma 2.4, $2E_i \in (L)_\nu$, for $i = 1, 2, 3, 4$. Because the residue field extension is nontrivial, there exists $u \in O_\nu^*$ such that $1-u \in O_\nu^*$. Then also $1-u^2 \in O_\nu^*$, for $1-u^2 = (1-u)(1+u)$, and $1 + u = (1 - u) + 2u$. Let $y_3 = (u)\left(\begin{smallmatrix} 1 & 0 \\ 0 & u \end{smallmatrix}\right) \cdot y_2 = (x_1', u, u^2, 0)$, and then let $y_4 = y_2 - y_3 = (0, 1 - u, 1 - u^2, 0)$. Observe that $(1 - u) + (1 - u^2) = (1 - u)(1 + (1 + u)) = (1 - u)(2 + u)$, which is a unit. So we have shown that $(L)_\nu$ contains an element $(0, a, b, 0)$ such that a, b, and $a + b$ are all units. $\Psi((0, a, b, 0)) = (a + b, 2b, 0, 0)$, and since we already know $2E_2 \in (L)_\nu$, we see that $(a + b, 0, 0, 0) \in (L)_\nu$. It follows easily that $E_1, E_4 \in (L)_\nu$.

$\left(\begin{smallmatrix}1&0\\0&u\end{smallmatrix}\right)\cdot(0,a,b,0)=(0,a,ub,0)$, and $(0,a,b,0)-(0,a,ub,0)=(0,0,(1-u)b,0)$. But we have chosen u so that $1-u$ is a unit, so we see $E_2,E_3\in(L)_\nu$. Hence $(L)_\nu=(L_1)_\nu$.

Next, assume x_1 and x_4 lie in $\pi_\nu O_\nu$. By primitivity, we have that $x_2\in O_\nu^*$ or $x_3\in O_\nu^*$. If $x_2+x_3\notin O_\nu^*$, then both x_2 and x_3 are units. There exits $v\in O_\nu^*$ such that $1+v\in O_\nu^*$, and applying $\left(\begin{smallmatrix}1&0\\0&v\end{smallmatrix}\right)$ to x gives $(v^{-1}x_1,x_2,vx_3,v^2x_4)$. Summing $(x_2+x_3)+(x_2+vx_3)$ shows that x_2+vx_3 is a unit, for we have assumed $x_2+x_3\in\pi_\nu O_\nu$. This reasoning lets us assume, without loss of generality, that $x=(x_1,x_2,x_3,x_4)$ is such that $x_2+x_3\in O_\nu^*$. But then we may apply Ψ to x, and we get a form with a unit in its first coordinate, thus reducing to the first case of this lemma.

Lemma 2.6 *If $\nu\mid 2$, and $[O_\nu/\pi_\nu O_\nu:\mathbb{Z}_2/2\mathbb{Z}_2]=1$, then $(L_5)_\nu\subseteq(L)_\nu$ or $(L_9)_\nu\subseteq(L)_\nu$. Here, $(L_5)_\nu=\{(a,b,c,d)\in O_\nu^4:\pi_\nu\mid a,d,b+c\}$, and $(L_9)_\nu=\{(a,b,c,d)\in O_\nu^4:\pi_\nu\mid a+b+d,a+c+d\}$.*

Proof: If ν is unramified over 2, the argument in [7] applies verbatim. So assume ν is ramified. Let $x\in L$ be primitive for ν. As usual, suppose that either $x_1\in O_\nu^*$ or $x_4\in O_\nu^*$. As in the preceeding lemmas, this reduces to the assumption that $x_4\in O_\nu^*$. This time, we choose the same y_2 as in the previous lemma; we have $y_2=(x_1',1,1,0)\in(L)_\nu$, and we saw that $2E_i\in(L)_\nu$ for $i=1,2,3,4$. Pick $u=1+\pi_\nu\in O_\nu^*$, so that $1-u$ has order 1 at π_ν. Following Lemma 2.5, let $y_3=(u)\left(\begin{smallmatrix}1&0\\0&u\end{smallmatrix}\right)\cdot y_2=(x_1',u,u^2,0)$. Now, $y_2-y_3=(0,1-u,1-u^2,0)$. Also, $1+u=1+1+\pi_\nu=2+\pi_\nu$, hence $1+u$ has order 1 at ν. But $(1-u)=(1+u)-2u$, and $2\in\pi_\nu^2O_\nu$, so $1-u$ also has order 1, and it follows that $1-u^2\in\pi_\nu^2O_\nu^*$. Applying matrices of the forms $\left(\begin{smallmatrix}v&0\\0&1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1&0\\0&v\end{smallmatrix}\right)$, with $(v\in O_\nu^*)$, to y_2-y_3, we find that $(0,\pi_\nu,\pi_\nu^2,0)=\pi_\nu E_2+\pi_\nu^2E_3\in(L)_\nu$. Now, $\Psi((0,\pi_\nu,\pi_\nu^2,0))=(\pi_\nu+\pi_\nu^2,2\pi_\nu^2,0,0)$. Since $2\pi_\nu^2E_2\in(L)_\nu$, $(\pi_\nu+\pi_\nu^2,0,0,0)$, and in turn, $\pi_\nu E_1,\pi_\nu E_4$ lie in $(L)_\nu$. Then also $(0,\pi_\nu,\pi_\nu^2,-\frac{1}{3}\pi_\nu^2)\in(L)_\nu$. Next, $u(1)\cdot(0,\pi_\nu,\pi_\nu^2,-\frac{1}{3}\pi_\nu^2)=(z,\pi_\nu+\pi_\nu^2,0,-\frac{1}{3}\pi_\nu^2)$. Here, z is a sum of integer multiples of the entries of $(0,\pi_\nu,\pi_\nu^2,-\frac{1}{3}\pi_\nu^2)$, so it lies in $\pi_\nu O_\nu$. Since $\pi_\nu E_1$ and $\pi_\nu E_4$ are in $(L)_\nu$, we can show that $(0,\pi_\nu+\pi_\nu^2,0,0)\in(L)_\nu$, and it follows easily that $\pi_\nu E_2$ and $\pi_\nu E_3$ are in $(L)_\nu$. We have thus shown that $\pi_\nu E_i \in (L)_\nu$ for $i = 1, 2, 3, 4$.

Now, consider the case where x_1 and x_4 are non-invertible, so that x_2 or x_3 is a unit. If $x_2 + x_3 \in O_\nu^*$, we can apply Ψ to x and use the above case to see that $\pi_\nu E_i \in (L)_\nu$ for $i = 1, 2, 3, 4$. Otherwise, we have that x_2 and x_3 are both units. Choose $v \in O_\nu^*$ so that $x_2 + vx_3 = 0$, and replace x by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & v \end{smallmatrix}\right) \cdot x$. The new $x = (x_1, x_2, x_3, x_4)$ satisfies $x_2 + x_3 = 0$. We proceed as follows: let $w^{'}(z) = -\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \cdot z$, that is, $w^{'}$ reverses the coordinates.

$$\Psi(x) = (x_2 + x_3 + x_4, 2x_3 + 3x_4, 3x_4, 0)$$
$$w^{'}(\Psi(x)) = (0, 3x_4, 2x_3 + 3x_4, x_2 + x_3 + x_4).$$

By choice of x, we find that $w^{'}(\Psi(x)) = (0, 3x_4, 2x_3 + 3x_4, x_4)$. Subtracting this result from x, we get $(z^{'}, a, b, 0)$, where $a, b \in O_\nu^*$. Using the matrices of the forms $\left(\begin{smallmatrix} v & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & v \end{smallmatrix}\right)$ (for units u, v), we may insist that $a = b = 1$. Lemma 2.4 now shows that $2E_i \in (L)_\nu$ for $i = 1, 2, 3, 4$, and from here a previous argument can be used to show that $\pi_\nu E_i \in (L)_\nu$ for $i = 1, 2, 3, 4$.

So regardless of which coordinates of x are initially taken to be units, we have seen that $(L)_\nu$ has an element of the form $(z, 1, 1, 0)$. (Here, we drop the notation from earlier in our proof). Now, observe that $(L_5)_\nu = O_\nu(\pi_\nu E_1) \bigoplus O_\nu(\pi_\nu E_4) \bigoplus O_\nu(E_2 + E_3) \bigoplus O_\nu(\pi_\nu E_2)$ and $(L_9)_\nu = O_\nu(E_1 + E_2 + E_3) \bigoplus O_\nu(E_2 + E_3 + E_4) \bigoplus O_\nu(\pi_\nu E_2) \bigoplus O_\nu(\pi_\nu E_3)$. In addition to $(z, 1, 1, 0)$, $(L)_\nu$ also contains $\pi_\nu E_i$, for $i = 1, 2, 3, 4$. If $z \notin O_\nu^*$, then $(z, 1, 1, 0) - (\pi_\nu^{-1} z)\pi_\nu E_1 = E_2 + E_3 \in (L)_\nu$, so $(L_5)_\nu \subseteq (L)_\nu$. If $z \in O_\nu^*$, then we can write $z = 1 + z^{'}$, where $z^{'} \in \pi_\nu O_\nu$, since $[O_\nu/\pi_\nu O_\nu : \mathbb{Z}_2/2\mathbb{Z}_2] = 1$. Then $(z, 1, 1, 0) - (\pi_\nu^{-1} z^{'})\pi_\nu E_1 = E_1 + E_2 + E_3 \in (L)_\nu$, and by applying $w^{'}$, we can conclude that $E_2 + E_3 + E_4 \in (L)_\nu$. Hence $(L_9)_\nu \subseteq (L)_\nu$. This completes the proof of the lemma.

Notation 2.2 *Up to this, we have defined* $(L)_\nu$, $(L_5)_\nu$, *and* $(L_9)_\nu$. *Now, we introduce* $(L_3)_\nu = \{(a, b, c, d) \in O_\nu^4 : \pi_\nu \mid b + c\}$ *and* $(L_7)_\nu = \{(a, b, c, d) \in O_\nu^4 : \pi_\nu \mid a + b + c, b + c + d\}$.

In view of Lemma 2.6, we can prove the following, more precise lemma.

Lemma 2.7 *If* $\nu \mid 2$, *and* $[O_\nu/\pi_\nu O_\nu : \mathbb{Z}_2/2\mathbb{Z}_2] = 1$, *then* $(L)_\nu \in \{(L_1)_\nu, (L_3)_\nu, (L_5)_\nu, (L_7)_\nu, (L_9)_\nu\}$.

Proof: By Lemma 2.6, either $(L_5)_\nu \subseteq (L)_\nu \subseteq (L_1)_\nu$ or $(L_9)_\nu \subseteq (L)_\nu \subseteq (L_1)_\nu$.

Case I: Assume that $(L_5)_\nu \subseteq (L)_\nu \subseteq (L_1)_\nu$. If $(L)_\nu = (L_5)_\nu$, there is nothing to prove, so assume that $(L)_\nu$ properly contains $(L_5)_\nu$. We will write $(L_1)_\nu = O_\nu E_1 \bigoplus O_\nu E_4 \bigoplus O_\nu(E_2 + E_3) \bigoplus O_\nu E_2$, and $(L_5)_\nu = O_\nu \pi_\nu E_1 \bigoplus O_\nu \pi_\nu E_4 \bigoplus O_\nu(E_2 + E_3) \bigoplus O_\nu \pi_\nu E_2$. Then $\{aE_1 + bE_4 + cE_2 : a, b \in \{0, 1\}\}$ is a set of coset representatives for $(L_1)_\nu/(L_5)_\nu$. Then the fact that $(L_5)_\nu \subseteq (L)_\nu \subseteq (L_1)_\nu$ implies that one of our coset representatives lies in $(L)_\nu$. 0 cannot be the only such representative, as we have assumed $(L)_\nu$ is not $(L_5)_\nu$. Suppose the representative that lies in $(L)_\nu$ is E_1, E_4, or $E_1 + E_4$. $(L)_\nu$ also contains $(L_5)_\nu$, so we can show $(L_3)_\nu \subseteq (L)_\nu$, for $(L_3)_\nu = O_\nu E_1 \bigoplus O_\nu E_4 \bigoplus O_\nu(E_2 + E_3) \bigoplus O_\nu \pi_\nu E_2$. Indeed, the E_1 and E_4 cases are obvious. If instead, we have $E_1 + E_4 \in (L)_\nu$, note that $\Psi(E_1 + E_4) = E_1 + 3(E_2 + E_3)$, which reduces this case to that of E_1. Hence $(L_3)_\nu \subseteq (L)_\nu$, as desired. But $(L_1)_\nu/(L_3)_\nu \cong \mathbb{Z}/2\mathbb{Z}$, and so no lattices lie (properly) between $(L_1)_\nu$ and $(L_3)_\nu$. Hence either $(L)_\nu = (L_3)_\nu$ or $(L)_\nu = (L_1)_\nu$. Next, suppose that the coset representative is E_2, $E_1 + E_2$, or $E_2 + E_4$. Recall that $2E_2 \in (L_5)_\nu \subseteq (L)_\nu$. Now, $E_1 = \Psi(E_2) = \Psi(E_1 + E_2) = \Psi(\omega(E_2 + E_4)) - 2E_2$, and we can easily show that both E_1 and E_2 are in $(L)_\nu$. In this case, $(L)_\nu = (L_1)_\nu$. To finish this case, suppose the coset representative that lies in $(L)_\nu$ is $E_1 + E_2 + E_4$. Write $(L_7)_\nu = O_\nu(E_1 + E_2 + E_4) \bigoplus O_\nu(E_1 + E_3 + E_4) \bigoplus O_\nu \pi_\nu E_1 \bigoplus O_\nu \pi_\nu E_4$. Now, $E_2 + E_3 \in (L_5)_\nu$, and clearly, $(E_1 + E_2 + E_4) - (E_2 + E_3) + 2E_3 = (E_1 + E_3 + E_4)$, so $(L_7)_\nu \subseteq (L)_\nu$. Moreover, $(L_1)_\nu/(L_7)_\nu$ has $\{0, E_1, E_4, E_1 + E_4\}$ as a set of coset representatives. $(L)_\nu$ must contain one of these, and if it contains any of the nonzero members of this set, it contains them all (see the argument above). We have seen that this forces $(L)_\nu = (L_1)_\nu$. Otherwise, $(L)_\nu = (L_7)_\nu$.

Case II: Assume that Assume that $(L_9)_\nu \subseteq (L)_\nu \subseteq (L_1)_\nu$. We have that $(L_9)_\nu = O_\nu(E_1 + E_2 + E_3) \bigoplus O_\nu(E_2 + E_3 + E_4) \bigoplus O_\nu \pi_\nu E_1 \bigoplus O_\nu \pi_\nu E_2$, and

$(L_1)_\nu = O_\nu(E_1+E_2+E_3) \bigoplus O_\nu(E_2+E_3+E_4) \bigoplus O_\nu E_2 \bigoplus O_\nu E_3$. A set of coset representatives for $(L_1)_\nu/(L_9)_\nu$ is given by $\{aE_2 + bE_3 : a, b \in \{0, 1\}\}$. Again, $(L)_\nu$ contains at least one coset representative, and if the only representative in $(L)_\nu$ is 0, then $(L)_\nu = (L_9)_\nu$. If $(L)_\nu$ contains E_2 or E_3, it clearly contains both, and $\Psi(E_2) = E_1$ implies that also $E_1 \in (L)_\nu$. Hence $(L)_\nu = (L_1)_\nu$. If instead the coset representative contained in $(L)_\nu$ is $E_2 + E_3$, then $(L)_\nu$ contains $(E_1 + E_2 + E_3) - (E_1 + E_2) = E_1$. It follows that $E_4 \in (L)_\nu$, and thus $(L_3)_\nu \subseteq (L)_\nu$. We have seen that this implies that either $(L)_\nu = (L_3)_\nu$ or $(L)_\nu = (L_1)_\nu$. This completes the proof.

We are now in position to state and prove our main theorem.

Theorem 2.2 *Let $\mathfrak{p}_1, \mathfrak{p}_2, ..., \mathfrak{p}_r$ be the prime ideal divisors of $3O_K$, and let $\mathfrak{p}_{r+1}, \mathfrak{p}_{r+2}, ..., \mathfrak{p}_t$ be the divisors of $2O_K$ with residue field $\mathbb{Z}/2\mathbb{Z}$. Then the primitive, $GL_2(O_K)$-invariant lattices of K^4 are of the form $\{(a, b, c, d) \in O_K^4 : b, c \in \mathfrak{p}_1^{m_1} \cdots \mathfrak{p}_r^{m_r}; (b+c) \in \mathfrak{p}_{r+1}^{n_{r+1,1}} \cdots \mathfrak{p}_t^{n_{t,1}}; a, d, (b+c) \in \mathfrak{p}_{r+1}^{n_{r+1,2}} \cdots \mathfrak{p}_t^{n_{t,2}}; (a+b+c), (b+c+d) \in \mathfrak{p}_{r+1}^{n_{r+1,3}} \cdots \mathfrak{p}_t^{n_{t,3}}; (a+b+d), (a+c+d) \in \mathfrak{p}_{r+1}^{n_{r+1,4}} \cdots \mathfrak{p}_t^{n_{t,4}}\}$, where $0 \le m_i \le ord_{\mathfrak{p}_i}(3)$, $0 \le n_{i,j} \le 1$, and $n_{i,j} = 1$ implies $n_{i,k} = 0$ for any $k > j$.*

Proof: Combine Lemma 2.2 through Lemma 2.7, and use the fact that a lattice $L \subseteq K^4$ is given by $\bigcap_{\nu \in M_0(K)} ((L)_\nu \bigcap K^4)$, where $(L)_\nu$ is the closure in K_ν^4 of L.

CHAPTER 3

Dirichlet Series and Functional Equation

In this section, we connect our invariant lattices to the adelic zeta function for the space of binary cubic forms [11]. We fix a number field, K, and let $\mathbb{A}$ denote the ring of adeles of K. We abbreviate $GL_2(\mathbb{A})$ simply as G, and write G_K for $GL_2(K)$, viewed as a subgroup of G. We introduce

$$G(\infty) = \prod_{\nu \in M_\infty(K)} GL_2(K_\nu) \times \prod_{\nu \in M_0(K)} GL_2(O_\nu).$$

Observe that the number of double cosets of $G(\infty)$
G/G_K is just h_K, the class number of K. We have the decomposition

$$G = \bigcup_{t \in \mathbb{A}^*/K^*\mathbb{A}^*(\infty)} G(\infty) \left(\begin{smallmatrix} 1 & 0 \\ 0 & t \end{smallmatrix}\right) G_K.$$

Observe also that $G(\infty) \left(\begin{smallmatrix} 1 & 0 \\ 0 & t \end{smallmatrix}\right) G_K = \{g \in G : \det(g) \in K^*\mathbb{A}^*(\infty)\}$.

Now, we fix a (primitive) invariant lattice $L \subseteq K^4$. We choose a Schwartz-Bruhat function, $\phi = \prod_\nu \phi_\nu$ whose finite components are the indicator maps for the respective closures of L. For notational convenience, we also write $\phi = \prod_{\nu \in M_\infty(K)} \phi_\nu \times 1_U$, where U is the product of the finite closures, and 1_U is the indicator map for U. Let $H = \mathbb{A}^*/K^*\mathbb{A}^*(\infty)$, and let $\hat{H}$ be the dual of H.

We now consider the sum

$$\frac{1}{h_K}\sum_{\chi\in\hat{H}} Z(\chi\omega_s,\phi) = \int_{G/G_K} \frac{1}{h_K}\sum_{\chi\in\hat{H}}\chi(\det(g))|\det(g)|^s \sum_{x\in V_K'}\phi(g\cdot x)dg.$$

This reduces to

$$\int_{G(\infty)G_K/G_K} |\det(g)|^s \sum_{x\in V_K'}\phi(g\cdot x)dg,$$

since $\sum_{\chi\in\hat{H}}\chi(\det(g)) = 0$ whenever $g\notin G(\infty)G_K/G_K$. By the isomorphism theorems, this integral is just

$$\int_{G(\infty)/G_K\bigcap G(\infty)} |\det(g)|^s \sum_{x\in V_K'}\phi(g\cdot x)dg.$$

Note that $G_K\bigcap G(\infty) = GL_2(O_K)$. Now if $g\in G(\infty)$, then $g=(g_\nu)_\nu$, where $g_\nu\in GL_2(O_\nu)$ for all finite primes ν. We have that $g_\nu\cdot x\in GL_2(O_\nu)$ iff $x\in g_\nu^{-1}U_\nu$, where U_ν is the local component of U. Now, by invariance of the lattice, we see that $g_\nu\cdot x\in GL_2(O_\nu)$ iff $x\in U_\nu$. If $x\in U_\nu$ for all finite ν, then we have $x\in K\bigcap \prod_{\nu\in M_0(K)} U_\nu = K\bigcap U = L$. Hence

$$\int_{G(\infty)/G_K\bigcap G(\infty)} |\det(g)|^s \sum_{x\in V_K'}\phi(g\cdot x)dg = \int_{G(\infty)/GL_2(O_K)} |\det(g)|^s \sum_{x\in L'}\phi(g\cdot x)dg,$$

where $L' = \{x\in L : P(x)\neq 0\}$. This integral can be rewritten as a sum, namely

$$\sum_{x\in GL_2(O_K)L'} \frac{1}{|G_x(O_K)|}\int_{G(\infty)} |\det(g)|^s\phi(g\cdot x)dg.$$

The integrals under the sum are of the form

$$\int_{G(\infty)} |\det(g)|^s\phi(g\cdot x)dg =$$
$$\prod_{\nu\in M_\infty(K)}\int_{GL_2(K_\nu)} |\det(g_\nu)|_\nu^s\phi_\nu(g_\nu\cdot x)dg_\nu \times \prod_{\nu\in M_0(K)}\int_{GL_2(O_\nu)} 1_{U_\nu}(g_\nu\cdot x)dg_\nu.$$

All of the finite local factors evaluate to 1. We denote the product of the infinite local factors by $Z_{x_\infty}(s,\phi_\infty)$. Our work and conventions thus show

$$\frac{1}{h_K}\sum_{\chi\in\hat{H}} Z(\chi\omega_s,\phi) = \sum_{x\in GL_2(O_K)L'} \frac{1}{|G_x(O_K)|} Z_{x_\infty}(s,\phi_\infty).$$

Write $\xi_L(s)$ for the right hand side of this equation. If we do a change of variables in each $Z_{x_\infty}(s,\phi_\infty)$ to replace x by its corresponding standard orbital representative, which, by abuse of notation, we will also call x, we get

$$\xi_L(s) = \sum_{x \in GL_2(O_K)L'} |G_x(O_K)|^{-1} Z_{x_\infty}(s,\phi_\infty) |N_{\mathbb{Q}}^K(P(x))|^{\frac{-s}{2}},$$

a "Dirichlet series" of discriminants, analogous to those in [4]. We conclude by deriving a functional equation for these series. Using the functional equation, for the adelic zeta function, we have

$$\xi_L(s) = \frac{1}{h_K} \sum_{\chi \in \hat{H}} Z(\chi\omega_s, \phi) = \frac{1}{h_K} \sum_{\chi \in \hat{H}} Z(\chi\omega_{2-s}, \hat{\phi}) = \xi_{\hat{L}}(2-s).$$

Here, we have used the fact that as χ ranges over $\hat{H}$, so does χ^{-1}, and we have the proviso that $\hat{L}$ may not be primitive.

REFERENCES

[1] M. Bhargava, Higher Composition Laws III: The Parameterization of Quartic Rings, Ann. of Math. 159 (2004) 1329-1360.

[2] M. Bhargava and M. Matchett-Wood, The Density of Discriminants of S_3-sextic Number Fields, Proc. Amer. Math. Soc. 136 (2008), 1581-1587.

[3] B. Datskovsky, On Zeta Functions Associated with the Space of Binary Cubic Forms with Coefficients in a Function Field, Ph. D. thesis, Harvard (1984).

[4] B. Datskovsky and D. J. Wright, The Adelic Zeta Function Associated to the Space of Binary Cubic Forms II: Local Theory, J. Reine Angew. Math. 367 (1986), 27-75.

[5] B. Datskovsky and D. J. Wright, Density of Discriminants of Cubic Extensions, J. Reine Angew. Math. 386 (1988), 116-138.

[6] J. I. Igusa, Some Results on p-adic Complex Powers, Amer. J. Math. 106 (1984), 1013-1032.

[7] Y. Ohno, T. Taniguchi, and S. Wakatsuki, Relations Among Dirichlet Series Whose Coefficients Are Class Numbers of Integral Binary Cubic Forms, Amer. J. Math. to appear.

[8] M. Sato and T. Shintani, On Zeta Functions Associated with Prehomogeneous Vector Spaces, Ann. of Math. (2) 100 (1974), 131-170.

[9] T. Shintani, On Dirichlet Series Whose Coefficients Are Class-Numbers of Integral Binary Cubic Forms, J. Math. Soc. Japan 24 (1972), 132-188.

[10] A. Weil, Basic Number Theory, 2nd edition, Springer-Verlag, (1974).

[11] D. J. Wright, The Adelic Zeta Function Associated to the Space of Binary Cubic Forms Part I: Global Theory, Math. Ann. 270 (1985), 503-534.

BIBLIOGRAPHY

K. Belabas and E. Fouvry, Discriminants Cubiques et Progressions Arithmtiques, preprint.

M. Bhargava, Higher Composition Laws III: The Parameterization of Quartic Rings, Ann. of Math. 159 (2004) 1329-1360.

M. Bhargava, The Density of Discriminants of Quartic Rings and Fields, Ann. of Math. 162 (2005) 1031-1063.

M. Bhargava and M. Matchett-Wood, The Density of Discriminants of S_3-sextic Number Fields, Proc. Amer. Math. Soc. 136 (2008), 1581-1587.

B. Datskovsky, On Zeta Functions Associated with the Space of Binary Cubic Forms with Coefficients in a Function Field, Ph. D. thesis, Harvard (1984).

B. Datskovsky and D. J. Wright, The Adelic Zeta Function Associated to the Space of Binary Cubic Forms II: Local Theory, J. Reine Angew. Math. 367 (1986), 27-75.

B. Datskovsky and D. J. Wright, Density of Discriminants of Cubic Extensions, J. Reine Angew. Math. 386 (1988), 116-138.

H. Davenport and H. Heilbronn, On the Density of Discriminants of Cubic Fields. II, Proc. Royal. Soc. A. 322 (1971), 405-420.

B. N. Delone and D. K. Fadeev, The Theory of Irrationalities of the Third Degree, Translations of Mathematical Monographs, vol. 10, American Mathematical Society, (1964).

J. I. Igusa, Some Results on p-adic Complex Powers, Amer. J. Math. 106 (1984), 1013-1032.

G. Janusz, Algebraic Number Fields, 2nd edition, American Mathematical Society, (1996).

E. Landau, ber die Anzahl der Gitterpunkte in gewissen Bereichen. II, Nachr. v. d. Gesellschaft d. Wiss. zu Gttingen, Math.-Phys. Klasse, (1915), 209-243.

S. Lang, Algebraic Number Theory, 2nd edition, Springer-Verlag, (1986).

J. Nakagawa, On the Relations Among the Class Numbers of Binary Cubic Forms, Invent. Math. 134 (1998), 101-138.

Y. Ohno, A Conjecture on Coincidence Among the Zeta Functions Associated with the Space of Binary Cubic Forms, Amer. J. Math. 119 (1997), no. 5, 1083-1094.

Y. Ohno, T. Taniguchi, and S. Wakatsuki, Relations Among Dirichlet Series Whose Coefficients Are Class Numbers of Integral Binary Cubic Forms, Amer. J. Math. to appear.

D. Roberts, Density of Cubic Field Discriminants, Math. Comp. 70 (2001), no. 236, 1699-1705 (electronic).

M. Sato and T. Shintani, On Zeta Functions Associated with Prehomogeneous Vector Spaces, Ann. of Math. (2) 100 (1974), 131-170.

T. Shintani, On Dirichlet Series Whose Coefficients Are Class-Numbers of Integral Binary Cubic Forms, J. Math. Soc. Japan 24 (1972), 132-188.

J. Tate, Fourier Analysis in Number Fields and Hecke's Zeta Functions, Ph. D. thesis, Princeton (1950).

A. Weil, Sur Certains Groupes d'Oprateurs Unitaires, Acta Math. 111 (1964), 143-211.

A. Weil, Sur la formule de Siegel dans la Thory des Groupes Classiques, Acta Math. 113, (1965), 1-87.

A. Weil, Basic Number Theory, 2nd edition, Springer-Verlag, (1974).

D. J. Wright, The Adelic Zeta Function Associated to the Space of Binary Cubic Forms Part I: Global Theory, Math. Ann. 270 (1985), 503-534.