DEFORMATION COMPLEXES FOR ALGEBRAIC OPERADS AND THEIR APPLICATIONS

A Dissertation
Submitted to
the Temple University Graduate Board

---

in Partial Fulfillment
of the Requirements for the Degree of
DOCTOR OF PHILOSOPHY

---

by
Brian Paljug
May 2015

Examining Committee Members:

Vasily Dolgushev, Advisory Chair, Mathematics
Dave Futer, Mathematics
Martin Lorenz, Mathematics
Jonathan Block, The University of Pennsylvania, Mathematics

ABSTRACT

DEFORMATION COMPLEXES FOR ALGEBRAIC OPERADS
AND THEIR APPLICATIONS

Brian Paljug
DOCTOR OF PHILOSOPHY

Temple University, May 2015

Dr. Vasily Dolgushev, Chair

Given a reduced cooperad $\mathcal{C}$, we consider the 2-colored operad $\mathrm{Cyl}(\mathcal{C})$ which governs diagrams $U : V \rightsquigarrow W$, where V, W are $\mathrm{Cobar}(\mathcal{C})$-algebras, and U is an ∞-morphism. We then investigate the deformation complexes of $\mathrm{Cyl}(\mathcal{C})$ and $\mathrm{Cobar}(\mathcal{C})$. Our main result is that the restriction maps between between the deformation complexes $\mathrm{Der}'(\mathrm{Cyl}(\mathcal{C}))$ and $\mathrm{Der}'(\mathrm{Cobar}(\mathcal{C}))$ are homotopic quasi-isomorphisms of filtered Lie algebras. We show how this result may be applied to modifying diagrams of homotopy algebras by derived automorphism.

We then recall that Tamarkin's construction gives us a map from the set of Drinfeld associators to the homotopy classes of L_∞-quasi-isomorphisms for Hochschild cochains of a polynomial algebra. Due to results of V. Drinfeld and T. Willwacher, both the source and the target of this map are equipped with natural actions of the Grothendieck-Teichmueller group GRT_1. We use our earlier results to prove that this map from the set of Drinfeld associators to the set of homotopy classes of L_∞-quasi-isomorphisms for Hochschild cochains is GRT_1-equivariant.

ACKNOWLEDGEMENTS

I would foremost like to thank my advisor Vasily Dolgushev, who has been a tremendous educator and inspiration to me in the years I've known him. I've been honored to learn from and work with Vasily, and thank him for all his patience, encouragement, explanations, and answers.

I would also like to thank my examining committee Jonathan Block, Martin Lorenz, and Dave Futer. I would especially like to thank Dave, who has been a good friend, mentor, and role model in my time at Temple.

I would like to thank the Temple University Mathematics Department, for being a wonderful place to spend the last several years and pursue mathematics. For that I thank the entire department faculty and students – especially my good friends Christian Millichap and Jessie Hamm, who I've had the very great pleasure of meeting and knowing.

Finally, I would like to thank my family and friends for their love and support over the years, especially my wife Kay, who was willing to move to the frozen northern tundra of Philadelphia to be with me.

To Kay

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

Homotopy algebras and morphisms appear in many areas throughout mathematics; in homological algebra in the form of algebraic transfer theorems, in geometry in the study of iterated loop spaces, in deformation quantization in Kontsevich's formality theorem, and so on. Much work has been done to find the correct framework in which to study homotopy algebras, and the theory of operads is one such attempt. The complicated coherence relations that define homotopy algebras are encoded in the language of operads, which are easily manipulated with homological or combinatorial techniques; see [27] for an excellent overview of these methods. This is the approach taken in this dissertation.

While there is a notion of morphisms between homotopy algebras of a specific type, in practice and theory one is interested in the looser notions of ∞-morphisms, which themselves satisfy some complicated system of coherence relations. Since homotopy algebras can be defined as algebras over a specific operad, it seems natural to ask if ∞-morphisms can be defined in the language of operads. The answer is provided in [19] via a 2-colored "cylinder construction" operad, which we restate and study further; similar ideas were also considered in [28], [5] and [16]. Specifically, given a cooperad $\mathcal{C}$ we construct and study a 2-colored operad $\mathrm{Cyl}(\mathcal{C})$ that governs pairs of homotopy algebras and ∞-morphisms between them.

Our main goal is to answer the following question; given a pair of homotopy algebras and an ∞-morphism between them, can we change the homotopy algebras and the ∞-morphism simultaneously to get new homotopy algebras and a new ∞-morphism (all of the same type)? More specifically, given a derivation of the operad $\mathrm{Cobar}(\mathcal{C})$ governing the homotopy algebras V and W, we can exponentiate that derivation to an automorphism of $\mathrm{Cobar}(\mathcal{C})$ and use that automorphism to define new $\mathrm{Cobar}(\mathcal{C})$-algebra structures on V and W via pullback; can we do the same to an ∞-morphism between V and W, to create a new ∞-morphism that respects the new $\mathrm{Cobar}(\mathcal{C})$-algebra structures? We show that this is possible, and moreover that the answer is unique up to homotopy, using the previously mentioned techniques of operadic homological algebra. In particular, we have Theorem 4.1, which says that the natural restriction maps between between the deformation complexes $\mathrm{Der}'(\mathrm{Cyl}(\mathcal{C}))$ and $\mathrm{Der}'(\mathrm{Cobar}(\mathcal{C}))$ are homotopic quasi-isomorphisms of filtered Lie algebras.

Theorem. *The maps*

$$\mathrm{res}_\alpha, \mathrm{res}_\beta : \quad \mathrm{Der}(\mathrm{Cyl}(\mathcal{C})) \quad \longrightarrow \quad \mathrm{Der}(\mathrm{Cobar}(\mathcal{C}))$$

given by restricting to a single color α or β are homotopic quasi-isomorphisms of dg Lie algebras at all filtration levels.

This leads us to Theorem 4.2:

Theorem. *The group homomorphisms*

$$\mathrm{res}_\alpha, \mathrm{res}_\beta : \quad \mathrm{Aut}'(\mathrm{Cyl}(\mathcal{C})) \quad \longrightarrow \quad \mathrm{Aut}'(\mathrm{Cobar}(\mathcal{C}))$$

induce identical isomorphisms on homotopy classes:

$$\mathrm{res} : \quad \mathrm{hAut}'(\mathrm{Cyl}(\mathcal{C})) \quad \longrightarrow \quad \mathrm{hAut}'(\mathrm{Cobar}(\mathcal{C})).$$

This then allows us to answer the motivating question, shown in Theorem 5.1:

Theorem. *Let V and W be $\mathrm{Cobar}(\mathcal{C})$-algebras for a cooperad $\mathcal{C}$, and let $U : V \rightsquigarrow W$ be an ∞-morphism between them. Given a degree 0 closed derivation $D \in \mathrm{Der}'(\mathrm{Cobar}(\mathcal{C}))$, there exists a degree 0 cocycle $\widetilde{D} \in \mathrm{Der}'(\mathrm{Cyl}(\mathcal{C}))$ such that D, $\widetilde{D}_\alpha$, and $\widetilde{D}_\beta$ are cohomologous in $\mathrm{Der}'(\mathrm{Cobar}(\mathcal{C}))$. Therefore we can construct*

$$U^{\widetilde{D}} : V^{\widetilde{D}_\alpha} \rightsquigarrow W^{\widetilde{D}_\beta}$$

such that $V^{\widetilde{D}_\alpha}$ is homotopy equivalent to V^D and $W^{\widetilde{D}_\beta}$ is homotopy equivalent to W^D, and so that the linear term of U is unchanged: $U^{\widetilde{D}}_{(0)} = U_{(0)}$.

We then provide an application of the previous results to a question in deformation quantization. Let $\Bbbk$ be a field of characteristic zero, $A = \Bbbk[x^1, x^2, \dots, x^d]$ be the algebra of functions on the affine space $\Bbbk^d$, and V_A be the algebra of polyvector fields on $\Bbbk^d$. Let us recall that Tamarkin's construction [21], [10] gives us a map from the set of Drinfeld associators to the set of homotopy classes of L_∞-quasi-isomorphisms from V_A to the Hochschild cochain complex $C^\bullet(A) := C^\bullet(A, A)$ of A.

In paper [38], among proving many other things, Thomas Willwacher constructed a natural action of the Grothendieck-Teichmueller group GRT_1 from [17] on the set of homotopy classes of L_∞-quasi-isomorphisms from V_A to $C^\bullet(A)$. On the other hand, it is known [17] that the group GRT_1 acts simply transitively on the set of Drinfeld associators.

The goal of the second half of this dissertation is to prove GRT_1-equivariance of the map resulting from Tamarkin's construction using Theorem 4.3 from [33], providing the necessary background and preliminary results. This is given in Theorem 7.1:

Theorem. *Let $\pi_0\big(\mathrm{Ger}_\infty \to \mathrm{Braces}\big)$ be the set of homotopy classes of operad maps* (6.1) *from the dg operad Ger_∞ governing homotopy Gerstenhaber algebras to the dg operad Braces of brace trees. Let $\pi_0\big(V_A \rightsquigarrow C^\bullet(A)\big)$ be the set of homotopy classes of $\Lambda\,\mathrm{Lie}_\infty$-quasi-isomorphisms from the algebra V_A of polyvector fields to the algebra $C^\bullet(A)$ of Hochschild cochains of a graded affine space. Then Tamarkin's construction of formality morphisms*

$$\mathfrak{T} : \pi_0\big(\mathrm{Ger}_\infty \to \mathrm{Braces}\big) \to \pi_0\big(V_A \rightsquigarrow C^\bullet(A)\big)$$

commutes with the action of the group $GRT_1 = \exp(H^0(\mathrm{Der}'(\mathrm{Ger}_\infty)))$.

We should remark that the statement about GRT_1-equivariance of Tamarkin's construction was made in [38] (see the last sentence of Section 10.2 in [38, Version 3]) in which the author stated that "it is easy to see". We also prove various statements related to Tamarkin's construction [21], [10] which are "known to specialists" but not proved in the literature in the desired generality. In fact, even the formulation of the problem of GRT_1-equivariance of Tamarkin's construction requires some additional work.

Here, Tamarkin's construction is presented in the slightly more general setting of graded affine space versus the particular case of the usual affine space. Thus, A is always the free (graded) commutative algebra over $\mathbb{k}$ in variables $x^1, x^2, \dots, x^d$ of (not necessarily zero) degrees $t_1, t_2, \dots, t_d$, respectively. Furthermore, V_A denotes the Gerstenhaber algebra of polyvector fields on the corresponding graded affine space, i.e.

$$V_A := S_A\big(\mathbf{s}\ \mathrm{Der}_{\mathbb{k}}(A)\big)\,,$$

where $\mathrm{Der}_{\mathbb{k}}(A)$ denotes the A-module of derivations of A, $\mathbf{s}$ is the operator which shifts the degree up by 1, and $S_A(M)$ denotes the free (graded) commutative algebra on the A-module M.

The dissertation is organized as follows.

In Chapter 2, we review the notation and background material needed for the remainder of the dissertation. This chapter tries to be somewhat self-contained with regards to content, if not detail.

In Chapter 3, we construct a 2-colored dg operad $\mathrm{Cyl}(\mathcal{C})$ for any cooperad $\mathcal{C}$ that governs pairs of $\mathrm{Cobar}(\mathcal{C})$-algebras and an ∞-morphism between them. We also investigate the cohomology of $\mathrm{Cyl}(\mathcal{C})$, and construct homotopic quasi-isomorphisms between $\mathrm{Cobar}(\mathcal{C})$ and the mixed-color part of $\mathrm{Cyl}(\mathcal{C})$ (Corollary 3.1).

In Chapter 4 we turn to studying the dg Lie algebra of derivations of $\mathrm{Cyl}(\mathcal{C})$, and how it relates to derivations of $\mathrm{Cobar}(\mathcal{C})$ – these are the titular "deformation complexes of operads." In particular we show in Theorem 4.1 that the obvious restriction maps between $\mathrm{Der}(\mathrm{Cyl}(\mathcal{C}))$ and $\mathrm{Der}(\mathrm{Cobar}(\mathcal{C}))$ are homotopic quasi-isomorphisms. We then conclude in Theorem 4.2 that, after exponentiating, they yield the same group isomorphism on homotopy classes of maps.

In Chapter 5 we show how the previous results may be applied to the motivating question, that is, how to modify ∞-morphisms of homotopy algebra via derived automorphism. This is the content of Theorem 5.1. We also discuss various ways in which this procedure is unique up to homotopy.

In Chapter 6, we briefly review the main part of Tamarkin's construction and prove that it gives us a map $\mathfrak{T}$ (see Eq. (6.21)) from the set of homotopy classes of certain quasi-isomorphisms of dg operads to the set of homotopy classes of L_∞-quasi-isomorphisms for Hochschild cochains of A.

In Chapter 7, we introduce a (prounipotent) group which is isomorphic (due to Willwacher's theorem [38, Theorem 1.2]) to the prounipotent part GRT_1 of the Grothendieck-Teichmüller group GRT introduced in [17] by V. Drinfeld. We recall from [38] the actions of the group (isomorphic to GRT_1) both on the source and the target of the map $\mathfrak{T}$ (6.21). Finally, we prove the main result of these chapters (see Theorem 7.1) which says that Tamarkin's map $\mathfrak{T}$ (see Eq. (6.21)) is GRT_1-equivariant.

In Chapter 8, we recall how to use the map $\mathfrak{T}$ (see Eq. (6.21) from Chapter 6), a solution of the Deligne conjecture on the Hochschild complex, and the formality of the operad of little discs [34] to construct a map from the set of Drinfeld associators to the set of homotopy classes of L_∞-quasi-isomorphisms for Hochschild cochains of A. Finally, we deduce, from Theorem 7.1, GRT_1-equivariance of the resulting map from the set of Drinfeld associators. The latter statement (see Corollary 8.1 in Chapter 8) can be deduced from what is written in [38] and Theorem 7.1 given in Chapter 7. However, we decided to add Chapter 8 just to make the story more complete.

Appendices, at the end of the dissertation, are devoted to proofs of various technical statements used in the body of the dissertation.

It should be noted that this dissertation combines the content of papers [33] and [10], rearranged, modified, and supplemented as necessary. While working on this dissertation, the author was partionally supported by NSF grants DMS-0856196 and DMS-1161867.

CHAPTER 2

PRELIMINARIES

2.1 Notation & conventions

We begin be establishing notation and conventions, before reviewing other foundational material. Throughout, the ground field $\Bbbk$ has characteristic zero. For most algebraic structures considered here, the underlying symmetric monoidal category is the category $\mathsf{Ch}_{\Bbbk}$ of unbounded cochain complexes of $\Bbbk$-vector spaces. We will frequently use the ubiquitous combination "dg" (differential graded) to refer to algebraic objects in $\mathsf{Ch}_{\Bbbk}$, and denote their differentials by d or ∂. For a cochain complex V we denote by $\mathrm{s}\, V$ (resp. by $\mathrm{s}^{-1}\, V$) the suspension (resp. the desuspension) of V. In other words,

$$\big(\mathrm{s}\, V\big)^{\bullet} = V^{\bullet - 1}\,, \qquad \big(\mathrm{s}^{-1}\, V\big)^{\bullet} = V^{\bullet + 1}\,.$$

Any $\mathbb{Z}$-graded vector space V is tacitly considered as the cochain complex with the zero differential. For a homogeneous vector v in a cochain complex or a graded vector space the notation $|v|$ is reserved for its degree.

The notation S_n is reserved for the symmetric group on n letters and $\mathrm{Sh}_{p_1,\dots,p_k}$ denotes the subset of $(p_1, \dots, p_k)$-shuffles in S_n, i.e. $\mathrm{Sh}_{p_1,\dots,p_k}$ consists of elements $\sigma \in S_n$, $n = p_1 + p_2 + \cdots + p_k$ such that

$$\sigma(1) < \sigma(2) < \dots < \sigma(p_1),$$
$$\sigma(p_1+1) < \sigma(p_1+2) < \dots < \sigma(p_1+p_2),$$
$$\dots$$
$$\sigma(n-p_k+1) < \sigma(n-p_k+2) < \dots < \sigma(n)\,.$$

We tacitly assume the Koszul sign rule. In particular,

$$(-1)^{\varepsilon(\sigma;v_1,\dots,v_m)}$$

will always denote the sign factor corresponding to the permutation $\sigma \in S_m$ of homogeneous vectors $v_1, v_2, \dots, v_m$. Namely,

$$(-1)^{\varepsilon(\sigma;v_1,\dots,v_m)} := \prod_{(i<j)} (-1)^{|v_i||v_j|}\,, \tag{2.1}$$

where the product is taken over all inversions $(i<j)$ of $\sigma \in S_m$.

For a pair V, W of $\mathbb{Z}$-graded vector spaces we denote by

$$\mathrm{Hom}(V,W)$$

the corresponding inner-hom object in the category of $\mathbb{Z}$-graded vector spaces, i.e.

$$\mathrm{Hom}(V,W) := \bigoplus_m \mathrm{Hom}^m_{\Bbbk}(V,W)\,, \tag{2.2}$$

where $\mathrm{Hom}^m_{\Bbbk}(V,W)$ consists of $\Bbbk$-linear maps $f : V \to W$ such that

$$f(V^{\bullet}) \subset W^{\bullet+m}\,.$$

For a commutative algebra B and a B-module M, the notation $S_B(M)$ (resp. $\underline{S}_B(M)$) is reserved for the symmetric B-algebra (resp. the truncated symmetric B-algebra) on M, i.e.

$$S_B(M) := B \oplus M \oplus S^2_B(M) \oplus S^3_B(M) \oplus \dots\,,$$

and

$$\underline{S}_B(M) := M \oplus S^2_B(M) \oplus S^3_B(M) \oplus \dots\,.$$

For an A_∞-algebra A, the notation $C^{\bullet}(A)$ is reserved for the Hochschild cochain complex of A with coefficients in A.

2.2 Operads

For a general introduction to the theory of operads, see [11] or [27]; we recall the basic definitions here.

Definition 2.1. *A* collection $\mathcal{O}$ *is a set of cochain complexes indexed by* $\mathbb{Z}_{\geq 0}$

$$\mathcal{O} = \{\mathcal{O}(n)\}_{n\geq 0}$$

such that $\mathcal{O}(n)$ *is a left* S_n*-representation. We say that* $\mathcal{O}$ *is an* operad *if it is equipped with linear composition maps*

$$\mu : \mathcal{O}(n) \otimes \mathcal{O}(k_1) \otimes ... \otimes \mathcal{O}(k_n) \to \mathcal{O}(k_1 + ... + k_n) \tag{2.3}$$

that satisfy appropriate associativity and symmetry axioms, and has a distinguished element id $\in \mathcal{O}(1)$ *that satisfies an appropriate unit axiom.*

The terms composition and multiplication are often used interchangeably. Morphisms of operads must respect composition, symmetric actions, and units in the obvious ways.

There are two equivalent ways of describing operadic multiplication. The first is in terms of the elementary insertion maps

$$\circ_i : \mathcal{O}(n) \otimes \mathcal{O}(k) \to \mathcal{O}(n + k - 1), \quad 1 \leq i \leq n \tag{2.4}$$

which will satisfy their own associativity axiom [11, Definition 3.2]. It is straightforward to see how elementary insertions and composition determine each other.

The second way highlights the combinatorial nature of operads. Following [11], let $\mathsf{Tree}(n)$ be the category of n-labeled rooted, planted, planar trees. That is, planar trees with a distinguished valency 1 vertex (the root), and such that the other valency 1 vertices (the leaves) carry labels 1 to n. Non-leaf, non-root vertices are called nodal or internal. For every collection $\mathcal{O}$, we have the functor

$$\underline{\mathcal{O}}_n : \mathsf{Tree}(n) \longrightarrow \mathsf{Ch}_{\Bbbk} \tag{2.5}$$

that maps a tree $\mathbf{t}$ to the tensor product of components of $\mathcal{O}$, according to the arity of vertices and planar structure. Explicitly,

$$\underline{\mathcal{O}}_n(\mathbf{t}) = \bigotimes_{x \in V_{\mathrm{nod}(\mathbf{t})}} \mathcal{O}(m(x)) \tag{2.6}$$

where $V_{\mathrm{nod}(\mathbf{t})}$ is the set of all nodal vertices of $\mathbf{t}$, and $m(x)$ is the number of incoming edges at the vertex x, and the order of factors in the right side of equation 2.6 agrees with the natural order on the set of nodal vertices coming from the planar structure.

With this in mind, a collection $\mathcal{O}$ is an operad if it is equipped with composition maps

$$\mu_{\mathbf{t}} : \underline{\mathcal{O}}_n(\mathbf{t}) \longrightarrow \mathcal{O}(n). \tag{2.7}$$

To recover the first notion of operadic composition and elementary insertions, we need two subcategories of $\mathsf{Tree}(n)$. Elementary insertations are obtained by focusing on $\mathsf{Tree}_2(n)$, the full subcategory of $\mathsf{Tree}(n)$ consisting of trees with exactly 2 internal vertices. The first notion of composition is obtained by focusing on $\mathsf{PF}_k(n)$ ("pitchforks"), the full subcategory of $\mathsf{Tree}(n)$ consisting of trees with exactly $k+1$ internal vertices, one of which has height 1, and the other k have height exactly 2. Some examples of such trees can be found in figures 2.1 and 2.2.

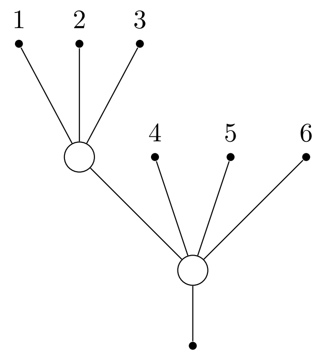


Figure 2.1: An element of $\mathsf{Tree}_2(6)$.

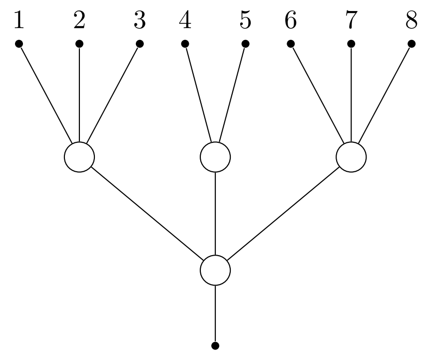


Figure 2.2: An element of $\mathsf{PF}_3(8)$.

Example 2.1. *Given a cochain complex V, the* endomorphism operad End_V *is defined by*

$$\mathrm{End}_V(n) = \mathrm{Hom}_{\Bbbk}(V^{\otimes n}, V)$$

with S_n acting on $\mathrm{End}_V(n)$ by permutation of arguments, operadic composition determined by function composition, and the unit the identity morphism on V.

If we have a map of operads

$$\mathcal{O} \longrightarrow \mathrm{End}_V \tag{2.8}$$

then we say that V *is an algebra over* $\mathcal{O}$, or an $\mathcal{O}$*-algebra*. Intuitively, this means that V possesses multi-ary operations governed by the elements of $\mathcal{O}$. That is, we have multiplication maps

$$\mu_n : \mathcal{O}(n) \otimes V^{\otimes n} \to V \tag{2.9}$$

for all $n \geq 0$, satisfying appropriate associativity, equivariance, and unit axioms [29]. With this in mind, we define the free $\mathcal{O}$-algebra to be

$$\mathcal{O}(V) = \bigoplus_{n \geq 0} \left(\mathcal{O}(n) \otimes V^{\otimes n}\right)_{S_n}$$

with differential coming from the differential on $\mathcal{O}$ and the differential on V.

Example 2.2. *The operad* Assoc *is defined by* $\mathrm{Assoc}(0) = \mathbf{0}$ *and*

$$\mathrm{Assoc}(n) = \Bbbk[S_n]$$

with S_n acting on $\mathrm{Assoc}(n)$ by composition of permutatons, operadic composition determined by insertion of permutations, and the unit the identity permutation on S_1.

Equivalently, we may define $\mathrm{Assoc}(n)$ to be spanned by associative words in noncommuting letters $x_1, x_2, ..., x_n$ such that each letter appears exactly once. The monomial $x_{i_1}...x_{i_n}$ corresponds to the permutation $(x_{i_1}...x_{i_n})$.

Algebras over Assoc are exactly non-unital associative algebras.

We may similarly define the operads Com, Lie, and Ger governing non-unital commutative, Lie, and Gerstenhaber algebras, respectively.

Example 2.3. *The* initial operad $*$ *is defined by*

$$*(n) = \begin{cases} \mathbb{k} & n = 1 \\ 0 & \textit{otherwise} \end{cases}$$

which uniquely possesses the structure of an operad, and is indeed the initial object is the category of operads.

With this example in mind, an operad is called *augmented* if there is a map of operads

$$\varepsilon : \mathcal{O} \longrightarrow *.$$

Given any augmented operad $\mathcal{O}$, we will denote the kernel of the augmentation map by $\mathcal{O}_\circ$, which carries the structure of an operad without unit (or *pseudo-operad*).

If we dualize the above we obtain the notion of *cooperads*, which are collections

$$\mathcal{C} = \{\mathcal{C}(n)\}_{n\geq 0}$$

with comultiplication maps

$$\Delta : \mathcal{C}(k_1 + ... + k_n) \to \mathcal{C}(n) \otimes \mathcal{C}(k_1) \otimes ... \otimes \mathcal{C}(k_n) \tag{2.10}$$

and counits, satisfying the appropriate dual axioms. As before, comultiplications are determined by the elementary coinsertaions

$$\Delta_i : \mathcal{C}(n + k - 1) \to \mathcal{C}(n) \otimes \mathcal{C}(k) \tag{2.11}$$

or the more general comultiplication maps

$$\Delta_{\mathbf{t}} : \mathcal{C}(n) \longrightarrow \underline{\mathcal{C}}_n(\mathbf{t}). \tag{2.12}$$

If an operad $\mathcal{O}$ consists of finite dimensional vector spaces, then the collection of linear duals naturally carries the structure of a cooperad, and will be denoted $\mathcal{O}^*$. In this way we obtain the cooperad $\mathrm{coCom} = \mathrm{Com}^*$ governing cocommutative coalgebras (without counit), for example.

Given that $*$ is uniquely a cooperad, a cooperad is said to be coaugmented if there is a coaugmentation map

$$* \longrightarrow \mathcal{C} \tag{2.13}$$

the cokernel of which will be denoted $\mathcal{C}_\circ$, naturally a cooperad without counit (or *pseudo-cooperad*). A coaugmented cooperad $\mathcal{C}$ is called *reduced* if

$$\mathcal{C}(0) = \{0\} \qquad\qquad \mathcal{C}(1) = \Bbbk$$

and hence $\mathcal{C}_\circ(0) = \mathcal{C}_\circ(1) = \{0\}$. We will henceforth assume that all cooperads are reduced.

We say that V is a coalgebra over the cooperad $\mathcal{C}$ if we have comultiplication maps

$$\Delta_n : V \to \mathcal{C}(n) \otimes V^{\otimes n} \tag{2.14}$$

satisfying appropriate coassicativity, equivariance, and counit axioms. The cofree $\mathcal{C}$-coalgebra is then defined to be

$$\mathcal{C}(V) = \bigoplus_{n\geq 0} \left(\mathcal{C}(n) \otimes V^{\otimes n}\right)^{S_n} \tag{2.15}$$

with differential coming from the differential on $\mathcal{C}$ and the differential on V[1].

Example 2.4. *We denote by Λ the underlying collection of the endomorphism operad*

$$\mathrm{End}_{\mathbf{s}\,\Bbbk}$$

of the 1-dimensional space $\mathbf{s}\,\Bbbk$ *placed in degree* 1*. The n-th space of Λ is*

$$\Lambda(n) = \mathrm{sgn}_n \otimes \mathbf{s}^{1-n}\,,$$

where sgn_n denotes the sign representation of the symmetric group S_n. Λ is naturally an operad and a cooperad.

The tensor product of (co)operads is obtained by taking the level-wise tensor product of cochain complexes, which will then naturally carry the structure of a (co)operad. Explicitly,

$$(\mathcal{P}_1 \otimes \mathcal{P}_2)(n) = \mathcal{P}_1(n) \otimes \mathcal{P}_2(n). \tag{2.16}$$

[1]Given that we are working over a field of characteristic 0, coinvariants and invariants are isomorphic. For example, we have as a result that $\mathrm{Com}(V) = \mathrm{coCom}(V) = \underline{S}(V)$.

Then for a (co)operad P, we denote by ΛP the (co)operad which is obtained from P by tensoring with Λ:

$$\Lambda P := \Lambda \otimes P\,. \tag{2.17}$$

It is clear that tensoring with

$$\Lambda^{-1} := \mathrm{End}_{\mathbf{s}^{-1}\,\Bbbk} \tag{2.18}$$

gives us the inverse of the operation $P \mapsto \Lambda P$.

Given a collection $\mathcal{Q}$, we can form the free operad $\mathbb{OP}(\mathcal{Q})$ [11, Section 3.6]. It is generally most convenient to think of elements of $\mathbb{OP}(\mathcal{Q})$ as rooted trees with internal vertices decorated by elements of $\mathcal{Q}$, subject to an appropriate symmetry relation; with this in mind, $(\mathbf{t}; x_1, ..., x_k)$ is the element of $\mathbb{OP}(\mathcal{Q})(n)$ where the n-labeled tree **t** has k internal vertices decorated by the elements $x_1, ..., x_k$ of $\mathcal{M}$, according to the total ordering on internal vertices. We will frequently identify an element $x \in \mathcal{Q}(n)$ with the standard n-corolla decorated by x in $\mathbb{OP}(\mathcal{Q})(n)$. It is occasionally useful to remember that at each level $\mathbb{OP}(\mathcal{Q})(n) = \mathrm{colim}\, \underline{\mathcal{Q}}_n$ (strictly, speaking, this only defines the free pseudo-operad, with the free operad then obtained by formally adjoining a unit). Since the S_n action on $\mathbb{OP}(\mathcal{Q})(n)$ permutes the labels, we will simplify diagrams by omitting these labels when drawing elements of $\mathbb{OP}(\mathcal{Q})$.

Definition 2.2. *Given a coaugmented cooperad $\mathcal{C}$, we define the* cobar construction $\mathrm{Cobar}(\mathcal{C})$ to be $\mathbb{OP}(\mathrm{s}\,\mathcal{C}_\circ)$, *with the differential defined on generators* $\mathrm{s}\,x \in \mathrm{s}\,\mathcal{C}_\circ$ *by*

$$\partial_{\mathrm{Cobar}}(\mathrm{s}\,x) = -\,\mathrm{s}\,\partial_{\mathcal{C}}(x) - \sum_{z \in \mathrm{Isom}(\mathsf{Tree}_2(n))} (-1)^{|x_1|}(\boldsymbol{t}_z; \mathrm{s}\,x_1, \mathrm{s}\,x_2) \tag{2.19}$$

where the sum is taken over all isomorphism classes of $\mathsf{Tree}_2(n)$, *with* $\boldsymbol{t}_z$ *a representative of the isomorphism class* $z \in \mathrm{Isom}(\mathsf{Tree}_2(n))$, *and* $\Delta_{\boldsymbol{t}_z}(x) = \sum x_1 \otimes x_2$.

Note that we use Sweedler-type notation in the above equation, and will continue to do so throughout the dissertation.

Much of the above extends immediately to the colored setting, so we will only focus on certain ideas and notation; our primary reference is [8]. We will focus on the 2-colored setting for now, and the 3-colored versions should be clear (and will only be minimally needed in what follows). We will refer to our 2 colors as α and β. Given a 2-colored collection $\mathcal{Q}$, $\mathcal{Q}(a, b; \alpha)$ denotes the level of $\mathcal{Q}$ with a inputs of color α, b inputs of color β, and output of color α. Similarly, $\mathcal{Q}(a, b; \beta)$ indicates that the output is of color β. As in the single-color case, we have a free colored operad construction, governed by colored trees. The category of colored n-labeled trees is defined similarly to $\mathsf{Tree}(n)$, except that the edges carry colors, and morphisms must respect the coloring. In figures, edges with color α will be represented by solid lines, while edges of color β will be represented by dashed lines.

As before, we will need two subcategories of colored n-labeled trees, but slightly more specialized than those given above. The first is $\mathsf{Tree}_2'(n)$, the full subcategory of $\mathsf{Tree}(n)$ consisting of trees with exactly 2 internal vertices, such that the root edge carries color β, and all other edges carry color α. The second is $\mathsf{PF}_k'(n)$, the full subcategory of colored $\mathsf{Tree}(n)$ consisting of trees with exactly $k + 1$ internal vertices, one of which has height 1, and the other k have height exactly 2, such that all leaf edges carry color α and all other edges carry color β. Some examples of such trees can be found in figures 2.3 and 2.4.

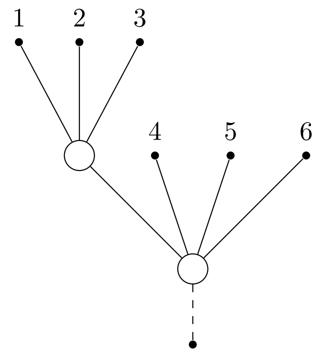


Figure 2.3: An element of $\mathsf{Tree}_2'(6)$.

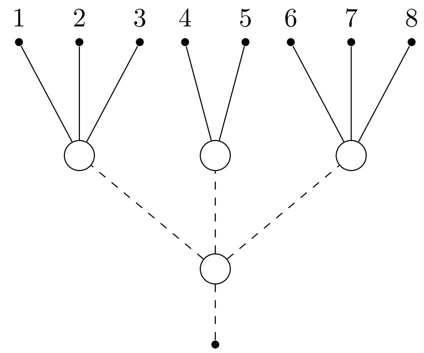


Figure 2.4: An element of $\mathsf{PF}_3'(8)$.

Example 2.5. *Given vector spaces V, W, their* 2-colored endomorphism operad $\mathrm{End}_{V,W}$ *is defined by*

$$\mathrm{End}_{V,W}(n, m, \alpha) = \mathrm{Hom}_{\Bbbk}(V^{\otimes n} \otimes W^{\otimes m}, V)$$

$$\mathrm{End}_{V,W}(n,m,\beta) = \mathrm{Hom}_{\Bbbk}(V^{\otimes n} \otimes W^{\otimes m}, W)$$

with the obvious operadic structure.

Just as before, algebras over colored operads are defined to be operad maps to $\mathrm{End}_{V,W}$.

2.3 Deformation complexes of operads

Definition 2.3. *[11, Section 6.1] Let $\mathcal{O}$ be an operad, and D a linear map on $\mathcal{O}$. We say that D is a derivation of $\mathcal{O}$ if*

$$D(x \circ_i y) = D(x) \circ_i y + (-1)^{|x|} x \circ_i D(y) \tag{2.20}$$

for all elementary insertions $\circ_i$, and all $x, y \in \mathcal{O}$.

More generally, derivations satisfy a Leibniz rule with respect to operadic composition. The space of derivations of an operad is denoted $\mathrm{Der}(\mathcal{O})$, and is a dg Lie algebra with the graded commutator bracket

$$[D_1, D_2] = D_1 \circ D_2 - (-1)^{|D_1||D_2|} D_2 \circ D_1 \tag{2.21}$$

and with differential given by the bracket with the internal differential $\partial_{\mathcal{O}}$

$$\partial(D) = [\partial_{\mathcal{O}}, D]. \tag{2.22}$$

Following [31, 32], when $\mathcal{O} = \mathrm{Cobar}(\mathcal{C})$ [2] for a reduced cooperad $\mathcal{C}$, we call $\mathrm{Der}(\mathrm{Cobar}(\mathcal{C}))$ the *deformation complex of* $\mathrm{Cobar}(\mathcal{C})$. [3]

2.4 Homotopy algebras

We will briefly recall the definitions of homotopy algebras and ∞-morphisms; see [11], [25], and [27] for more thorough introductions to the subject. Given a

[2] Or more generally, any operad that is cofibrant in the model category structure inherited from the standard model category structure on $\mathsf{Ch}_{\Bbbk}$.

[3] Papers [31, 32] actually introduce and study the deformation complex of morphisms of PROPs, and therefore treat this subject in greater generality. What we call the deformation complex of an operad $\mathcal{O}$, Merkulov and Vallette call the deformation complex of the identity morphism on $\mathcal{O}$.

coaugmented cooperad $\mathcal{C}$, we will use the following "pedestrian" definition of homotopy algebras.

Definition 2.4. *A* homotopy algebra of type $\mathcal{C}$ *is an algebra* V *over* $\mathrm{Cobar}(\mathcal{C})$. *That is, we have a map of operads*

$$\mathrm{Cobar}(\mathcal{C}) \longrightarrow \mathrm{End}_V . \tag{2.23}$$

This is equivalent [11] to a coderivation Q_V on $\mathcal{C}(V)$, the cofree coalgebra generated by V over the cooperad $\mathcal{C}$, that satisfies the Maurer-Cartan equation

$$\partial(Q_V) + Q_V \circ Q_V = 0 \tag{2.24}$$

(equivalently, $d_{\mathcal{C}(V)} + Q_V$ is a differential on $\mathcal{C}(V)$). In the case that $\mathcal{C}$ is the Koszul dual of an operad $\mathcal{O}$, it is common to call V an $\mathcal{O}_\infty$-algebra.

Example 2.6. A_∞*-algebras (or homotopy associative algebras) are algebras over* $\mathrm{Cobar}(\Lambda\,\mathrm{coAssoc})$. *It is possible to unravel what this means in terms of operations and relations [27, Section 10.1.10]; an* A_∞*-algebra possesses* n*-ary higher multiplications for* $n \geq 0$*, that are "associative up to homotopy" in a precise sense.*

Example 2.7. L_∞*-algebras (or homotopy Lie algebras) are algebras over*

$$\mathrm{Lie}_\infty = \mathrm{Cobar}(\Lambda\,\mathrm{coCom}).$$

It is possible to unravel what this means in terms of operations and relations [27, Section 10.1.12]; an L_∞*-algebra possesses* n*-ary higher brackets for* $n \geq 0$*, that are skew-symmetric and satisfy a generalized Jacobi identity "up to homotopy" in a precise sense. For technical reasons, it is often easier to work with* $\Lambda\,\mathrm{Lie}_\infty$*-algebras, which are algebras over* $\mathrm{Cobar}(\Lambda^2\,\mathrm{coCom})$.

Example 2.8. Ger_∞*-algebras (or homotopy Gerstenhaber algebras) are algebras over*

$$\mathrm{Ger}_\infty = \mathrm{Cobar}(\mathrm{Ger}^\vee)$$

where $\mathrm{Ger}^\vee = (\Lambda^{-2}\,\mathrm{Ger})^*$. Ger_∞*-algebras possess both higher multiplications and brackets. See the following section for a further discussion on* $\mathrm{Ger}^\vee$.

While there is the natural notion of morphisms of $\mathcal{O}$-algebras for any operad $\mathcal{O}$, in this setting we have the richer notion of ∞-morphisms. An ∞-*morphism* between two homotopy algebras V, W of type $\mathcal{C}$ is a map of dg coalgebras

$$U : (\mathcal{C}(V), d_{\mathcal{C}(V)} + Q_V) \longrightarrow (\mathcal{C}(W), d_{\mathcal{C}(W)} + Q_W). \tag{2.25}$$

We denote an ∞-morphism by $U : V \rightsquigarrow W$. An ∞-isomorphism (resp. quasi-isomorphism) is an ∞-morphism such that the linear term $U_{(0)} : V \to W$ is an isomorphism (resp. quasi-isomorphism) of complexes. In the event that V and W are ∞-quasi-isomorphic, we will say that V and W are homotopy equivalent. It is possible to describe ∞-morphisms more directly in terms of diagrams [19]; equivalently, the results of Chapters 3 and 5 show how one may describe ∞-morphisms as an algebra over a certain 2-colored operad.

2.5 A basis for the operad $\Lambda^{-2}\,\mathrm{Ger}$

Recall that Ger_∞-algebras (or homotopy Gerstenhaber algebras) are governed by the dg operad

$$\mathrm{Cobar}(\mathrm{Ger}^\vee)\,, \tag{2.26}$$

where $\mathrm{Ger}^\vee$ is the cooperad which is obtained by taking the linear dual of $\Lambda^{-2}\,\mathrm{Ger}$.

For the purposes of conveniently describing elements of $\Lambda^{-2}\,\mathrm{Ger}$, we introduce the free $\Lambda^{-2}\,\mathrm{Ger}$-algebra $\Lambda^{-2}\,\mathrm{Ger}(b_1, b_2, \dots, b_n)$ in n auxiliary variables $b_1, b_2, \dots, b_n$ of degree 0 and identify the n-th space $\Lambda^{-2}\,\mathrm{Ger}(n)$ of $\Lambda^{-2}\,\mathrm{Ger}$ with the subspace of $\Lambda^{-2}\,\mathrm{Ger}(b_1, b_2, \dots, b_n)$ spanned by $\Lambda^{-2}\,\mathrm{Ger}$-monomials in which each variable b_j appears exactly once. For example, $\Lambda^{-2}\,\mathrm{Ger}(2)$ is spanned by the monomials $b_1 b_2$ and $\{b_1, b_2\}$ of degrees 2 and 1, respectively.

Let us consider the ordered partitions of the set $\{1, 2, \dots, n\}$

$$\{i_{11}, i_{12}, \dots, i_{1p_1}\} \sqcup \{i_{21}, i_{22}, \dots, i_{2p_2}\} \sqcup \dots \sqcup \{i_{t1}, i_{t2}, \dots, i_{tp_t}\} \tag{2.27}$$

satisfying the following properties:

- for each $1 \le \beta \le t$ the index $i_{\beta p_\beta}$ is the biggest among $i_{\beta 1}, \dots, i_{\beta p_\beta}$

- $i_{1p_1} < i_{2p_2} < \cdots < i_{tp_t}$ (in particular, $i_{tp_t} = n$).

It is clear that the monomials

$$\{b_{i_{11}}, \dots, \{b_{i_{1(p_1-1)}}, b_{i_{1p_1}}\} \dots\} \dots \{b_{i_{t1}}, \dots, \{b_{i_{t(p_t-1)}}, b_{i_{tp_t}}\} \dots\} \tag{2.28}$$

corresponding to all ordered partitions (2.27) satisfying the above properties form a basis of the space $\Lambda^{-2}\,\mathrm{Ger}(n)$.

In this dissertation, we use the notation

$$\big(\{b_{i_{11}}, \dots, \{b_{i_{1(p_1-1)}}, b_{i_{1p_1}}\} \dots\} \dots \{b_{i_{t1}}, \dots, \{b_{i_{t(p_t-1)}}, b_{i_{tp_t}}\} \dots\}\big)^* \tag{2.29}$$

for the elements of the dual basis in $\mathrm{Ger}^{\vee}(n) = \big(\Lambda^{-2}\,\mathrm{Ger}(n)\big)^*$.

2.6 The dg operad of brace trees

In this brief section, we recall the dg operad Braces from [6, Section 9] and [24][4].

Following [6], we introduce, for every $n \geq 1$, the auxiliary set $\mathcal{T}(n)$. An element of $\mathcal{T}(n)$ is a planted[5] planar tree T with the following data

- a partition of the set $V(T)$ of vertices

  $$V(T) = V_{\mathrm{lab}}(T) \sqcup V_{\nu}(T) \sqcup V_{root}(T)$$

  into the singleton $V_{root}(T)$ consisting of the root vertex, the set $V_{\mathrm{lab}}(T)$ consisting of n vertices called *labeled*, and the set $V_{\nu}(T)$ consisting of vertices which we call *neutral*;

- a bijection between the set $V_{\mathrm{lab}}(T)$ and the set $\{1, 2, \dots, n\}$.

We also require that each element T of $\mathcal{T}(n)$ satisfies the condition that every neutral vertex of T has at least 2 incoming edges (that is, are of valency at least 3). Elements of $\mathcal{T}(n)$ are called *brace trees*.

[4]In paper [24], the dg operad Braces is called the "minimal operad".

[5]Recall that a *planted* tree is a rooted tree whose root vertex has valency 1 .

For $n \geq 1$, the vector space $\mathrm{Braces}(n)$ consists of all finite linear combinations of brace trees in $\mathcal{T}(n)$. To define a structure of a graded vector space on $\mathrm{Braces}(n)$, we declare that each brace tree $T \in \mathcal{T}(n)$ carries degree

$$|T| = 2|V_\nu(T)| - |E(T)| + 1\,, \tag{2.30}$$

where $|V_\nu(T)|$ denotes the total number of neutral vertices of T and $|E(T)|$ denotes the total number of edges of T.

Examples of brace trees in $\mathcal{T}(2)$ (and hence vectors in $\mathrm{Braces}(2)$) are shown on figures 2.5, 2.6, 2.7, 2.8.

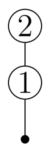


Figure 2.5: A brace tree $T \in \mathcal{T}(2)$

Figure 2.6: A brace tree $T_{21} \in \mathcal{T}(2)$

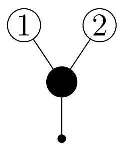


Figure 2.7: A brace tree $T_{\cup} \in \mathcal{T}(2)$

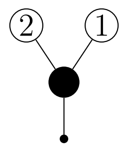


Figure 2.8: A brace tree $T_{\cup^{opp}} \in \mathcal{T}(2)$

According to (2.30), the brace trees T and T_{21} on figures 2.5 and 2.6, respectively, carry degree -1 and the brace trees $T_{\cup}$, $T_{\cup^{opp}}$ on figures 2.7, 2.8, respectively, carry degree 0.

Since neutral vertices of elements of $\mathcal{T}$ have valency at least 3, this implies that $\mathcal{T}(1)$ consists of exactly one brace tree T_{id} shown on figure 2.9. Hence we have

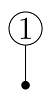


Figure 2.9: The brace tree $T_{\mathrm{id}} \in \mathcal{T}(1)$

$\mathrm{Braces}(1) = \mathbb{k}$.

Finally, we set $\mathrm{Braces}(0) = \mathbf{0}$.

For the definition of the operadic multiplications on Braces, we refer the reader to[6] [6, Section 8] and, in particular, Example 8.2. For the definition of the differential on Braces, we refer the reader to [6, Section 8.1] and, in particular, Example 8.4.

Let us also recall that the dg operad Braces acts naturally on the Hochschild cochain complex $C^\bullet(\mathcal{A})$ of any A_∞-algebra $\mathcal{A}$. For example, if T (resp. T_{21}) is the brace tree shown on figure 2.5 (resp. figure 2.6), then the expression

$$T(P_1, P_2) + T_{21}(P_1, P_2)\,, \qquad P_1, P_2 \in C^\bullet(\mathcal{A})$$

coincides (up to a sign factor) with the Gerstenhaber bracket of P_1 and P_2. Similarly, if $T_\cup$ is the brace tree shown on figure 2.7, then the expression

$$T_\cup(P_1, P_2)\,, \qquad P_1, P_2 \in C^\bullet(\mathcal{A})$$

coincides (up to a sign factor) with the cup product of P_1 and P_2.

For the precise construction of the action of Braces on $C^\bullet(\mathcal{A})$, we refer the reader to [6, Appendix B].

[6] Strictly speaking Braces is a suboperad of the dg operad defined in [6, Section 8].

CHAPTER 3

THE 2-COLORED CYLINDER OPERAD

As mentioned in the introduction, the goal of this dissertation is to study diagrams of homotopy algebras, and give an application to Tamarkin's construction of formality morphisms. We begin by defining and studying the 2-colored operad governing pairs of $\mathrm{Cobar}(\mathcal{C})$-algebras and an ∞-morphism $V \rightsquigarrow W$ (Proposition 5.1), which will play a key role in formulating and proving our later results.

3.1 Basic definition and properties

First, given a coaugmented cooperad $\mathcal{C}$, define the 2-colored collection $\widetilde{\mathcal{C}}$ as follows:

$$\widetilde{\mathcal{C}}(n,0;\alpha) = \widetilde{\mathcal{C}}(0,n;\beta) = \mathbf{s}\,\mathcal{C}_\circ(n)$$

$$\widetilde{\mathcal{C}}(n,0;\beta) = \mathcal{C}(n)$$

$$\widetilde{\mathcal{C}} = \mathbf{0} \quad \text{otherwise.}$$

Then form the free 2-colored operad $\mathbb{OP}(\widetilde{\mathcal{C}})$. We think of elements of $\mathbb{OP}(\widetilde{\mathcal{C}})$ as 2-colored trees with vertices decorated by elements of $\mathbf{s}\,\mathcal{C}_\circ$ and $\mathcal{C}$, such that vertices have incoming edges only of a single color, and vertices have input color α and output color β exactly when the vertex is decorated by an element of $\mathcal{C}$.

Note that unlike $\mathrm{Cobar}(\mathcal{C})$, elements of $\mathbb{OP}(\widetilde{\mathcal{C}})$ may have mixed-color vertices decorated by unsuspended elements of $\mathcal{C}$, and since $\mathcal{C} = \Bbbk \oplus \mathcal{C}_\circ$, in particular may have vertices with a single input and output. We denote these "trivial vertices" by $1^{\alpha\beta} \in \mathbb{OP}(\widetilde{\mathcal{C}})(1,0;\beta)$. This leads us to an alternate notion of degree: given $X \in \mathbb{OP}(\widetilde{\mathcal{C}})$ (or $X \in \mathrm{Cobar}(\mathcal{C})$), we will say that the *weight* of X, or $\mathrm{wt}(X)$, is the number of internal vertices of X not of the form $1^{\alpha\beta}$ (in $\mathrm{Cobar}(\mathcal{C})$, weight is just the number of internal vertices). We will say that a map F has weight m if it raises weight by exactly m. Note finally that we have $\mathrm{Cobar}(\mathcal{C}) \subseteq \mathbb{OP}(\widetilde{\mathcal{C}})$ by declaring that an element $X \in \mathrm{Cobar}(\mathcal{C})$ has edges only of color α or only of color β; we will denote these assignments by X^α and X^β, respectively. Similarly, given $x \in \mathcal{C}(n)$, we will write $x^{\alpha\beta} \in \mathbb{OP}(\widetilde{\mathcal{C}})(n,0;\beta)$ to indicate the corolla with n incoming edges of color α and outgoing edge of color β; this is consistent with our earlier notation for $1^{\alpha\beta} \in \mathbb{OP}(\widetilde{\mathcal{C}})(1,0;\beta)$. We will often mark vectors with superscripts in this way to clearly indicate their input and output colors.

On $\mathbb{OP}(\widetilde{\mathcal{C}})$, define a derivation ∂ on generators as follows:

$$\begin{array}{rclr}
\partial(\mathbf{s}\,x^\alpha) & = & \partial_{\mathrm{Cobar}}(\mathbf{s}\,x)^\alpha & \mathbf{s}\,x \in \widetilde{\mathcal{C}}(n,0;\alpha) = \mathbf{s}\,\mathcal{C}_\circ(n) \\
\partial(\mathbf{s}\,x^\beta) & = & \partial_{\mathrm{Cobar}}(\mathbf{s}\,x)^\beta & \mathbf{s}\,x \in \widetilde{\mathcal{C}}(0,n;\beta) = \mathbf{s}\,\mathcal{C}_\circ(n) \\
\partial(1^{\alpha\beta}) & = & 0 & 1 \in \widetilde{\mathcal{C}}(n,0;\beta) = \mathcal{C}(n) \\
\partial(x^{\alpha\beta}) & = & \partial_{\mathcal{C}}(x)^{\alpha\beta} + \partial'(x^{\alpha\beta}) + \partial''(x^{\alpha\beta}) & 1 \neq x \in \widetilde{\mathcal{C}}(n,0;\beta) = \mathcal{C}(n)
\end{array}$$

where ∂' is defined by

$$\partial'(x^{\alpha\beta}) = \sum_{z \in \mathrm{Isom}(\mathsf{Tree}'_2(n,0;\beta))} (-1)^{|x_1|}(\mathbf{t}_z; x_1, \mathbf{s}\,x_2) \tag{3.1}$$

with $\mathbf{t}_z$ a representative of the isomorphism class $z \in \mathrm{Isom}(\mathsf{Tree}'_2(n))$ and $\Delta_{\mathbf{t}_z}(x) = \sum x_1 \otimes x_2$, and where ∂'' is defined by

$$\partial''(x^{\alpha\beta}) = -\sum_k \sum_{z \in \mathrm{Isom}(\mathsf{PF}'_k(n,0;\beta))} (\mathbf{t}_z; \mathbf{s}\,x_0, x_1, ..., x_k) \tag{3.2}$$

with $\mathbf{t}_z$ a representative of the isomorphism class $z \in \mathrm{Isom}(\mathsf{PF}'_k(n,0;\beta))$ and $\Delta_{\mathbf{t}_z}(x) = \sum x_0 \otimes x_1 \otimes ... \otimes x_k$, and where both comultiplications are taken by forgetting the coloring on $\mathbf{t}_z$. A visual interpretation of $\partial(x^{\alpha\beta})$ is found in Figure 3.1.

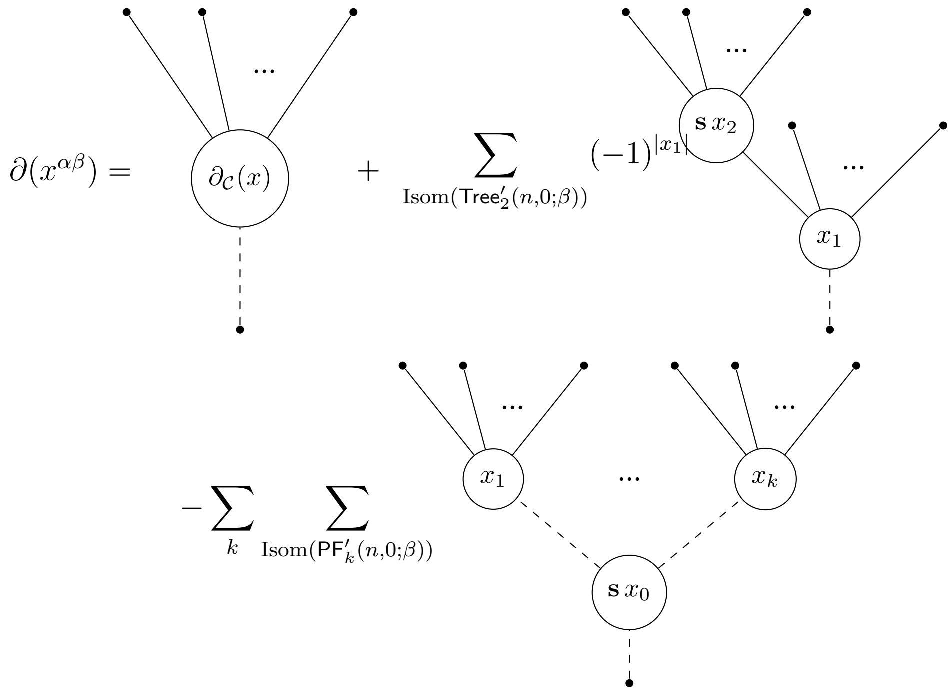


Figure 3.1: The differential on $\mathrm{Cyl}(\mathcal{C})$.

∂ visibly has degree 1, and so with the following proposition, we see that ∂ gives $\mathbb{OP}(\widetilde{\mathcal{C}})$ the structure of a dg operad. Following [19], we will call this operad $\mathrm{Cyl}(\mathcal{C})$.

Proposition 3.1. $\partial^2 = 0$.

Proof. The proof is a technical computation in the same spirit as showing $\partial^2_{\text{Cobar}} = 0$. It suffices to show that $\partial^2 = 0$ on corollas. Since $\partial = \partial_{\text{Cobar}}$ on single-color corollas, it remains to justify that $\partial^2 = 0$ on mixed-color corollas; we will give the general ideas behind this computation. Since $\partial = \partial_{\mathcal{C}} + \partial' + \partial''$, we have that $\partial^2 = 0$ from the following observations:

1. $\partial^2_{\mathcal{C}} = 0$ because $\partial_{\mathcal{C}}$ is a differential on $\mathcal{C}$;

2. $\partial_{\mathcal{C}} \circ \partial' + \partial' \circ \partial_{\mathcal{C}} = \partial_{\mathcal{C}} \circ \partial'' + \partial'' \circ \partial_{\mathcal{C}} = 0$ because $\partial_{\mathcal{C}}$ is as a coderivation;

3. $\partial' \circ \partial' = 0$ because of coassociativity;

4. $\partial' \circ \partial'' + \partial'' \circ \partial' + \partial'' \circ \partial'' = 0$ because of coassociativity and elementary combinatorial identities.

□

3.2 Cohomological properties

As mentioned earlier, the significance of $\mathrm{Cyl}(\mathcal{C})$ is that it governs pairs of homotopy algebras and ∞-morphisms between them, which we will prove in Chapter 4; for now, we proceed to study $\mathrm{Cyl}(\mathcal{C})$ in more depth. Given that $\mathrm{Cyl}(\mathcal{C})$ is essentially a 2-colored modification of $\mathrm{Cobar}(\mathcal{C})$, one would expect their cohomology to be related somehow. This is indeed the case, at least if we restrict our attention to the weight 0 components of their respective differentials. The weight 0 component of $\partial_{\mathrm{Cobar}}$ is just $\partial_{\mathcal{C}}$; explicitly,

$$\partial_{\mathcal{C}}(\mathbf{s}\, x) = -\,\mathbf{s}\, \partial_{\mathcal{C}}(x) \tag{3.3}$$

for $\mathbf{s}\, x \in \mathbf{s}\, \mathcal{C}_\circ$. On $\mathrm{Cyl}(\mathcal{C})$, the weight 0 part of ∂, to be denoted ∂_0, is given explicitly by

$$\begin{array}{rcll}
\partial_0(\mathbf{s}\, x^\alpha) & = & \partial_{\mathcal{C}}(\mathbf{s}\, x)^\alpha & \mathbf{s}\, x \in \widetilde{\mathcal{C}}(n,0;\alpha) = \mathbf{s}\, \mathcal{C}_\circ(n) \\
\partial_0(\mathbf{s}\, x^\beta) & = & \partial_{\mathcal{C}}(\mathbf{s}\, x)^\beta & \mathbf{s}\, x \in \widetilde{\mathcal{C}}(0,n;\beta) = \mathbf{s}\, \mathcal{C}_\circ(n) \\
\partial(1^{\alpha\beta}) & = & 0 & 1 \in \widetilde{\mathcal{C}}(n,0;\beta) = \mathcal{C}(n) \\
\partial_0(x^{\alpha\beta}) & = & \partial_{\mathcal{C}}(x)^{\alpha\beta} + \partial_0'(x^{\alpha\beta}) + \partial_0''(x^{\alpha\beta}) & 1 \neq x \in \widetilde{\mathcal{C}}(n,0;\beta) = \mathcal{C}(n)
\end{array}$$

where

$$\partial_0'(x^{\alpha\beta}) = 1^{\alpha\beta} \circ_1 \mathbf{s}\, x^\alpha \tag{3.4}$$

and where

$$\partial_0''(x^{\alpha\beta}) = -\mu(\mathbf{s}\, x^\beta; 1^{\alpha\beta}, ..., 1^{\alpha\beta}). \tag{3.5}$$

It may seem as though we have overrused the notation $\partial_{\mathcal{C}}$ by now, but all such uses are really just the original $\partial_{\mathcal{C}}$ acting as a derivation on a free operad, respecting suspensions and/or coloring. A visual representation of the action of ∂_0 on mixed-color generators is found in figure 3.2.

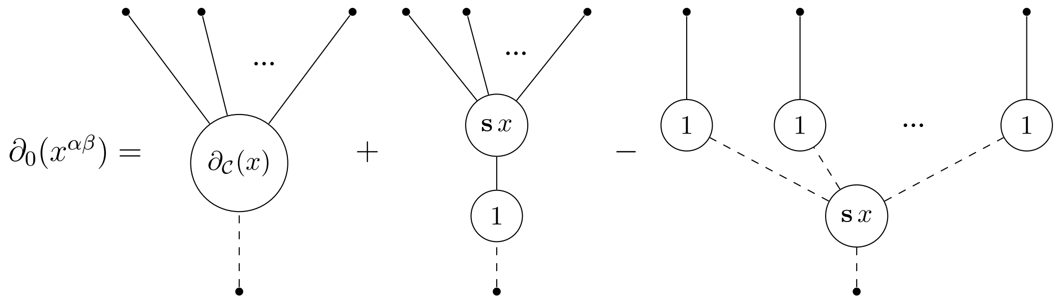


Figure 3.2: The weight 0 component of the differential on $\mathrm{Cyl}(\mathcal{C})$.

From weight considerations (or directly checking), both $\partial_{\mathcal{C}}^2 = 0$ and $\partial_0^2 = 0$, so we may consider $\mathrm{Cobar}(\mathcal{C})$ and $\mathrm{Cyl}(\mathcal{C})$ with respect to these simpler differentials. Then we have:

Theorem 3.1. *The inclusion maps*

$$\begin{array}{cccccc} \iota_\alpha,\, \iota_\beta & : & (\mathrm{Cobar}(\mathcal{C})(n), \partial_{\mathcal{C}}) & \longrightarrow & (\mathrm{Cyl}(\mathcal{C})(n, 0; \beta), \partial_0) \\ \iota_\alpha & : & X & \mapsto & 1^{\alpha\beta} \circ_1 X^\alpha \\ \iota_\beta & : & X & \mapsto & \mu(X^\beta; 1^{\alpha\beta}, ..., 1^{\alpha\beta}) \end{array}$$

are quasi-isomorphisms for all $n \geq 0$*, and furthermore, are homotopic.*

Proof. Given that we will show that ι_α is homotopic to ι_β, it suffices to show that ι_β is a quasi-isomorphism; we will begin with this. Introduce the following filtrations on $\mathrm{Cobar}(\mathcal{C})(n)$ and $\mathrm{Cyl}(\mathcal{C})(n, 0; \beta)$:

$$\mathcal{F}^m \mathrm{Cobar}(\mathcal{C})(n) = \left\{ \begin{array}{c} X \in \mathrm{Cobar}(\mathcal{C})(n) \;\mid \\ \text{(the number of edges in } X) - |X| \leq m \end{array} \right\}$$

$$\mathcal{F}^m \,\mathrm{Cyl}(\mathcal{C})(n, 0; \beta) = \left\{ \begin{array}{c} X \in \mathrm{Cyl}(\mathcal{C})(n, 0; \beta) \;\mid \\ \text{(the number of edges of color } \alpha \text{ in } X) - |X| \leq m \end{array} \right\}.$$

These filtrations are ascending, cocomplete, and compatible with ι_β (since they are essentially the same filtration). They also respect $\partial_{\mathcal{C}}$ and ∂_0; in particular, note that $\partial_{\mathcal{C}}$ and ∂_0'' raise internal degree without changing the number of (straight) edges, so they lower the filtration index, while ∂_0' raises internal degree and the number of straight edges, so it preserves the filtration index. Consequently, when we consider the associated graded complexes, we have

$$\mathrm{Gr}_{\mathcal{F}}\, \mathrm{Cobar}(\mathcal{C})(n) = (\mathrm{Cobar}(\mathcal{C})(n), 0) \tag{3.6}$$

$$\mathrm{Gr}_{\mathcal{F}}\,\mathrm{Cyl}(\mathcal{C})(n,0;\beta) = (\mathrm{Cyl}(\mathcal{C})(n,0;\beta),\partial_0') \tag{3.7}$$

By Appendix A of [11], it suffices to show that

$$\iota_\beta : (\mathrm{Cobar}(\mathcal{C})(n),0) \to (\mathrm{Cyl}(\mathcal{C})(n,0;\beta),\partial_0') \tag{3.8}$$

is a quasi-isomorphism. For the remainder of this first section of the proof, when we refer to those complexes, they will carry those differentials.

For this, we need an auxiliary construction. Define the 3-colored collection $\mathcal{Q}$, with colors α, β, γ, by

$$\begin{array}{ccccc} \mathcal{Q}(a,0,0;\alpha) & = & \mathbf{s}\,\mathcal{C}_\circ(a) & \text{with } \partial_{\mathcal{Q}} = 0 \\ \mathcal{Q}(0,b,c;\beta) & = & \mathbf{s}\,\mathcal{C}_\circ(b+c) & \text{with } \partial_{\mathcal{Q}} = 0 \\ \mathcal{Q}(a,0,0;\beta) & = & \mathcal{C}_\circ(a)\oplus \mathbf{s}\,\mathcal{C}_\circ(a) & \text{with } \partial_{\mathcal{Q}} : x \to \mathbf{s}\,x \\ \mathcal{Q} & = & 0 & \text{otherwise.} \end{array}$$

Note that

$$H^\bullet(\mathcal{Q}(0,b,c;\beta)) = H^\bullet(\mathcal{Q}(b+c,0,0;\alpha)) = \mathbf{s}\,\mathcal{C}_\circ(b+c) \tag{3.9}$$

while

$$H^\bullet(\mathcal{Q}(a,0,0;\beta)) = 0. \tag{3.10}$$

When we form $\mathbb{OP}(\mathcal{Q})$, we have that

$$\mathrm{Cyl}(\mathcal{C})(n,0;\beta) \cong \bigoplus_{m=0}^{n} \mathbb{OP}(\mathcal{Q})(m,0,n-m;\beta) \tag{3.11}$$

via the (backwards) identification $\mathbb{OP}(\mathcal{Q})(m,0,n-m;\beta) \to \mathrm{Cyl}(\mathcal{C})(n,0;\beta)$ determined by the following rules. First, send edges of color γ to the element $1^{\alpha\beta} \in \mathrm{Cyl}(\mathcal{C})(1,0;\beta)$. Then perform the following identifications:

$$\begin{array}{ccc} \mathbf{s}\,x^\alpha \in \mathcal{Q}(a,0,0;\alpha) & \mapsto & \mathbf{s}\,x^\alpha \in \widetilde{\mathcal{C}}(a,0;\alpha) \\ \mathbf{s}\,x^\beta \in \mathcal{Q}(0,b,0;\beta) & \mapsto & \mathbf{s}\,x^\beta \in \widetilde{\mathcal{C}}(0,b;\beta) \\ x^{\alpha\beta} \in \mathcal{C}_\circ \subseteq \mathcal{Q}(a,0,0;\beta) & \mapsto & x^{\alpha\beta} \in \widetilde{\mathcal{C}}(a,0;\beta) \\ \mathbf{s}\,x^{\alpha\beta} \in \mathbf{s}\,\mathcal{C}_\circ \subseteq \mathcal{Q}(a,0,0;\beta) & \mapsto & 1^{\alpha\beta}\circ_1 \mathbf{s}\,x^\alpha \in \mathrm{Cyl}(\mathcal{C})(a,0;\beta) \end{array}$$

An example of this identification is shown in Figure 3.3.

It is not hard to check that this identification is an isomorphism of cochain complexes, and consequently

$$H^\bullet(\mathrm{Cyl}(\mathcal{C})(n,0;\beta)) \cong \bigoplus_{m=0}^{n} H^\bullet(\mathbb{OP}(\mathcal{Q})(m,0,n-m;\beta)) \tag{3.12}$$

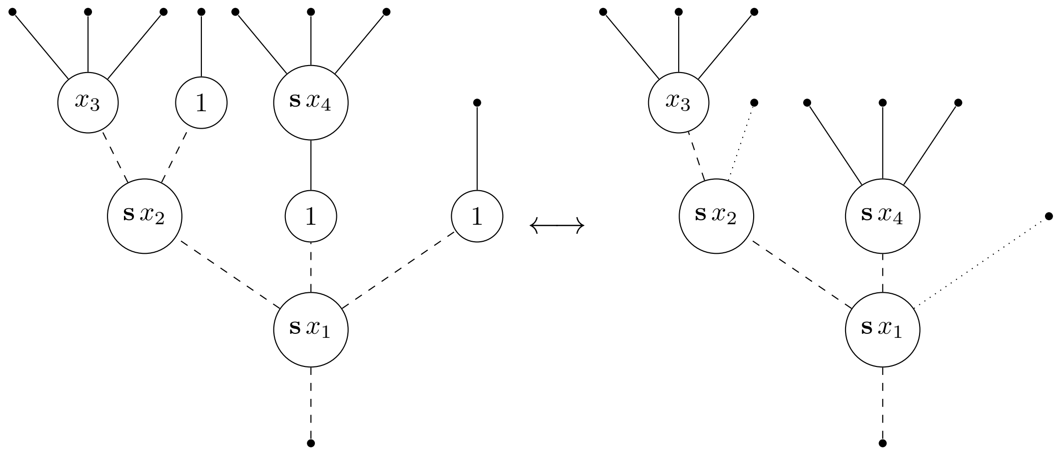


Figure 3.3: An element of $\mathrm{Cyl}(\mathcal{C})$ (on the left) identified with an element of $\mathbb{OP}(\mathcal{Q})$ (on the right). The dotted lines indicate edges of color γ.

Since $\mathbb{OP}(\mathcal{Q})(m, 0, n-m; \beta)$ is colim from a finite, disjoint union of connected groupoids (specifically, the groupoids consisting of members of isomorphism classes of 3-colored n-labeled planar trees), and carries only the differential structure coming from $\mathcal{Q}$, Lemma A.1 applies. In particular, since taking coinvariants is exact when working over a field of characteristic 0, we have from Lemma A.1 that

$$H^\bullet(\mathbb{OP}(\mathcal{Q})(m, 0, n-m; \beta)) = \mathbb{OP}(H^\bullet(\mathcal{Q}))(m, 0, n-m; \beta). \tag{3.13}$$

But if $m > 0$, any element of $\mathbb{OP}(\mathcal{Q})(m, 0, n-m; \beta)$ must contain at least one vertex decorated by an element of $\mathcal{Q}(a, 0, 0; \beta)$. Since $H^\bullet(\mathcal{Q}(a, 0, 0; \beta)) = 0$, we have in this case that $\mathbb{OP}(H^\bullet(\mathcal{Q}))(m, 0, n-m; \beta) = 0$ also. On the other hand, if $m = 0$, all vertices are decorated by elements of $\mathcal{Q}(0, b, c; \beta)$, and in this case we have

$$H^\bullet(\mathcal{Q}(0, b, c; \beta)) = \mathcal{Q}(0, b, c; \beta). \tag{3.14}$$

Consequently,

$$H^\bullet(\mathrm{Cyl}(\mathcal{C})(n, 0; \beta)) \cong H^\bullet(\mathbb{OP}(\mathcal{Q})(0, 0, n; \beta)) \tag{3.15}$$

$$= \mathbb{OP}(H^\bullet(\mathcal{Q}))(0, 0, n; \beta) \tag{3.16}$$

$$= \mathbb{OP}(\mathcal{Q})(0, 0, n; \beta). \tag{3.17}$$

Passing back to $\mathrm{Cyl}(\mathcal{C})(n, 0; \beta)$ via the earlier isomorphism, we see that

$$\mathbb{OP}(\mathcal{Q})(0, 0, n; \beta) \cong \iota_\beta(\mathrm{Cobar}(\mathcal{C})(n)) \subseteq \mathrm{Cyl}(\mathcal{C})(n, 0; \beta) \tag{3.18}$$

which shows that ι_β is a quasi-isomorphism; therefore ι_β is a quasi-isomorphism for the original complexes, as desired.

It remains to show that ι_α is homotopic to ι_β in $\mathrm{Cyl}(\mathcal{C})$ with the original differential ∂_0. Observe that in $\mathrm{Cyl}(\mathcal{C})$, the presence of vertices of type $1^{\alpha\beta}$ is determined completely by the coloring of adjacent vertices, and whether they are decorated by suspended vectors or not. Therefore, given $X \in \mathrm{Cobar}(\mathcal{C})$, we may define $X_i \in \mathrm{Cyl}(\mathcal{C})$ by declaring that X_i has the same underlying tree as X, it has the same internal vectors as X but that the ith (nontrivial) vertex is no longer suspended (using the total order on vertices), that the edges before (nontrivial) vertex i are of color β and the edges after are of color α (using the total order on edges), and then finally adding trivial vertices $1^{\alpha\beta}$ and edges of color β as necessary to make X_i a valid element of $\mathrm{Cyl}(\mathcal{C})$. Figure 3.4 provides an example of this construction.

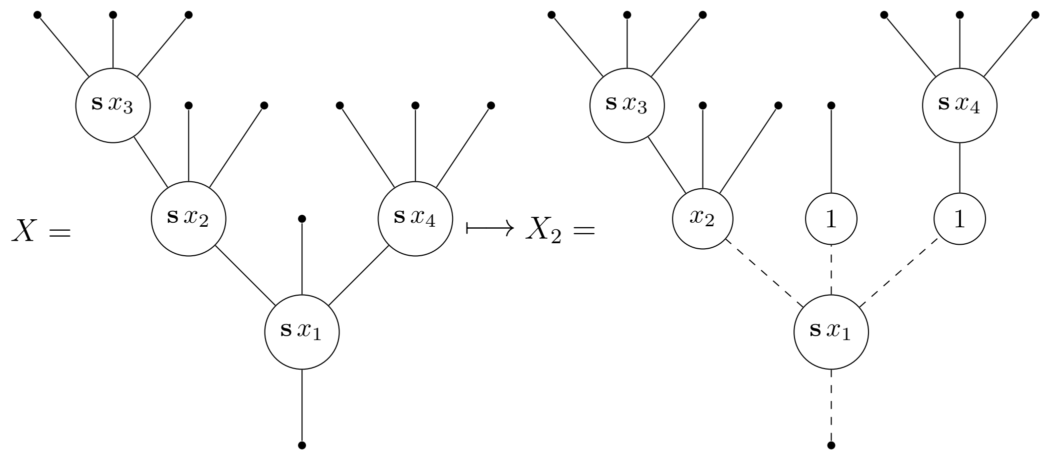


Figure 3.4: The construction of $X_2 \in \mathrm{Cyl}(\mathcal{C})$ given $X \in \mathrm{Cobar}(\mathcal{C})$.

We may now construct the necessary homotopy between the maps ι_α and ι_β. Given $X = (\mathbf{t}; \mathrm{s}\, x_1, ..., \mathrm{s}\, x_k)$ an element of $\mathrm{Cobar}(\mathcal{C})$, define $h : \mathrm{Cobar}(\mathcal{C}) \to \mathrm{Cyl}(\mathcal{C})$ by

$$h(X) = \sum_{i=1}^{k} (-1)^{|\,\mathrm{s}\, x_1| + \ldots + |\,\mathrm{s}\, x_{i-1}|}\, X_i. \tag{3.19}$$

We may think of the sign in the above term coming from the suspension decorating the ith nodal vertex x_i "jumping over" the vertices $\mathrm{s}\, x_1, ..., \mathrm{s}\, x_{i-1}$ to leave the tree. Since $\partial_{\mathcal{C}}(\mathrm{s}\, x) = -\,\mathrm{s}\, \partial_{\mathcal{C}}(x)$ for $x \in \mathcal{C}$, we have that

$$\partial_{\mathcal{C}} \circ h + h \circ \partial_{\mathcal{C}} = 0. \tag{3.20}$$

It is also easy to check that

$$\iota_\alpha - \iota_\beta = (\partial_0' + \partial_0'') \circ h; \tag{3.21}$$

∂_0' applied to the the first term of $h(X)$ yields $\iota_\alpha(X)$, ∂_0'' applied to the last term of $h(X)$ yields $-\iota_\beta(X)$ (the sign from h will cancel with the sign coming from ∂_0'' "jumping over" the nontrivial vertices before the final vertex x_k), and all middle terms cancel from similar sign considerations. We therefore have in general that

$$\iota_\alpha - \iota_\beta = \partial_0 \circ h + h \circ \partial_{\mathcal{C}}, \tag{3.22}$$

which shows that ι_α and ι_β are homotopic, which completes the proof.

□

In fact, Theorem 3.1 is true with respect to the full differentials on $\mathrm{Cobar}(\mathcal{C})$ and $\mathrm{Cyl}(\mathcal{C})$, not simply the weight 0 parts.

Corollary 3.1. *The inclusion maps*

$$\begin{array}{cccc} \iota_\alpha,\, \iota_\beta & : & (\mathrm{Cobar}(\mathcal{C})(n), \partial_{\mathrm{Cobar}}) & \longrightarrow & (\mathrm{Cyl}(\mathcal{C})(n, 0; \beta), \partial) \\ \iota_\alpha & : & X & \mapsto & 1^{\alpha\beta} \circ_1 X^\alpha \\ \iota_\beta & : & X & \mapsto & \mu(X^\beta; 1^{\alpha\beta}, ..., 1^{\alpha\beta}) \end{array}$$

are quasi-isomorphisms for all $n \geq 0$*, and furthermore, are homotopic.*

Proof. The argument that ι_β is a quasi-isomorphism is very similar, but requires an initial modification. First filter $\mathrm{Cyl}(\mathcal{C})(n, 0; \beta)$ and $\mathrm{Cobar}(\mathcal{C})(n)$ by weight and form the associated graded complexes; this then gives us the exact situation of Theorem 3.1, and the result holds.

A different argument is needed to show that ι_α and ι_β are homotopic. For this we introduce the map

$$\Pi : \mathrm{Cyl}(\mathcal{C})(n, 0; \beta) \longrightarrow \mathrm{Cobar}(\mathcal{C})(n) \tag{3.23}$$

defined as follows. If $X \in \mathrm{Cyl}(\mathcal{C})(n, 0; \beta)$ contains any nontrivial mixed-color vertices (that is, vertices decorated by elements of $\mathcal{C}_\circ$), $\Pi(X) = 0$. Otherwise, define $\Pi(X)$ by changing all edges to color α and delete all trivial mixed vertices $1^{\alpha\beta}$, merging the adjacent edges; the result is an element of $\mathrm{Cobar}(\mathcal{C})$ because X contained no nontrivial mixed vertices. Figure 3.5 gives an example of this. It is easy to check that Π is a map of cochain complexes and that Π is a one-sided inverse to both ι_α and ι_β:

$$\Pi \circ \iota_\alpha = 1_{\mathrm{Cobar}(\mathcal{C})} = \Pi \circ \iota_\beta. \tag{3.24}$$

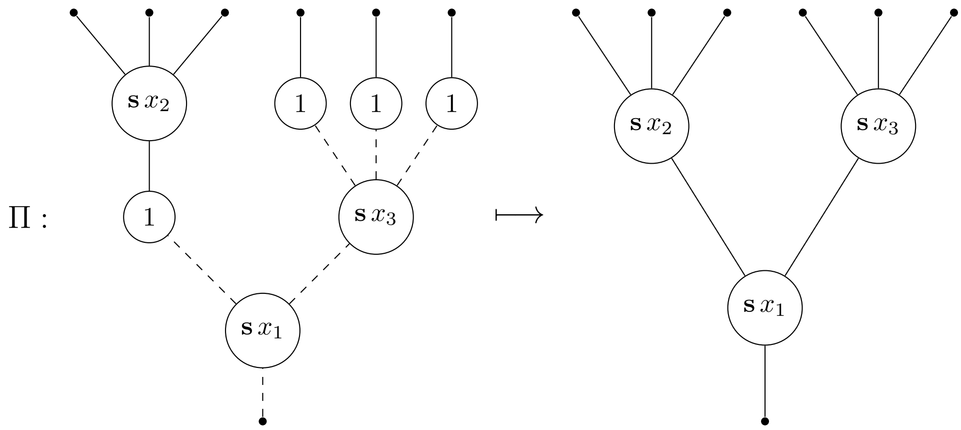


Figure 3.5: A nontrivial example of the map $\Pi : \mathrm{Cyl}(\mathcal{C})(n, 0; \beta) \to \mathrm{Cobar}(\mathcal{C})(n)$.

Since we already know that ι_β is a quasi-isomorphism, it follows that ι_α and ι_β induce the same map on cohomology, and therefore are homotopic.

□

CHAPTER 4

DERIVATIONS AND DERIVED AUTOMORPHISMS OF THE CYLINDER OPERAD

We turn our attention to the dg Lie algebras $\mathrm{Der}(\mathrm{Cobar}(\mathcal{C}))$ and $\mathrm{Der}(\mathrm{Cyl}(\mathcal{C}))$, and investigate how they relate to each other. Our ultimate goal is to show that given any derivation of $\mathrm{Cobar}(\mathcal{C})$, we can extend it to a derivation of $\mathrm{Cyl}(\mathcal{C})$; the ramifications of this in terms of ∞-morphisms will be discussed in the next chapter. We will also discuss derivations that can be exponentiated to operad automorphisms, which will be of particular importance in applications.

4.1 Derivations of $\mathrm{Cyl}(\mathcal{C})$

We begin with a technical lemma, to be used several times in the remainder of the dissertation.

Lemma 4.1. *Let* $\mathrm{Hom}(\widetilde{\mathcal{C}}, \mathrm{Cyl}(\mathcal{C}))$ *be the cochain complex of maps of colored collections, with differential*

$$\partial(F) = \partial_0 \circ F - (-1)^{|F|} F \circ \partial_0 \tag{4.1}$$

for $F \in \mathrm{Hom}(\widetilde{\mathcal{C}}, \mathrm{Cyl}(\mathcal{C}))$*. Define the cochain complex* $\mathrm{Hom}(\mathbf{s}\,\mathcal{C}_\circ, \mathrm{Cobar}(\mathcal{C}))$ *similarly, with differential*

$$\partial(F) = \partial_{\mathcal{C}} \circ F - (-1)^{|F|} F \circ \partial_{\mathcal{C}} \tag{4.2}$$

Then the maps

$$\mathrm{res}_\alpha, \mathrm{res}_\beta : \quad \mathrm{Hom}(\widetilde{\mathcal{C}}, \mathrm{Cyl}(\mathcal{C})) \quad \longrightarrow \quad \mathrm{Hom}(\mathbf{s}\,\mathcal{C}_\circ, \mathrm{Cobar}(\mathcal{C}))$$

given by restricting to a single color α *or* β *are quasi-isomorphisms of complexes.*

Remark 4.1. *By* $F \circ \partial_0$*, we mean that* $F \in \mathrm{Hom}(\widetilde{\mathcal{C}}, \mathrm{Cyl}(\mathcal{C}))$ *acts on the nontrivial vertices that are present after applying* ∂_0*. Explicitly, for a mixed-color generator* $x^{\alpha\beta} \in \mathcal{C}$*,*

$$(F \circ \partial_0)(x^{\alpha\beta}) = F(\partial_{\mathcal{C}}(x)^{\alpha\beta}) + 1^{\alpha\beta} \circ_1 F(\mathbf{s}\,x^\alpha) - \mu(F(\mathbf{s}\,x^\beta); 1^{\alpha\beta}, ..., 1^{\alpha\beta}) \tag{4.3}$$

Proof. We restrict our attention to level n, and decompose $\mathrm{Hom}(\widetilde{\mathcal{C}}, \mathrm{Cyl}(\mathcal{C}))$ into subspaces (not subcomplexes) based on how a derivation acts on different color generators:

$$\mathrm{Hom}(\mathbf{s}\,\mathcal{C}_\circ(n), \mathrm{Cobar}(\mathcal{C})(n))^\alpha \oplus \mathrm{Hom}(\mathcal{C}(n), \mathrm{Cyl}(\mathcal{C})(n,0;\beta))^{\alpha\beta}$$
$$\oplus \mathrm{Hom}(\mathbf{s}\,\mathcal{C}_\circ(n), \mathrm{Cobar}(\mathcal{C})(n))^\beta.$$

Here, the first summand gives the action of a derivation on corollas purely of color α, the second summand on mixed-color corollas, and the third summand on corollas of color β; the superscripts make this explicit. Before we can state how the differential structure respects this decomposition, we need to recall the earlier maps

$$\begin{array}{cccc} \iota_\alpha, \iota_\beta & : & \mathrm{Cobar}(\mathcal{C})(n) & \longrightarrow & \mathrm{Cyl}(\mathcal{C})(n,0;\beta) \\ \iota_\alpha & : & X & \mapsto & 1^{\alpha\beta} \circ_1 X^\alpha \\ \iota_\beta & : & X & \mapsto & \mu(X^\beta; 1^{\alpha\beta}, ..., 1^{\alpha\beta}) \end{array}$$

and use them to define new, degree 1 maps:

$$\mathrm{incl}_\alpha, \mathrm{incl}_\beta : \mathrm{Hom}(\mathbf{s}\,\mathcal{C}_\circ(n), \mathrm{Cobar}(\mathcal{C})(n)) \longrightarrow \mathrm{Hom}(\mathcal{C}(n), \mathrm{Cyl}(\mathcal{C})(n,0;\beta))$$
$$\mathrm{incl}_\alpha(F)(x) = (-1)^{|F|} \iota_\alpha(F(\mathbf{s}\,x))$$
$$\mathrm{incl}_\beta(F)(x) = (-1)^{|F|} \iota_\beta(F(\mathbf{s}\,x))$$

for $F \in \mathrm{Hom}(\mathbf{s}\,\mathcal{C}_\circ(n), \mathrm{Cobar}(\mathcal{C})(n))$ and $x \in \mathcal{C}(n)$. It is then straightforward to check that with respect to the above decomposition of $\mathrm{Hom}(\widetilde{\mathcal{C}}, \mathrm{Cyl}(\mathcal{C}))$, ∂ acts as follows:

$$\partial(F + F' + F'') = \partial(F) - \mathrm{incl}_\alpha(F) + \partial(F') + \mathrm{incl}_\beta(F'') + \partial(F'') \tag{4.4}$$

where

$$F \in \mathrm{Hom}(\mathbf{s}\,\mathcal{C}_\circ(n), \mathrm{Cobar}(\mathcal{C})(n))^\alpha$$
$$F' \in \mathrm{Hom}(\mathcal{C}(n), \mathrm{Cyl}(\mathcal{C})(n, 0; \beta))^{\alpha\beta}$$
$$F'' \in \mathrm{Hom}(\mathbf{s}\,\mathcal{C}_\circ(n), \mathrm{Cobar}(\mathcal{C})(n))^\beta$$

and where

$$\partial(F) = \partial_{\mathcal{C}} \circ F - (-1)^{|F|} F \circ \partial_{\mathcal{C}}$$
$$\partial(F') = \partial_0 \circ F' - (-1)^{|F'|} F' \circ \partial_0$$
$$\partial(F'') = \partial_{\mathcal{C}} \circ F'' - (-1)^{|F''|} F'' \circ \partial_{\mathcal{C}}.$$

As a cochain complex, $\mathrm{Hom}(\widetilde{\mathcal{C}}, \mathrm{Cyl}(\mathcal{C}))$ is therefore a "cylinder-type construction" as described in [14, Appendix A], and the maps res_α and res_β are the natural projections onto the first and third summands. By the same reference, it is enough to show that the maps

$$\mathbf{s}^{-1}\,\mathrm{incl}_\alpha, \mathbf{s}^{-1}\,\mathrm{incl}_\beta : \mathrm{Hom}(\mathbf{s}\,\mathcal{C}_\circ(n), \mathrm{Cobar}(\mathcal{C})(n)) \to \mathbf{s}^{-1}\,\mathrm{Hom}(\mathcal{C}(n), \mathrm{Cyl}(\mathcal{C})(n, 0; \beta))$$

are quasi-isomorphisms. But this is precisely the situation obtained by applying the functor $\mathrm{Hom}(\mathbf{s}\,\mathcal{C}_\circ, -)$ to the maps

$$\iota_\alpha, \iota_\beta : \mathrm{Cobar}(\mathcal{C})(n) \to \mathrm{Cyl}(\mathcal{C})(n, 0; \beta) \tag{4.5}$$

and we know those maps are quasi-isomorphisms from Theorem 3.1. Since Hom_{S_n} is exact when working over a field of characteristic 0, incl_α and incl_β are also quasi-isomorphisms. From the results of [14, Appendix A], we conclude that the maps res_α, res_β are quasi-isomorphisms. □

Proposition 4.1. *Introduce the following descending filtration on* $\operatorname{Hom}(\widetilde{\mathcal{C}}, \operatorname{Cyl}(\mathcal{C}))$*:*

$$\mathcal{F}_m \operatorname{Hom}(\widetilde{\mathcal{C}}, \operatorname{Cyl}(\mathcal{C})) = \{F \in \operatorname{Hom}(\widetilde{\mathcal{C}}, \operatorname{Cyl}(\mathcal{C})) \mid \operatorname{wt}(F) \geq m\}.$$

Introduce the same filtration on $\operatorname{Hom}(\mathbf{s}\,\mathcal{C}_\circ, \operatorname{Cobar}(\mathcal{C}))$*. These filtrations are complete and compatible with the appropriate differentials and the restriction maps* $\operatorname{res}_\alpha$*,* res_β*. Then*

$$\operatorname{res}_\alpha, \operatorname{res}_\beta : \quad \mathcal{F}_m \operatorname{Hom}(\widetilde{\mathcal{C}}, \operatorname{Cyl}(\mathcal{C})) \quad \longrightarrow \quad \mathcal{F}_m \operatorname{Hom}(\mathbf{s}\,\mathcal{C}_\circ, \operatorname{Cobar}(\mathcal{C}))$$

remain quasi-isomorphisms.

Proof. The proof is exactly the same as that of Lemma 4.1, restricting to the appropriate filtration levels. □

With the above results, we can now show our first main result, that restricting derivations of $\operatorname{Cyl}(\mathcal{C})$ to derivations of $\operatorname{Cobar}(\mathcal{C})$ yields quasi-isomorphisms of Lie algebras. This is the key statement needed for later results concerning derived automorphisms and their action on ∞-morphisms.

Theorem 4.1. *The maps*

$$\operatorname{res}_\alpha, \operatorname{res}_\beta : \quad \operatorname{Der}(\operatorname{Cyl}(\mathcal{C})) \quad \longrightarrow \quad \operatorname{Der}(\operatorname{Cobar}(\mathcal{C}))$$

given by restricting to a single color α *or* β *are homotopic quasi-isomorphisms of dg Lie algebras at all filtration levels.*

Proof. We will show the result for the entire derivation algebras; the argument for a specific filtration level is almost identical, the only difference being restricting to the appropriate filtration level and using Proposition 4.1.

It is clear that the above restriction maps are morphisms of dg Lie algebras. Since derivations are uniquely determined by their action on generators, we may equivalently consider the maps

$$\operatorname{res}_\alpha, \operatorname{res}_\beta : \quad \operatorname{Hom}(\widetilde{\mathcal{C}}, \operatorname{Cyl}(\mathcal{C})) \quad \longrightarrow \quad \operatorname{Hom}(\mathbf{s}\,\mathcal{C}_\circ, \operatorname{Cobar}(\mathcal{C}))$$

still determined by restricting to a single color α or β. Here, the differentials on $\mathrm{Hom}(\widetilde{\mathcal{C}}, \mathrm{Cyl}(\mathcal{C}))$ and $\mathrm{Hom}(\mathbf{s}\,\mathcal{C}_\circ, \mathrm{Cobar}(\mathcal{C}))$ take the following form:

$$\partial(F) = \partial \circ F - (-1)^{|F|} \widehat{F} \circ \partial \tag{4.6}$$

where the map F defined on generators extends uniquely to the derivation $\widehat{F}$. This way, the identification from derivations to morphisms respects the differential structure.

Filter $\mathrm{Hom}(\widetilde{\mathcal{C}}, \mathrm{Cyl}(\mathcal{C}))$ and $\mathrm{Hom}(\mathbf{s}\,\mathcal{C}_\circ, \mathrm{Cobar}(\mathcal{C}))$ by weight, and note that we could have equivalently defined these filtrations on $\mathrm{Der}(\mathrm{Cyl}(\mathcal{C}))$ and $\mathrm{Der}(\mathrm{Cobar}(\mathcal{C}))$. These filtrations are complete and compatible with the appropriate differentials and the restriction maps res_α, res_β. As before, we will move to the associated graded complexes which carry simpler differentials; from Lemma E.1 of [6], it suffices to show that the restriction maps are quasi-isomorphisms in this simpler setting. When we move to the associated graded complexes for this filtration, only the part of the differentials coming from the weight 0 part of the internal differentials survives. Explicitly, $\mathrm{Hom}(\widetilde{\mathcal{C}}, \mathrm{Cyl}(\mathcal{C}))$ carries the reduced differential

$$\partial(F) = \partial_0 \circ F - (-1)^{|F|} \widehat{F} \circ \partial_0 \tag{4.7}$$

for $F \in \mathrm{Hom}(\widetilde{\mathcal{C}}, \mathrm{Cyl}(\mathcal{C}))$, and $\mathrm{Hom}(\mathbf{s}\,\mathcal{C}_\circ, \mathrm{Cobar}(\mathcal{C}))$ carries the differential

$$\partial(F) = \partial_{\mathcal{C}} \circ F - (-1)^{|F|} \widehat{F} \circ \partial_{\mathcal{C}} \tag{4.8}$$

for $F \in \mathrm{Hom}(\mathbf{s}\,\mathcal{C}_\circ, \mathrm{Cobar}(\mathcal{C}))$. This is exactly the same situation as in Lemma 4.1; and so $\mathrm{res}_\alpha, \mathrm{res}_\beta$ are quasi-isomorphisms on the associated graded level, and therefore in general as well.

To show that res_α and res_β are homotopic, we will show that they induce the same map on cohomology. Recall the map

$$\Pi : \mathrm{Cyl}(\mathcal{C})(n, 0; \beta) \to \mathrm{Cobar}(\mathcal{C})(n) \tag{4.9}$$

from the proof of Corollary 3.1. Given a closed derivation $D \in \mathrm{Der}(\mathrm{Cyl}(\mathcal{C}))$, define $T \in \mathrm{Der}(\mathrm{Cobar}(\mathcal{C}))$ on generators by

$$T(\mathbf{s}\,x) = (-1)^{|D|}(\Pi \circ D)(x^{\alpha\beta}). \tag{4.10}$$

D is closed, so in particular

$$0 = [\partial, D](x^{\alpha\beta}) = (\partial \circ D)(x^{\alpha\beta}) - (-1)^{|D|}(D \circ \partial)(x^{\alpha\beta}). \tag{4.11}$$

If we rearrange the above terms and apply Π we obtain the equation

$$(\Pi \circ D \circ \partial)(x^{\alpha\beta}) = (-1)^{|D|}(\Pi \circ \partial \circ D)(x^{\alpha\beta}) \tag{4.12}$$

$$= (-1)^{|D|}(\partial \circ \Pi \circ D)(x^{\alpha\beta}) \tag{4.13}$$

$$= (\partial \circ T)(\mathbf{s}\, x) \tag{4.14}$$

recalling that Π is a cochain map. It is straightforward to check that

$$(\Pi \circ D \circ \partial)(x^{\alpha\beta}) = (\operatorname{res}_\alpha D - \operatorname{res}_\beta D - (-1)^{|D|}(T \circ \partial))(sx) \tag{4.15}$$

and so we substitute this into the previous equation and rearrange terms to see that

$$(\operatorname{res}_\alpha D - \operatorname{res}_\beta D)(sx) = (\partial \circ T + (-1)^{|D|} T \circ \partial)(sx) = \partial(T)(sx). \tag{4.16}$$

Thus $\operatorname{res}_\alpha$ and res_β induce the same map on cohomology, and hence are homotopic.

□

4.2 Derived automorphisms of $\operatorname{Cyl}(\mathcal{C})$

We now turn our attention to derivations that may be exponentiated to operad automorphisms. Define

$$\operatorname{Der}'(\operatorname{Cobar}(\mathcal{C})) = \mathcal{F}_1 \operatorname{Der}(\operatorname{Cobar}(\mathcal{C})) \tag{4.17}$$

to be the dg Lie algebra of derivations that raise the number of internal vertices by at least one. Equivalently, $\operatorname{Der}'(\operatorname{Cobar}(\mathcal{C}))$ may be defined as consisting of derivations D that satisfy

$$p_{\mathbf{s}\mathcal{C}_\circ} \circ D = 0 \tag{4.18}$$

where $p_{\mathbf{s}\mathcal{C}_\circ}$ is the canonical projection $\operatorname{Cobar}(\mathcal{C}) \to \mathbf{s}\,\mathcal{C}_\circ$. This definition, in addition to our standing assumption of working with reduced cooperads $\mathcal{C}$, ensures that the dg Lie algebra $\operatorname{Der}'(\mathcal{C})$ is pronilpotent.

Proposition 4.2. *Given a degree 0 derivation $D \in \mathrm{Der}'(\mathrm{Cobar}(\mathcal{C}))$ for a cooperad $\mathcal{C}$, D is* locally nilpotent*: for all $X \in \mathrm{Cobar}(\mathcal{C})$, $D^m(X) = 0$ for some $m \geq 0$. Consequently, assuming D is a cocycle, we may exponentiate D to an automorphism of $\mathrm{Cobar}(\mathcal{C})$,*

$$\exp(D) = \sum_{m=0}^{\infty} \frac{1}{m!} D^m.$$

Here, we use the same filtration as in the proof of Theorem 4.1.

Proof. It is a standard result that a locally nilpotent derivation exponentiates to an automorphism, so it is enough to show that, under our assumptions, all derivations $D \in \mathrm{Der}'(\mathrm{Cobar}(\mathcal{C}))$ are locally nilpotent. This follows from straightforward weight considerations, since D raises weight by at least 1, and since all nodal vertices of $\mathrm{Cobar}(\mathcal{C})$ have at least 2 incoming edges for reduced $\mathcal{C}$. □

We have an identical result for derivations of $\mathrm{Cyl}(\mathcal{C})$.

Proposition 4.3. *Given a degree 0 cocycle $\widetilde{D} \in \mathrm{Der}'(\mathrm{Cyl}(\mathcal{C}))$ for a cooperad $\mathcal{C}$, $\widetilde{D}$ is locally nilpotent, and therefore may be exponentiated to an automorphism of $\mathrm{Cyl}(\mathcal{C})$.*

Proof. The same weight argument works here as for $\mathrm{Cobar}(\mathcal{C})$, with the observation that for a vertex to have only a single incoming edge, it must be of type $1^{\alpha\beta} \in \mathrm{Cyl}(\mathcal{C})(1, 0; \beta)$. □

See Appendix B for a discussion of these notions for the more general case of quasi-free operads. Following this Appendix, define

$$\mathrm{Aut}'(\mathrm{Cobar}(\mathcal{C})) = \{\varphi \in \mathrm{Aut}(\mathrm{Cobar}(\mathcal{C})) \mid \pi \circ \varphi|_{\mathbf{s}\mathcal{C}} = \mathrm{id}_{\mathbf{s}\mathcal{C}}\} \tag{4.19}$$

and

$$\mathrm{Aut}'(\mathrm{Cyl}(\mathcal{C})) = \{\varphi \in \mathrm{Aut}(\mathrm{Cyl}(\mathcal{C})) \mid \pi \circ \varphi|_{\widetilde{\mathcal{C}}} = \mathrm{id}_{\widetilde{\mathcal{C}}}\}, \tag{4.20}$$

which are the groups obtained by exponentiating the Lie algebras $Z^0 \mathrm{Der}'(\mathrm{Cobar}(\mathcal{C}))$ and $Z^0 \mathrm{Der}'(\mathrm{Cyl}(\mathcal{C}))$, respectively. The earlier restriction maps induce obvious group homomorphisms, and the results from Appendix B show that these group homomorphisms are isomorphisms on homotopy classes of automorphisms.

Theorem 4.2. *The group homomorphisms*

$$\mathrm{res}_\alpha, \mathrm{res}_\beta : \quad \mathrm{Aut}'(\mathrm{Cyl}(\mathcal{C})) \quad \longrightarrow \quad \mathrm{Aut}'(\mathrm{Cobar}(\mathcal{C}))$$

induce identical isomorphisms on homotopy classes:

$$\mathrm{res} : \quad \mathrm{hAut}'(\mathrm{Cyl}(\mathcal{C})) \quad \longrightarrow \quad \mathrm{hAut}'(\mathrm{Cobar}(\mathcal{C})).$$

Proof. Since res_α and res_β are homotopic dg Lie algebra quasi-isomorphisms from Theorem 4.1, they induce a single group homomorphism

$$\mathrm{res} : H^0(\mathrm{Der}'(\mathrm{Cyl}(\mathcal{C}))) \longrightarrow H^0(\mathrm{Der}'(\mathrm{Cobar}(\mathcal{C}))) \tag{4.21}$$

(where the group structure is given my the Campbell-Hausdorff formula, as explained in Appendix B). Using the isomorphisms of Proposition B.2, we immediately get the induced isomorphism

$$\mathrm{res} : \mathrm{hAut}(\mathrm{Cyl}(\mathcal{C})) \longrightarrow \mathrm{hAut}(\mathrm{Cobar}(\mathcal{C})). \tag{4.22}$$

□

CHAPTER 5

ACTING ON INFINITY MORPHISMS

We will now turn our attention to homotopy algebras, and show how the operadic techniques and results developed earlier may be applied to their study. In particular, we will use $\mathrm{Cyl}(\mathcal{C})$ to study ∞-morphisms.

5.1 $\mathrm{Cyl}(\mathcal{C})$ and diagrams of $\mathrm{Cobar}(\mathcal{C})$-algebras

Proposition 5.1. *A* $\mathrm{Cyl}(\mathcal{C})$*-algebra structure on a pair of dg vector spaces* (V, W) *is equivalent to the following triple of data:*

1. *a map* $\mathrm{Cobar}(\mathcal{C}) \to \mathrm{End}_V$*;*

2. *a map* $\mathrm{Cobar}(\mathcal{C}) \to \mathrm{End}_W$*;*

3. *an* ∞*-morphism* $V \rightsquigarrow W$.

We will often call this triple a diagram of $\mathrm{Cobar}(\mathcal{C})$-algebras.

Proof. Following [8], given cochain complexes V and W, let $\mathrm{End}_{V,W}$ be the 2-colored endomorphism operad. The pair V, W, being algebras over $\mathrm{Cyl}(\mathcal{C})$ means that there is a map of colored operads

$$F : \mathrm{Cyl}(\mathcal{C}) \longrightarrow \mathrm{End}_{V,W} . \tag{5.1}$$

By construction, the single-color portions of the above map correspond exactly to maps

$$F_\alpha : \mathrm{Cobar}(\mathcal{C}) \longrightarrow \mathrm{End}_V \tag{5.2}$$

$$F_\beta : \mathrm{Cobar}(\mathcal{C}) \longrightarrow \mathrm{End}_W\,. \tag{5.3}$$

Observe next that the mixed-color portion of F can be expressed in terms of its components

$$F_{\alpha\beta}(n) : \mathrm{Cyl}(\mathcal{C})(n, 0; \beta) \longrightarrow \mathrm{End}_{V,W}(n, 0; \beta). \tag{5.4}$$

Equivalently,

$$F_{\alpha\beta}(n) : \mathcal{C}(n) \longrightarrow \mathrm{Hom}_{\mathbb{k}}(V^{\otimes n}, W). \tag{5.5}$$

This is, in turn, equivalent to a map

$$U_n : (\mathcal{C}(n) \otimes V^{\otimes n})^{S_n} \longrightarrow W. \tag{5.6}$$

which extends uniquely to (and is uniquely determined by) a coalgebra map

$$U : \mathcal{C}(V) \longrightarrow \mathcal{C}(W). \tag{5.7}$$

Finally, it is a straightforward check that the compatibility of F with the differentials on $\mathrm{Cyl}(\mathcal{C})$ and $\mathrm{End}_{V,W}$ is equivalent to U being a dg coalgebra map, respecting the coderivations Q_V and Q_W (which correspond to F_α and F_β). □

It is easy to see that, given an operad $\mathcal{O}$, an $\mathcal{O}$-algebra V with algebra structure map $F : \mathcal{O} \to \mathrm{End}_V$, and an endomorphism φ of $\mathcal{O}$, the composite $F \circ \varphi$ defines a new $\mathcal{O}$-algebra structure on V via pullback, which we will often denote V^φ. This also holds true for colored operads and algebras over them. Thus, given a pair (V, W) that is an algebra over $\mathrm{Cyl}(\mathcal{C})$ via the operad morphism $\widetilde{F} : \mathrm{Cyl}(\mathcal{C}) \to \mathrm{End}_{V,W}$, and given an endomorphism $\widetilde{\varphi}$ of $\mathrm{Cyl}(\mathcal{C})$, the morphism $\widetilde{F}^{\widetilde{\varphi}} = \widetilde{F} \circ \widetilde{\varphi}$ defines a new $\mathrm{Cyl}(\mathcal{C})$-algebra structure on (V, W); the result is encoded in the diagram

$$U^{\widetilde{\varphi}} : V^{\widetilde{\varphi}_\alpha} \rightsquigarrow W^{\widetilde{\varphi}_\beta} \tag{5.8}$$

where $\widetilde{\varphi}_\alpha$ and $\widetilde{\varphi}_\beta$ denote the obvious restriction maps of $\widetilde{\varphi}$ onto the first and second single-colored components, respectively (we will begin using this notation more generally, keeping in mind the earlier results about res_α and res_β for derivations). Keeping with the above notation for consistency, we will henceforth decorate colored derivations, automorphisms, maps, etc. by tildes (e.g. $\widetilde{\varphi}$), and omit such decoration for derivations etc. on single-color objects.

This procedure is very general, but requires that we start with an endomorphism of $\mathrm{Cyl}(\mathcal{C})$, which may be difficult to construct. The goal of this section is to show how, in certain instances, to extend an automorphism of $\mathrm{Cobar}(\mathcal{C})$ to an automorphism of $\mathrm{Cyl}(\mathcal{C})$ in a well-controlled way. Therefore, given a diagram $U : V \rightsquigarrow W$, we will not only be able to modify V and W via the automorphism φ, but the ∞-morphism U as well, in some coherent way.

Given an algebra structure map F as above, we will often abuse notation slightly and write F^D instead of $F^{\exp(D)}$ (likewise for the colored setting). We will similarly abbreviate the decoration of homotopy algebras, writing V^D instead of $V^{\exp(D)}$, etc. At this point, we can state and prove the main theorem of these chapters.

Theorem 5.1. *Let V and W be $\mathrm{Cobar}(\mathcal{C})$-algebras for a given cooperad $\mathcal{C}$, and let $U : V \rightsquigarrow W$ be an ∞-morphism between them. Given a degree 0 closed derivation $D \in \mathrm{Der}'(\mathrm{Cobar}(\mathcal{C}))$, there exists a degree 0 cocycle $\widetilde{D} \in \mathrm{Der}'(\mathrm{Cyl}(\mathcal{C}))$ such that D, $\widetilde{D}_\alpha$, and $\widetilde{D}_\beta$ are cohomologous in $\mathrm{Der}'(\mathrm{Cobar}(\mathcal{C}))$. Therefore we can construct*

$$U^{\widetilde{D}} : V^{\widetilde{D}_\alpha} \rightsquigarrow W^{\widetilde{D}_\beta}$$

such that $V^{\widetilde{D}_\alpha}$ is homotopy equivalent to V^D and $W^{\widetilde{D}_\beta}$ is homotopy equivalent to W^D, and so that the linear term of U is unchanged: $U^{\widetilde{D}}_{(0)} = U_{(0)}$.

Proof. The existence of $\widetilde{D}$ satisfying the above properties is provided by Theorem 4.1, which says that the maps res_α, res_β are homotopic quasi-isomorphisms. Therefore the automorphism $\exp(\widetilde{D}) \in \mathrm{Aut}(\mathrm{Cyl}(\mathcal{C}))$ may be used to modify $U : V \rightsquigarrow W$ to $U^{\widetilde{D}} : V^{\widetilde{D}_\alpha} \rightsquigarrow W^{\widetilde{D}_\beta}$, as explained earlier.

For the statement about homotopy equivalence, observe first that Proposition B.2 says that exponentiating cohomologous derivations of $\mathrm{Cobar}(\mathcal{C})$ yields homotopic automorphisms (here, we use a path object notion of homotopy). Thus,

$\exp(D)$ and $\exp(\widetilde{D})_\alpha$ are homotopic. Using [15] to link various notions of homotopy, we see that $\exp(D)$ and $\exp(\widetilde{D})_\alpha$ are cylinder homotopic in the sense of [19]. Therefore, Theorem 5.2.1 of [19] says that there is an ∞-quasi-isomorphism

$$\Phi_V : V \rightsquigarrow V^{\widetilde{D}_\alpha}. \tag{5.9}$$

Similarly, using that D and $\widetilde{D}_\beta$ are cohomologous, we deduce the existence of an ∞-quasi-isomorphism

$$\Phi_W : W \rightsquigarrow W^{\widetilde{D}_\beta}. \tag{5.10}$$

Finally, the statement that the linear terms of $U^{\widetilde{D}}$ and U coincide follows from the fact that we are modifying U via exponentiated derivations, which necessarily start with the identity. Since all derivations considered raise weight by at least 1, all linear terms remain unchanged. □

In [10], we need the above theorem exactly, in particular that the ∞-morphism $U^{\widetilde{D}}$ comes from an exponentiated derivation. The following corollary may be useful in other applications, and may be seen as a more full answer to the motivating question.

Corollary 5.1. *Let V and W be $\mathrm{Cobar}(\mathcal{C})$-algebras for a given cooperad $\mathcal{C}$, and let $U : V \rightsquigarrow W$ be an ∞-morphism between them. Given a degree 0 closed derivation $D \in \mathrm{Der}'(\mathrm{Cobar}(\mathcal{C}))$, there exists an ∞-morphism*

$$U' : V^D \rightsquigarrow W^D$$

Proof. Using the same notation as in the proof of Theorem 5.1, let

$$U' = {\Phi_W}^{-1} \circ U^{\widetilde{D}} \circ \Phi_V : V^D \rightsquigarrow W^D, \tag{5.11}$$

where ${\Phi_W}^{-1} : W^{\widetilde{D}_\beta} \rightsquigarrow W^D$ is a homotopy inverse to Φ_W in the sense of Section 10.4 of [27]. □

We may also study how the construction behaves when iterated, and see that the result is straightforward. For simplicity of notation, we will just focus on the ∞-morphisms - the changes on the source/target algebras should be clear.

Proposition 5.2. *Let $U : V \rightsquigarrow W$ be as above, and $D_1, D_2 \in \mathrm{Der}'(\mathrm{Cobar}(\mathcal{C}))$ degree zero closed derivations that give $\widetilde{D}_1, \widetilde{D}_2 \in \mathrm{Der}'(\mathrm{Cyl}(\mathcal{C}))$. Then, using the above notation,*

$$(U^{\widetilde{D}_1})^{\widetilde{D}_2} = U^{\mathrm{CH}(\widetilde{D}_1, \widetilde{D}_2)}$$

where $\mathrm{CH}(x, y)$ denotes the Campbell-Hausdorff series in the symbols x and y.

Proof. If $U : V \rightsquigarrow W$ is an algebra over $\mathrm{Cyl}(\mathcal{C})$ via $F : \mathrm{Cyl}(\mathcal{C}) \to \mathrm{End}_{V,W}$, we have

$$F \circ \exp(\widetilde{D}_1) \circ \exp(\widetilde{D}_2) = F \circ \exp(\mathrm{CH}(\widetilde{D}_1, \widetilde{D}_2)) \tag{5.12}$$

which gives the desired formula. □

As an example of a situation in which $H^0(\mathrm{Der}'(\mathrm{Cobar}(\mathcal{C})))$ is known to be nonzero, results from [38] show that $H^0(\mathrm{Der}'(\mathrm{Cobar}(\mathrm{Ger}^\vee))) \cong \mathfrak{grt}_1$, the Grothendieck-Teichmueller Lie algebra. This leads to an application of Theorem 5.1 to justify a statement made in Section 10.2 of [38], concerning GRT_1-equivariance of Tamarkin's construction of formality morphisms. We will deal with this question in the remaining chapters [1].

5.2 Homotopy uniqueness

Given that the only choices in the proof of Theorem 5.1 involve cohomologous derivations, the result should be "unique up to homotopy" in some sense. We give two characterizations of this uniqueness in this section. [2]

First, and most obviously, we can reinterpret Theorem 4.2 as saying the following:

Proposition 5.3. *Any automorphism $\widetilde{\varphi} \in \mathrm{Aut}'(\mathrm{Cyl}(\mathcal{C}))$ is uniquely determined up to homotopy by its restriction onto either color. In particular, if Theorem 5.1*

[1] Alternately, in the stable setting, this question will be addressed in [7].

[2] A full answer would rely on a more appropriate theoretical framework, such as the theory of model categories or ∞-categories, that are outside the scope of this dissertation.

produces $\exp(\widetilde{D})$ *such that* $\exp(D)$ *is homotopic to either* $\widetilde{\varphi}_\alpha$ *or* $\widetilde{\varphi}_\beta$*, then* $\exp(\widetilde{D})$ *is homotopic to* $\widetilde{\varphi}$*.*

Homotopic automorphisms will yield homotopic operad maps $\mathrm{Cyl}(\mathcal{C}) \to \mathrm{End}_{V,W}$, and so the second characterization of homotopy uniqueness involves unraveling exactly what results from homotopic structure maps, in terms of the resulting homotopy algebras and ∞-morphisms. This should be viewed as a 2-colored extension of the result from [19] that homotopic structure maps $\mathrm{Cobar}(\mathcal{C}) \to \mathrm{End}_V$ yield homotopy equivalent algebras.

Proposition 5.4. *Let* $F, G : \mathrm{Cyl}(\mathcal{C}) \to \mathrm{End}_{V,W}$ *be maps of operads, corresponding respectively to the homotopy algebras and* ∞*-morphisms*

$$U_F : V_F \rightsquigarrow W_F$$

$$U_G : V_G \rightsquigarrow W_G.$$

Suppose F *is homotopic to* G*. Then we obtain* ∞*-quasi-isomorphisms*

$$\Phi : V_F \rightsquigarrow V_G$$

$$\Psi : W_F \rightsquigarrow W_G$$

such that $\Psi \circ U_F$ *is homotopic to* $U_G \circ \Phi$*. That is, the following diagram of* ∞*-morphisms commutes up to homotopy:*

$$\begin{array}{ccc} V_F & \xrightarrow{U_F} & W_F \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Psi} \\ V_G & \xrightarrow{U_G} & W_G \end{array}$$

Proof. Let

$$\mathcal{H} : \mathrm{Cyl}(\mathcal{C}) \to \mathrm{End}_{V,W} \otimes \Omega^\bullet(\Bbbk) \tag{5.13}$$

be the operadic homotopy between F and G:

$$\mathcal{H}|_{t=0,dt=0} = F, \quad \mathcal{H}|_{t=1,dt=0} = G. \tag{5.14}$$

By restricting $\mathcal{H}$ to color α, we see that F_α is homotopic to G_α. As explained more fully in the proof of Theorem 5.1, this yields homotopy equivalent algebras $\Phi : V_F \rightsquigarrow W_F$ [19]. Restricting to color β, we similarly obtain Ψ.

Let us express

$$\mathcal{H}(t, dt) = \mathcal{H}_0(t) + \mathcal{H}_1(t)dt \tag{5.15}$$

so that $\mathcal{H}$ being a map of operads is equivalent to the following:

1. For all t, $\mathcal{H}_0$ is a map of operads $\mathrm{Cyl}(\mathcal{C}) \to \mathrm{End}_{V,W}$
2. For all t, $\mathcal{H}_1$ is a derivation relative to $\mathcal{H}_0$
3. $\frac{d}{dt}\mathcal{H}_0 = \partial_{\mathrm{End}_{V,W}} \circ \mathcal{H}_1 + \mathcal{H}_1 \circ \partial_{\mathrm{Cyl}(\mathcal{C})}$

(see also Proposition B.2). We will use the above data to construct an explicit homotopy between $\Psi \circ U_F$ and $U_G \circ \Phi$, that is, a map of cofree $\mathcal{C}$-coalgebras

$$H : \mathcal{C}(V_F) \longrightarrow \mathcal{C}(W_G \otimes \Omega^\bullet(\Bbbk)) \tag{5.16}$$

such that

$$H|_{t=0,dt=0} = \Psi \circ U_F, \quad H|_{t=1,dt=0} = U_G \circ \Phi \tag{5.17}$$

Here, we are using the notation $\mathcal{C}(V_F)$ to denote the cofree $\mathcal{C}$-coalgebra on V, with differential coming from $F_\alpha : \mathrm{Cobar}(\mathcal{C}) \to \mathrm{End}_V$, and similarly elsewhere. If we also write

$$H(t, dt) = H_0(t) + H_1(t)dt, \tag{5.18}$$

we see the H must satisfy appropriate parallel conditions as $\mathcal{H}$ listed above; H_0 is a map of coalgebras, H_1 is a coderivation relative to H_0, and the correct similar condition on $\frac{d}{dt}H_0$.

For all t, $\mathcal{H}$ yields the following: a diagram of homotopy algebras

$$U_t : V_t \rightsquigarrow W_t \tag{5.19}$$

along with ∞-quasi-isomorphisms

$$\Phi_t : V_F \rightsquigarrow V_t \tag{5.20}$$

$$\Psi_{1-t} : W_t \rightsquigarrow W_G. \tag{5.21}$$

Just as for coderivations, relative coderivations are uniquely determined by their composition with the canonical projection, so we may define a coderivation relative to U_t by the following formula:

$$P_t : \mathcal{C}(V_t) \rightsquigarrow W_t \tag{5.22}$$

$$P_t(X; v_1, ..., v_n) = \mathcal{H}_1(X^{\alpha\beta})(v_1, ..., v_n) \tag{5.23}$$

where $(X; v_1, ..., v_n) \in \mathcal{C}(V_t)$.

Since P_t is a coderivation relative to the coalgebra map U_t, $\Psi_{1-t} \circ P_t \circ \Phi_t$ is a coderivation relative to $\Psi_{1-t} \circ U_t \circ \Phi_t$. It is then easy to check that

$$H_0(t) = \Psi_{1-t} \circ U_t \circ \Phi_t \tag{5.24}$$

$$H_1(t) = \Psi_{1-t} \circ P_t \circ \Phi_t \tag{5.25}$$

satisfy the required properties for $H = H_0 + H_1 dt$ to be the desired homotopy between $\Psi \circ U_F$ and $U_G \circ \Phi$. $\square$

CHAPTER 6

TAMARKIN'S CONSTRUCTION OF FORMALITY MORPHISMS

We now proceed to show how the previous operadic results may be applied to show that Tamarkin's construction is equivariant with respect to the action of the Grothendieck-Teichmueller group. Various solutions of the Deligne conjecture on thr Hochschild cochain complex [3], [4], [13], [24], [30], [35], [37] imply that the dg operad Braces is quasi-isomorphic to the dg operad

$$C_{-\bullet}(E_2, \Bbbk)$$

of singular chains for the little disc operad E_2 .

Combining this statement with the formality [23], [34] for the dg operad $C_{-\bullet}(E_2, \Bbbk)$, we conclude that the dg operad Braces is quasi-isomorphic to the operad Ger. Hence there exists a quasi-isomorphism of dg operads

$$\Psi : \mathrm{Ger}_\infty \to \mathrm{Braces} \tag{6.1}$$

for which the vector[1] $\Psi(\mathbf{s}(b_1 b_2)^*)$ is cohomologous to the sum $T + T_{21}$ and the vector $\Psi(\mathbf{s}\{b_1, b_2\}^*)$ is cohomologous to

$$\frac{1}{2}(T_\cup + T_{\cup^{opp}})\,,$$

[1] Here, we use basis (2.29) in $\mathrm{Ger}^\vee(n)$.

where T (resp. T_{21}, $T_{\cup}$, $T_{\cup^{opp}}$) is the brace tree depicted on figure 2.5 (resp. figure 2.6, 2.7, 2.8).

Replacing Ψ by a homotopy equivalent map we may assume, without loss of generality, that

$$\Psi(\mathbf{s}(b_1 b_2)^*) = T + T_{21}\,, \qquad \Psi(\mathbf{s}\{b_1, b_2\}^*) = \frac{1}{2}(T_{\cup} + T_{\cup^{opp}})\,. \tag{6.2}$$

So from now on we will assume that the map Ψ (6.1) satisfies conditions (6.2).

Since the dg operad Braces acts on the Hochschild cochain complex $C^{\bullet}(A)$ of an A_{∞}-algebra A, the map Ψ equips the Hochschild cochain complex $C^{\bullet}(A)$ with a structure of a Ger_{∞}-algebra. We will call it *Tamarkin's* Ger_{∞}*-structure* and denote by

$$C^{\bullet}(A)^{\Psi}$$

the Hochschild cochain complex of A with the Ger_{∞}-structure coming from Ψ.

The choice of the homotopy class of Ψ (6.1) (and hence the choice of Tamarkin's Ger_{∞}-structure) is far from unique. In fact, it follows from [38, Theorem 1.2] that the set of homotopy classes of maps (6.1) satisfying conditions (6.2) form a torsor for an infinite dimensional pro-algebraic group.

A simple degree bookkeeping in Braces shows that for every $n \geq 3$

$$\Psi(\mathbf{s}(b_1 b_2 \dots b_n)^*) = 0\,. \tag{6.3}$$

Combining this observation with (6.2) we see that any Tamarkin's Ger_{∞}-structure on $C^{\bullet}(A)$ satisfies the following remarkable property:

Property 6.1. *The* $\Lambda\,\mathrm{Lie}_{\infty}$ *part of Tamarkin's* Ger_{∞}*-structure on* $C^{\bullet}(A)$ *coincides with the* $\Lambda\,\mathrm{Lie}$*-structure given by the Gerstenhaber bracket on* $C^{\bullet}(A)$.

From now on, we only consider the case when $A = \Bbbk[x^1, ..., x^d]$, i.e. the free (graded) commutative algebra over $\Bbbk$ in variables $x^1, x^2, \dots, x^d$ of (not necessarily zero) degrees $t_1, t_2, \dots, t_d$, respectively. Furthermore, V_A denotes the Gerstenhaber algebra of polyvector fields on the corresponding graded affine space, i.e.

$$V_A := S_A\big(\mathbf{s}\ \mathrm{Der}_{\Bbbk}(A)\big)\,.$$

It is known[2] [22] that the canonical embedding

$$V_A \hookrightarrow C^\bullet(A) \tag{6.4}$$

is a quasi-isomorphism of cochain complexes, where V_A is considered with the zero differential. We refer to (6.4) as the *Hochschild-Kostant-Rosenberg embedding*.

Let us now consider the Ger_∞-algebra $C^\bullet(A)^\Psi$ for a chosen map Ψ (6.1). By the first claim of Corollary D.2 from Appendix D, there exists a Ger_∞-quasi-isomorphism

$$U_{\mathrm{Ger}} : V_A \rightsquigarrow C^\bullet(A)^\Psi \tag{6.5}$$

whose linear term coincides with the Hochschild-Kostant-Rosenberg embedding.

Restricting U_{Ger} to the $\Lambda^2\,\mathrm{coCom}$-coalgebra

$$\Lambda^2\,\mathrm{coCom}(V_A)$$

and taking into account Property 6.1 we get a $\Lambda\,\mathrm{Lie}_\infty$-quasi-isomorphism

$$U_{\mathrm{Lie}} : V_A \rightsquigarrow C^\bullet(A) \tag{6.6}$$

of (dg) $\Lambda\,\mathrm{Lie}$-algebras.

Thus we have obtained the main statement of Tamarkin's construction [36] which can be summarized as

Theorem 6.1 (D. Tamarkin, [36])**.** *Let A (resp. V_A) be the algebra of functions (resp. the algebra of polyvector fields) on a graded affine space. Let us consider the Hochschild cochain complex $C^\bullet(A)$ with the standard $\Lambda\,\mathrm{Lie}$-algebra structure. Then, for every map of dg operads Ψ* (6.1)*, there exists a $\Lambda\,\mathrm{Lie}_\infty$-quasi-isomorphism*

$$U_{\mathrm{Lie}} : V_A \rightsquigarrow C^\bullet(A) \tag{6.7}$$

which can be extended to a Ger_∞-quasi-isomorphism

$$U_{\mathrm{Ger}} : V_A \rightsquigarrow C^\bullet(A)^\Psi$$

where V_A carries the standard Gerstenhaber algebra structure. □

[2] Paper [22] treats only the case of usual (not graded) affine algebras. However, the proof of [22] can be generalized to the graded setting in a straightforward manner.

Remark 6.1. *We tacitly assume that the linear part of every* $\Lambda\,\mathrm{Lie}_\infty$ *(resp.* Ger_∞*)-quasi-isomorphism from* V_A *to* $C^\bullet(A)$ *(resp.* $C^\bullet(A)^\Psi$*) coincides with the Hochschild-Kostant-Rosenberg embedding of polyvector fields into Hochschild cochains.*

Since the above construction involves several choices, we are left with the following obvious questions:

Question A. Is it possible to construct two homotopy inequivalent $\Lambda\,\mathrm{Lie}_\infty$-quasi-isomorphisms (6.6) corresponding to the same map Ψ (6.1)? And if no then

Question B. Are $\Lambda\,\mathrm{Lie}_\infty$-quasi-isomorphisms U_{Lie} and $\widetilde{U}_{\mathrm{Lie}}$ (6.6) homotopy equivalent if so are the corresponding maps of dg operads Ψ and $\widetilde{\Psi}$ (6.1)?

The (expected) answer (NO) to Question A is given in the following proposition:

Proposition 6.1. *Let* Ψ *a map of dg operads* (6.1) *satisfying* (6.2) *and*

$$U_{\mathrm{Lie}},\, \widetilde{U}_{\mathrm{Lie}} \,:\, V_A \leadsto C^\bullet(A) \tag{6.8}$$

be $\Lambda\,\mathrm{Lie}_\infty$*-quasi-morphisms which extend to* Ger_∞*-quasi-isomorphisms*

$$U_{\mathrm{Ger}},\, \widetilde{U}_{\mathrm{Ger}} \,:\, V_A \leadsto C^\bullet(A)^\Psi \tag{6.9}$$

respectively. Then U_{Lie} *is homotopy equivalent to* $\widetilde{U}_{\mathrm{Lie}}$*.*

Proof. This statement is essentially a consequence of general Corollary D.2 from Appendix D.2.

Indeed, the second claim of Corollary D.2 implies that Ger_∞-morphisms (6.9) are homotopy equivalent. Hence so are their restrictions to the $\Lambda^2\,\mathrm{coCom}$-coalgebra

$$\Lambda^2\,\mathrm{coCom}(V_A)$$

which coincide with U_{Lie} and $\widetilde{U}_{\mathrm{Lie}}$, respectively. □

The expected answer (YES) to Question B is given in the following addition to Theorem 6.1:

Theorem 6.2. *The homotopy type of* $U_{\rm Lie}$ (6.6) *depends only on the homotopy type of the map* Ψ (6.1).

Proof. Let Ψ and $\widetilde{\Psi}$ be maps of dg operads (6.1) satisfying (6.2) and let

$$U_{\rm Lie} \,:\, V_A \rightsquigarrow C^\bullet(A) \tag{6.10}$$

$$\widetilde{U}_{\rm Lie} \,:\, V_A \rightsquigarrow C^\bullet(A) \tag{6.11}$$

be $\Lambda\,{\rm Lie}_\infty$-quasi-morphisms which extend to ${\rm Ger}_\infty$-quasi-isomorphisms

$$U_{\rm Ger} \,:\, V_A \rightsquigarrow C^\bullet(A)^{\Psi}\,, \qquad \text{and} \qquad \widetilde{U}_{\rm Ger} \,:\, V_A \rightsquigarrow C^\bullet(A)^{\widetilde{\Psi}} \tag{6.12}$$

respectively. Our goal is to show that if Ψ is homotopy equivalent to $\widetilde{\Psi}$ then $U_{\rm Lie}$ is homotopy equivalent to $\widetilde{U}_{\rm Lie}$.

Let us denote by $\Omega^\bullet(\Bbbk)$ the dg commutative algebra of polynomial forms on the affine line with the canonical coordinate t.

Since quasi-isomorphisms $\Psi, \widetilde{\Psi} \,:\, {\rm Ger}_\infty \to {\rm Braces}$ are homotopy equivalent, we have[3] a map of dg operads

$$\mathfrak{H} : {\rm Ger}_\infty \to {\rm Braces} \otimes \Omega^\bullet(\Bbbk) \tag{6.13}$$

such that

$$\Psi = p_0 \circ \mathfrak{H}\,, \qquad \text{and} \qquad \widetilde{\Psi} = p_1 \circ \mathfrak{H}\,,$$

where p_0 and p_1 are the canonical maps (of dg operads)

$$p_0, p_1 \,:\, {\rm Braces} \otimes \Omega^\bullet(\Bbbk) \to {\rm Braces}\,,$$

$$p_0(v) := v\Big|_{dt=0,\ t=0}\,, \qquad p_1(v) := v\Big|_{dt=0,\ t=1}\,.$$

The map $\mathfrak{H}$ induces a ${\rm Ger}_\infty$-structure on $C^\bullet(A)\otimes\Omega^\bullet(\Bbbk)$ such that the evaluation maps (which we denote by the same letters)

$$\begin{aligned} &p_0 : C^\bullet(A) \otimes \Omega^\bullet(\Bbbk) \to C^\bullet(A)^{\Psi}\,, \qquad p_0(v) := v\big|_{dt=0,\ t=0}\,, \\ &p_1 : C^\bullet(A) \otimes \Omega^\bullet(\Bbbk) \to C^\bullet(A)^{\widetilde{\Psi}}\,, \qquad p_1(v) := v\big|_{dt=0,\ t=1}\,. \end{aligned} \tag{6.14}$$

[3] For justification of this step see, for example, [11, Section 5.1].

are strict quasi-isomorphisms of the corresponding Ger_∞-algebras.

So, in this proof, we consider the cochain complex $C^\bullet(A) \otimes \Omega^\bullet(\Bbbk)$ with the Ger_∞-structure coming from $\mathfrak{H}$ (6.13). The same degree bookkeeping argument in Braces shows that[4]

$$\mathfrak{H}(\mathbf{s}(b_1 b_2 \dots b_n)^*) = 0\,. \tag{6.15}$$

Hence, the $\Lambda\,\mathrm{Lie}_\infty$ part of the Ger_∞-structure on $C^\bullet(A)\otimes\Omega^\bullet(\Bbbk)$ coincides with the $\Lambda\,\mathrm{Lie}$-structure given by the Gerstenhaber bracket extended from $C^\bullet(A)$ to $C^\bullet(A)\otimes\Omega^\bullet(\Bbbk)$ to by $\Omega^\bullet(\Bbbk)$-linearity.

Since the canonical embedding

$$P \mapsto P\otimes 1 : C^\bullet(A) \hookrightarrow C^\bullet(A)\otimes\Omega^\bullet(\Bbbk) \tag{6.16}$$

is a quasi-isomorphism of cochain complexes, Corollary D.2 from Appendix D.2 implies that there exists a Ger_∞-quasi-isomorphism

$$U^{\mathfrak{H}}_{\mathrm{Ger}} : V_A \rightsquigarrow C^\bullet(A)\otimes\Omega^\bullet(\Bbbk)\,, \tag{6.17}$$

where V_A is considered with the standard Gerstenhaber structure.

Since the $\Lambda\,\mathrm{Lie}_\infty$ part of the Ger_∞-structure on $C^\bullet(A)\otimes\Omega^\bullet(\Bbbk)$ coincides with the standard $\Lambda\,\mathrm{Lie}$-structure, the restriction of $U^{\mathfrak{H}}_{\mathrm{Ger}}$ to the $\Lambda^2\,\mathrm{coCom}$-coalgebra $\Lambda^2\,\mathrm{coCom}(V_A)$ gives us a homotopy connecting the $\Lambda\,\mathrm{Lie}_\infty$-quasi-isomorphism

$$p_0 \circ U^{\mathfrak{H}}_{\mathrm{Ger}}\Big|_{\Lambda^2\,\mathrm{coCom}(V_A)} \;:\; V_A \rightsquigarrow C^\bullet(A) \tag{6.18}$$

to the $\Lambda\,\mathrm{Lie}_\infty$-quasi-isomorphism

$$p_1 \circ U^{\mathfrak{H}}_{\mathrm{Ger}}\Big|_{\Lambda^2\,\mathrm{coCom}(V_A)} \;:\; V_A \rightsquigarrow C^\bullet(A)\,, \tag{6.19}$$

where p_0 and p_1 are evaluation maps (6.14).

Let us now observe that $\Lambda\,\mathrm{Lie}_\infty$-quasi-isomorphisms (6.18) and (6.19) extend to Ger_∞-quasi-isomorphisms

$$p_0\circ U^{\mathfrak{H}}_{\mathrm{Ger}} \;:\; V_A \rightsquigarrow C^\bullet(A)^{\Psi}\,, \qquad \text{and} \qquad p_1\circ U^{\mathfrak{H}}_{\mathrm{Ger}} \;:\; V_A \rightsquigarrow C^\bullet(A)^{\widetilde{\Psi}} \tag{6.20}$$

[4] Here, we use basis (2.29) in $\mathrm{Ger}^\vee(n)$.

respectively. Hence, by Proposition 6.1, $\Lambda\,\mathrm{Lie}_\infty$-quasi-isomorphism (6.18) is homotopy equivalent to (6.10) and $\Lambda\,\mathrm{Lie}_\infty$-quasi-isomorphism (6.19) is homotopy equivalent to (6.11).

Thus $\Lambda\,\mathrm{Lie}_\infty$-quasi-isomorphisms (6.10) and (6.11) are indeed homotopy equivalent. □

The general conclusion of this chapter is that Tamarkin's construction [21], [36] gives us a map

$$\mathfrak{T} : \pi_0\big(\,\mathrm{Ger}_\infty \to \mathrm{Braces}\,\big) \to \pi_0\big(V_A \rightsquigarrow C^\bullet(A)\big) \qquad (6.21)$$

from the set $\pi_0\big(\,\mathrm{Ger}_\infty \to \mathrm{Braces}\,\big)$ of homotopy classes of operad morphisms (6.1) satisfying conditions (6.2) to the set $\pi_0\big(V_A \rightsquigarrow C^\bullet(A)\big)$ of homotopy classes of $\Lambda\,\mathrm{Lie}_\infty$-morphisms from V_A to $C^\bullet(A)$ whose linear term is the Hochschild-Kostant-Rosenberg embedding.

CHAPTER 7

ACTIONS OF THE GROTHENDIECK-TEICHMUELLER GROUP ON TAMARKIN'S CONSTRUCTION

Let $\mathcal{C}$ be a coaugmented cooperad in the category of graded vector spaces and $\mathcal{C}_\circ$ be the cokernel of the coaugmentation. Recall the standing assumption that all cooperads are reduced, that is, that $\mathcal{C}(0) = \mathbf{0}$ and $\mathcal{C}(1) = \mathbb{k}$.

Let us denote by

$$\mathrm{Der}'\big(\mathrm{Cobar}(\mathcal{C})\big) \tag{7.1}$$

the dg Lie algebra of derivation D of $\mathrm{Cobar}(\mathcal{C})$ satisfying the condition

$$p_{\mathbf{s}\,\mathcal{C}_\circ} \circ D = 0\,, \tag{7.2}$$

where $p_{\mathbf{s}\,\mathcal{C}_\circ}$ is the canonical projection $\mathrm{Cobar}(\mathcal{C}) \to \mathbf{s}\,\mathcal{C}_\circ$. Since $\mathcal{C}$ is reduced, (7.2) implies that $\mathrm{Der}'\big(\mathrm{Cobar}(\mathcal{C})\big)^0$ and $H^0\big(\mathrm{Der}'(\mathrm{Cobar}(\mathcal{C}))\big)$ are pronilpotent Lie algebras.

In this dissertation, we are mostly interested in the case when $\mathcal{C} = \Lambda^2\,\mathrm{coCom}$ and $\mathcal{C} = \mathrm{Ger}^\vee$. The corresponding dg operads $\Lambda\,\mathrm{Lie}_\infty = \mathrm{Cobar}(\Lambda^2\,\mathrm{coCom})$ and $\mathrm{Ger}_\infty = \mathrm{Cobar}(\mathrm{Ger}^\vee)$ govern $\Lambda\,\mathrm{Lie}_\infty$ and Ger_∞ algebras, respectively.

A simple degree bookkeeping shows that

$$\mathrm{Der}'(\Lambda\,\mathrm{Lie}_\infty)^{\le 0} = \mathbf{0}\,, \tag{7.3}$$

i.e. the dg Lie algebra $\mathrm{Der}'(\Lambda\,\mathrm{Lie}_\infty)$ does not have non-zero elements in degrees ≤ 0. In particular, the Lie algebra $H^0\big(\,\mathrm{Der}'(\Lambda\,\mathrm{Lie}_\infty)\big)$ is zero.

On the other hand, the Lie algebra

$$\mathfrak{g} = H^0\big(\,\mathrm{Der}'(\mathrm{Ger}_\infty)\big) \tag{7.4}$$

is much more interesting. According to Willwacher's theorem [38, Theorem 1.2], this Lie algebra is isomorphic to the pro-nilpotent part $\mathfrak{grt}_1$ of the Grothendieck-Teichmüller Lie algebra $\mathfrak{grt}$ [1, Section 4.2]. Hence, the group $\exp(\mathfrak{g})$ is isomorphic to the group $\mathrm{GRT}_1 = \exp(\mathfrak{grt}_1)$.

Let us now describe how the group $\exp(\mathfrak{g}) \cong \mathrm{GRT}_1$ acts both on the source and the target of Tamarkin's map $\mathfrak{T}$ (6.21).

7.1 The action of GRT_1 on $\pi_0\big(\,\mathrm{Ger}_\infty \to \mathrm{Braces}\,\big)$

Let v be a vector of $\mathfrak{g}$ represented by a (degree zero) cocycle $D \in \mathrm{Der}'(\mathrm{Ger}_\infty)$. Since the Lie algebra $\mathrm{Der}'(\mathrm{Ger}_\infty)^0$ is pro-nilpotent, D gives us an automorphism

$$\exp(D) \tag{7.5}$$

of the operad Ger_∞.

Let Ψ be a quasi-isomorphism of dg operads (6.1). Due to Proposition B.2, the homotopy type of the composition

$$\Psi \circ \exp(D) \tag{7.6}$$

does not depend on the choice of the cocycle D in the cohomology class v. Furthermore, for every pair of (degree zero) cocycles $D, \widetilde{D} \in \mathrm{Der}'(\mathrm{Ger}_\infty)$ we have

$$\Psi \circ \exp(D) \circ \exp(\widetilde{D}) = \Psi \circ \exp\big(\,\mathrm{CH}(D, \widetilde{D})\big)\,, \tag{7.7}$$

where $\mathrm{CH}(x, y)$ denotes the Campbell-Hausdorff series in symbols x, y.

Thus the assignment

$$\Psi \to \Psi \circ \exp(D) \tag{7.8}$$

induces a *right* action of the group $\exp(\mathfrak{g})$ on the set $\pi_0\big(\mathrm{Ger}_\infty \to \mathrm{Braces}\big)$ of homotopy classes of operad morphisms (6.1).

7.2 The action of GRT_1 on $\pi_0\big(V_A \rightsquigarrow C^\bullet(A)\big)$

Let us now show that $\exp(\mathfrak{g}) \cong \mathrm{GRT}_1$ also acts on the set $\pi_0\big(V_A \rightsquigarrow C^\bullet(A)\big)$ of homotopy classes of $\Lambda\,\mathrm{Lie}_\infty$-morphisms from V_A to $C^\bullet(A)$.

For this purpose, we denote by

$$\mathrm{Act}_{stan} : \mathrm{Ger}_\infty \to \mathrm{End}_{V_A} \tag{7.9}$$

the operad map corresponding to the standard Gerstenhaber algebra structure on V_A.

Then, given a degree 0 cocycle $D \in \mathrm{Der}'(\mathrm{Ger}_\infty)$ representing $v \in \mathfrak{g}$, we may precompose map (7.9) by the automorphism $\exp(D)$. This way, we equip the graded vector space V_A with a new Ger_∞-structure $Q^{\exp(D)}$ whose binary operations are the standard ones. Therefore, by Corollary D.1 from Appendix D.1, there exists a Ger_∞-quasi-isomorphism

$$U_{\mathrm{corr}} : V_A \to V_A^{Q^{\exp(D)}} \tag{7.10}$$

from V_A with the standard Gerstenhaber structure to V_A with the Ger_∞-structure $Q^{\exp(D)}$.

Due to observation (7.3), the restriction of D onto the suboperad $\mathrm{Cobar}(\Lambda^2\,\mathrm{coCom}) \subset \mathrm{Cobar}(\mathrm{Ger}^\vee)$ is zero. Hence, for every degree zero cocycle $D \in \mathrm{Der}'(\mathrm{Ger}_\infty)$, we have

$$\exp(D)\Big|_{\mathrm{Cobar}(\Lambda^2\,\mathrm{coCom})} = \mathrm{Id} : \mathrm{Cobar}(\Lambda^2\,\mathrm{coCom}) \to \mathrm{Cobar}(\Lambda^2\,\mathrm{coCom})\,. \tag{7.11}$$

Therefore the $\Lambda\,\mathrm{Lie}_\infty$-part of the Ger_∞-structure $Q^{\exp(D)}$ coincides with the standard $\Lambda\,\mathrm{Lie}$-structure on V_A given by the Schouten bracket. Hence the restriction of the Ger_∞-quasi-isomorphism U_{corr} onto the $\Lambda^2\,\mathrm{coCom}$-coalgebra $\Lambda^2\,\mathrm{coCom}(V_A)$ gives us a $\Lambda\,\mathrm{Lie}_\infty$-automorphism

$$U^D : V_A \rightsquigarrow V_A\,. \tag{7.12}$$

Note that, for a fixed Ger_∞-structure $Q^{\exp(D)}$, Ger_∞-quasi-isomorphism (7.10) is far from unique. However, the second statement of Corollary D.2 implies that the homotopy class of (7.10) *is* unique. Therefore, the assignment

$$D \mapsto \big[U^D\big]$$

is a well defined map from the set of degree zero cocycles of $\mathrm{Der}'(\mathrm{Ger}_\infty)$ to homotopy classes of $\Lambda\,\mathrm{Lie}_\infty$-automorphisms of V_A.

This statement can be strengthened further:

Proposition 7.1. *The homotopy type of U^D does not depend on the choice of the representative D of the cohomology class v. Furthermore, for any pair of degree zero cocycles $D_1, D_2 \in \mathrm{Der}'(\mathrm{Ger}_\infty)$, the composition $U^{D_1} \circ U^{D_2}$ is homotopy equivalent to $U^{\mathrm{CH}(D_1,D_2)}$, where $\mathrm{CH}(x,y)$ denotes the Campbell-Hausdorff series in symbols x, y.*

Let us postpone the technical proof of Proposition 7.1 to Section 7.4 and observe that this proposition implies the following statement:

Corollary 7.1. *Let D be a degree zero cocycle in $\mathrm{Der}'(\mathrm{Ger}_\infty)$ representing a cohomology class $v \in \mathfrak{g}$ and let U_{Lie} be a $\Lambda\,\mathrm{Lie}_\infty$-quasi-isomorphism from V_A to $C^\bullet(A)$. The assignment*

$$U_{\mathrm{Lie}} \mapsto U_{\mathrm{Lie}} \circ U^D \tag{7.13}$$

induces a right action of the group $\exp(\mathfrak{g})$ on the set $\pi_0\big(V_A \rightsquigarrow C^\bullet(A)\big)$ of homotopy classes of $\Lambda\,\mathrm{Lie}_\infty$-morphisms from V_A to $C^\bullet(A)$. □

From now on, by abuse of notation, we denote by U^D any representative in the homotopy class of $\Lambda\,\mathrm{Lie}_\infty$-automorphism (7.12).

7.3 The theorem on GRT_1-equivariance

The following theorem is the main result of these chapters:

Theorem 7.1. *Let* $\pi_0\big(\mathrm{Ger}_\infty \to \mathrm{Braces}\big)$ *be the set of homotopy classes of operad maps* (6.1) *from the dg operad* Ger_∞ *governing homotopy Gerstenhaber algebras to the dg operad* Braces *of brace trees. Let* $\pi_0\big(V_A \rightsquigarrow C^\bullet(A)\big)$ *be the set of homotopy classes of* $\Lambda\,\mathrm{Lie}_\infty$*-quasi-isomorphisms*[1] *from the algebra* V_A *of polyvector fields to the algebra* $C^\bullet(A)$ *of Hochschild cochains of a graded affine space. Then Tamarkin's map* $\mathfrak{T}$ (6.21) *commutes with the action of the group* $\exp(\mathfrak{g})$ *which corresponds to Lie algebra* (7.4).

Proof. Following the notation of earlier chapters, we will denote by $\mathrm{Cyl}(\mathrm{Ger}^\vee)$ the 2-colored dg operad whose algebras are pairs (V, W) with the data

1. a Ger_∞-structure on V,
2. a Ger_∞-structure on W, and
3. a Ger_∞-morphism F from V to W, i.e. a homomorphism of corresponding dg $\mathrm{Ger}^\vee$-coalgebras $\mathrm{Ger}^\vee(V) \to \mathrm{Ger}^\vee(W)$.

Recall that if we forget about the differential, then the operad $\mathrm{Cyl}(\mathrm{Ger}^\vee)$ is a free operad on a certain 2-colored collection $\mathcal{M}(\mathrm{Ger}^\vee)$ naturally associated to $\mathrm{Ger}^\vee$.

Recall that we denote by

$$\mathrm{Der}'(\mathrm{Cyl}(\mathrm{Ger}^\vee)) \tag{7.14}$$

the dg Lie algebra of derivations D of $\mathrm{Cyl}(\mathrm{Ger}^\vee)$ subject to the condition[2]

$$p \circ D = 0\,, \tag{7.15}$$

[1] We tacitly assume that operad maps (6.1) satisfies conditions (6.2) and $\Lambda\,\mathrm{Lie}_\infty$ quasi-isomorphisms $V_A \rightsquigarrow C^\bullet(A)$ extend the Hochschild-Kostant-Rosenberg embedding.

[2] Recall that it is condition (7.15) which guarantees that any degree zero cocycle in $\mathrm{Der}'(\mathrm{Cyl}(\mathrm{Ger}^\vee))$ can be exponentiated to an automorphism of $\mathrm{Cyl}(\mathrm{Ger}^\vee)$.

where p is the canonical projection from $\mathrm{Cyl}(\mathrm{Ger}^\vee)$ onto $\mathcal{M}(\mathrm{Ger}^\vee)$.

The restrictions to the first color part and the second color part of $\mathrm{Cyl}(\mathrm{Ger}^\vee)$, respectively, give us natural maps of dg Lie algebras

$$\mathrm{res}_\alpha, \ \ \mathrm{res}_\beta : \mathrm{Der}'(\mathrm{Cyl}(\mathrm{Ger}^\vee)) \to \mathrm{Der}'(\mathrm{Ger}_\infty), \tag{7.16}$$

and, due to 4.1, res_α and res_β are chain homotopic quasi-isomorphisms.

Therefore, for every $v \in \mathfrak{g}$ there exists a degree zero cocycle

$$D \in \mathrm{Der}'(\mathrm{Cyl}(\mathrm{Ger}^\vee)) \tag{7.17}$$

such that both $\mathrm{res}_\alpha(D)$ and $\mathrm{res}_\beta(D)$ represent the cohomology class v.

Let

$$U_{\mathrm{Ger}} : V_A \rightsquigarrow C^\bullet(A)^\Psi \tag{7.18}$$

be a Ger_∞-morphism from V_A to $C^\bullet(A)$ which restricts to a $\Lambda\,\mathrm{Lie}_\infty$-morphism

$$U_{\mathrm{Lie}} : V_A \rightsquigarrow C^\bullet(A)\,. \tag{7.19}$$

The triple consisting of

- the standard Gerstenhaber structure on V_A,
- the Ger_∞-structure on $C^\bullet(A)$ coming from a map Ψ, and
- Ger_∞-morphism (7.18)

gives us a map of dg operads

$$U_{\mathrm{Cyl}} : \mathrm{Cyl}(\mathrm{Ger}^\vee) \to \mathrm{End}_{V_A, C^\bullet(A)} \tag{7.20}$$

from $\mathrm{Cyl}(\mathrm{Ger}^\vee)$ to the 2-colored endomorphism operad $\mathrm{End}_{V_A, C^\bullet(A)}$ of the pair $(V_A, C^\bullet(A))$.

Precomposing U_{Cyl} with the endomorphism

$$\exp(D) : \mathrm{Cyl}(\mathrm{Ger}^\vee) \to \mathrm{Cyl}(\mathrm{Ger}^\vee)$$

we get another operad map

$$U_{\mathrm{Cyl}} \circ \exp(D) : \mathrm{Cyl}(\mathrm{Ger}^\vee) \to \mathrm{End}_{V_A, C^\bullet(A)} \tag{7.21}$$

which corresponds to the triple consisting of

- the new Ger_∞-structure $Q^{\exp(\mathrm{res}_\alpha(D))}$ on V_A,
- the Ger_∞-structure on $C^\bullet(A)$ corresponding to $\Psi \circ \exp(\mathrm{res}_\beta(D))$, and
- a Ger_∞-quasi-isomorphism

$$\widetilde{U}_{\mathrm{Ger}} : V_A^{Q^{\exp(\mathrm{res}_\alpha(D))}} \rightsquigarrow C^\bullet(A)^{\Psi \circ \exp(\mathrm{res}_\beta(D))} \tag{7.22}$$

Due to technical Proposition E.1 proved in Appendix E below, the restriction of the Ger_∞-quasi-isomorphism $\widetilde{U}_{\mathrm{Ger}}$ (7.22) to $\Lambda^2 \operatorname{coCom}(V_A)$ gives us the same $\Lambda \operatorname{Lie}_\infty$-morphism (7.19).

On the other hand, by Corollary D.1 from Appendix D.1, there exists a Ger_∞-quasi-isomorphism

$$U_{\mathrm{corr}} : V_A \rightsquigarrow V_A^{Q^{\exp(\mathrm{res}_\alpha(D))}} \tag{7.23}$$

from V_A with the standard Gerstenhaber structure to V_A with the new Ger_∞-structure $Q^{\exp(\mathrm{res}_\alpha(D))}$.

Thus, composing U_{corr} with $\widetilde{U}_{\mathrm{Ger}}$ (7.22), we get a Ger_∞-quasi-isomorphism

$$U_{\mathrm{Ger}}^{\exp(D)} : V_A \rightsquigarrow C^\bullet(A)^{\Psi \circ \exp(\mathrm{res}_\beta(D))} \tag{7.24}$$

from V_A with the standard Gerstenhaber structure to $C^\bullet(A)$ with the Ger_∞-structure coming from $\Psi \circ \exp(\mathrm{res}_\beta(D))$.

The restriction of this Ger_∞-morphism $U_{\mathrm{Ger}}^{\exp(D)}$ to $\Lambda^2 \operatorname{coCom}(V_A)$ gives us the $\Lambda \operatorname{Lie}_\infty$-morphism

$$U_{\mathrm{Lie}} \circ U^{\mathrm{res}_\alpha(D)} \tag{7.25}$$

where $U^{\mathrm{res}_\alpha(D)}$ is the $\Lambda \operatorname{Lie}_\infty$-automorphism of V_A obtained by restricting (7.23) to $\Lambda^2 \operatorname{coCom}(V_A)$.

Since both cocycles $\mathrm{res}_\alpha(D)$ and $\mathrm{res}_\beta(D)$ of $\mathrm{Der}'(\mathrm{Ger}_\infty)$ represent the same cohomology class $v \in \mathfrak{g}$, Theorem 7.1 follows. □

7.4 The proof of Proposition 7.1

Let D and $\widetilde{D}$ be two cohomologous cocycles in $\mathrm{Der}'(\mathrm{Ger}_\infty)$ and let $Q^{\exp(D)}$, $Q^{\exp(\widetilde{D})}$ be Ger_∞-structures on V_A corresponding to the operad maps

$$\mathrm{Act}_{stan} \circ \exp(D) : \mathrm{Ger}_\infty \to \mathrm{End}_{V_A} \,, \tag{7.26}$$

$$\mathrm{Act}_{stan} \circ \exp(\widetilde{D}) : \mathrm{Ger}_\infty \to \mathrm{End}_{V_A} \,, \tag{7.27}$$

respectively. Here Act_{stan} is the map $\mathrm{Ger}_\infty \to \mathrm{End}_{V_A}$ corresponding to the standard Gerstenhaber structure on V_A.

Due to Proposition B.2, operad maps (7.26) and (7.27) are homotopy equivalent. Hence there exists a Ger_∞-structure Q_t on $V_A \otimes \Omega^\bullet(\Bbbk)$ such that the evaluation maps

$$\begin{array}{ll} p_0 : V_A \otimes \Omega^\bullet(\Bbbk) \to V_A^{Q^{\exp(D)}} \,, & p_0(v) := v\big|_{dt=0,\; t=0} \,, \\ p_1 : V_A \otimes \Omega^\bullet(\Bbbk) \to V_A^{Q^{\exp(\widetilde{D})}} \,, & p_1(v) := v\big|_{dt=0,\; t=1} \,. \end{array} \tag{7.28}$$

are strict quasi-isomorphisms of the corresponding Ger_∞-algebras.

Furthermore, observation (7.3) implies that the restriction of a homotopy connecting the automorphisms $\exp(D)$ and $\exp(\widetilde{D})$ of Ger_∞ to the suboperad $\Lambda\,\mathrm{Lie}_\infty$ coincides with the identity map on $\Lambda\,\mathrm{Lie}_\infty$ for every t. Therefore, the $\Lambda\,\mathrm{Lie}_\infty$-part of the Ger_∞-structure Q_t on $V_A \otimes \Omega^\bullet(\Bbbk)$ coincides with the standard $\Lambda\,\mathrm{Lie}$-structure given by the Schouten bracket.

Since tensoring with $\Omega^\bullet(\Bbbk)$ does not change cohomology, Corollary D.2 from Appendix D.2 implies that the canonical embedding $V_A \hookrightarrow V_A \otimes \Omega^\bullet(\Bbbk)$ can be extended to a Ger_∞ quasi-isomorphism

$$U^{\mathfrak{H}}_{\mathrm{corr}} : V_A \rightsquigarrow V_A \otimes \Omega^\bullet(\Bbbk) \tag{7.29}$$

from V_A with the standard Gerstenhaber structure to $V_A \otimes \Omega^\bullet(\Bbbk)$ with the Ger_∞-structure Q_t.

Since the $\Lambda\,\mathrm{Lie}_\infty$-part of the Ger_∞-structure Q_t on $V_A \otimes \Omega^\bullet(\Bbbk)$ coincides with the standard $\Lambda\,\mathrm{Lie}$-structure given by the Schouten bracket, the restriction of $U^{\mathfrak{H}}_{\mathrm{corr}}$ onto $\Lambda^2\,\mathrm{coCom}(V_A)$ gives us a homotopy connecting the $\Lambda\,\mathrm{Lie}_\infty$-automorphisms

$$p_0 \circ U^{\mathfrak{H}}_{\mathrm{corr}}\Big|_{\Lambda^2\,\mathrm{coCom}(V_A)} : V_A \rightsquigarrow V_A \tag{7.30}$$

and

$$p_1 \circ U^{\mathfrak{H}}_{\rm corr} \Big|_{\Lambda^2 \,{\rm coCom}(V_A)} : V_A \rightsquigarrow V_A\,. \tag{7.31}$$

Due to the second part of Corollary D.2, $\Lambda\,{\rm Lie}_\infty$-automorphism (7.30) is homotopy equivalent to U^D and $\Lambda\,{\rm Lie}_\infty$-automorphism (7.31) is homotopy equivalent to $U^{\widetilde{D}}$.

Thus the homotopy type of U^D is indeed independent of the representative D of the cohomology class.

To prove the second claim of Proposition 7.1, we will need to use the 2-colored dg operad ${\rm Cyl}({\rm Ger}^\vee)$ recalled in the proof of Theorem 7.1 above. Moreover, we need [33, Theorem 4.3] which implies that restrictions (7.16) are homotopic quasi-isomorphisms of cochain complexes.

Let D_1 and D_2 be degree zero cocycles in ${\rm Der}'({\rm Ger}_\infty)$ and let $Q^{\exp(D_1)}$ be the ${\rm Ger}_\infty$-structure on V_A which comes from the composition

$$ {\rm Act}_{stan} \circ \exp(D_1) : {\rm Ger}_\infty \to {\rm End}_{V_A}\,, \tag{7.32}$$

where ${\rm Act}_{stan}$ denotes the map ${\rm Ger}_\infty \to {\rm End}_{V_A}$ corresponding to the standard Gerstenhaber structure on V_A.

Let $U_{{\rm Ger},1}$ be a ${\rm Ger}_\infty$-quasi-isomorphism

$$U_{{\rm Ger},1} : V_A \rightsquigarrow V_A^{Q^{\exp(D_1)}}\,, \tag{7.33}$$

where the source is considered with the standard Gerstenhaber structure.

By construction, the $\Lambda\,{\rm Lie}_\infty$-automorphism

$$U^{D_1} : V_A \rightsquigarrow V_A$$

is the restriction of $U_{{\rm Ger},1}$ onto $\Lambda^2\,{\rm coCom}(V_A)$.

Let us denote by $U^{V_A}_{\rm Cyl}$ the operad map

$$U^{V_A}_{\rm Cyl} : {\rm Cyl}({\rm Ger}^\vee) \to {\rm End}_{V_A, V_A}$$

which corresponds to the triple:

- the standard Gerstenhaber structure on the first copy of V_A,

- the Ger_∞-structure $Q^{\exp(D_1)}$ on the second copy of V_A, and
- the chosen Ger_∞ quasi-isomorphism in (7.33).

Due to [33, Theorem 4.3], there exists a degree zero cocycle D_{Cyl} in $\mathrm{Der}'\big(\mathrm{Cyl}(\mathrm{Ger}^\vee)\big)$ for which the cocycles

$$D := \mathrm{res}_\alpha(D_{\mathrm{Cyl}}) \,, \qquad D' := \mathrm{res}_\beta(D_{\mathrm{Cyl}}) \tag{7.34}$$

are both cohomologous to the given cocycle D_2.

Precomposing the map $U^{V_A}_{\mathrm{Cyl}}$ with the automorphism $\exp(D_{\mathrm{Cyl}})$ we get a new $\mathrm{Cyl}(\mathrm{Ger}^\vee)$-algebra structure on the pair (V_A, V_A) which corresponds to the triple

- the Ger_∞-structure $Q^{\exp(D)}$ on the first copy of V_A,
- the Ger_∞-structure $Q^{\exp(\mathrm{CH}(D_1,D'))}$ on the second copy of V_A, and
- a Ger_∞ quasi-isomorphism

$$\widetilde{U}_{\mathrm{Ger}} : V_A^{Q^{\exp(D)}} \rightsquigarrow V_A^{Q^{\exp(\mathrm{CH}(D_1,D'))}} \,. \tag{7.35}$$

Let us observe that, due to Proposition E.1 from Appendix E, the restriction of $\widetilde{U}_{\mathrm{Ger}}$ onto $\Lambda^2\,\mathrm{coCom}(V_A)$ coincides with the restriction of (7.33) onto $\Lambda^2\,\mathrm{coCom}(V_A)$. Hence,

$$\widetilde{U}_{\mathrm{Ger}}\Big|_{\Lambda^2\,\mathrm{coCom}(V_A)} = U^{D_1} \,, \tag{7.36}$$

where U^{D_1} is a $\Lambda\,\mathrm{Lie}_\infty$-automorphism of V_A corresponding[3] to D_1.

Recall that there exists a Ger_∞-quasi-isomorphism

$$U_{\mathrm{Ger}} : V_A \rightsquigarrow V_A^{Q^{\exp(D)}} \,. \tag{7.37}$$

where the source is considered with the standard Gerstenhaber structure. Furthermore, since D is cohomologous to D_2, the first claim of Proposition 7.1 implies that the restriction of U_{Ger} onto $\Lambda^2\,\mathrm{coCom}(V_A)$ gives us a $\Lambda\,\mathrm{Lie}_\infty$-automorphism U^D of V_A which is homotopy equivalent to U^{D_2}.

[3] Strictly speaking, only the homotopy class of the $\Lambda\,\mathrm{Lie}_\infty$-automorphism U^{D_1} is uniquely determined by D_1.

Let us also observe that the composition $\widetilde{U}_{\mathrm{Ger}} \circ U_{\mathrm{Ger}}$ gives us a Ger_∞-quasi-isomorphism

$$\widetilde{U}_{\mathrm{Ger}} \circ U_{\mathrm{Ger}} : V_A \rightsquigarrow V_A^{Q^{\exp(\mathrm{CH}(D_1, D'))}} \tag{7.38}$$

Hence, the restriction of $\widetilde{U}_{\mathrm{Ger}} \circ U_{\mathrm{Ger}}$ gives us a $\Lambda\,\mathrm{Lie}_\infty$-automorphism of V_A corresponding to $\mathrm{CH}(D_1, D')$. Due to (7.36), this $\Lambda\,\mathrm{Lie}_\infty$-automorphism coincides with

$$U^{D_1} \circ U^{D}\,.$$

Since D and D' are both cohomologous to D_2, the second claim of Proposition 7.1 follows. $\square$

Remark 7.1. *The second claim of Proposition 7.1 can probably be deduced from [38, Proposition 5.4] and some other statements in [38]. However, this would require a digression to "stable setting" which we avoid in this dissertation. For this reason, we decided to present a complete proof of Proposition 7.1 which is independent of any intermediate steps in [38].*

CHAPTER 8

CONNECTING DRINFELD ASSOCIATORS TO FORMALITY MORPHISMS

In this chapter we recall how to construct a GRT_1-equivariant map $\mathfrak{B}$ from the set $\mathrm{DrAssoc}_1$ of Drinfeld associators to the set

$$\pi_0\big(\,\mathrm{Ger}_\infty \to \mathrm{Braces}\,\big)$$

of homotopy classes of operad morphisms (6.1) satisfying conditions (6.2).

Composing $\mathfrak{B}$ with the map $\mathfrak{T}$ (6.21), we get the desired map

$$\mathfrak{T} \circ \mathfrak{B} : \mathrm{DrAssoc}_1 \to \pi_0\big(V_A \rightsquigarrow C^\bullet(A)\big) \tag{8.1}$$

from the set $\mathrm{DrAssoc}_1$ to the set of homotopy classes of $\Lambda\,\mathrm{Lie}_\infty$-morphisms from V_A to $C^\bullet(A)$ whose linear term is the Hochschild-Kostant-Rosenberg embedding.

Theorem 7.1 will then imply that map (8.1) is GRT_1-equivariant.

8.1 The sets $\mathrm{DrAssoc}_\kappa$ of Drinfeld associators

In this short section, we briefly recall Drinfeld's associators and the Grothendieck-Teichmueller group GRT_1 . For more details we refer the reader to [1], [2], or [17].

Let m be an integer ≥ 2. We denote by $\mathfrak{t}_m$ the Lie algebra generated by symbols $\{t^{ij} = t^{ji}\}_{1\leq i\neq j\leq m}$ subject to the following relations:

$$[t^{ij}, t^{ik} + t^{jk}] = 0 \qquad \text{for any triple of distinct indices } i, j, k\,,$$
$$[t^{ij}, t^{kl}] = 0 \qquad \text{for any quadruple of distinct indices } i, j, k, l\,. \tag{8.2}$$

The notation $\mathcal{A}^{\mathrm{pb}}_m$ is reserved for the associative algebra (over $\mathbb{k}$) of formal power series in noncommutative symbols $\{t^{ij} = t^{ji}\}_{1\leq i\neq j\leq m}$ subject to the same relations (8.2). Let us recall [34, Section 4] that the collection $\mathcal{A}^{\mathrm{pb}} := \{\mathcal{A}^{\mathrm{pb}}_m\}_{m\geq 1}$ with $\mathcal{A}^{\mathrm{pb}}_1 := \mathbb{k}$ forms an operad in the category of associative $\mathbb{k}$-algebras.

Let $\mathfrak{lie}(x, y)$ be the degree completion of the free Lie algebra in two symbols x and y and let κ be any element of $\mathbb{k}$.

The set $\mathrm{DrAssoc}_\kappa$ consists of elements $\Phi \in \exp\big(\mathfrak{lie}(x, y)\big)$ which satisfy the equations

$$\Phi(y, x)\Phi(x, y) = 1\,, \tag{8.3}$$

$$\Phi(t^{12}, t^{23} + t^{24})\, \Phi(t^{13} + t^{23}, t^{34}) = \Phi(t^{23}, t^{34})\, \Phi(t^{12} + t^{13}, t^{24} + t^{34})\, \Phi(t^{12}, t^{23})\,, \tag{8.4}$$

$$e^{\kappa(t^{13}+t^{23})/2} = \Phi(t^{13}, t^{12}) e^{\kappa t^{13}/2} \Phi(t^{13}, t^{23})^{-1} e^{\kappa t^{23}/2} \Phi(t^{12}, t^{23})\,, \tag{8.5}$$

and

$$e^{\kappa(t^{12}+t^{13})/2} = \Phi(t^{23}, t^{13})^{-1} e^{\kappa t^{13}/2} \Phi(t^{12}, t^{13}) e^{\kappa t^{12}/2} \Phi(t^{12}, t^{23})^{-1}\,. \tag{8.6}$$

For $\kappa \neq 0$, elements Φ of $\mathrm{DrAssoc}_\kappa$ are called Drinfeld associators. However, for our purposes, we only need the set $\mathrm{DrAssoc}_1$ and the set $\mathrm{DrAssoc}_0$.

According to [17, Section 5], the set

$$\mathrm{DrAssoc}_0 \tag{8.7}$$

forms a prounipotent group and, by [17, Proposition 5.5], this group acts simply transitively on the set of associators in $\mathrm{DrAssoc}_1$. Following [17], we denote the group $\mathrm{DrAssoc}_0$ by GRT_1.

8.2 A map $\mathfrak{B}$ from $\mathrm{DrAssoc}_1$ to $\pi_0\big(\mathrm{Ger}_\infty \to \mathrm{Braces}\big)$

Let us recall [2], [34] that collections of all braid groups can be assembled into the operad PaB in the category of $\Bbbk$-linear categories. Similarly, the collection of algebras $\{\mathcal{A}^{\mathrm{pb}}_m\}_{m\ge 1}$ can be "upgraded" to the operad PaCD also in the category of $\Bbbk$-linear categories. Every associator $\Phi \in \mathrm{DrAssoc}_1$ gives us an isomorphism of these operads

$$I_\Phi : \mathrm{PaB} \stackrel{\cong}{\longrightarrow} \mathrm{PaCD}\,. \tag{8.8}$$

The group GRT_1 acts on the operad PaCD in such a way that, for every pair $g \in \mathrm{GRT}_1,\ \ \Phi \in \mathrm{DrAssoc}_1$, the diagram

$$\begin{array}{ccc} \mathrm{PaB} & \stackrel{I_\Phi}{\longrightarrow} & \mathrm{PaCD} \\ \Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow{\scriptstyle g} \\ \mathrm{PaB} & \stackrel{I_{g(\Phi)}}{\longrightarrow} & \mathrm{PaCD} \end{array} \tag{8.9}$$

commutes.

Applying to PaB and PaCD the functor $C_{-\bullet}(-,\Bbbk)$, where $C_\bullet(-,\Bbbk)$ denotes the Hochschild chain complex with coefficients in $\Bbbk$, we get dg operads

$$C_{-\bullet}(\mathrm{PaB},\Bbbk) \tag{8.10}$$

and

$$C_{-\bullet}(\mathrm{PaCD},\Bbbk)\,. \tag{8.11}$$

By naturality of $C_{-\bullet}(\ ,\Bbbk)$, diagram (8.9) gives us the commutative diagram

$$\begin{array}{ccc} C_{-\bullet}(\mathrm{PaB},\Bbbk) & \stackrel{I_\Phi}{\longrightarrow} & C_{-\bullet}(\mathrm{PaCD},\Bbbk) \\ \Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow{\scriptstyle g} \\ C_{-\bullet}(\mathrm{PaB},\Bbbk) & \stackrel{I_{g(\Phi)}}{\longrightarrow} & C_{-\bullet}(\mathrm{PaCD},\Bbbk), \end{array} \tag{8.12}$$

where, for simplicity, the maps corresponding to I_Φ, $I_{g(\Phi)}$ and g are denoted by the same letters, respectively.

Recall that Eq. (5) from [34] gives us the canonical quasi-isomorphism from the operad Ger to $C_{-\bullet}(\mathcal{A}^{\mathrm{pb}}, \Bbbk)$. The latter operad, in turn, receives the natural map

$$C_{-\bullet}(\mathrm{PaCD}, \Bbbk) \to C_{-\bullet}(\mathcal{A}^{\mathrm{pb}}, \Bbbk)$$

from $C_{-\bullet}(\mathrm{PaCD}, \Bbbk)$ which is also known to be a quasi-isomorphism.

Thus, using the lifting property (see [11, Corollary 5.8]) for maps from the operad $\mathrm{Ger}_\infty = \mathrm{Cobar}(\mathrm{Ger}^\vee)$, we get the quasi-isomorphism[1]

$$\mathrm{Ger}_\infty \ \stackrel{\sim}{\longrightarrow}\ C_{-\bullet}(\mathrm{PaCD}, \Bbbk)\,. \tag{8.13}$$

Using this quasi-isomorphism and [11, Corollary 5.8], one can construct (see [38, Section 6.3.1]) a group homomorphism

$$\mathrm{GRT}_1 \to \exp(\mathfrak{g})\,, \tag{8.14}$$

where the Lie algebra $\mathfrak{g}$ is defined in (7.4). By [38, Theorem 1.2], homomorphism (8.14) is an isomorphism.

Any specific solution of Deligne's conjecture on the Hochschild complex (see, for example, [4], [13], or [30]) combined with Fiedorowicz's recognition principle [18] provides us with a sequence of quasi-isomorphisms

$$\mathrm{Braces} \stackrel{\sim}{\leftarrow} \bullet \stackrel{\sim}{\rightarrow} \bullet \stackrel{\sim}{\leftarrow} \bullet \ \dots\ \bullet \stackrel{\sim}{\rightarrow} C_{-\bullet}(\mathrm{PaB}, \Bbbk) \tag{8.15}$$

which connects the dg operad Braces to $C_{-\bullet}(\mathrm{PaB}, \Bbbk)$.

Hence, every associator $\Phi \in \mathrm{DrAssoc}_1$ gives us a sequence of quasi-isomorphisms

$$\mathrm{Braces} \stackrel{\sim}{\leftarrow} \bullet \stackrel{\sim}{\rightarrow} \bullet \stackrel{\sim}{\leftarrow} \bullet \ \dots\ \bullet \stackrel{\sim}{\rightarrow} C_{-\bullet}(\mathrm{PaB}, \Bbbk) \xrightarrow{I_\Phi} C_{-\bullet}(\mathrm{PaCD}, \Bbbk) \ \stackrel{\sim}{\longleftarrow}\ \mathrm{Ger}_\infty \tag{8.16}$$

connecting the dg operads Braces to Ger_∞.

Using [11, Corollary 5.8] once again, we conclude that the sequence of quasi-isomorphisms (8.16) determines a unique homotopy class of quasi-isomorphisms (of dg operads)

$$\Psi : \mathrm{Ger}_\infty \to \mathrm{Braces}\ . \tag{8.17}$$

[1] By the same lifting property (see [11, Corollary 5.8]), we know that the homotopy type of the quasi-isomorphism (8.13) is uniquely determined by the operad map $\mathrm{Ger} \to C_{-\bullet}(\mathcal{A}^{\mathrm{pb}}, \Bbbk)$ from [34, Eq. (5)].

Thus we get a well defined map

$$\mathfrak{B} : \mathrm{DrAssoc}_1 \to \pi_0\big(\,\mathrm{Ger}_\infty \to \mathrm{Braces}\,\big)\,. \tag{8.18}$$

In view of isomorphism (8.14), the set of homotopy classes $\pi_0\big(\,\mathrm{Ger}_\infty \to \mathrm{Braces}\,\big)$ is equipped with a natural action of GRT_1. Moreover, the commutativity of diagram (8.12) implies that the map $\mathfrak{B}$ is GRT_1-equivariant.

Thus, combining this observation with Theorem 7.1 we deduce the following corollary:

Corollary 8.1. *Let $\pi_0\big(V_A \rightsquigarrow C^\bullet(A)\big)$ be the set of homotopy classes of $\Lambda\,\mathrm{Lie}_\infty$ quasi-isomorphisms which extend the Hochschild-Kostant-Rosenberg embedding of polyvector fields into Hochschild cochains. If we consider $\pi_0\big(V_A \rightsquigarrow C^\bullet(A)\big)$ as a set with the GRT_1-action induced by isomorphism* (8.14) *then the composition*

$$\mathfrak{T} \circ \mathfrak{B} : \mathrm{DrAssoc}_1 \to \pi_0\big(V_A \rightsquigarrow C^\bullet(A)\big) \tag{8.19}$$

is GRT_1-equivariant. □

Remark 8.1. *Any sequence of quasi-isomorphisms of dg operads* (8.15) *gives us an isomorphism between the objects corresponding to $C_{-\bullet}(\mathrm{PaB}, \Bbbk)$ and* Braces *in the homotopy category of dg operads. However, there is no reason to expect that different solutions of the Deligne conjecture give the same isomorphisms from $C_{-\bullet}(\mathrm{PaB}, \Bbbk)$ to* Braces *in the homotopy category. Hence the resulting composition in* (8.19) *may depend on the choice of a specific solution of Deligne's conjecture on the Hochschild complex.*

REFERENCES

[1] A. Alekseev and C. Torossian. The Kashiwara-Vergne conjecture and Drinfeld's associators. *Ann. of Math.*, 175(2):415–463, 2012. Also available as arXiv:0802.4300 .

[2] D. Bar-Natan. On associators and the Grothendieck-Teichmueller group i. *Selecta Math. (N.S.)*, 4:183–212, 1998. Also available as arXiv:q-alg/9606021.

[3] M. Batanin and M. Markl. Crossed interval groups and operations on the Hochschild cohomology. *J. Noncommut. Geom.*, 8:655–693, 2014. Also available as arXiv:0803.2249.

[4] C. Berger and B. Fresse. Combinatorial operad actions on cochains. *Math. Proc. Cambridge Philos. Soc.*, 137:135–174, 2004. Also available as arXiv:math/0109158.

[5] D.V. Borisov. Formal deformations of morphisms of associative algebras. *Int. Math. Res. Notices*, 41:2499–2523, 2005. Also available as arXiv:math/0506486.

[6] V. Dolgushev and T. Willwacher. Operadic twisting – with an application to Deligne's conjecture. *J. of Pure and Applied Algebra*, 219(5):1349–1428, 2015. Also available as arXiv:1207.2180.

[7] V.A. Dolgushev. Stable formality quasi-isomorphisms for Hochschild cochains II. In preparation.

[8] V.A. Dolgushev. Stable formality quasi-isomorphisms for Hochschild cochains I. Available as arXiv:1109.6031, 2011.

[9] V.A. Dolgushev, A.E. Hoffnung, and C.L. Rogers. What do homotopy algebras form? Available as arXiv:1406.1751, 2014.

[10] V.A. Dolgushev and B. Paljug. Tamarkin's construction is equivariant with respect to the action of the Grothendieck-Teichmueller group. Submitted to Journal of Homotopy and Related Structures. Available as arXiv:1305.4699, 2014.

[11] V.A. Dolgushev and C.L. Rogers. Notes on algebraic operads, graph complexes, and Willwacher's construction. In *Mathematical aspects of quantization*, volume 583 of *Contemp. Math.*, pages 25–145. 2012. Also available as arXiv:1202.2937.

[12] V.A. Dolgushev and C.L. Rogers. On an enhancement of the category of shifted L_∞ algebras. Available as arXiv:1406.1744, 2014.

[13] V.A. Dolgushev, D.E. Tamarkin, and B.L. Tsygan. Proof of swiss cheese version of Deligne's conjecture. *Int. Math. Res. Notices*, pages 4666–4746, 2011. Also available as arXiv:0904.2753.

[14] V.A. Dolgushev and T. Willwacher. The deformation complex is a homotopy invariant of a homotopy algebra. In *Developments and Retrospectives in Lie Theory*, volume 38 of *Developments in Mathematics*, pages 137–158. Springer International Publishing, 2014. Also available as arXiv:1305.4165.

[15] V. Dotsenko and N. Poncin. A tale of three homotopies. Available as arXiv:1208.4695, 2012.

[16] M. Doubek. On resolutions of diagrams of algebras. Available as arXiv:1107.1408, 2011.

[17] V.G. Drinfeld. On quasitriangular quasi-Hopf algebras and on a group that is closely connected with $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. *Algebra i Analiz*, 2:149–181, 1990. Russian. *Translation in Leningrad Math. J.*, 2:829–860, 1991.

[18] Z. Fiedorowicz. The symmetric bar construction. Available at https://people.math.osu.edu/fiedorowicz.1.

[19] B. Fresse. Operadic cobar constructions, cylinder objects, and homotopy morphisms of algebras over operads. In *Alpine Perspectives on Algebraic Topology*, Contemp. Math., pages 125–188. 2009. Also available as arXiv:0902.0177.

[20] E. Getzler. Lie theory for nilpotent L_∞-algebras. *Ann. of Math.*, 170:271–301, 2009. Also available as arXiv:math/0404003.

[21] V. Hinich. Tamarkin's proof of Kontsevich formality theorem. *Forum Math.*, 15:591–614, 2003. Also available as arXiv:math/0003052.

[22] G. Hochschild, B. Kostant, and A. Rosenberg. Differential forms on regular affine algebras. *Trans. Amer. Math. Soc.*, 102:383–408, 1962.

[23] M. Kontsevich. Operads and motives in deformation quantization. *Lett. Math. Phys.*, 48:35–72, 1999. Also available as arXiv:math/9904055.

[24] M. Kontsevich and Y. Soibelman. Deformations of algebras over operads and the Deligne conjecture. In *Proceedings of the Moshé Flato Conference*, volume 21 of *Math. Phys. Stud.*, pages 255–307. Kluwer Acad. Publ., Dordrecht, 2000. Also available as arXiv:math/0001151.

[25] T. Leinster. Homotopy algebras for operads. Available as arXiv:math/0002180, 2000.

[26] J.-L. Loday. *Cyclic Homology*, volume 301 of *Grundlehren Math. Wiss.* Springer-Verlag, 1992. Appendix E by Maria O. Ronco.

[27] J.-L. Loday and B. Vallette. *Algebraic Operads*, volume 346 of *Grundlehren Math. Wiss.* Springer-Verlag, 2012.

[28] M. Markl. Homotopy diagrams of algebras. *Rend. Circ. Mat. Palermo*, 69(2 supp.):161–180, 2002. Also available as arXiv:math/0103052.

[29] M. Markl. Operads and PROPs. *Handbook of Algebra*, 5:87–140, 2008. Also available as arXiv:math/0601129.

[30] J.E. McClure and J.H. Smith. A solution of Deligne's conjecture. *Contemp. Math.*, 293:153–193, 2002. Also available as arXiv:math/9910126.

[31] S. Merkulov and title= B. Vallette.

[32] S. Merkulov and title= B. Vallette.

[33] B. Paljug. Action of derived automorphisms on infinity morphisms. Submitted to Journal of Homotopy and Related Structures. Available as arXiv:1305.4699, 2013.

[34] D. Tamarkin. Formality of chain operad of little discs. *Lett. Math. Phys.*, 66:65–72, 2003.

[35] D. Tamarkin. What do DG categories form? *Compos. Math.*, 143:1335–1358, 2007. Also available as arXiv:math/0606553.

[36] D.E. Tamarkin. Another proof of M. Kontsevich formality theorem. Available as arXiv:math/9803025, 1998.

[37] A.A. Voronov. Homotopy Gerstenhaber algebras. In *Proceedings of the Moshé Flato Conference*, volume 22 of *Math. Phys. Stud.*, pages 307–331. Kluwer Acad. Publ., Dordrecht, 2000. Also available as arXiv:math/9908040.

[38] T. Willwacher. M. Kontsevich's graph complex and the Grothendieck-Teichmüller Lie algebra. *Inventiones mathematicae*, pages 1–90, 2014. Also available as arXiv:1009.1654.

[39] T. Willwacher. Stable cohomology of polyvector fields. Available as arXiv:1110.3762, 2014.

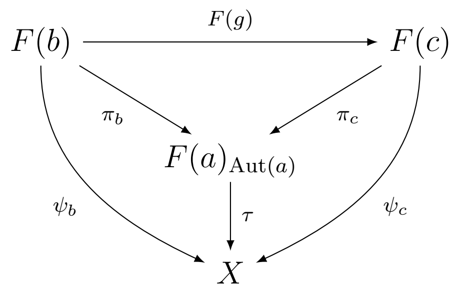
APPENDIX A

A LEMMA ON COLIMITS FROM CONNECTED GROUPOIDS

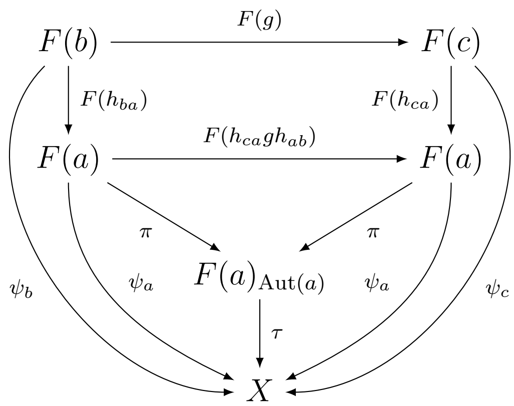
The following result seems to be well-known, but we take take the opportunity to record its statement and proof for use in this dissertation, and for future reference.

Lemma A.1. *Let* $F : \mathfrak{g} \to \mathsf{Ch}_{\Bbbk}$ *be a functor from a connected groupoid* $\mathfrak{g}$*. Then* $\operatorname{colim} F = F(a)_{\mathrm{Aut}(a)}$*, for any object* $a \in \mathfrak{g}$*.*

Proof. Choose $a \in \mathfrak{g}$; we need to show that $F(a)_{\mathrm{Aut}(a)}$ is a co-cone for $F : \mathfrak{g} \to \mathsf{Ch}_{\Bbbk}$, and that it is universal. That is, for any other co-cone X for F, there is a unique map $\tau : F(a)_{\mathrm{Aut}(a)} \to X$, and for any $a, b \in \mathfrak{g}$ there are maps $\pi_b : F(b) \to F(a)_{\mathrm{Aut}(a)}$ and $\pi_c : F(c) \to F(a)_{\mathrm{Aut}(a)}$, such that the following diagram commutes:



Note that we trivially have this for $b = c = a$, where g is any automorphism of a. Then $\pi_b = \pi_c = \pi$, the canonical projection $F(a) \twoheadrightarrow F(a)_{\mathrm{Aut}(a)}$, and τ exists and is unique because of the universal property of quotients. Since $\mathfrak{g}$ is a connected groupoid we have maps $h_{ba} : b \to a$ and $h_{ca} : c \to a$ (for simplicity, let the inverses of these maps be denoted h_{ab} and h_{ac}, respectively). Then we have the commuting diagram



where the left and right triangles commute because X is a co-cone for F, and the top rectangle commutes by construction. This then gives us the first diagram with $\pi_b = \pi \circ F(h_{ba})$ and $\pi_c = \pi \circ F(h_{ca})$, and therefore $\operatorname{colim} F = F(a)_{\mathrm{Aut}(a)}$. $\square$

APPENDIX B

ON COHOMOLOGOUS DERIVATIONS AND HOMOTOPIC AUTOMORPHISMS

In this appendix we investigate the relationship between cohomologous derivations and homotopic automorphisms of operads. The techniques and results of this appendix were inspired by Appendix A of [39], where a similar situation was considered in the specific setting of L_∞-algebras.

All operads will be (possibly colored) quasi-free dg operads $\mathcal{O} = \mathbb{OP}(\mathcal{M})$ ($\mathcal{M}$ a collection), equipped with the weight filtration as in Section 3 (our motivating exampes are $\mathrm{Cobar}(\mathcal{C})$ and $\mathrm{Cyl}(\mathcal{C})$ for a reduced cooperad $\mathcal{C}$). Let π be the projection $\mathcal{O} \to \mathcal{M}$. We then have the filtered dg Lie algebra $\mathrm{Der}(\mathcal{O})$ of operad derivations of $\mathcal{O}$, which contains the subalgebra

$$\mathrm{Der}'(\mathcal{O}) = \{D \in \mathrm{Der}(\mathcal{O}) \mid \pi \circ D|_{\mathcal{M}} = 0\}. \tag{B.1}$$

We also have the group $\mathrm{Aut}(\mathcal{O})$ of operad automorphisms of $\mathcal{O}$, with subgroup

$$\mathrm{Aut}'(\mathcal{O}) = \{\varphi \in \mathrm{Aut}(\mathcal{O}) \mid \pi \circ \varphi|_{\mathcal{M}} = \mathrm{id}_{\mathcal{M}}\}. \tag{B.2}$$

With these conditions, we have well-defined maps

$$D \mapsto \exp(D) = \sum_{k \geq 0} \frac{D^n}{n!} : Z^0(\mathrm{Der}'(\mathcal{O})) \to \mathrm{Aut}'(\mathcal{O}) \tag{B.3}$$

and

$$\varphi \mapsto \log(\varphi) = \sum_{k\geq 1}(-1)^{n-1}\frac{(\varphi - \mathrm{id})^n}{n} : \mathrm{Aut}'(\mathcal{O}) \to Z^0(\mathrm{Der}'(\mathcal{O})) \tag{B.4}$$

that are inverse to each other (it is straightforward to show that degree 0 closed derivations exponentiate to operad automorphisms, and vice versa).

We can make $Z^0(\mathrm{Der}'(\mathcal{O}))$ into a group with composition given by the Campbell-Hausdorff formula

$$\mathrm{CH}(X,Y) = \log(\exp(X)\exp(Y)) \tag{B.5}$$

and identity 0 (it is an easy exercise that if X, Y are degree 0 and closed, so too is $\mathrm{CH}(X,Y)$). It will be clear from context whether we consider $Z^0(\mathrm{Der}'(\mathcal{O}))$ as a dg Lie algebra, or as a group. The following proposition is well known:

Proposition B.1. *The maps*

$$\exp : Z^0(\mathrm{Der}'(\mathcal{O})) \to \mathrm{Aut}'(\mathcal{O})$$

$$\log : \mathrm{Aut}'(\mathcal{O}) \to Z^0(\mathrm{Der}'(\mathcal{O}))$$

are inverse group isomorphisms.

All of the above constructions and results are preserved when considering cohomologous derivations and homotopic automorphisms. It is straightforward to see that the group structure on $Z^0\,\mathrm{Der}'(\mathcal{O})$ induces the group structure on $H^0(\mathrm{Der}'(\mathcal{O}))$. Recall [11, Section 5.1] that two automorphisms $\varphi_1, \varphi_2 \in \mathrm{Aut}'(\mathcal{O})$ are homotopic if there exists an operad map

$$\mathcal{H} : \mathcal{O} \to \mathcal{O} \otimes \Omega^\bullet(\Bbbk) \tag{B.6}$$

such that

$$\mathcal{H}|_{t=0,dt=0} = \varphi_1, \quad \mathcal{H}|_{t=1,dt=0} = \varphi_2 \tag{B.7}$$

where $\Omega^\bullet(\Bbbk)$ denotes the algebra of polynomial differential forms on $\Bbbk$. We will also insist that such homotopies occur in $\mathrm{Aut}'(\mathcal{O})$, in the following sense. It is easy to see that for any specific choice of t, $\mathcal{H}|_{dt=0}$ is an operad endomorphism of $\mathcal{O}$ (see proof below); we additionally require that it be in $\mathrm{Aut}'(\mathcal{O})$. Let $\mathrm{hAut}'(\mathcal{O})$ denote the group of homotopy classes of automorphisms in $\mathrm{Aut}'(\mathcal{O})$. Then we have the following version of Proposition B.1:

Proposition B.2. *The induced maps*

$$\exp : H^0(\mathrm{Der}'(\mathcal{O})) \to \mathrm{hAut}'(\mathcal{O})$$

$$\log : \mathrm{hAut}'(\mathcal{O}) \to H^0(\mathrm{Der}'(\mathcal{O}))$$

are inverse group isomorphisms.

Proof. This proof is essentially borrowed from Appendix A of [39]. It must be shown that the above maps are well-defined; that they are inverse group isomorphisms is then essentially obvious.

To show that $\exp$ is well-defined, we will show that $\exp(\partial(P))$ is homotopic to the identity in $\mathrm{Aut}'(\mathcal{O})$ for every degree -1 derivation $P \in \mathrm{Der}'(\mathcal{O})$. Let us denote by t an auxiliary variable and consider the following map of dg operads:

$$\exp(t\partial(P)) : \mathcal{O} \to \mathcal{O}[t] \tag{B.8}$$

(this map lands in $\mathcal{O}[t]$ for the same reasons that $\exp$ is well-defined in this situation). We have

$$\frac{d}{dt}\exp(t\partial(P)) = \partial(P) \circ \exp(t\partial(P)) \tag{B.9}$$

and hence the sum

$$\mathcal{H}_P = \exp(t\partial(P)) + dt\, P \circ \exp(t\partial(P)) \tag{B.10}$$

is a map of dg operads

$$\mathcal{H}_P : \mathcal{O} \to \mathcal{O} \otimes \Omega^\bullet(\Bbbk). \tag{B.11}$$

It is clear that

$$\mathcal{H}_P|_{t=0,dt=0} = \mathrm{id}_{\mathcal{O}}, \quad \mathcal{H}_P|_{t=1,dt=0} = \exp(\partial(P)) \tag{B.12}$$

and therefore $\mathcal{H}_P$ is the desired homotopy connecting $\mathrm{id}_{\mathcal{O}}$ to the automorphism $\exp(\partial(P))$.

To show that $\log$ is well-defined, we will show that if $\varphi \in \mathrm{Aut}'(\mathcal{O})$ is homotopic to $\mathrm{id}_{\mathcal{O}}$ in $\mathrm{Aut}'(\mathcal{O})$, then $\log(\varphi)$ is exact. So assume that there is a homotopy

$$\mathcal{H} : \mathcal{O} \to \mathcal{O} \otimes \Omega^{\bullet}(\Bbbk) \tag{B.13}$$

such that

$$\mathcal{H}|_{t=0,dt=0} = \mathrm{id}_{\mathcal{O}}, \quad \mathcal{H}|_{t=1,dt=0} = \varphi. \tag{B.14}$$

$\mathcal{H}$ necessarily has the form

$$\mathcal{H} = \mathcal{H}_0 + dt\, \mathcal{H}_1, \ \text{ where } \ \mathcal{H}_0, \mathcal{H}_1 : \mathcal{O} \to \mathcal{O}[t]. \tag{B.15}$$

The compatibility of $\mathcal{H}$ with the operadic multiplications is equivalent to the equations

$$\mathcal{H}_0(x \circ_i y) = \mathcal{H}_0(x) \circ_i \mathcal{H}_0(y) \tag{B.16}$$

$$\mathcal{H}_1(x \circ_i y) = \mathcal{H}_1(x) \circ_i \mathcal{H}_0(y) + (-1)^{|x|}\mathcal{H}_0(x) \circ_i \mathcal{H}_1(y) \tag{B.17}$$

and compatibility with the differentials is equivalent to the equations

$$\partial \circ \mathcal{H}_0 = \mathcal{H}_0 \circ \partial \tag{B.18}$$

$$\frac{d}{dt}\mathcal{H}_0 = \partial \circ \mathcal{H}_1 + \mathcal{H}_1 \circ \partial. \tag{B.19}$$

Extending $\mathcal{H}_0, \mathcal{H}_1$ linearly across $\Bbbk[t]$ to get maps $\mathcal{H}_0, \mathcal{H}_1 : \mathcal{O}[t] \to \mathcal{O}[t]$, we see that the previous equation implies that

$$\frac{d}{dt}\log(\mathcal{H}_0) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sum_{m=0}^{n-1} (\mathcal{H}_0 - \mathrm{id})^m \circ [\partial, \mathcal{H}_1] \circ (\mathcal{H}_0 - \mathrm{id})^{n-m-1} \tag{B.20}$$

which can be rewritten as

$$\frac{d}{dt}\log(\mathcal{H}_0) = [\partial, Z] \tag{B.21}$$

where

$$Z = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sum_{m=0}^{n-1} (\mathcal{H}_0 - \mathrm{id})^m \circ \mathcal{H}_1 \circ (\mathcal{H}_0 - \mathrm{id})^{n-m-1}. \tag{B.22}$$

By integrating, this implies that $\log(\varphi) = \log(\mathcal{H}_0)|_{t=1}$ is exact, provided that Z is a derivation of $\mathcal{O}[t]$. To show this, recall that for fixed t, $\mathcal{H}_0$ is in $\mathrm{Aut}'(\mathcal{O})$. Consequently we can construct $\mathcal{H}_0^{-1}$. Since $\mathcal{H}_1$ is a derivation relative to $\mathcal{H}_0$, $\mathcal{H}_0^{-1}\mathcal{H}_1$ is a derivation of $\mathcal{O}[t]$. Now consider

$$\Psi(u) = \log(\mathcal{H}_0 \exp(u\mathcal{H}_0^{-1}\mathcal{H}_1)) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(\mathcal{H}_0 \exp(u\mathcal{H}_0^{-1}\mathcal{H}_1) - \mathrm{id})^n, \tag{B.23}$$

which is an element of $\mathrm{Der}'(\mathcal{O}[t])\widehat{\otimes}\Bbbk[u]$. Therefore $\frac{d}{dt}\Psi(u)|_{u=0}$ is a derivation of $\mathcal{O}[t]$, and it is straightforward to check that

$$\frac{d}{dt}\Psi(u)|_{u=0} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sum_{m=0}^{n-1} (\mathcal{H}_0 - \mathrm{id})^m \circ \mathcal{H}_1 \circ (\mathcal{H}_0 - \mathrm{id})^{n-m-1} = Z. \tag{B.24}$$

So Z is indeed a derivation.

□

APPENDIX C

FILTERED HOMOTOPY LIE ALGEBRAS

Let L be a cochain complex with the differential ∂. Recall that a $\Lambda^{-1}\mathrm{Lie}_\infty$-structure on L is a sequence of degree 1 multi-brackets

$$\{\ ,\ ,\dots,\ \}_m : S^m(L) \to L\,, \qquad m \ge 2 \tag{C.1}$$

satisfying the relations

$$\partial\{v_1, v_2, \dots, v_m\} + \sum_{i=1}^{m} (-1)^{|v_1|+\dots+|v_{i-1}|}\{v_1, \dots, v_{i-1}, \partial v_i, v_{i+1}, \dots, v_m\}$$
$$+ \sum_{k=2}^{m-1} \sum_{\sigma \in \mathrm{Sh}_{k,m-k}} (-1)^{\varepsilon(\sigma; v_1,\dots,v_m)} \big\{\{v_{\sigma(1)}, \dots, v_{\sigma(k)}\}, v_{\sigma(k+1)}, \dots, v_{\sigma(m)}\big\} = 0\,, \tag{C.2}$$

where $(-1)^{\varepsilon(\sigma; v_1,\dots,v_m)}$ is the Koszul sign factor (see eq. (2.1)).

We say that a $\Lambda^{-1}\mathrm{Lie}_\infty$-algebra L is *filtered* if it is equipped with a complete descending filtration

$$L = \mathcal{F}_1 L \supset \mathcal{F}_2 L \supset \mathcal{F}_3 L \supset \dots\,. \tag{C.3}$$

For such filtered $\Lambda^{-1}\mathrm{Lie}_\infty$-algebras we may define a Maurer-Cartan element as a degree zero element α satisfying the equation

$$\partial\alpha + \sum_{m\geq 2} \frac{1}{m!}\{\alpha, \alpha, \dots, \alpha\}_m = 0\,. \tag{C.4}$$

Note that this equation makes sense for any degree 0 element α because $L = \mathcal{F}_1 L$ and L is complete with respect to filtration (C.3). Let us denote by $\mathrm{MC}(L)$ the set of Maurer-Cartan elements of a filtered $\Lambda^{-1}\mathrm{Lie}_\infty$-algebra L.

According to[1] [20], the set $\mathrm{MC}(L)$ can be upgraded to an ∞-groupoid $\mathfrak{MC}(L)$ (i.e. a simplicial set satisfying the Kan condition). To introduce the ∞-groupoid $\mathfrak{MC}(L)$, we denote by $\Omega^\bullet(\Delta_n)$ the dg commutative $\Bbbk$-algebra of polynomial forms [20, Section 3] on the n-th geometric simplex Δ_n. Next, we declare that set of n-simplices of $\mathfrak{MC}(L)$ is

$$\mathrm{MC}\left(L \,\hat{\otimes}\, \Omega^\bullet(\Delta_n)\right), \tag{C.5}$$

where L is considered with the topology coming from filtration (C.3) and $\Omega^\bullet(\Delta_n)$ is considered with the discrete topology. The structure of the simplicial set is induced from the structure of a simplicial set on the sequence $\{\Omega^\bullet(\Delta_n)\}_{n\geq 0}$.

For example, 0-cells of $\mathfrak{MC}(L)$ are precisely Maurer-Cartan elements of L and 1-cells are sums

$$\alpha' + dt\,\alpha''\,, \qquad \alpha' \in L^0 \,\hat{\otimes}\, \Bbbk[t]\,, \qquad \alpha'' \in L^{-1} \,\hat{\otimes}\, \Bbbk[t] \tag{C.6}$$

satisfying the pair of equations

$$\partial\alpha' + \sum_{m\geq 2} \frac{1}{m!}\{\alpha', \alpha', \dots, \alpha'\}_m = 0\,, \tag{C.7}$$

$$\frac{d}{dt}\alpha' = \partial\alpha'' + \sum_{m\geq 1} \frac{1}{m!}\{\alpha', \alpha', \dots, \alpha', \alpha''\}_{m+1}\,. \tag{C.8}$$

Thus, two 0-cells α_0, α_1 of $\mathfrak{MC}(L)$ (i.e. Maurer-Cartan elements of L) are isomorphic if there exists an element (C.6) satisfying (C.7) and (C.8) and such that

$$\alpha_0 = \alpha'\Big|_{t=0} \qquad \text{and} \qquad \alpha_1 = \alpha'\Big|_{t=1}\,. \tag{C.9}$$

We say that a 1-cell (C.6) connects α_0 and α_1.

[1]A version of the Deligne-Getzler-Hinich ∞-groupoid for pro-nilpotent $\Lambda^{-1}\mathrm{Lie}_\infty$-algebras is introduced in [12, Section 4].

C.1 A lemma on adjusting Maurer-Cartan elements

Let α be a Maurer-Cartan element of a filtered $\Lambda^{-1}\mathrm{Lie}_\infty$-algebra and ξ be a degree -1 element in $\mathcal{F}_n L$ for some integer $n \ge 1$.

Let us consider the following sequence $\{\alpha'_k\}_{k\ge 0}$ of degree zero elements in $L \,\hat{\otimes}\, \Bbbk[t]$

$$\alpha'_0 := \alpha\,, \qquad \alpha'_{k+1}(t) := \alpha + \int_0^t dt_1 \Big(\partial \xi + \sum_{m \ge 1} \frac{1}{m!} \{\alpha'_k(t_1), \dots, \alpha'_k(t_1), \xi\}_{m+1} \Big)\,. \tag{C.10}$$

Since L is complete with respect to filtration (C.3), the sequence $\{\alpha'_k\}_{k\ge 0}$ convergences to a (degree 0) element $\alpha' \in L \,\hat{\otimes}\, \Bbbk[t]$ which satisfies the integral equation

$$\alpha'(t) = \alpha + \int_0^t dt_1 \Big(\partial \xi + \sum_{m \ge 1} \frac{1}{m!} \{\alpha'(t_1), \dots, \alpha'(t_1), \xi\}_{m+1} \Big)\,. \tag{C.11}$$

We claim that

Lemma C.1. *If, as above, ξ is a degree -1 element in $\mathcal{F}_n L$ and α' is an element of $L \,\hat{\otimes}\, \Bbbk[t]$ obtained by recursive procedure* (C.10) *then the sum*

$$\alpha' + dt\, \xi \tag{C.12}$$

is a 1-cell of $\mathfrak{MC}(L)$ which connects α to another Maurer-Cartan element $\widetilde{\alpha}$ of L such that

$$\alpha' - \alpha \in \mathcal{F}_n L \,\hat{\otimes}\, \Bbbk[t]\,, \tag{C.13}$$

and

$$\widetilde{\alpha} - \alpha - \partial \xi \in \mathcal{F}_{n+1} L\,. \tag{C.14}$$

If the element ξ satisfies the additional condition

$$\partial \xi \in \mathcal{F}_{n+1} L \tag{C.15}$$

then

$$\alpha' - \alpha \in \mathcal{F}_{n+1} L \,\hat{\otimes}\, \Bbbk[t]\,, \tag{C.16}$$

and

$$\widetilde{\alpha} - \alpha - \partial \xi - \{\alpha, \xi\} \in \mathcal{F}_{n+2} L\,. \tag{C.17}$$

Proof. Equation (C.11) implies that α' satisfies the differential equation

$$\frac{d}{dt}\alpha' = \partial\xi + \sum_{m\geq 1}\frac{1}{m!}\{\alpha', \dots, \alpha', \xi\}_{m+1} \tag{C.18}$$

with the initial condition

$$\alpha'\Big|_{t=0} = \alpha\,. \tag{C.19}$$

Let us denote by Ξ the following degree 1 element of $L \,\hat{\otimes}\, \Bbbk[t]$

$$\Xi := \partial\alpha' + \sum_{m\geq 2}\frac{1}{m!}\{\alpha', \alpha', \dots, \alpha'\}_m\,. \tag{C.20}$$

A direct computation shows that Ξ satisfies the following differential equation

$$\frac{d}{dt}\Xi = -\sum_{m\geq 0}\frac{1}{m!}\{\alpha', \dots, \alpha', \Xi, \xi\}_{m+2}\,. \tag{C.21}$$

Furthermore, since α is a Maurer-Cartan element of L, the element Ξ satisfies the condition

$$\Xi\Big|_{t=0} = 0$$

and hence Ξ satisfies the integral equation

$$\Xi(t) = -\int_0^t dt_1\Big(\sum_{m\geq 0}\frac{1}{m!}\{\alpha'(t_1), \dots, \alpha'(t_1), \Xi(t_1), \xi\}_{m+2}\Big)\,. \tag{C.22}$$

Equation (C.22) implies that

$$\Xi \in \bigcap_{n\geq 1}\mathcal{F}_n L \,\hat{\otimes}\, \Bbbk[t]\,.$$

Therefore $\Xi = 0$ and hence the limiting element α' of sequence (C.10) is a Maurer-Cartan element of $L \,\hat{\otimes}\, \Bbbk[t]$.

Combining this observation with differential equation (C.18), we conclude that the element $\alpha' + dt\,\xi \in L \,\hat{\otimes}\, \Omega^\bullet(\Delta_1)$ is indeed a 1-cell in $\mathfrak{MC}(L)$ which connects the Maurer-Cartan element α to the Maurer-Cartan element

$$\widetilde{\alpha} := \alpha + \int_0^1 dt\Big(\partial\xi + \sum_{m\geq 1}\frac{1}{m!}\{\alpha'(t), \dots, \alpha'(t), \xi\}_{m+1}\Big)\,. \tag{C.23}$$

Since $\xi \in \mathcal{F}_n L$ and $L = \mathcal{F}_1 L$, equation (C.11) implies that

$$\alpha' - \alpha \in \mathcal{F}_n L \,\hat{\otimes}\, \Bbbk[t]$$

and equation (C.23) implies that

$$\widetilde{\alpha} - \alpha - \partial \xi \in \mathcal{F}_{n+1} L\,.$$

Thus, the first part of Lemma C.1 is proved.

If $\xi \in \mathcal{F}_n L$ and $\partial \xi \in \mathcal{F}_{n+1} L$ then, again, it is clear from (C.11) that inclusion (C.16) holds.

Finally, using inclusion (C.16) and equation (C.23), it is easy to see that

$$\widetilde{\alpha} - \alpha - \partial \xi - \{\alpha, \xi\} \in \mathcal{F}_{n+2} L\,.$$

Lemma C.1 is proved. □

C.2 Convolution $\Lambda^{-1}\mathrm{Lie}_\infty$-algebra, ∞-morphisms and their homotopies

As usual, let $\mathcal{C}$ be a coaugmented, reduced cooperad, and let V be a cochain complex. (In our applications, $\mathcal{C}$ is usually the cooperad $\mathrm{Ger}^\vee$.)

Following Chapter 2, we say that V is a homotopy algebra of type $\mathcal{C}$ if V carries $\mathrm{Cobar}(\mathcal{C})$-algebra structure, or equivalently the $\mathcal{C}$-coalgebra $\mathcal{C}(V)$ has a degree 1 coderivation Q satisfying $Q\Big|_V = 0$ and the Maurer-Cartan equation

$$\partial(Q) + \frac{1}{2}[Q, Q] = 0 \tag{C.24}$$

where

$$\partial(Q) = [d_{\mathcal{C}(V)}, Q]. \tag{C.25}$$

For two homotopy algebras (V, Q_V) and (W, Q_W) of type $\mathcal{C}$, we consider the graded vector space

$$\mathrm{Hom}(\mathcal{C}(V), W) \tag{C.26}$$

with the differential ∂

$$\partial(f) := d_W \circ f - (-1)^{|f|} f \circ (d_{\mathcal{C}(V)} + Q_V) \tag{C.27}$$

and the multi-brackets (of degree 1)

$$\{\ ,\ ,\dots,\ \}_m : S^m\big(\mathrm{Hom}(\mathcal{C}(V), W)\big) \to \mathrm{Hom}(\mathcal{C}(V), W)\,, \qquad m \ge 2$$

$$\{f_1, \dots, f_m\}(X) = p_W \circ Q_W\big(1 \otimes f_1 \otimes \cdots \otimes f_m(\Delta_m(X))\big)\,, \tag{C.28}$$

where Δ_m is the m-th component of the comultiplication

$$\Delta_m : \mathcal{C}(V) \to \Big(\mathcal{C}(m) \otimes \mathcal{C}(V)^{\otimes\, m}\Big)^{S_m}$$

and p_W is the canonical projection

$$p_W : \mathcal{C}(W) \to W\,.$$

According to [9] or [15, Section 1.3], equation (C.28) define a $\Lambda^{-1}\,\mathrm{Lie}_\infty$-structure on the cochain complex $\mathrm{Hom}(\mathcal{C}(V), W)$ with the differential ∂ (C.27). The $\Lambda^{-1}\mathrm{Lie}_\infty$-algebra

$$\mathrm{Hom}(\mathcal{C}(V), W) \tag{C.29}$$

is called the *convolution* $\Lambda^{-1}\mathrm{Lie}_\infty$*-algebra* of the pair V, W.

The convolution $\Lambda^{-1}\mathrm{Lie}_\infty$-algebra $\mathrm{Hom}(\mathcal{C}(V), W)$ carries the obvious descending filtration "by arity"

$$\mathcal{F}_n\,\mathrm{Hom}(\mathcal{C}(V), W) \;=\; \{f \in \mathrm{Hom}(\mathcal{C}(V), W) \mid f\big|_{\mathcal{C}(m)\otimes_{S_m} V^{\otimes\, m}} = 0\ \ \forall\, m < n\}. \tag{C.30}$$

$\mathrm{Hom}(\mathcal{C}(V), W)$ is obviously complete with respect to this filtration and

$$\mathrm{Hom}(\mathcal{C}(V), W) = \mathcal{F}_1\,\mathrm{Hom}(\mathcal{C}(V), W). \tag{C.31}$$

In other words, under our assumption on the cooperad $\mathcal{C}$, the convolution $\Lambda^{-1}\mathrm{Lie}_\infty$-algebra $\mathrm{Hom}(\mathcal{C}(V), W)$ is pronilpotent.

According to [15, Proposition 3], ∞-morphisms from V to W are in bijection with Maurer-Cartan elements of $\mathrm{Hom}(\mathcal{C}(V), W)$ i.e. 0-cells of the Deligne-Getzler-Hinich ∞-groupoid corresponding to $\mathrm{Hom}(\mathcal{C}(V), W)$. Furthermore, due to [15, Corollary 2], two ∞-morphisms from V to W are homotopic if and only if the corresponding Maurer-Cartan elements are isomorphic 0-cells in the Deligne-Getzler-Hinich ∞-groupoid of $\mathrm{Hom}(\mathcal{C}(V), W)$.

APPENDIX D

TAMARKIN'S RIGIDITY

Let V_A denote the Gerstenhaber algebra of polyvector fields on the graded affine space corresponding to $A = \mathbb{k}[x^1, x^2, \dots, x^d]$ with

$$|x^i| = t_i \,.$$

As a graded commutative algebra over $\mathbb{k}$, V_A is freely generated by variables

$$x^1, x^2, \dots, x^d, \theta_1, \theta_2, \dots, \theta_d,$$

where θ_i carries degree $1 - t_i$.

$$V_A = \mathbb{k}[x^1, x^2, \dots, x^d, \theta_1, \theta_2, \dots, \theta_d] \,. \tag{D.1}$$

Let us denote by $\mu_\wedge$ and $\mu_{\{\ ,\ \}}$ the vectors in $\mathrm{End}_{V_A}(2)$ corresponding to the multiplication and the Schouten bracket $\{\ ,\ \}$ on V_A, respectively.

The composition of the canonical quasi-isomorphism

$$\mathrm{Cobar}(\mathrm{Ger}^\vee) \to \mathrm{Ger} \tag{D.2}$$

and the map $\mathrm{Ger} \to \mathrm{End}_{V_A}$ corresponds to the following Maurer-Cartan element

$$\alpha := \mu_\wedge \otimes \{b_1, b_2\} + \mu_{\{\ ,\ \}} \otimes b_1 b_2 \tag{D.3}$$

in the graded Lie algebra

$$\mathrm{Conv}^\oplus(\mathrm{Ger}^\vee, \mathrm{End}_{V_A}) := \bigoplus_{n \ge 1} \mathrm{Hom}_{S_n}\left(\mathrm{Ger}^\vee(n), \mathrm{End}_{V_A}(n)\right) \tag{D.4}$$

for which we frequently use the obvious identification[1]

$$\mathrm{Conv}^{\oplus}(\mathrm{Ger}^{\vee}, \mathrm{End}_{V_A}) \cong \bigoplus_{n \ge 1} \big(\mathrm{End}_{V_A}(n) \otimes \Lambda^{-2}\,\mathrm{Ger}(n)\big)^{S_n} . \tag{D.5}$$

In this appendix, we consider $\mathrm{Conv}^{\oplus}(\mathrm{Ger}^{\vee}, \mathrm{End}_{V_A})$ as the cochain complex with the following differential

$$\partial := [\alpha, \ ]\,. \tag{D.6}$$

We observe that $\mathrm{Conv}^{\oplus}(\mathrm{Ger}^{\vee}, \mathrm{End}_{V_A})$ carries the natural descending filtration "by arity":

$$\mathrm{Conv}^{\oplus}(\mathrm{Ger}^{\vee}, \mathrm{End}_{V_A}) = \mathcal{F}_0\,\mathrm{Conv}^{\oplus}(\mathrm{Ger}^{\vee}, \mathrm{End}_{V_A}) \supset \mathcal{F}_1\,\mathrm{Conv}^{\oplus}(\mathrm{Ger}^{\vee}, \mathrm{End}_{V_A}) \supset \dots$$

$$\mathcal{F}_m\,\mathrm{Conv}^{\oplus}(\mathrm{Ger}^{\vee}, \mathrm{End}_{V_A}) := \bigoplus_{n \ge m+1} \big(\mathrm{End}_{V_A}(n) \otimes \Lambda^{-2}\,\mathrm{Ger}(n)\big)^{S_n} . \tag{D.7}$$

More precisely,

$$\partial \big(\mathrm{End}_{V_A}(n) \otimes \Lambda^{-2}\,\mathrm{Ger}(n)\big)^{S_n} \ \subset \ \big(\mathrm{End}_{V_A}(n+1) \otimes \Lambda^{-2}\,\mathrm{Ger}(n+1)\big)^{S_{n+1}} . \tag{D.8}$$

In particular, every cocycle $X \in \mathrm{Conv}^{\oplus}(\mathrm{Ger}^{\vee}, \mathrm{End}_{V_A})$ is a finite sum

$$X = \sum_{n \ge 1} X_n\,, \qquad X_n \in \big(\mathrm{End}_{V_A}(n) \otimes \Lambda^{-2}\,\mathrm{Ger}(n)\big)^{S_n} \tag{D.9}$$

where each individual term X_n is a cocycle.

In this dissertation, we need the following version of Tamarkin's rigidity

Theorem D.1. *If n is an integer ≥ 2 then for every cocycle*

$$X \ \in \ \big(\mathrm{End}_{V_A}(n) \otimes \Lambda^{-2}\,\mathrm{Ger}(n)\big)^{S_n} \ \subset \ \mathrm{Conv}^{\oplus}(\mathrm{Ger}^{\vee}, \mathrm{End}_{V_A})$$

there exists a cochain $Y \in (\mathrm{End}_{V_A}(n-1) \otimes \Lambda^{-2}\,\mathrm{Ger}(n-1))^{S_{n-1}}$ such that

$$X = \partial Y\,.$$

[1] Recall from Chapter 2 that the cooperad $\mathrm{Ger}^{\vee}$ is the linear dual of the operad $\Lambda^{-2}\,\mathrm{Ger}$.

Remark D.1. *Note that the above statement is different from Tamarkin's rigidity in the "stable setting" [11, Section 12]. According to [11, Corollary 12.2], one may think that the vector*

$$\mu_{\{\ ,\ \}} \otimes b_1 b_2$$

is a non-trivial cocycle in (D.4). *In fact,*

$$\mu_{\{\ ,\ \}} \otimes b_1 b_2 = [\alpha, P \otimes b_1]\,,$$

where P is the following version of the "Euler derivation" of V_A:

$$P(v) := \sum_{i=1}^{d} \theta_i \frac{\partial}{\partial \theta_i}\,.$$

Proof of Theorem D.1. Theorem D.1 is only a slight generalization of the statement proved in Section 5.4 of [21] and, in the proof given here, we pretty much follow the same line of arguments as in [21, Section 5.4].

First, we introduce an additional set of auxiliary variables

$$\check{x}_1, \check{x}_2, \dots, \check{x}_d,\ \check{\theta}^1, \check{\theta}^2, \dots, \check{\theta}^d \tag{D.10}$$

of degrees

$$|\check{x}_i| = 2 - t_i\,, \qquad |\check{\theta}^i| = t_i + 1\,.$$

Second, we consider the de Rham complex of V_A:

$$\Omega^{\bullet}_{\Bbbk} V_A := V_A[\check{x}_1, \check{x}_2, \dots, \check{x}_d, \check{\theta}_1, \check{\theta}_2, \dots, \check{\theta}_d] \tag{D.11}$$

with the differential

$$D = \sum_{i=1}^{d} \check{x}_i \frac{\partial}{\partial \theta_i} + \sum_{i=1}^{d} \check{\theta}^i \frac{\partial}{\partial x^i} \tag{D.12}$$

and equip it with the following descending filtration:

$$\mathcal{F}_m \Omega^{\bullet}_{\Bbbk} V_A := \Big\{ P \in V_A[\check{x}_1, \check{x}_2, \dots, \check{x}_d, \check{\theta}_1, \check{\theta}_2, \dots, \check{\theta}_d] \;\Big|\; \text{the total degree of } P \text{ in } \check{x}_1, \dots, \check{x}_d, \check{\theta}_1, \dots, \check{\theta}_d \text{ is } \geq m+1 \Big\}\,. \tag{D.13}$$

Next, we observe that every homogeneous vector[2]

$$P = P^{i_1 i_2 \dots i_k}_{j_1 j_2 \dots j_q} \, \check{x}_{i_1} \dots \check{x}_{i_k} \check{\theta}^{j_1} \dots \check{\theta}^{j_q} \in V_A[\check{x}_1, \check{x}_2, \dots, \check{x}_d, \check{\theta}_1, \check{\theta}_2, \dots, \check{\theta}_d]$$

defines an element $P^{\mathrm{End}} \in \mathrm{End}_{V_A}(k+q)$:

$$P^{\mathrm{End}}(v_1, v_2, \dots, v_{k+q}) := \sum_{\sigma \in S_{k+q}} \pm P^{i_1 i_2 \dots i_k}_{j_1 j_2 \dots j_q} \partial_{x^{i_1}} v_{\sigma(1)} \, \partial_{x^{i_2}} v_{\sigma(2)} \, \dots \partial_{x^{i_k}} v_{\sigma(k)} \, \partial_{\theta_{j_1}} v_{\sigma(k+1)} \, \partial_{\theta_{j_2}} v_{\sigma(k+2)} \, \dots \partial_{\theta_{j_q}} v_{\sigma(k+q)} \,, \quad \text{(D.14)}$$

where the sign factors $\pm$ are determined by the usual Koszul rule.

Finally, we claim that the formula

$$\mathrm{VH}(P) := P^{\mathrm{End}} \otimes b_1 b_2 \dots b_{k+q} \quad \text{(D.15)}$$

defines a degree zero injective map

$$\mathrm{VH} : \mathbf{s}^{-2} \, \mathcal{F}_0 \Omega^{\bullet}_{\Bbbk} V_A \;\to\; \mathrm{Conv}^{\oplus}(\mathrm{Ger}^{\vee}, \mathrm{End}_{V_A}) \quad \text{(D.16)}$$

which is compatible with filtrations (D.7) and (D.13). A direct computation shows that VH intertwines differentials (D.6) and (D.12).

Let m be an integer and

$$\mathcal{G}^m \, \mathrm{Conv}^{\oplus}(\mathrm{Ger}^{\vee}, \mathrm{End}_{V_A}) \quad \text{(D.17)}$$

be the subspace of $\mathrm{Conv}^{\oplus}(\mathrm{Ger}^{\vee}, \mathrm{End}_{V_A})$ of sums

$$\sum_i M_i \otimes q_i \;\in\; \bigoplus_{n \ge 1} \big(\mathrm{End}_{V_A}(n) \otimes \Lambda^{-2} \, \mathrm{Ger}(n) \big)^{S_n} \quad \text{(D.18)}$$

satisfying the condition

$$\text{the number of Lie brackets in } q_i \;-\; |\, M_i \otimes q_i \,| \;\le\; m \,. \quad \text{(D.19)}$$

It is easy to see that the sequence of subspaces (D.17)

$$\dots \subset \mathcal{G}^{-1} \, \mathrm{Conv}^{\oplus}(\mathrm{Ger}^{\vee}, \mathrm{End}_{V_A}) \subset \mathcal{G}^{0} \, \mathrm{Conv}^{\oplus}(\mathrm{Ger}^{\vee}, \mathrm{End}_{V_A}) \subset \mathcal{G}^{1} \, \mathrm{Conv}^{\oplus}(\mathrm{Ger}^{\vee}, \mathrm{End}_{V_A}) \subset \dots$$

[2] Summation over repeated indices is assumed.

form an ascending filtration on the cochain complex $\mathrm{Conv}^{\oplus}(\mathrm{Ger}^{\vee}, \mathrm{End}_{V_A})$ and the associated graded cochain complex

$$\mathrm{Gr}_{\mathcal{G}}\,\mathrm{Conv}^{\oplus}(\mathrm{Ger}^{\vee}, \mathrm{End}_{V_A}) \tag{D.20}$$

is isomorphic to

$$\bigoplus_{n\ge 1} \big(\,\mathrm{End}_{V_A}(n) \otimes \Lambda^{-2}\,\mathrm{Ger}(n)\big)^{S_n}$$

with the differential

$$\partial^{\mathrm{Gr}} = [\,\mu_{\wedge} \otimes \{b_1, b_2\},\ \ ]\,, \tag{D.21}$$

where $\mu_{\wedge}$ is the vector in $\mathrm{End}_{V_A}(2)$ which corresponds to the multiplication on V_A.

Let us observe that (D.20) is naturally a V_A-module (where V_A is viewed as a graded commutative algebra), differential (D.21) is V_A-linear, and since

$$\mathrm{Ger}^{\vee}(V_A) = \Lambda^2\,\mathrm{coCom}(\Lambda\,\mathrm{coLie}(V_A))\,,$$

cochain complex (D.20) is isomorphic to

$$\mathrm{Hom}_{V_A}\,\big(\,\mathbf{s}^2\,\underline{S}_{V_A}(\mathbf{s}^{-1}\,V_A \otimes_{\Bbbk} \mathrm{coLie}(\mathbf{s}^{-1}\,V_A)), V_A\big) \tag{D.22}$$

with the differential coming from the one on the Harrison homological[3] complex [26, Section 4.2.10]

$$V_A \otimes_{\Bbbk} \mathrm{coLie}(\mathbf{s}^{-1}\,V_A) \tag{D.23}$$

of the graded commutative algebra V_A with coefficients in V_A.

Since V_A is freely generated by elements $x^1, \dots, x^d, \theta_1, \dots, \theta_d$, Theorem 3.5.6 and Proposition 4.2.11 from [26] imply that the embedding

$$I_{\mathrm{Harr}}\ :\ \bigoplus_{i=1}^{d} V_A e^i\ \oplus\ \bigoplus_{i=1}^{d} V_A f_i\ \to\ V_A \otimes \mathrm{coLie}(\mathbf{s}^{-1}\,V_A) \tag{D.24}$$

$$I_{\mathrm{Harr}}(e^i) := 1 \otimes \mathbf{s}^{-1}\,x^i\,, \qquad I_{\mathrm{Harr}}(f_i) := 1 \otimes \mathbf{s}^{-1}\,\theta_i$$

from the free V_A-module

$$\bigoplus_{i=1}^{d} V_A e^i\ \oplus\ \bigoplus_{i=1}^{d} V_A f_i\,, \qquad |e^i| := t_i - 1, \qquad |f_i| := -t_i \tag{D.25}$$

[3]The cochain complex in (D.23) is obtained from the conventional Harrison homological complex from [26, Section 4.2.10] by reversing the grading.

is a quasi-isomorphism of cochain complexes of V_A-modules from (D.25) with the zero differential to (D.23) with the Harrison differential.

Since (D.24) is a quasi-isomorphism of cochain complexes of free V_A-modules, it induces a quasi-isomorphism of cochain complexes of (free) V_A-modules:

$$\mathbf{s}^2\, V_A[\mathbf{s}^{-1}\, e^1, \dots, \mathbf{s}^{-1}\, e^d, \mathbf{s}^{-1}\, f_1, \dots, \mathbf{s}^{-1}\, f_d] \to \mathbf{s}^2\, S_{V_A}(\mathbf{s}^{-1}\, V_A \otimes_{\Bbbk} \mathrm{coLie}(\mathbf{s}^{-1}\, V_A))\,, \tag{D.26}$$

where the source carries the zero differential.

Therefore, map (D.16) induces a quasi-isomorphism of cochain complexes

$$\mathbf{s}^{-2}\, \mathcal{F}_0\Omega^{\bullet}_{\Bbbk} V_A \;\to\; \mathrm{Gr}_{\mathcal{G}}\, \mathrm{Conv}^{\oplus}(\mathrm{Ger}^{\vee}, \mathrm{End}_{V_A})\,,$$

where the source is considered with the zero differential. Thus, by Lemma A.3 from [11], map (D.16) is a quasi-isomorphism of cochain complexes.

Let $n \ge 2$ and

$$X \;\in\; \big(\mathrm{End}_{V_A}(n) \otimes \Lambda^{-2}\, \mathrm{Ger}(n)\big)^{S_n} \;\subset\; \mathrm{Conv}^{\oplus}(\mathrm{Ger}^{\vee}, \mathrm{End}_{V_A}) \tag{D.27}$$

be a cocycle. Since (D.16) is a quasi-isomorphism of cochain complexes, there exists a cocycle

$$\widetilde{X} \in \mathbf{s}^{-2}\, \mathcal{F}_0\Omega^{\bullet}_{\Bbbk} V_A \tag{D.28}$$

such that X is cohomologous to $\mathrm{VH}(\widetilde{X})$.

Let us observe that de Rham differential D (D.12) satisfies the property

$$D\big(\mathcal{F}_0\Omega^{\bullet}_{\Bbbk} V_A\big) \subset \mathcal{F}_1\Omega^{\bullet}_{\Bbbk} V_A\,.$$

Hence, since VH is injective, we conclude that

$$\widetilde{X} \in \mathbf{s}^{-2}\, \mathcal{F}_1\Omega^{\bullet}_{\Bbbk} V_A\,. \tag{D.29}$$

It is obvious that every cocycle in $\mathcal{F}_1\Omega^{\bullet}_{\Bbbk} V_A$ is exact in $\mathcal{F}_0\Omega^{\bullet}_{\Bbbk} V_A$. Therefore $\widetilde{X}$ is exact and so is cocycle (D.27).

Combining this statement with property (D.8) we easily deduce Theorem D.1.

□

D.1 The standard Gerstenhaber structure on V_A is "rigid"

The first consequence of Theorem D.1 is the following corollary:

Corollary D.1. *Let V_A be, as above, the algebra of polyvector fields on a graded affine space and Q be a Ger_∞-structure on V_A whose binary operations are the Schouten bracket and the usual multiplication. Then the identity map* $\mathrm{id} : V_A \to V_A$ *can be extended to a* Ger_∞ *morphism*

$$U_{\mathrm{corr}} : V_A \rightsquigarrow V_A^Q \tag{D.30}$$

from V_A with the standard Gerstenhaber structure to V_A with the Ger_∞*-structure* Q.

Proof. To prove this statement, we consider the graded space

$$\mathrm{Hom}(\mathrm{Ger}^\vee(V_A), V_A) \tag{D.31}$$

with two different algebraic structures. First, (D.31) is identified with the convolution Lie algebra[4]

$$\mathrm{Conv}(\mathrm{Ger}^\vee, \mathrm{End}_{V_A}) \tag{D.32}$$

with the Lie bracket $[\ ,\ ]$ defined in terms of the binary (degree zero) operation $\bullet$ from [11, Section 4, Eq. (4.2)].

To introduce the second algebraic structure on (D.31), we recall that a Ger_∞-structure on V_A is precisely a degree 1 element

$$Q = Q_2 + \sum_{n \ge 3} Q_n \qquad Q_n \in \mathrm{Hom}_{S_n}(\mathrm{Ger}^\vee(n) \otimes V_A^{\otimes n}, V_A) \tag{D.33}$$

in (D.32) satisfying the Maurer-Cartan equation

$$[Q, Q] = 0 \tag{D.34}$$

[4] In our case, Lie algebra (D.32) carries the zero differential.

and the above condition on the binary operations is equivalent to the requirement

$$Q_2 = \alpha\,, \tag{D.35}$$

where α is Maurer-Cartan element (D.3) of (D.32).

Given such a Ger_∞-structure Q on V_A, we get the convolution $\Lambda^{-1}\,\mathrm{Lie}_\infty$-algebra

$$\mathrm{Hom}(\mathrm{Ger}^\vee(V_A), V_A^Q) \tag{D.36}$$

corresponding to the pair (V_A, V_A^Q), where the first entry V_A is considered with the standard Gerstenhaber structure and the second entry is considered with the above Ger_∞-structure Q.

As a graded vector space, $\Lambda^{-1}\,\mathrm{Lie}_\infty$-algebra (D.36) coincides with (D.31). However, it carries a non-zero differential d_α given by the formula

$$d_\alpha(P) = -(-1)^{|P|} P \bullet \alpha\,, \tag{D.37}$$

and the corresponding (degree 1) brackets

$$\{\ ,\ ,\ldots,\ \}_k : S^k\big(\,\mathrm{Hom}(\mathrm{Ger}^\vee(V_A), V_A^Q)\big) \to \mathrm{Hom}(\mathrm{Ger}^\vee(V_A), V_A^Q)$$

are defined by general formula (C.28) in terms of the $\mathrm{Ger}^\vee$-coalgebra structure on $\mathrm{Ger}^\vee(V_A)$ and the Ger_∞-structure Q on V_A.

Let us recall [9], [15] that Ger_∞-morphisms from V_A to V_A^Q are in bijection with Maurer-Cartan elements[5]

$$\beta = \sum_{n \ge 1} \beta_n\,, \qquad \beta_n \in \mathrm{Hom}_{S_n}(\mathrm{Ger}^\vee(n) \otimes V_A^{\otimes\, n}, V_A) \tag{D.38}$$

of $\Lambda^{-1}\,\mathrm{Lie}_\infty$-algebra (D.36) such that β_1 corresponds to the linear term of the corresponding Ger_∞-morphism.

Thus our goal is to prove that, for every Maurer-Cartan element Q (D.33) of Lie algebra (D.32) satisfying condition (D.35), there exists a Maurer-Cartan element β (see (D.38)) of $\Lambda^{-1}\,\mathrm{Lie}_\infty$-algebra (D.36) such that

$$\beta_1 = \mathrm{id} : V_A \to V_A\,. \tag{D.39}$$

[5]Recall that Maurer-Cartan elements of a $\Lambda^{-1}\,\mathrm{Lie}_\infty$-algebra have degree 0.

Condition (D.35) implies that the element

$$\beta^{(1)} := \mathrm{id} \in \mathrm{Hom}(\mathrm{Ger}^\vee(V_A), V_A^Q)$$

satisfies the equation (in the $\Lambda^{-1}\,\mathrm{Lie}_\infty$-algebra $\mathrm{Hom}(\mathrm{Ger}^\vee(V_A), V_A^Q)$)

$$\Big(d_\alpha(\beta^{(1)}) + \sum_{k\ge 2} \frac{1}{k!}\{\beta^{(1)}, \dots, \beta^{(1)}\}_k \Big)(X) = 0 \tag{D.40}$$

for every $X \in (\mathrm{Ger}^\vee(m) \otimes V_A^{\otimes\, m})_{S_m}$ with $m \le 2$.

Let us assume that we constructed (by induction) a degree zero element

$$\beta^{(n-1)} = \mathrm{id} + \beta_2 + \beta_3 + \dots + \beta_{n-1}\,, \qquad \beta_j \in \mathrm{Hom}_{S_j}(\mathrm{Ger}^\vee(j) \otimes V_A^{\otimes\, j}, V_A) \tag{D.41}$$

such that

$$\Big(d_\alpha(\beta^{(n-1)}) + \sum_{k\ge 2} \frac{1}{k!}\{\beta^{(n-1)}, \dots, \beta^{(n-1)}\}_k \Big)(X) = 0 \tag{D.42}$$

for every $X \in (\mathrm{Ger}^\vee(m) \otimes V_A^{\otimes\, m})_{S_m}$ with $m \le n$.

We will try to find an element

$$\beta_n \in \mathrm{Hom}_{S_n}(\mathrm{Ger}^\vee(n) \otimes V_A^{\otimes\, n}, V_A) \tag{D.43}$$

such that the sum

$$\beta^{(n)} := \mathrm{id} + \beta_2 + \beta_3 + \dots + \beta_{n-1} + \beta_n \tag{D.44}$$

satisfies the equation

$$\Big(d_\alpha(\beta^{(n)}) + \sum_{k\ge 2} \frac{1}{k!}\{\beta^{(n)}, \dots, \beta^{(n)}\}_k \Big)(X) = 0 \tag{D.45}$$

for every $X \in (\mathrm{Ger}^\vee(m) \otimes V_A^{\otimes\, m})_{S_m}$ with $m \le n+1$.

Since $\beta_n \in \mathrm{Hom}_{S_n}(\mathrm{Ger}^\vee(n) \otimes V_A^{\otimes\, n}, V_A)$ and (D.42) is satisfied for every $X \in (\mathrm{Ger}^\vee(m) \otimes V_A^{\otimes\, m})_{S_m}$ with $m \le n$, equation (D.45) is also satisfied for every $X \in (\mathrm{Ger}^\vee(m) \otimes V_A^{\otimes\, m})_{S_m}$ with $m \le n$.

For $X \in (\mathrm{Ger}^\vee(n+1) \otimes V_A^{\otimes\,(n+1)})_{S_{n+1}}$, equation (D.45) can be rewritten as

$$-\beta_n \bullet \alpha(X) + \alpha \bullet \beta_n(X) = -\sum_{k\ge 2} \frac{1}{k!}\{\beta^{(n-1)}, \dots, \beta^{(n-1)}\}_k(X)\,. \tag{D.46}$$

Let us denote by γ the element in $\mathrm{Hom}_{S_{n+1}}(\mathrm{Ger}^\vee(n+1)\otimes V_A^{\otimes\,(n+1)}, V_A)$ defined as

$$\gamma := \sum_{k\geq 2} \frac{1}{k!}\{\beta^{(n-1)},\dots,\beta^{(n-1)}\}_k \Big|_{\mathrm{Ger}^\vee(n+1)\otimes V_A^{\otimes\,(n+1)}} \tag{D.47}$$

Evaluating the Bianchi type identity [20, Lemma 4.5]

$$\sum_{k\geq 2}\frac{1}{k!} d_\alpha\{\beta^{(n-1)},\dots,\beta^{(n-1)}\}_k + \sum_{k\geq 1}\frac{1}{k!}\{\beta^{(n-1)},\dots,\beta^{(n-1)}, d_\alpha\beta^{(n-1)}\}_{k+1}$$
$$+ \sum_{\substack{k\geq 2\\ t\geq 1}} \frac{1}{k!t!}\{\beta^{(n-1)},\dots,\beta^{(n-1)},\{\beta^{(n-1)},\dots,\beta^{(n-1)}\}_k\}_{t+1} = 0 \tag{D.48}$$

on an arbitrary element

$$Y \in (\mathrm{Ger}^\vee(n+2)\otimes V_A^{\otimes\,(n+2)})_{S_{n+2}}$$

and using the fact that

$$\beta^{(n-1)}(X) = 0\,, \qquad \forall\ X \in (\mathrm{Ger}^\vee(m)\otimes V_A^{\otimes\, m})_{S_m} \ \text{ with } \ m \geq n$$

we deduce that element γ (D.47) is a cocycle in cochain complex (D.4) with differential (D.6). Thus Theorem D.1 implies that equation (D.46) can always be solved for β_n, which concludes the proof of Corollary D.1. □

D.2 The Gerstenhaber algebra V_A is intrinsically formal

Let $(C^\bullet, \mathfrak{d})$ be an arbitrary cochain complex whose cohomology is isomorphic to V_A

$$H^\bullet(C^\bullet) \cong V_A\,. \tag{D.49}$$

Let us consider V_A as the cochain complex with the zero differential and choose[6] a quasi-isomorphism of cochain complexes

$$I : V_A \to C^\bullet\,. \tag{D.50}$$

[6] Such a quasi-isomorphism exists since we are dealing with cochain complexes of vector spaces over a field.

Let us assume that $C^\bullet$ carries a Ger_∞-structure such that the map I induces an isomorphism of Gerstenhaber algebras $V_A \cong H^\bullet(C^\bullet)$.

Then Theorem D.1 gives us the following remarkable corollary:

Corollary D.2. *There exists a* Ger_∞*-morphism*

$$U : V_A \rightsquigarrow C^\bullet \tag{D.51}$$

whose linear term coincides with I (D.50)*. Moreover, any two such* Ger_∞*-morphisms*

$$U,\ \widetilde{U}\ :\ V_A \rightsquigarrow C^\bullet \tag{D.52}$$

are homotopy equivalent.

Remark D.2. *The above statement is a slight refinement of one proved in [21, Section 5]. Following V. Hinich, we say that the Gerstenhaber algebra* V_A *is intrinsically formal.*

Proof of Corollary D.2. By the Homotopy Transfer Theorem [9, Section 5], [27, Section 10.3], there exists a Ger_∞-structure Q on V_A and a Ger_∞-quasi-isomorphism

$$U' : V_A^Q \rightsquigarrow C^\bullet\,, \tag{D.53}$$

such that

- the binary operations of the Ger_∞-structure Q on V_A are the Schouten bracket and the usual multiplication of polyvector fields,
- the linear term of U' coincides with I.

Corollary D.1 implies that there exists a Ger_∞-morphism

$$U_{\mathrm{corr}} : V_A \rightsquigarrow V_A^Q\,, \tag{D.54}$$

whose linear term is the identity map $\mathrm{id} : V_A \to V_A$.

Hence the composition

$$U = U' \circ U_{\mathrm{corr}} : V_A \rightsquigarrow C^\bullet \tag{D.55}$$

is a desired Ger_∞-morphism.

To prove the second claim, we need the $\Lambda^{-1}\mathrm{Lie}_\infty$-algebra

$$\mathrm{Hom}(\mathrm{Ger}^\vee(V_A), C^\bullet) \tag{D.56}$$

corresponding to the Gerstenhaber algebra V_A and the Ger_∞-algebra $C^\bullet$. The differential $\mathcal{D}$ on (D.56) is given by the formula

$$\mathcal{D}(\Psi) := \mathfrak{d} \circ \Psi - (-1)^{|\Psi|} \Psi \circ Q_{\wedge, \{\ ,\ \}}\,, \qquad \Psi \in \mathrm{Hom}(\mathrm{Ger}^\vee(V_A), C^\bullet)\,, \tag{D.57}$$

where $\mathfrak{d}$ is the differential on $C^\bullet$ and $Q_{\wedge,\{\ ,\ \}}$ is the differential on the $\mathrm{Ger}^\vee$-coalgebra $\mathrm{Ger}^\vee(V_A)$ corresponding to the standard Gerstenhaber structure on V_A.

The multi-brackets $\{\ ,\ ,\dots,\ \}_m$ are defined by the general formula (see eq. (C.28)) in terms of the $\mathrm{Ger}^\vee$-coalgebra structure on $\mathrm{Ger}^\vee(V_A)$ and the Ger_∞-structure on $C^\bullet$.

Let us recall (see Appendix C.2 for more details) that Ger_∞-morphisms from V_A to $C^\bullet$ are in bijection with Maurer-Cartan elements of $\Lambda^{-1}\,\mathrm{Lie}_\infty$-algebra (D.56) and Ger_∞-morphisms (D.52) are homotopy equivalent if and only if the corresponding Maurer-Cartan elements P and $\widetilde{P}$ in (D.56) are isomorphic 0-cells in the Deligne-Getzler-Hinich ∞-groupoid [20] of (D.56).

So our goal is to prove that any two Maurer-Cartan elements P and $\widetilde{P}$ in (D.56) satisfying

$$P\Big|_{V_A} = \widetilde{P}\Big|_{V_A} = I\ :\ V_A\ \to\ C^\bullet \tag{D.58}$$

are isomorphic.

Condition (D.58) implies that

$$\widetilde{P} - P \in \mathcal{F}_2\,\mathrm{Hom}(\mathrm{Ger}^\vee(V_A), C^\bullet)\,,$$

where $\mathcal{F}_\bullet\,\mathrm{Hom}(\mathrm{Ger}^\vee(V_A), C^\bullet)$ is the arity filtration (C.30) on $\mathrm{Hom}(\mathrm{Ger}^\vee(V_A), C^\bullet)$.

Let us assume that we constructed a sequence of Maurer-Cartan elements

$$P = P_2, P_3, P_4, \dots, P_{n+1} \tag{D.59}$$

such that for every $2 \le m \le n+1$

$$\widetilde{P} - P_m \in \mathcal{F}_m \operatorname{Hom}(\mathrm{Ger}^\vee(V_A), C^\bullet) \tag{D.60}$$

and for every $2 \le m \le n$ there exists 1-cell

$$P'_m(t) + dt\, \xi_{m-1} \in \operatorname{Hom}(\mathrm{Ger}^\vee(V_A), C^\bullet) \,\hat{\otimes}\, \Omega^\bullet(\Delta_1)$$

which connects P_m to P_{m+1} and such that

$$\xi_{m-1} \in \mathcal{F}_{m-1} \operatorname{Hom}(\mathrm{Ger}^\vee(V_A), C^\bullet)\,, \tag{D.61}$$

and

$$P'_m(t) - P_m \in \mathcal{F}_m \operatorname{Hom}(\mathrm{Ger}^\vee(V_A), C^\bullet) \,\hat{\otimes}\, \Bbbk[t]\,. \tag{D.62}$$

Let us now prove that one can construct a 1-cell

$$P'_{n+1}(t) + dt\, \xi_n \in \operatorname{Hom}(\mathrm{Ger}^\vee(V_A), C^\bullet) \,\hat{\otimes}\, \Omega^\bullet(\Delta_1) \tag{D.63}$$

such that

$$P'_{n+1}(t)\Big|_{t=0} = P_{n+1}\,,$$

$$\xi_n \in \mathcal{F}_n \operatorname{Hom}(\mathrm{Ger}^\vee(V_A), C^\bullet)\,, \tag{D.64}$$

$$P'_{n+1}(t) - P_{n+1} \in \mathcal{F}_{n+1} \operatorname{Hom}(\mathrm{Ger}^\vee(V_A), C^\bullet) \,\hat{\otimes}\, \Bbbk[t]\,, \tag{D.65}$$

and the Maurer-Cartan element

$$P_{n+2} := P'_{n+1}(t)\Big|_{t=1} \tag{D.66}$$

satisfies the condition

$$\widetilde{P} - P_{n+2} \in \mathcal{F}_{n+2} \operatorname{Hom}(\mathrm{Ger}^\vee(V_A), C^\bullet)\,. \tag{D.67}$$

Let us denote the difference $\widetilde{P} - P_{n+1}$ by K. Since $\widetilde{P} - P_{n+1} \in \mathcal{F}_{n+1} \operatorname{Hom}(\mathrm{Ger}^\vee(V_A), C^\bullet)$,

$$K = \sum_{m \ge n+1} K_m\,, \qquad K_m \in \operatorname{Hom}_{S_m}(\mathrm{Ger}^\vee(m) \otimes V_A^{\otimes m}, C^\bullet)\,. \tag{D.68}$$

Subtracting the left hand side of the Maurer-Cartan equation

$$\mathcal{D}(P_{n+1}) + \sum_{m\ge 2} \frac{1}{m!}\{P_{n+1}, P_{n+1}, \dots, P_{n+1}\}_m = 0 \tag{D.69}$$

from the left hand side of the Maurer-Cartan equation

$$\mathcal{D}(\widetilde{P}) + \sum_{m\ge 2} \frac{1}{m!}\{\widetilde{P}, \widetilde{P}, \dots, \widetilde{P}\}_m = 0 \tag{D.70}$$

we see that element (D.68) satisfies the equation

$$\mathcal{D}(K) + \sum_{m\ge 1} \frac{1}{m!}\{P_{n+1}, \dots, P_{n+1}, K\}_{m+1} + \sum_{m\ge 2} \frac{1}{m!}\{K, K, \dots, K\}^{P_{n+1}}_m = 0\,, \tag{D.71}$$

where the multi-bracket $\{K, K, \dots, K\}^{P_{n+1}}_m$ is defined by the formula

$$\{X_1, X_2, \dots, X_m\}^{P_{n+1}}_m := \sum_{q\ge 0} \frac{1}{q!}\{P_{n+1}, \dots, P_{n+1}, X_1, X_2, \dots, X_m\}_{q+m} \tag{D.72}$$

Evaluating (D.71) on $\mathrm{Ger}^\vee(n+1) \otimes V_A^{\otimes\,(n+1)}$ and using the fact that

$$K \in \mathcal{F}_{n+1}\,\mathrm{Hom}(\mathrm{Ger}^\vee(V_A), C^\bullet)\,, \tag{D.73}$$

we conclude that

$$\mathfrak{d} \circ K_{n+1} = 0\,, \tag{D.74}$$

where $\mathfrak{d}$ is the differential on $C^\bullet$.

Hence there exist elements

$$K^{V_A}_{n+1} \in \mathrm{Hom}_{S_{n+1}}(\mathrm{Ger}^\vee(n+1) \otimes V_A^{\otimes\,(n+1)}, V_A)$$

and

$$K'_{n+1} \in \mathrm{Hom}_{S_{n+1}}(\mathrm{Ger}^\vee(n+1) \otimes V_A^{\otimes\,(n+1)}, C^\bullet)$$

such that

$$K_{n+1} = I \circ K^{V_A}_{n+1} + \mathfrak{d} \circ K'_{n+1}\,. \tag{D.75}$$

Next, evaluating (D.71) on $Y \in \mathrm{Ger}^\vee(n+2) \otimes V_A^{\otimes\,(n+2)}$ and using inclusion (D.73) again, we get the following identity

$$\mathfrak{d} \circ K_{n+2}(Y) - K_{n+1} \circ Q_{\wedge,\{\,,\,\}}(Y) + \{P_{n+1}, K_{n+1}\}_2(Y) = 0\,. \tag{D.76}$$

Unfolding $\{P_{n+1}, K_{n+1}\}_2(Y)$ we get

$$\{P_{n+1}, K_{n+1}\}_2(Y) = \sum_{i=1}^{n+2} Q_{C^\bullet}\Big((\mathrm{id}_{\mathrm{Ger}^\vee(2)} \otimes K_{n+1} \otimes I) \circ \big(\Delta_{\mathbf{t}_i} \otimes \mathrm{id}^{\otimes\,(n+2)}\big)(Y)\Big)\,, \tag{D.77}$$

where $Q_{C^\bullet}$ is the Ger_∞-structure on $C^\bullet$, $\mathbf{t}_i$ is the $(n+2)$-labeled planar tree shown on figure (D.1), and $\Delta_{\mathbf{t}_i}$ is the corresponding component of the comultiplication

$$\Delta_{\mathbf{t}_i} : \mathrm{Ger}^\vee(n+2) \to \mathrm{Ger}^\vee(2) \otimes \mathrm{Ger}^\vee(n+1)\,. \tag{D.78}$$

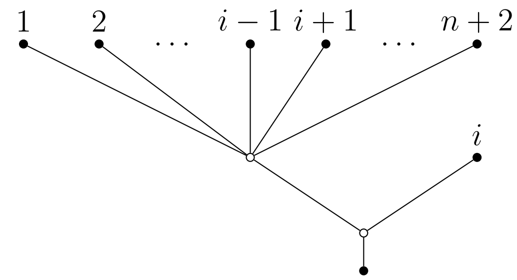


Figure D.1: The $(n+2)$-labeled planar tree $\mathbf{t}_i$

Now using (D.75) and (D.77), we rewrite (D.76) as follows

$$\begin{aligned}
&\mathfrak{d} \circ K_{n+2}(Y) - I \circ (K^{V_A}_{n+1} \bullet \alpha)(Y) \\
&\qquad + \sum_{i=1}^{n+2} Q_{C^\bullet}\Big((\mathrm{id}_{\mathrm{Ger}^\vee(2)} \otimes (\mathfrak{d} \circ K'_{n+1}) \otimes I) \circ \big(\Delta_{\mathbf{t}_i} \otimes \mathrm{id}^{\otimes\,(n+2)}\big)(Y)\Big) \\
&\quad + \sum_{i=1}^{n+2} Q_{C^\bullet}\Big((\mathrm{id}_{\mathrm{Ger}^\vee(2)} \otimes (I \circ K^{V_A}_{n+1}) \otimes I) \circ \big(\Delta_{\mathbf{t}_i} \otimes \mathrm{id}^{\otimes\,(n+2)}\big)(Y)\Big) = 0\,,
\end{aligned} \tag{D.79}$$

where α is defined in (D.3).

Since the last two sums in (D.79) involve only binary Ger_∞-operations on $C^\bullet$ and these binary operations induce the usual multiplication and the Schouten bracket on V_A, we conclude that each term in the first sum in (D.79) is $\mathfrak{d}$-exact and the second sum in (D.79) is cohomologous to

$$I \circ (\alpha \bullet K^{V_A}_{n+1})(Y)$$

Therefore, identity (D.79) implies that for every $Y \in \mathrm{Ger}^\vee(n+2) \otimes V_A^{\otimes (n+2)}$ the expression

$$I \circ (\alpha \bullet K^{V_A}_{n+1} - K^{V_A}_{n+1} \bullet \alpha)(Y) \tag{D.80}$$

is $\mathfrak{d}$-exact. Thus

$$\alpha \bullet K^{V_A}_{n+1} - K^{V_A}_{n+1} \bullet \alpha = 0 \tag{D.81}$$

or, in other words, the element $K^{V_A}_{n+1}$ is a cocycle in complex (D.4) with differential (D.6).

Hence, by Theorem D.1, there exists a degree -1 element

$$\widetilde{K}^{V_A}_n \in \mathrm{Hom}_{S_n}(\mathrm{Ger}^\vee(n) \otimes V_A^{\otimes (n)}, V_A) \tag{D.82}$$

such that

$$K^{V_A}_{n+1} = [\alpha, \widetilde{K}^{V_A}_n]\,. \tag{D.83}$$

Let us now consider the degree -1 element

$$\xi_n = I \circ \widetilde{K}^{V_A}_n + K''_{n+1} \in \mathcal{F}_n \,\mathrm{Hom}(\mathrm{Ger}^\vee(V_A), C^\bullet)\,, \tag{D.84}$$

where $\widetilde{K}^{V_A}_n$ is element (D.82) entering equation (D.83) and K''_{n+1} is an element in

$$\mathrm{Hom}_{S_{n+1}}\big(\mathrm{Ger}^\vee(n+1) \otimes V_A^{\otimes (n+1)}, C^\bullet\big)$$

which will be determined later.

Using ξ_n, we define $P'_{n+1}(t) \in \mathrm{Hom}(\mathrm{Ger}^\vee(V_A), C^\bullet)\,\hat{\otimes}\,\Bbbk[t]$ as the limiting element of the recursive procedure

$$\begin{aligned}
&(P')^{(0)} := P_{n+1}\,,\\
&(P')^{(k+1)}(t) := P_{n+1} + \int_0^t dt_1 \Big(\mathcal{D}(\xi_n) + \sum_{m\ge 1} \frac{1}{m!}\{(P')^{(k)}(t_1), \dots, (P')^{(k)}(t_1), \xi_n\}_{m+1}\Big)\,.
\end{aligned} \tag{D.85}$$

Since

$$\mathfrak{d}\big(I \circ \widetilde{K}^{V_A}_n\big) = 0 \tag{D.86}$$

the element ξ_n satisfies the condition

$$\mathcal{D}(\xi_n) \in \mathcal{F}_{n+1} \,\mathrm{Hom}(\mathrm{Ger}^\vee(V_A), C^\bullet)\,. \tag{D.87}$$

Hence, by Lemma C.1, the sum

$$P'_{n+1}(t) + dt\xi_n \in \mathrm{Hom}(\mathrm{Ger}^\vee(V_A), C^\bullet)\,\hat{\otimes}\,\Omega^\bullet(\Delta_1) \tag{D.88}$$

is a 1-cell in the ∞-groupoid corresponding to $\mathrm{Hom}(\mathrm{Ger}^\vee(V_A), C^\bullet)$ satisfying (D.65) and such that the Maurer-Cartan element P_{n+2} (D.66) satisfies the condition

$$P_{n+2} - P_{n+1} - \mathcal{D}(\xi_n) - \{P_{n+1}, \xi_n\}_2 \in \mathcal{F}_{n+2}\,\mathrm{Hom}(\mathrm{Ger}^\vee(V_A), C^\bullet)\,. \tag{D.89}$$

Let us now show that, by choosing the element K''_{n+1} in (D.84) appropriately, we can get desired inclusion (D.67).

For this purpose we unfold $\{P_{n+1}, \xi_n\}_2(Y)$ for an arbitrary $Y \in \mathrm{Ger}^\vee(n+1) \otimes V_A^{\otimes\,(n+1)}$ and get

$$\{P_{n+1}, \xi_n\}_2(Y) = \sum_{i=1}^{n+1} Q_{C^\bullet}\Big((\mathrm{id}_{\mathrm{Ger}^\vee(2)} \otimes (I \circ \widetilde{K}_n^{V_A}) \otimes I) \circ \big(\Delta_{\mathbf{t}'_i} \otimes \mathrm{id}^{\otimes\,(n+1)}\big)(Y)\Big)\,, \tag{D.90}$$

where $Q_{C^\bullet}$ is the Ger_∞-structure on $C^\bullet$, $\mathbf{t}'_i$ is the $(n+1)$-labeled planar tree shown on figure (D.91), and $\Delta_{\mathbf{t}'_i}$ is the corresponding component of the comultiplication

$$\Delta_{\mathbf{t}'_i} : \mathrm{Ger}^\vee(n+1) \to \mathrm{Ger}^\vee(2) \otimes \mathrm{Ger}^\vee(n)\,. \tag{D.91}$$

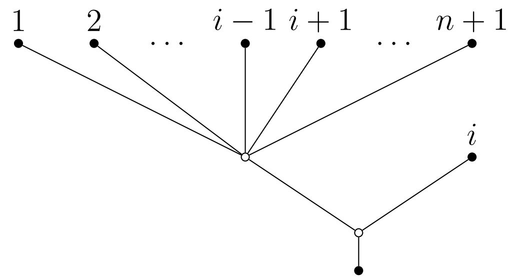


Figure D.2: The $(n+1)$-labeled planar tree $\mathbf{t}'_i$

Since the right hand side of (D.90) involves only binary Ger_∞-operations on $C^\bullet$ and these binary operations induce the usual multiplication and the Schouten bracket on V_A, we conclude that $\{P_{n+1}, \xi_n\}_2(Y)$ is cohomologous (in $C^\bullet$) to

$$I \circ (\alpha \bullet \widetilde{K}_n^{V_A})(Y)\,,$$

where α is defined in (D.3).

In other words, there exists an element

$$\phi \in \mathrm{Hom}_{S_{n+1}} \big(\mathrm{Ger}^{\vee}(n+1) \otimes V_A^{\otimes (n+1)}, C^{\bullet} \big) \tag{D.92}$$

such that

$$\{P_{n+1}, \xi_n\}_2(Y) = I \circ (\alpha \bullet \widetilde{K}_n^{V_A})(Y) + \mathfrak{d} \circ \phi(Y). \tag{D.93}$$

Hence the expression $\big(\mathcal{D}(\xi_n) + \{P_{n+1}, \xi_n\}_2\big)(Y)$ can be rewritten as

$$\big(\mathcal{D}(\xi_n) + \{P_{n+1}, \xi_n\}_2\big)(Y) = \mathfrak{d} \circ K''_{n+1}(Y) + \mathfrak{d} \circ \phi(Y) + I \circ [\alpha, \widetilde{K}_n^{V_A}](Y)\,. \tag{D.94}$$

Thus if

$$K''_{n+1} = K'_{n+1} - \phi \tag{D.95}$$

then equations (D.75), (D.83), and inclusion (D.89) imply that (D.67) holds, as desired.

Thus we showed that one can construct an infinite sequence of Maurer-Cartan elements

$$P = P_2, P_3, P_4, \dots \tag{D.96}$$

and an infinite sequence of 1-cells $(m \geq 2)$

$$P'_m(t) + dt\, \xi_{m-1} \in \mathrm{Hom}(\mathrm{Ger}^{\vee}(V_A), C^{\bullet}) \,\hat{\otimes}\, \Omega^{\bullet}(\Delta_1) \tag{D.97}$$

such that for every $m \geq 2$

$$\widetilde{P} - P_m \in \mathcal{F}_m \,\mathrm{Hom}(\mathrm{Ger}^{\vee}(V_A), C^{\bullet})\,, \tag{D.98}$$

the 1-cell $P'_m(t) + dt\, \xi_{m-1}$ connects P_m to P_{m+1}

$$\xi_{m-1} \in \mathcal{F}_{m-1} \,\mathrm{Hom}(\mathrm{Ger}^{\vee}(V_A), C^{\bullet})\,, \tag{D.99}$$

and

$$P'_m(t) - P_m \in \mathcal{F}_m \,\mathrm{Hom}(\mathrm{Ger}^{\vee}(V_A), C^{\bullet}) \,\hat{\otimes}\, \Bbbk[t]\,. \tag{D.100}$$

Since the $\Lambda^{-1}\mathrm{Lie}_{\infty}$-algebra $\mathrm{Hom}(\mathrm{Ger}^{\vee}(V_A), C^{\bullet})$ is complete with respect to "arity" filtration (C.30), inclusions (D.99) and (D.100) imply that we can form the infinite composition[7] of all 1-cells (D.97) and get a 1-cell which connects the Maurer-Cartan element $P = P_2$ to the Maurer-Cartan element $\widetilde{P}$. Thus, Corollary D.2 is proved. □

[7]Note that the composition of 1-cells in an infinity groupoid is not unique but this does not create a problem.

APPENDIX E

ON DERIVATIONS OF HOMOTOPY DIAGRAMS

Let $\mathcal{C}$ be a coaugmented, reduced cooperad in the category of graded vector spaces and $\mathcal{C}_\circ$ be the cokernel of the coaugmentation.

Following Chapter 3, we will denote by $\mathrm{Cyl}(\mathcal{C})$ the 2-colored dg operad whose algebras are pairs (V, W) with the data

1. a $\mathrm{Cobar}(\mathcal{C})$-algebra structure on V,
2. a $\mathrm{Cobar}(\mathcal{C})$-algebra structure on W, and
3. an ∞-morphism F from V to W, i.e. a homomorphism of corresponding dg $\mathcal{C}$-coalgebras $\mathcal{C}(V) \to \mathcal{C}(W)$.

In fact, if we forget about the differential, then $\mathrm{Cyl}(\mathcal{C})$ is a free operad on a certain 2-colored collection $\mathcal{M}(\mathcal{C})$ naturally associated to $\mathcal{C}$.

Following the conventions of Chapters 4 and 7, we denote by

$$\mathrm{Der}'\big(\mathrm{Cyl}(\mathcal{C})\big) \tag{E.1}$$

the dg Lie algebra of derivations D of $\mathrm{Cyl}(\mathcal{C})$ subject to the condition

$$p \circ D = 0\,, \tag{E.2}$$

where p is the canonical projection from $\mathrm{Cyl}(\mathcal{C})$ onto $\mathcal{M}(\mathcal{C})$.

We have the following generalization of (7.3):

Proposition E.1. *The dg Lie algebra* $\mathrm{Der}'\big(\mathrm{Cyl}(\Lambda^2\,\mathrm{coCom})\big)$ *does not have non-zero elements in degrees* ≤ 0*, i.e.*

$$\mathrm{Der}'\big(\mathrm{Cyl}(\Lambda^2\,\mathrm{coCom})\big)^{\le 0} = \mathbf{0}\,.$$

Proof. Let us denote by α and β, respectively, the first and the second color for the collection $\mathcal{M}(\Lambda^2\,\mathrm{coCom})$ and the operad $\mathrm{Cyl}(\Lambda^2\,\mathrm{coCom})$.

Recall from [33] that $\mathrm{Cyl}(\Lambda^2\,\mathrm{coCom})$ is generated by the collection $\mathcal{M} = \mathcal{M}(\Lambda^2\,\mathrm{coCom})$ with

$$\begin{aligned}\mathcal{M}(n,0;\alpha) &= \mathbf{s}\,\Lambda^2\,\mathrm{coCom}_\circ(n) = \mathbf{s}^{3-2n}\,\Bbbk\,,\\ \mathcal{M}(0,n;\beta) &= \mathbf{s}\,\Lambda^2\,\mathrm{coCom}_\circ(n) = \mathbf{s}^{3-2n}\,\Bbbk\,,\\ \mathcal{M}(n,0;\beta) &= \Lambda^2\,\mathrm{coCom}(n) = \mathbf{s}^{2-2n}\,\Bbbk\,,\end{aligned}$$

and with all the remaining spaces being zero. Let D be a derivation of $\mathrm{Cyl}(\Lambda^2\,\mathrm{coCom})$ of degree ≤ 0.

Since

$$\mathrm{Cyl}\big(\Lambda^2\,\mathrm{coCom}\big)(n,0,\alpha) = \Lambda\,\mathrm{Lie}_\infty(n) \tag{E.3}$$

$$\mathrm{Cyl}\big(\Lambda^2\,\mathrm{coCom}\big)(0,n,\beta) = \Lambda\,\mathrm{Lie}_\infty(n) \tag{E.4}$$

observation (7.3) implies that

$$D\Big|_{\mathcal{M}(n,0;\alpha)} = D\Big|_{\mathcal{M}(0,n;\beta)} = 0\,. \tag{E.5}$$

Hence, it suffices to show that

$$D\Big|_{\mathcal{M}(n,0;\beta)} = 0\,. \tag{E.6}$$

Let us denote by $\pi_0(\mathsf{Tree}_k(n))$ the set of isomorphism classes of labeled 2-colored planar trees corresponding to corolla $(n,0;\beta)$ with k internal vertices. Figure E.1 show two examples of such trees with $n=5$ leaves. The left tree has $k=2$ internal vertices and the right tree has $k=3$ internal vertices.

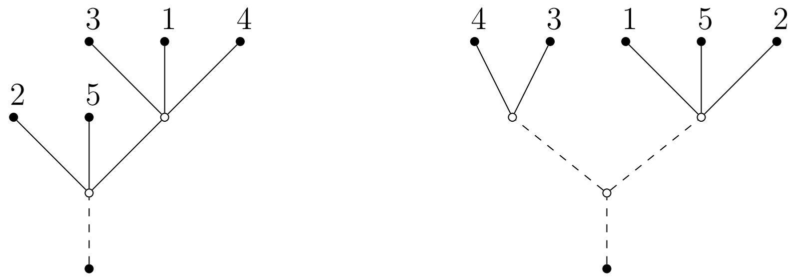


Figure E.1: Solid edges carry the color α and dashed edges carry the color β; internal vertices are denoted by small white circles; leaves and the root vertex are denoted by small black circles

For a generator $X \in \mathcal{M}(n, 0; \beta) = \mathbf{s}^{2-2n}\,\Bbbk$, the element $D(X) \in \mathrm{Cyl}(\Lambda^2\,\mathrm{coCom})$ takes the form

$$D(X) \;=\; \sum_{k\geq 2}\;\sum_{z\in\pi_0(\mathsf{Tree}_k(n))} (\mathbf{t}_z; X_1, ..., X_k) \tag{E.7}$$

where $\mathbf{t}_z$ is a representative of an isomorphism class $z \in \pi_0(\mathsf{Tree}_k(n))$ and X_i are the corresponding elements of $\mathcal{M}$.

For every term in sum (E.7), we have k_1 X_i's in $\mathbf{s}\,\Lambda^2\,\mathrm{coCom}_\circ$ (call them X_{i_a}), and k_2 X_i's in $\Lambda^2\,\mathrm{coCom}$ (call them X_{j_b}).

We obviously have that $k = k_1 + k_2$ and

$$|D| = \sum_{a=1}^{k_1} |X_{i_a}| + \sum_{b=1}^{k_2} |X_{j_b}| - |X| \tag{E.8}$$

or equivalently

$$|D| = 2(n-1) + \sum_{a=1}^{k_1}(3 - 2n_{i_a}) + \sum_{b=1}^{k_2}(2 - 2n_{j_b})\,, \tag{E.9}$$

where n_{i_a} (resp. n_{j_b}) is the number of incoming edges of the vertex corresponding to X_{i_a} (resp. X_{j_b}).

On the other hand, simple combinatorics of trees shows that

$$n - 1 = \sum_{a=1}^{k_1}(n_{i_a} - 1) + \sum_{b=1}^{k_2}(n_{j_b} - 1) \tag{E.10}$$

and hence

$$|D| = k_1\,. \tag{E.11}$$

Since $|D| \leq 0$ the latter is possible only if $k_1 = 0 = |D|$, i.e. every tree in the sum $D(X)$ is assembled exclusively from mixed colored corollas. That would force every tree **t** to have only one internal vertex which contradicts to the fact that the summation in (E.7) starts at $k = 2$.

Therefore (E.6) holds, and the proposition follows. □