The Arithmetic of Rational Polytopes

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Abstract

We study the number of integer points ("lattice points") in rational polytopes. We use an associated generating function in several variables, whose coefficients are the lattice point enumerators of the dilates of a polytope. We focus on applications of this theory to several problems in combinatorial number theory.

In chapter 2, we present a new method of deriving the lattice point count operators for rational polytopes. In particular, we show how various generalizations of Dedekind sums appear naturally in the lattice point count formulas, and give geometric interpretations of reciprocity laws for these sums.

In chapter 3, we use our methods to obtain new results on the Frobenius problem: namely, given positive integers $a_1, \ldots, a_n$ with $\gcd(a_1, \ldots, a_n) = 1$, find the largest integer that cannot be represented as a linear combination of $a_1, \ldots, a_n$ with nonnegative coefficients. We transfer this problem into our geometric setting and deduce and extend from this point of view some classical results on this problem.

In our formulas, the following generalization of the Dedekind sum appears naturally: Let $c_1, \ldots, c_n \in \mathbb{Z}$ be relatively prime to $c \in \mathbb{Z}$, and let $t \in \mathbb{Z}$. Define the *Fourier-Dedekind sum* as

$$\sigma_t(c_1, \ldots, c_n; c) = \frac{1}{c} \sum_{\lambda^c = 1 \neq \lambda} \frac{\lambda^t}{(\lambda^{c_1} - 1) \cdots (\lambda^{c_n} - 1)} \, .$$

We discuss these sums in depth; in particular, we prove two reciprocity laws for them: a rederivation of the reciprocity law for Zagier's higher-dimensional Dedekind sums, and a new reciprocity law that generalizes a theorem of Gessel.

In chapter 4, we generalize Ehrhart's idea of counting lattice points in dilated rational polytopes: instead of just a single dilation factor, we allow different dilation factors for each of the facets of the polytope. We prove that the lattice point counts in the interior and closure of such a *vector-dilated* polytope are quasipolynomials satisfying an Ehrhart-type reciprocity law. Our theorem generalizes the classical reciprocity law for rational polytopes.

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To Elisabeth and Hermann,
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and to Tendai,
with all my love.

Contents

Chapter 1

Introduction

Ubi materia, ibi geometria. (Where there is matter, there is geometry.)
Johannes Kepler

1.1 Pick's Theorem

We study the number of integer points in polytopes, contained in some real space $\mathbb{R}^n$. Since the integer points $\mathbb{Z}^n$ form a lattice in $\mathbb{R}^n$, we frequently call them **lattice points**. The first interesting case is dimension $n = 2$. Consider a simple, closed polygon whose vertices have integer coordinates. Denote the number of integer points inside the polygon by I, and the number of integer points on the polygon by B. In 1899, Pick ([Pi]) discovered the astonishing fact that the area A inside the polygon can be computed simply by counting lattice points:

Theorem 1.1 (Pick)

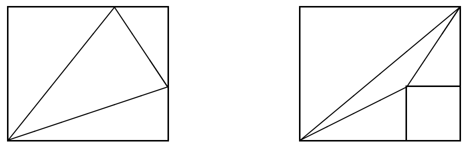
$$A = I + \frac{1}{2}B - 1 \ .$$

We give an elementary proof of Pick's theorem; the main ideas are from [Va].

Proof. We start by proving that Pick's identity has an additive character: suppose our polygon has more than 3 vertices. Then we can write the 2-dimensional polytope $\mathcal{P}$ bounded by our polygon as the union of two 2-dimensional polytopes $\mathcal{P}_1$ and $\mathcal{P}_2$, such that the interiors of $\mathcal{P}_1$ and $\mathcal{P}_2$ do not meet. Both have fewer vertices than $\mathcal{P}$. We claim that the validity of Pick's identity for $\mathcal{P}$ is equivalent to the validity of Pick's identity for $\mathcal{P}_1$ and $\mathcal{P}_2$. Denote the area, number of interior lattice points, and number of boundary lattice points of $\mathcal{P}_k$ by A_k, I_k, and B_k, respectively, for $k = 1, 2$. Clearly,

$$A = A_1 + A_2 \ .$$

Figure 1.1: Embedding of triangles



Furthermore, if we denote the number of lattice points on the edges common to $\mathcal{P}_1$ and $\mathcal{P}_2$ by L, then

$$I = I_1 + I_2 + L - 2 \qquad \text{and} \qquad B = B_1 + B_2 - 2L + 2\ .$$

Hence

$$\begin{aligned} I + \frac{1}{2}B - 1 &= I_1 + I_2 + L - 2 + \frac{1}{2}B_1 + \frac{1}{2}B_2 - L + 1 - 1 \\ &= I_1 + \frac{1}{2}B_1 - 1 + I_2 + \frac{1}{2}B_2 - 1\ . \end{aligned}$$

This proves the claim. Therefore, we can triangulate $\mathcal{P}$, and it suffices to prove Pick's theorem for triangles. Moreover, by further triangulations, we may assume that there are no lattice points on the boundary of the triangle other than the vertices. To prove Pick's theorem for such triangles, embed them into rectangles, as shown in figure 1.1.

Again by additivity, we conclude that it suffices to prove Pick's theorem for rectangles and rectangular triangles, which have no lattice points on the hypothenuse, and whose other two sides are parallel to a coordinate axis. If these two sides have lengths a and b, respectively, we have

$$A = \frac{1}{2}ab \qquad \text{and} \qquad B = a + b + 1\ .$$

Furthermore, by thinking of the triangle as 'half' of a rectangle, we obtain

$$I = \frac{1}{2}(a-1)(b-1)\ .$$

(Here it is crucial that there are no points on the hypothenuse.) Pick's identity is now a straightforward consequence for these triangles. Finally, for a rectangle whose sides have length a and b, it is easy to see that

$$A = ab,\ I = (a-1)(b-1),\ B = 2a + 2b,$$

and Pick's theorem follows for rectangles, which finishes our proof. □

1.2 Ehrhart Theory

In which ways does Pick's theorem extend to higher dimensions, and to polytopes whose vertices are not on the lattice? To study rigorously the lattice point count in polytopes, Ehrhart ([Eh]) initiated in the 1960's (when he was a high school teacher) the very useful notion of lattice point enumeration in *dilated* polytopes. Let's start with some terminology.

A **convex polytope** is the convex hull of finitely many points in some real vector space. A **polytope** is the union of finitely many overlapping convex polytopes. Note that this implies that our polytopes are always compact. Equivalently, we can define a polytope to be the union of overlapping sets which are determined by a bounded intersection of halfspaces.

Next, we define the notion of a face of a polytope. We will do this for *convex* polytopes; the definitions extend very naturally to general polytopes. Given a convex polytope $\mathcal{P} \in \mathbb{R}^n$, we say that the linear inequality $\mathbf{a} \cdot \mathbf{x} \leq b$ is **valid** for $\mathcal{P}$ if it holds for all $\mathbf{x} \in \mathcal{P}$; here *cdot* denotes the usual scalar product in $\mathbb{R}^n$. A **face** of $\mathcal{P}$ is a set of the form $\mathcal{P} \cap \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a} \cdot \mathbf{x} = b\}$, where $\mathbf{a} \cdot \mathbf{x} \leq b$ is a valid inequality for $\mathcal{P}$. Note that both $\mathcal{P}$ itself and the empty set are faces of $\mathcal{P}$. The $(n-1)$-dimensional faces are called **facets**, the 1-dimensional faces **vertices** of $\mathcal{P}$. Ehrhart restricted himself, for reasons that will become obvious soon, to **rational polytopes**, that is, polytopes whose vertices have rational coordinates. For positive integers t, we define $t\mathcal{P} = \{tx : x \in \mathcal{P}\}$. This allows to make the following

Definition 1.1 *Let $\mathcal{P} \subset \mathbb{R}^n$ be a rational polytope, and t a positive integer. We denote the number of lattice points in the dilates of $\mathcal{P}$ and its interior by*

$$L(\overline{\mathcal{P}}, t) = \# \left(t\mathcal{P} \cap \mathbb{Z}^n \right) \qquad \textit{and} \qquad L(\mathcal{P}^\circ, t) = \# \left(t\mathcal{P}^\circ \cap \mathbb{Z}^n \right) \ ,$$

respectively.

Pick's Theorem, written in these terms, reads now

$$L(\mathcal{P}^\circ, 1) = A - \frac{1}{2} B + 1 \ ,$$

which holds for any two-dimensional **lattice polytope**, that is, whose vertices are on the integer lattice. It is not hard to modify our proof of Pick's theorem to arbitrary dilates of such a polytope:

$$L(\mathcal{P}^\circ, t) = At^2 - \frac{1}{2} Bt + 1 \ .$$

It is this kind of formula that we aim to achieve for more general polytopes. Two fundamental results will prove very helpful in this process. The first one is due to Ehrhart himself ([Eh]), and shows in what ways we can expect Pick's theorem to generalize. Before stating Ehrhart's theorem, we need the

Definition 1.2 *A* **quasipolynomial** *is an expression of the form*

$$c_n(t)\ t^n + \cdots + c_1(t)\ t + c_0(t) \ ,$$

where $c_0, \ldots, c_n$ are periodic functions in the integer variable t.

Theorem 1.2 (Ehrhart) *Let $\mathcal{P}$ be a rational polytope. Then $L(\overline{\mathcal{P}}, t)$ and $L(\mathcal{P}^\circ, t)$ are quasipolynomials in the integer variable t. The leading term of $L(\overline{\mathcal{P}}, t)$ is the volume of $\mathcal{P}$. Moreover, if $\mathcal{P}$ is a lattice polytope then $L(\overline{\mathcal{P}}, t)$ is a polynomial in t. In this case, the second leading term of $L(\overline{\mathcal{P}}, t)$ is the (relative) volume of the boundary of $\mathcal{P}$, normalized with respect to the sublattice on each facet of $\mathcal{P}$, and the constant term of $L(\overline{\mathcal{P}}, t)$ is the Euler characteristic of $\mathcal{P}$.* □

The **Euler characteristic** of an n-dimensional polytope $\mathcal{P}$ can be defined as

$$\chi(\mathcal{P}) = \sum_{k=0}^{n} (-1)^k f_k \ ,$$

where f_k denotes the number of k-dimensional faces of $\mathcal{P}$. We note that most of the polytopes we consider here will be convex, and hence have Euler characteristic 1; this is the content of the famous *Euler-Poincaré formula* ([Poi]).

The normalization for the contribution of a facet $\mathcal{F}$ to the second term of $L(\overline{\mathcal{P}}, t)$ can be visualized as follows: $\mathcal{F}$ lives on a hyperplane H. Now $H \cap Z^n$ forms an abelian group of rank $n-1$, that is, $H \cap Z^n \simeq Z^{n-1}$, via a bijective affine transformation ϕ. The contribution of $\mathcal{F}$ to the second term of $L(\overline{\mathcal{P}}, t)$ is the volume (in $\mathbb{R}^{n-1}$) of $\phi(\mathcal{F})$.

We will see the validity of Theorem 1.2 in all the polytopes we discuss here. Ehrhart also conjectured the following fundamental theorem, which establishes an algebraic connection between our two lattice point count operators. Its original proof is due to Macdonald ([Ma]). Recall that two subsets of $\mathbb{R}^n$ are **homeomorphic** if there exists a continuous bijection mapping one into the other, whose inverse is also continuous.

Theorem 1.3 (Ehrhart-Macdonald reciprocity law) *Suppose the rational polytope $\mathcal{P}$ is homeomorphic to an n-manifold. Then*

$$L(\mathcal{P}^\circ, -t) = (-1)^n L(\overline{\mathcal{P}}, t) \ .$$

□

In particular, this result holds for convex rational polytopes. We postpone a new proof, which will at the same time generalize this reciprocity law, to chapter 4.

Since Ehrhart's initiative, formulas for the coefficients of the lattice point count operators for rational polytopes have long been sought. It is interesting to note that the first formulas for such *Ehrhart quasipolynomials* came up as recently as 1993, in a paper by Pommersheim ([Pom]), who generalized a result of Mordell ([Mo]). Other recent work on lattice point enumeration in polytopes can be found in [Ba], [BV], [CS], [DR], [Gu], [KK], [KP]. In this thesis, we will present a new approach to this problem, which works in particular for a wide class of rational polytopes. We emphasize new connections to generalizations of Dedekind sums (which will be introduced in the next section), and applications of our formulas to the linear diophantine problem of Frobenius (chapter 3).

1.3 Dedekind Sums

According to Riemann's will, it was his wish that Dedekind should get Riemann's unpublished notes and manuscripts ([RG]). Among these was a discussion of the important function

$$\eta(z) = e^{\frac{\pi i z}{12}} \prod_{n \geq 1} \left(1 - e^{2\pi i n z}\right) ,$$

which Dedekind took up and eventually published in Riemann's collected works ([De]). Through the study of the transformation properties of η under $\mathrm{SL}_2(\mathbb{Z})$, he naturally arrived at the following expression.

Definition 1.3 *Let $((x))^\star$ be the sawtooth function defined by*

$$((x))^\star = \begin{cases} x - [x] - \frac{1}{2} & \text{if } x \notin \mathbb{Z} \\ 0 & \text{if } x \in \mathbb{Z} . \end{cases}$$

For two integers a and b, we define the **Dedekind sum** *as*

$$\mathfrak{s}(a,b) = \sum_{k \bmod b} \left(\left(\frac{ka}{b}\right)\right)^\star \left(\left(\frac{k}{b}\right)\right)^\star .$$

This expression has since appeared in various contexts in Number Theory, Combinatorics, and Topology. The classic introduction to the arithmetic properties of the Dedekind sum is [RG]. The most important of these, already proved by Dedekind ([De]), is the famous reciprocity law

Theorem 1.4 (Dedekind) *If a and b are relatively prime then*

$$\mathfrak{s}(a,b) + \mathfrak{s}(b,a) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{b} + \frac{1}{ab} + \frac{b}{a}\right) .$$

□

This reciprocity law is easily seen to be equivalent to the transformation law of the η-function ([De]). We note that, among other things, Theorem 1.4 allows us to compute $\mathfrak{s}(a,b)$ in polynomial time, similar in spirit to the Euclidean algorithm. This is due to the periodicity of $((x))^\star$: we can reduce a modulo b in $\mathfrak{s}(a,b)$. The Dedekind sum $\mathfrak{s}(a,b)$ has various generalizations, of which we introduce two here. The first one is due to Rademacher ([Ra]), who generalized sums introduced by Meyer ([Me]) and Dieter ([Di]):

Definition 1.4 *For $a, b \in \mathbb{Z}$, $x, y \in \mathbb{R}$, the* **Dedekind-Rademacher sum** *is defined by*

$$\mathfrak{s}(a,b;x,y) = \sum_{k \bmod b} \left(\left(\frac{(k+y)a}{b} + x\right)\right)^\star \left(\left(\frac{k+y}{b}\right)\right)^\star .$$

This sum posseses again a reciprocity law:

Theorem 1.5 (Rademacher) *If a and b are relatively prime and x and y are not both integers, then*

$$\mathfrak{s}(a,b;x,y)+\mathfrak{s}(b,a;y,x)=((x))^{\star}((y))^{\star}+\frac{1}{2}\left(\frac{a}{b}B_2(y)+\frac{1}{ab}B_2(ay+bx)+\frac{b}{a}B_2(x)\right) .$$

Here

$$B_2(x):=(x-[x])^2-(x-[x])+\frac{1}{6}$$

is the periodized second Bernoulli polynomial. □

If x and y are both integers, the Dedekind-Rademacher sum is simply the classical Dedekind sum, whose reciprocity law we already stated.

The second generalization of the Dedekind sum we mention here is due to Zagier ([Za2]). From topological considerations, he arrived naturally at expressions of the following kind:

Definition 1.5 *Let $a_1,\ldots,a_n$ be integers relatively prime to $a_0\in\mathbb{N}$. Define the* **higher-dimensional Dedekind sum** *as*

$$\mathfrak{s}(a_0;a_1,\ldots,a_n)=\frac{(-1)^{n/2}}{a_0}\sum_{k=1}^{a_0-1}\cot\frac{\pi ika_1}{a_0}\cdots\cot\frac{\pi ika_n}{a_0} .$$

This sum vanishes if n is odd. It is not hard to see ([RG]) that this indeed generalizes the classical Dedekind sum:

$$\mathfrak{s}(a,b)=\frac{1}{4b}\sum_{k \bmod b}\cot\frac{\pi ika}{b}\cot\frac{\pi ik}{b}=-\frac{1}{4}\mathfrak{s}(b;a,1) .$$

Again, there exists a reciprocity law for Zagier's sums:

Theorem 1.6 (Zagier) *If $a_0,\ldots,a_n$ are pairwise relatively prime positive integers then*

$$\sum_{j=0}^{n}\mathfrak{s}(a_j;a_0,\ldots,\hat{a_j},\ldots,a_n)=\phi(a_0,\ldots,a_n) .$$

Here ϕ is a rational function in $a_0,\ldots,a_n$, which can be expressed in terms of Hirzebruch L-functions ([Za2])*.* □

It should be mentioned that a version of the higher-dimensional Dedekind sums was already introduced by Carlitz ([Ca]) via sawtooth functions.

In the process of obtaining formulas for the lattice point count in various classes of polytopes, we will give geometric proofs of Dedekind's and Zagier's reciprocity laws, as well as a reciprocity law for Dedekind-Rademacher sums due to Gessel ([Ge]).

Chapter 2

Lattice Points in Rational Polytopes

The full beauty of the subject of generating functions emerges only from tuning in on both channels: the discrete and the continuous.
Herb Wilf ([Wi])

In this chapter we introduce a new method of computing formulas for the lattice point count in rational polytopes. We use generating functions whose coefficients are the lattice point counts of different dilates of the polytope. We present two ways of extracting this information from the generating function: *partial fractions* and the *residue theorem*. Both are inspired by works on generalized Dedekind sums, the first one by Gessel ([Ge]), the latter one by Zagier ([Za2]). In fact, the two ways are completely equivalent, since our generating functions are rational. We will illustrate both methods below; in section 2.1 we will use the residue theorem, in section 2.2 partial fractions.

2.1 The Mordell-tetrahedron in n Dimensions

We start with a tetrahedron which has integer vertices: let

$$\mathcal{P} = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n : \ x_k \geq 0, \ \sum_{k=1}^{n} \frac{x_k}{a_k} \leq 1 \right\} ,$$

where $a_1, \dots, a_n$ are positive integers. We present an elementary method for computing the Ehrhart polynomials $L(\overline{\mathcal{P}}, t)$ and $L(\mathcal{P}^\circ, t)$ using the residue theorem. We verify the Ehrhart-Macdonald reciprocity law for these n-dimensional tetrahedra. To illustrate our method, we compute the first nontrivial coefficient, c_{n-2}, of the Ehrhart polynomial.

2.1.1 Generating Functions and the Residue Theorem

Let us begin with $L(\overline{\mathcal{P}},t)$. We introduce the notation

$$A := a_1 \cdots a_n, \quad A_k := a_1 \cdots \hat{a}_k \cdots a_n,$$

where $\hat{a}_k$ means we omit the factor a_k. We can write

$$\begin{aligned} L(\overline{\mathcal{P}},t) &= \# \left\{ (m_1,\dots,m_n) \in \mathbb{Z}^n : \ m_k \geq 0, \ \sum_{k=1}^{n} \frac{m_k}{a_k} \leq t \right\} \\ &= \# \left\{ (m_1,\dots,m_n,m) \in \mathbb{Z}^{n+1} : \ m_k, m \geq 0, \ \sum_{k=1}^{n} m_k A_k + m = tA \right\}, \end{aligned}$$

so that we have cleared the denominators, and introduced a 'slack' variable m. Throughout this chapter, it is important to keep in mind that t is a positive integer. One can interpret $L(\overline{\mathcal{P}},t)$ as the Taylor coefficient of z^{tA} for the function

$$\begin{aligned} &\left(1 + z^{A_1} + z^{2A_1} + \dots\right)\left(1 + z^{A_2} + z^{2A_2} + \dots\right)\cdots \\ &\quad \cdot \left(1 + z^{A_n} + z^{2A_n} + \dots\right)\left(1 + z + z^2 + \dots\right) \\ &= \frac{1}{1 - z^{A_1}} \frac{1}{1 - z^{A_2}} \cdots \frac{1}{1 - z^{A_n}} \frac{1}{1 - z} \ . \end{aligned}$$

Equivalently,

$$L(\overline{\mathcal{P}},t) = \operatorname{Res}\left(\frac{z^{-tA-1}}{\left(1 - z^{A_1}\right)\left(1 - z^{A_2}\right)\cdots\left(1 - z^{A_n}\right)(1 - z)}, z = 0 \right).$$

It is convenient to change this function slightly; this residue is clearly equal to

$$\operatorname{Res}\left(\frac{z^{-tA} - 1}{\left(1 - z^{A_1}\right)\left(1 - z^{A_2}\right)\cdots\left(1 - z^{A_n}\right)(1 - z)\, z}, z = 0 \right) + 1.$$

This trick allows us to reduce the number of poles, and hence simplifies the computation. If this expression counts the number of lattice points in $t\mathcal{P}$, then all we have to do is compute the other residues of

$$f_{-t}(z) := \frac{z^{-tA} - 1}{\left(1 - z^{A_1}\right)\left(1 - z^{A_2}\right)\cdots\left(1 - z^{A_n}\right)(1 - z)\, z}$$

and use the residue theorem for the compact sphere $\mathbb{C} \cup \{\infty\}$. In this notation,

$$L(\overline{\mathcal{P}},t) = \operatorname{Res}\left(f_{-t}(z), z = 0\right) + 1 \ . \tag{2.1}$$

Alternatively, one could have expanded f_{-t} into partial fractions. We will illustrate this equivalent method in section 2.2.

The only poles of f_{-t} are at 0, 1 and the roots of unity in

$$\Omega := \left\{ z \in \mathbb{C} \setminus \{1\} : z^{\frac{A}{a_k a_j}} = 1, 1 \leq k < j \leq n \right\} \ .$$

Note that $\operatorname{Res}(f_{-t}, z = \infty) = 0$, so that the residue theorem gives us our first result:

Theorem 2.1

$$L(\overline{\mathcal{P}}, t) = 1 - \operatorname{Res}\left(f_{-t}(z), z = 1\right) - \sum_{\lambda \in \Omega} \operatorname{Res}\left(f_{-t}(z), z = \lambda\right)$$

□

The residue at $z = 1$ can be calculated easily:

$$\begin{aligned} \operatorname{Res}\left(f_{-t}(z), z = 1\right) &= \operatorname{Res}\left(e^z f_{-t}(e^z), z = 0\right) \\ &= \operatorname{Res}\left(\frac{e^{-tAz} - 1}{\left(1 - e^{A_1 z}\right)\left(1 - e^{A_2 z}\right)\cdots\left(1 - e^{A_n z}\right)\left(1 - e^z\right)}, z = 0\right). \end{aligned}$$

This enables us to use the Laurent expansion

$$\frac{1}{e^z - 1} = \frac{1}{z} \sum_{k \geq 0} \frac{B_k}{k!} z^k \ ,$$

where B_k denotes the k'th Bernoulli number.

To facilitate the computation in higher dimensions, one can use mathematics software such as `Maple`, `Mathematica`, or `Derive`. It is easy to see that $\operatorname{Res}(f_{-t}(z), z = 1)$ is a polynomial in t whose coefficients are rational expressions in $a_1, \ldots, a_n$. The first few are as follows:

For dimension $n = 2$:

$$-\frac{a_1 a_2}{2} t^2 - \frac{t}{2}(a_1 + a_2 + 1)$$

$n = 3$:

$$\begin{aligned} -\frac{a_1 a_2 a_3}{6} t^3 - \frac{t^2}{4}(a_1 a_2 + a_1 a_3 + a_2 a_3 + 1) - \frac{t}{4}\left(a_1 + a_2 + a_3 + \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}\right) \\ -\frac{t}{12}\left(\frac{a_1 a_2}{a_3} + \frac{a_1 a_3}{a_2} + \frac{a_2 a_3}{a_1} + \frac{1}{a_1 a_2 a_3}\right) \ . \end{aligned}$$

$n = 4$:

$$\begin{aligned} &-\frac{a_1 a_2 a_3 a_4}{24} t^4 - \frac{t^3}{12}(a_1 a_2 a_3 + a_1 a_2 a_4 + a_1 a_3 a_4 + a_2 a_3 a_4 + 1) \\ &\quad -\frac{t^2}{8}\left(a_1 a_2 + a_1 a_3 + a_1 a_4 + a_2 a_3 + a_2 a_4 + a_3 a_4 + \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4}\right) \\ &\quad -\frac{t^2}{24}\left(\frac{a_1 a_2 a_3}{a_4} + \frac{a_1 a_2 a_4}{a_3} + \frac{a_1 a_3 a_4}{a_2} + \frac{a_2 a_3 a_4}{a_1} + \frac{1}{a_1 a_2 a_3 a_4}\right) \\ &\quad -\frac{t}{8}\left(a_1 + a_2 + a_3 + a_4 + \frac{1}{a_1 a_2} + \frac{1}{a_1 a_3} + \frac{1}{a_1 a_4} + \frac{1}{a_2 a_3} + \frac{1}{a_2 a_4} + \frac{1}{a_3 a_4}\right) \\ &\quad -\frac{t}{24}\left(\frac{a_1 a_2}{a_3} + \frac{a_1 a_2}{a_4} + \frac{a_1 a_3}{a_2} + \frac{a_1 a_3}{a_4} + \frac{a_1 a_4}{a_2} + \frac{a_1 a_4}{a_3} + \frac{a_2 a_3}{a_1} + \frac{a_2 a_3}{a_4}\right. \\ &\qquad + \frac{a_2 a_4}{a_1} + \frac{a_2 a_4}{a_3} + \frac{a_3 a_4}{a_1} + \frac{a_3 a_4}{a_2} + \frac{1}{a_1^2} + \frac{1}{a_2^2} + \frac{1}{a_3^2} + \frac{1}{a_4^2} \\ &\qquad \left. + \frac{1}{a_1^2 a_2 a_3 a_4} + \frac{1}{a_1 a_2^2 a_3 a_4} + \frac{1}{a_1 a_2 a_3^2 a_4} + \frac{1}{a_1 a_2 a_3 a_4^2}\right) \ . \end{aligned}$$

The residues at the roots of unity in Ω are in general not as easy to compute. They give rise to Dedekind sums and their higher dimensional analogues, as we will illustrate below. There is, however, one feature we can read off from these residues immediately, the dependency on the dilation parameter t: as promised in the introduction, we have

Corollary 2.2 *$L(\overline{\mathcal{P}}, t)$ is a polynomial in t.*

With Corollary 2.4 below, this will also imply that $L(\mathcal{P}^\circ, t)$ is a polynomial.

Proof. Let $\lambda \in \Omega$ be a B'th root of unity, where B is the product of some of the a_k. Now express z^{-tA} in terms of its power series about $z = \lambda$. The coefficients of this power series involve various derivatives of z^{-tA}, evaluated at $z = \lambda$. Here we can introduce a change of variable: $z = w^{\frac{1}{B}} = \exp\left(\frac{1}{B}\log w\right)$, where a suitable branch of the logarithm is chosen such that $\exp\left(\frac{1}{B}\log(1)\right) = \lambda$. The terms depending on t in the power series of z^{-tA} consist therefore of derivatives of the function $z^{-tA/B}$, evaluated at $z = 1$. From this it is easy to see that the coefficients of the power series of z^{-tA} are polynomials in t. Finally, the fact that $L(\overline{\mathcal{P}}, t)$ is simply the sum of all of these residues gives the statement. □

We remark that this is *not* the simplest way to prove that $L(\overline{\mathcal{P}}, t)$ is a polynomial. In fact, we could have proved this fact right after introducing our generating function, without the use of residues. However, the proof given here ties in naturally with the residue methods introduced earlier.

For the computation of $L(\mathcal{P}^\circ, t)$ (the number of lattice points in the *interior* of our tetrahedron $t\mathcal{P}$), we write, similarly,

$$\begin{aligned} L(\mathcal{P}^\circ, t) &= \#\left\{(m_1, \ldots, m_n) \in \mathbb{Z}^n : \ m_k > 0, \ \sum_{k=1}^n \frac{m_k}{a_k} < t\right\} \\ &= \#\left\{(m_1, \ldots, m_n, m) \in \mathbb{Z}^{n+1} : \ m_k, m > 0, \ \sum_{k=1}^n m_k A_k + m = tA\right\}. \end{aligned}$$

Now $L(\mathcal{P}^\circ, t)$ can be interpreted as the Taylor coefficient of z^{tA} for the function

$$\begin{aligned} &\left(z^{A_1} + z^{2A_1} + \ldots\right) \cdots \left(z^{A_n} + z^{2A_n} + \ldots\right) \left(z + z^2 + \ldots\right) \\ &= \frac{z^{A_1}}{1 - z^{A_1}} \frac{z^{A_2}}{1 - z^{A_2}} \cdots \frac{z^{A_n}}{1 - z^{A_n}} \frac{z}{1 - z}, \end{aligned}$$

or equivalently as

$$\begin{aligned} &\operatorname{Res}\left(\frac{z^{A_1}}{1 - z^{A_1}} \frac{z^{A_2}}{1 - z^{A_2}} \cdots \frac{z^{A_n}}{1 - z^{A_n}} \frac{z}{1 - z} \, z^{-tA-1}, z = 0\right) \\ &= \operatorname{Res}\left(\frac{z^{A_1}}{1 - z^{A_1}} \frac{z^{A_2}}{1 - z^{A_2}} \cdots \frac{z^{A_n}}{1 - z^{A_n}} \frac{z}{1 - z} \frac{z^{-tA} - 1}{z}, z = 0\right) \\ &= \operatorname{Res}\left(\frac{-1}{z^2} \frac{1}{z^{A_1} - 1} \frac{1}{z^{A_2} - 1} \cdots \frac{1}{z^{A_n} - 1} \frac{1}{z - 1} \, z\left(z^{tA} - 1\right), z = \infty\right). \end{aligned}$$

Again we used the function

$$-\frac{z^{A_1}}{1-z^{A_1}}\,\frac{z^{A_2}}{1-z^{A_2}}\cdots\frac{z^{A_n}}{1-z^{A_n}}\,\frac{z}{1-z}\,\frac{1}{z}$$

with residue 0 at $z=0$ to cancel some of the poles. To be able to use the residue theorem, this time we have to consider the function

$$-\frac{1}{z^{A_1}-1}\,\frac{1}{z^{A_2}-1}\cdots\frac{1}{z^{A_n}-1}\,\frac{1}{z-1}\,\frac{z^{tA}-1}{z}=(-1)^n f_t(z)\ ,$$

so that

$$L(\mathcal{P}^\circ,t)=(-1)^n\ \mathrm{Res}\left(f_t(z),z=\infty\right). \tag{2.2}$$

The finite poles of f_t are at 0 (with residue -1), 1, and the roots of unity in Ω as before. This gives us, by the residue theorem,

Theorem 2.3

$$L(\mathcal{P}^\circ,t)=(-1)^n\left(1-\mathrm{Res}\left(f_t(z),z=1\right)-\sum_{\lambda\in\Omega}\mathrm{Res}\left(f_t(z),z=\lambda\right)\right)\ .$$

□

As an immediate consequence we get the first instance of the Ehrhart-Macdonald reciprocity law:

Corollary 2.4

$$L(\mathcal{P}^\circ,-t)=(-1)^n L(\overline{\mathcal{P}},t)\ .$$

Proof. Compare the statements of Theorems 2.1 and 2.3. □

2.1.2 The Ehrhart Coefficients

With a small modification of $f_t(z)$, we can actually derive a formula for each coefficient of the Ehrhart polynomial

$$L(\overline{\mathcal{P}},t)=c_nt^n+\cdots+c_0\ .$$

Consider the function

$$\begin{aligned} g_k(z) &:= \frac{\left(z^{-tA}-1\right)^k}{\left(1-z^{A_1}\right)\left(1-z^{A_2}\right)\cdots\left(1-z^{A_n}\right)(1-z)\,z} \\ &= \frac{\sum_{j=0}^{k}\binom{k}{j}z^{-tA(k-j)}(-1)^j}{\left(1-z^{A_1}\right)\left(1-z^{A_2}\right)\cdots\left(1-z^{A_n}\right)(1-z)\,z}. \end{aligned}$$

If we insert $-\sum_{j=0}^{k}\binom{k}{j}(-1)^j = 0$ in the numerator, this becomes

$$\begin{aligned} g_k(z) &= \sum_{j=0}^{k}\binom{k}{j}(-1)^j \frac{z^{-t(k-j)A}-1}{\left(1-z^{A_1}\right)\left(1-z^{A_2}\right)\cdots\left(1-z^{A_n}\right)(1-z)\,z} \\ &= \sum_{j=0}^{k-1}\binom{k}{j}(-1)^j f_{-t(k-j)}(z)\ . \end{aligned}$$

Recall that (2.1) gave us $L(\overline{\mathcal{P}},t) = \operatorname{Res}\left(f_{-t}(z), z=0\right)+1$. Using this relation, we obtain

$$\begin{aligned} \operatorname{Res}\left(g_k(z), z=0\right) &= \sum_{j=0}^{k-1}\binom{k}{j}(-1)^j \operatorname{Res}\left(f_{-t(k-j)}(z), z=0\right) \\ &= \sum_{j=0}^{k-1}\binom{k}{j}(-1)^j\Big(L\left(\overline{\mathcal{P}},(k-j)t\right)-1\Big) \\ &= \sum_{j=0}^{k-1}\binom{k}{j}(-1)^j L\left(\overline{\mathcal{P}},(k-j)t\right) \ + \ (-1)^k\ . \end{aligned}$$

We claim that this polynomial has no terms with exponent smaller than k:

Lemma 2.5 *Suppose $L(\overline{\mathcal{P}},t) = c_n t^n + \cdots + c_0$. Then for $1 \le k \le n$*

$$\operatorname{Res}\left(g_k(z), z=0\right) = k!\sum_{m=k}^{n} S(m,k)\ c_m\ t^m\ , \tag{2.3}$$

where $S(m,k)$ denotes the Stirling number of the second kind.

Proof. Suppose

$$\sum_{j=0}^{k-1}\binom{k}{j}(-1)^j L\left(\overline{\mathcal{P}},(k-j)t\right) = \sum_{m=0}^{n} b_{k,m}t^m\ , \tag{2.4}$$

so that for $m > 0$

$$b_{k,m} = \sum_{j=0}^{k-1}\binom{k}{j}(-1)^j c_m (k-j)^m = c_m \sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j} j^m\ .$$

The Stirling number of the second kind $S(m,k)$ is the number of partitions of an m-set into k subsets. We are interested in these numbers because ([St2])

$$S(m,k) = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j} j^m,$$

so that $b_{k,m} = c_m\ k!\ S(m,k)$ for $m > 0$. Some of the elementary properties of $S(m,k)$ are ([St2])

$$S(m,k) = 0 \quad \text{if } k > m \tag{2.5}$$
$$S(m,1) = 1 \tag{2.6}$$
$$S(m,m) = 1 \tag{2.7}$$
$$S(m,k) = k\ S(n-1,k) + S(n-1,k-1)\ .$$

By (2.5), we conclude that $b_{k,m} = 0$ for $1 \leq m < k$. The constant term in (2.4) is

$$b_{k,0} = \sum_{j=0}^{k-1} \binom{k}{j} (-1)^j c_0 = -c_0(-1)^k.$$

Since $c_0 = \chi(\mathcal{P}) = 1$ for our tetrahedron (in fact, $c_0 = 1$ for any convex lattice polytope [Eh]), (2.3) follows. □

The other poles of g_k are at 1 and the roots of unity in

$$\Omega_k := \left\{ z \in \mathbb{C} \setminus \{1\} : z^{\frac{A}{a_{j_1} \cdots a_{j_{k+1}}}} = 1, 1 \leq j_1 < j_2 < \cdots < j_{k+1} \leq n \right\}.$$

Note that as k gets larger, Ω_k gets smaller. That is, we have fewer residues to consider. This is consistent with the notion that the computational complexity increases with each additional coefficient, that is, the computation of c_k is more complicated than that of c_{k+1}. Using the residue theorem, we can rewrite (2.3) as

Theorem 2.6 *Suppose* $L(\overline{\mathcal{P}}, t) = c_n t^n + \cdots + c_0$. *Then for* $1 \leq k \leq n$

$$\sum_{m=k}^{n} S(m,k)\ c_m\ t^m = \frac{-1}{k!} \left(\operatorname{Res}\left(g_k(z), z=1\right) + \sum_{\lambda \in \Omega_k} \operatorname{Res}\left(g_k(z), z=\lambda\right) \right).$$

□

Remarks. 1. For $k = 1$, (2.6) yields a reformulation of Theorem 2.1.

2. The coefficients of $L(\mathcal{P}^\circ, t)$ are the same as those of $L(\overline{\mathcal{P}}, t)$, up to the sign: By Corollary 2.4,

$$L(\mathcal{P}^\circ, t) = c_n t^n - c_{n-1} t^{n-1} + \cdots + (-1)^n c_0\ .$$

3. $\operatorname{Res}(g_k(z), z = 1)$ can be computed as easily as before; the slightly more difficult task is to get the residues at the roots of unity (see also remark 2 following Theorem 2.1). However, with increasing k, we have to consider fewer of them, so that there is less to calculate. If we want to compute the Ehrhart coefficient c_m, we only have to consider the roots of unity in Ω_m. We can make this more precise: With (2.7), we obtain

Corollary 2.7 *For $m > 0$, c_m is the coefficient of t^m in*

$$\frac{-1}{m!}\left(\operatorname{Res}\left(g_m(z), z=1\right) + \sum_{\lambda\in\Omega_m} \operatorname{Res}\left(g_m(z), z=\lambda\right)\right) .$$

□

2.1.3 An Example

As an application, we will compute the first nontrivial Ehrhart coefficient c_{n-2} for the n-dimensional tetrahedron $\mathcal{P}$ ($n \geq 3$) under the additional assumption that $a_1, \ldots, a_n$ are pairwise relatively prime integers ≥ 2. This case was first explored by Pommersheim ([Pom]) in 1993.

Theorem 2.8 *Under the above assumptions,*

$$c_{n-2} = \frac{1}{(n-2)!}\,\Big(C_n - \mathfrak{s}(A_1, a_1) - \cdots - \mathfrak{s}(A_n, a_n)\Big),$$

where $\mathfrak{s}(a, b)$ denotes the Dedekind sum, and

$$C_n := \frac{1}{4}\left(n + A_{1,2} + \cdots + A_{n-1,n}\right) + \frac{1}{12}\left(\frac{1}{A} + \frac{A_1}{a_1} + \cdots + \frac{A_n}{a_n}\right) .$$

Here $A_{j,k}$ denotes $a_1 \cdots \hat{a}_j \cdots \hat{a}_k \cdots a_n$.

Proof. We have to consider

$$g_{n-2}(z) = \frac{\left(z^{-tA}-1\right)^{n-2}}{\left(1-z^{A_1}\right)\left(1-z^{A_2}\right)\cdots\left(1-z^{A_n}\right)(1-z)\,z} .$$

Because $a_1, \ldots, a_n$ are pairwise relatively prime, g_{n-2} has *simple* poles at all the $a_1, \ldots, a_n$'th roots of unity. Let $\lambda^{a_1} = 1 \neq \lambda$. Then

$$\operatorname{Res}\left(g_{n-2}(z), z=\lambda\right) = \frac{1}{\left(1-\lambda^{A_1}\right)(1-\lambda)\,\lambda}\operatorname{Res}\left(\frac{\left(z^{-tA}-1\right)^{n-2}}{\left(1-z^{A_2}\right)\cdots\left(1-z^{A_n}\right)}, z=\lambda\right) .$$

Using the methods that allowed us to arrive at Corollary 2.2, we make a change of variables $z = w^{1/a_1} = \exp\left(\frac{1}{a_1}\log w\right)$, where we choose a suitable branch of the logarithm such that $\exp\left(\frac{1}{a_1}\log(1)\right) = \lambda$. We thus obtain

$$\operatorname{Res}\left(g_{n-2}(z), z=\lambda\right) = \frac{1}{\left(1-\lambda^{A_1}\right)(1-\lambda)\,\lambda}\,\frac{\lambda}{a_1}\operatorname{Res}\left(\frac{\left(w^{-tB}-1\right)^{n-2}}{\left(1-w^{B_2}\right)\cdots\left(1-w^{B_n}\right)}, w=1\right) ,$$

where $B := a_2 \cdots a_n,\ B_k := a_2 \cdots \hat{a}_k \cdots a_n$. We claim that

$$\operatorname{Res}\left(\frac{\left(z^{-tB}-1\right)^{n-2}}{\left(1-z^{B_2}\right)\cdots\left(1-z^{B_n}\right)}, z=1 \right) = -t^{n-2}.$$

To prove this, first note that

$$\left(z^{-tB}-1\right)^{n-2} = (-tB)^{n-2}(z-1)^{n-2} + O\left((z-1)^{n-1}\right).$$

Now for $m \in \mathbb{N}$,

$$\operatorname{Res}\left(\frac{1}{1-z^m}, z=1 \right) = \lim_{z\to 1} \frac{z-1}{1-z^m} = -\frac{1}{m}\ .$$

Putting all of this together, we obtain

$$\operatorname{Res}\left(\frac{\left(z^{-tB}-1\right)^{n-2}}{\left(1-z^{B_2}\right)\cdots\left(1-z^{B_n}\right)}, z=1 \right) = \frac{(-tB)^{n-2}}{(-B_2)\cdots(-B_n)} = -\frac{t^{n-2}a_2^{n-2}\cdots a_n^{n-2}}{a_2^{n-2}\cdots a_n^{n-2}}$$
$$= -t^{n-2},$$

as desired. Therefore

$$\operatorname{Res}\left(g_{n-2}(z), z=\lambda\right) = \frac{-t^{n-2}}{a_1\left(1-\lambda^{A_1}\right)(1-\lambda)}\ .$$

Adding up all the a_1'th roots of unity $\neq 1$, we get

$$\sum_{\lambda^{a_1}=1\neq\lambda} \operatorname{Res}\left(g_{n-2}(z), z=\lambda\right) = \frac{-t^{n-2}}{a_1} \sum_{\lambda^{a_1}=1\neq\lambda} \frac{1}{\left(1-\lambda^{A_1}\right)(1-\lambda)}$$
$$= \frac{-t^{n-2}}{a_1} \sum_{k=1}^{a_1-1} \frac{1}{\left(1-\xi^{kA_1}\right)\left(1-\xi^k\right)},$$

where ξ is a primitive a_1'th root of unity. This finite sum is practically a Dedekind sum:

$$\frac{1}{a_1}\sum_{k=1}^{a_1-1} \frac{1}{\left(1-\xi^{kA_1}\right)\left(1-\xi^k\right)} = \frac{1}{4a_1}\sum_{k=1}^{a_1-1}\left(1+\frac{1+\xi^{kA_1}}{1-\xi^{kA_1}}\right)\left(1+\frac{1+\xi^k}{1-\xi^k}\right)$$
$$= \frac{1}{4a_1}(a_1-1) - \frac{i}{4a_1}\sum_{k=1}^{a_1-1}\left(\cot\frac{\pi k A_1}{a_1} + \cot\frac{\pi k}{a_1}\right) - \frac{1}{4a_1}\sum_{k=1}^{a_1-1}\cot\frac{\pi k A_1}{a_1}\cot\frac{\pi k}{a_1}$$
$$= \frac{1}{4} - \frac{1}{4a_1} - \mathfrak{s}(A_1, a_1).$$

The imaginary terms disappear here, since the sum on the left hand side and $\mathfrak{s}(A_1, a_1)$ are rational: Both are elements of the cyclotomic field of a_1'th roots of unity $\mathbb{Q}\left(e^{\frac{2\pi i}{a_1}}\right)$, and invariant under all Galois transformations of this field.

Hence we obtain

$$\sum_{\lambda^{a_1}=1\neq\lambda} \operatorname{Res}\left(g_{n-2}(z), z=\lambda\right) = -t^{n-2}\left(\frac{1}{4} - \frac{1}{4a_1} - \mathfrak{s}(A_1, a_1)\right).$$

We get similar expressions for the residues at the other roots of unity, so that Corollary 2.7 gives us for $n \geq 3$

$$c_{n-2} = \frac{1}{(n-2)!}\left(\frac{n}{4} - C - \frac{1}{4}\left(\frac{1}{a_1} + \cdots + \frac{1}{a_n}\right) \right. \left. - \mathfrak{s}(A_1, a_1) - \cdots - \mathfrak{s}(A_n, a_n)\right), \tag{2.8}$$

where C is the coefficient of t^{n-2} of $\operatorname{Res}\left(g_{n-2}(z), z=1\right)$. We can actually obtain a closed form for C: As before,

$$\begin{aligned} \operatorname{Res}\left(g_{n-2}(z), z=1\right) &= \operatorname{Res}\left(e^z g_{n-2}(e^z), z=0\right) \\ &= \operatorname{Res}\left(\frac{\left(e^{-tAz}-1\right)^{n-2}}{\left(1-e^{A_1 z}\right)\left(1-e^{A_2 z}\right)\cdots\left(1-e^{A_n z}\right)\left(1-e^z\right)}, z=0\right). \end{aligned}$$

Now with

$$\left(e^{-tAz}-1\right)^{n-2} = (-tAz)^{n-2} + O\left((tz)^{n-1}\right)$$

and

$$\frac{1}{1-e^z} = -z^{-1} + \frac{1}{2} - \frac{1}{12}z + O\left(z^3\right),$$

the coefficient of t^{n-2} of $\operatorname{Res}\left(g_{n-2}(z), z=1\right)$ turns out to be

$$\begin{aligned} C = (-A)^{n-2} &\left[\frac{1}{12}\left(\frac{(-1)^{n+1}}{A_1\cdots A_n} + \frac{(-1)^{n+1}A_1}{A_2\cdots A_n} + \cdots + \frac{(-1)^{n+1}A_n}{A_1\cdots A_{n-1}}\right) + \right. \\ &+ \frac{1}{4}\left(\frac{(-1)^{n-1}}{A_2\cdots A_n} + \cdots + \frac{(-1)^{n-1}}{A_1\cdots A_{n-1}} + \right. \\ &\left.\left. + \frac{(-1)^{n-1}}{A_3\cdots A_n} + \frac{(-1)^{n-1}}{A_2 A_4\cdots A_n} + \cdots + \frac{(-1)^{n-1}}{A_1\cdots A_{n-2}}\right)\right] \\ = -\frac{1}{12}&\left(\frac{1}{A} + \frac{A_1}{a_1} + \cdots + \frac{A_n}{a_n}\right) - \frac{1}{4}\left(\frac{1}{a_1} + \cdots + \frac{1}{a_n} + A_{1,2} + \cdots + A_{n-1,n}\right). \end{aligned}$$

Substituting this into (2.8) yields the statement. □

The other Ehrhart-coefficients for this tetrahedron can be derived in a similar fashion, although the computation gets more and more complicated, as noted in the previous section. The coefficient c_k contains information about the k-skeleton of our polytope.

2.2 Rational Polygons

We turn now to the first interesting case of polytopes with *rational* vertices, namely of dimension 2. We give explicit, polynomial-time computable (in the logarithm of the coordinates of the vertices) formulas for the number of integer points in any two-dimensional rational polytope and its integral dilations. Our formulas bear new connections between Ehrhart theory and the Dedekind-Rademacher sum $\mathfrak{s}(a, b; x, y)$ introduced in the first chapter.

2.2.1 Generating Functions and Partial Fractions

Since we can triangulate any polytope, it suffices to consider rational triangles. We can further simplify the picture by embedding an arbitrary rational triangle in a rational rectangle, a fact we already used to prove Pick's theorem in the first chapter. Since rectangles are easy to deal with, the problem reduces to finding a formula for a right-angled rational triangle.

Such a rectangular triangle $\mathcal{T}$ is given as a subset of $\mathbb{R}^2$ consisting of all points (x, y) satisfying

$$x \geq \frac{a}{d},\ y \geq \frac{b}{d},\ ex + fy \leq r$$

for some integers a, b, d, e, f, r with $ea + fb \leq rd$. Because the lattice point count is invariant under horizontal and vertical integer translation and under flipping about x- or y-axis, we may assume that $a, b, d, e, f, r \geq 0$ and $a, b < d$. Let's further factor out the greatest common divisor c of e and f, so that $e = cp$ and $f = cq$, where p and q are relatively prime. Hence

$$\mathcal{T} = \left\{ (x, y) \in \mathbb{R}^2 :\ x \geq \frac{a}{d},\ y \geq \frac{b}{d},\ cpx + cqy \leq r \right\} . \tag{2.9}$$

To derive a formula for $L\left(\overline{\mathcal{T}}, t\right)$ we interpret the lattice point enumerator, as in the previous section,

$$L\left(\overline{\mathcal{T}}, t\right) = \#\left\{ (m, n) \in \mathbb{Z}^2 :\ m \geq \frac{ta}{d},\ n \geq \frac{tb}{d},\ cpm + cqn \leq tr \right\}$$

as the Taylor coefficient of z^{tr} of the function

$$\begin{aligned}
&\left(\sum_{m \geq \left[\frac{ta-1}{d}\right]+1} z^{cpm} \right) \left(\sum_{n \geq \left[\frac{tb-1}{d}\right]+1} z^{cqn} \right) \left(\sum_{k \geq 0} z^k \right) \\
&= \frac{z^{\left(\left[\frac{ta-1}{d}\right]+1\right)cp}}{1 - z^{cp}} \frac{z^{\left(\left[\frac{tb-1}{d}\right]+1\right)cq}}{1 - z^{cq}} \frac{1}{1-z} \\
&= \frac{z^{u+v}}{\left(1 - z^{cp}\right)\left(1 - z^{cq}\right)(1 - z)} ,
\end{aligned} \tag{2.10}$$

where we introduced, for ease of notation,

$$u := \left(\left[\frac{ta-1}{d} \right] + 1 \right) cp \qquad \text{and} \qquad v := \left(\left[\frac{tb-1}{d} \right] + 1 \right) cq . \tag{2.11}$$

Again it is crucial that t is a positive integer. We could now shift the Taylor coefficient we are interested in to a residue and use the methods of the previous section. This time, we will use a partial fraction approach, which is completely equivalent, since our generating function is rational. We will show

Theorem 2.9 *For the rectangular rational triangle $\mathcal{T}$ given by* (2.9),

$$\begin{aligned} L\left(\overline{\mathcal{T}},t\right) = &\frac{1}{2c^2pq}\left(tr-u-v\right)^2 + \frac{1}{2}\left(tr-u-v\right)\left(\frac{1}{cp}+\frac{1}{cq}+\frac{1}{c^2pq}\right) \\ &+\frac{1}{4}\left(1+\frac{1}{cp}+\frac{1}{cq}\right)+\frac{1}{12}\left(\frac{p}{q}+\frac{q}{p}+\frac{1}{c^2pq}\right) \\ &+\left(\frac{1}{2cp}+\frac{1}{2cq}-\frac{u+v-tr}{c^2pq}\right)\sum_{\lambda^c=1\neq\lambda}\frac{\lambda^{-tr}}{1-\lambda}-\frac{1}{c^2pq}\sum_{\lambda^c=1\neq\lambda}\frac{\lambda^{-tr+1}}{(1-\lambda)^2} \\ &+\frac{1}{cp}\sum_{\lambda^{cp}=1\neq\lambda^c}\frac{\lambda^{v-tr}}{\left(1-\lambda^{cq}\right)\left(1-\lambda\right)}+\frac{1}{cq}\sum_{\lambda^{cq}=1\neq\lambda^c}\frac{\lambda^{u-tr}}{\left(1-\lambda^{cp}\right)\left(1-\lambda\right)}\ , \end{aligned}$$

where u and v are given by (2.11).

It will be useful to have the Laurent expansion of the factors of our generating function. The following lemma will provide a bridge between the residue method and the partial fraction method.

Lemma 2.10 *Let a,b be positive integers, and $\lambda^a = 1$. Then*

$$\frac{1}{1-z^{ab}} = -\frac{\lambda}{ab}(z-\lambda)^{-1}+\frac{ab-1}{2ab}+O(z-\lambda)\ .$$

Proof. First,

$$\operatorname{Res}\left(\frac{1}{1-z^{ab}},z=\lambda\right)=\lim_{z\to\lambda}\frac{z-\lambda}{1-z^{ab}}=-\frac{\lambda}{ab}\ .$$

For $ab=1$, the statement is trivial, so we may assume $ab\geq 2$. Then the constant term of the Laurent series of $\frac{1}{1-z^{ab}}$ can be computed as

$$\begin{aligned} \lim_{z\to\lambda}\left(\frac{1}{1-z^{ab}}+\frac{\lambda}{ab(z-\lambda)}\right) &= \lim_{z\to\lambda}\frac{ab(z-\lambda)+\lambda\left(1-z^{ab}\right)}{ab(z-\lambda)\left(1-z^{ab}\right)} \\ &= \lim_{z\to\lambda}\frac{ab-ab\lambda z^{ab-1}}{ab\left(1-z^{ab}-(z-\lambda)abz^{ab-1}\right)} \\ &= \lim_{z\to\lambda}\frac{-\lambda(ab-1)z^{ab-2}}{-2abz^{ab-1}-(z-\lambda)ab(ab-1)z^{ab-2}}=\frac{ab-1}{2ab}\ . \end{aligned}$$

Proof of Theorem 2.9. To make life easier, we translate the coefficient of z^{tr} of our generating function, which yields the lattice point count, to the constant coefficient of the function

$$\frac{z^{u+v-tr}}{\left(1-z^{cp}\right)\left(1-z^{cq}\right)\left(1-z\right)}\ . \tag{2.12}$$

By expanding (2.12) into partial fractions

$$\frac{z^{u+v-tr}}{(1-z^{cp})(1-z^{cq})(1-z)} = \sum_{\lambda^{cp}=1\neq\lambda^c} \frac{A_\lambda}{z-\lambda} + \sum_{\lambda^{cq}=1\neq\lambda^c} \frac{B_\lambda}{z-\lambda}$$
$$+ \sum_{\lambda^c=1\neq\lambda} \left(\frac{C_\lambda}{z-\lambda} + \frac{D_\lambda}{(z-\lambda)^2} \right) + \sum_{k=1}^{3} \frac{E_k}{(z-1)^k} + \sum_{k=1}^{tr-u-v} \frac{F_k}{z^k} ,$$

we can compute $L\left(\overline{\mathcal{T}}, t\right)$ as the constant coefficient of the right-hand side:

$$L\left(\overline{\mathcal{T}}, t\right) = -\sum_{\lambda^{cp}=1\neq\lambda^c} \frac{A_\lambda}{\lambda} - \sum_{\lambda^{cq}=1\neq\lambda^c} \frac{B_\lambda}{\lambda} + \sum_{\lambda^c=1\neq\lambda} \left(-\frac{C_\lambda}{\lambda} + \frac{D_\lambda}{\lambda^2} \right)$$
$$-E_1 + E_2 - E_3 . \tag{2.13}$$

The computation of the coefficients A_λ for $\lambda^{cp} = 1 \neq \lambda^c$ is straightforward:

$$A_\lambda = \lim_{z\to\lambda} \frac{(z-\lambda)z^{u+v-tr}}{(1-z^{cp})(1-z^{cq})(1-z)} = \frac{\lambda^{v-tr}}{(1-\lambda^{cq})(1-\lambda)} \lim_{z\to\lambda} \frac{(z-\lambda)}{1-z^{cp}}$$
$$= -\frac{\lambda^{v-tr+1}}{cp\,(1-\lambda^{cq})(1-\lambda)} .$$

Similarly, we obtain, for the cq'th roots of unity $\lambda^{cq} = 1 \neq \lambda^c$,

$$B_\lambda = -\frac{\lambda^{u-tr+1}}{cq\,(1-\lambda^{cp})(1-\lambda)} .$$

The coefficients D_λ and C_λ are the two leading coefficients of the Laurent series of (2.12) about a nontrivial c'th root of unity λ. By Lemma 2.10, they are easily seen to be

$$D_\lambda = \frac{\lambda^{-tr+2}}{c^2pq(1-\lambda)}$$

and

$$C_\lambda = \left(-\frac{1}{2cp} - \frac{1}{2cq} + \frac{u+v-tr+1}{c^2pq} \right) \frac{\lambda^{-tr+1}}{1-\lambda} + \frac{\lambda^{-tr+2}}{c^2pq(1-\lambda)^2} .$$

Finally, we obtain the coefficients E_k from the Laurent series of (2.12) about $z = 1$ (by hand or, preferably, using a computer algebra system) as

$$E_3 = -\frac{1}{c^2pq} , \qquad E_2 = -\frac{u+v-tr+1}{c^2pq} + \frac{1}{2cp} + \frac{1}{2cq} ,$$

and

$$E_1 = -\frac{(u+v-tr)^2}{2c^2pq} + \frac{u+v-tr}{2} \left(-\frac{1}{c^2pq} + \frac{1}{cp} + \frac{1}{cq} \right)$$
$$+\frac{1}{4} \left(\frac{1}{cp} + \frac{1}{cq} - 1 \right) - \frac{1}{12} \left(\frac{p}{q} + \frac{1}{c^2pq} + \frac{q}{p} \right) .$$

Substituting all of these expressions into (2.13) yields the statement. □

In the following section, we will further analyze the finite sums appearing in the lattice point count operators; consequently, we will be able to make statements about their computational complexity.

2.2.2 Using the Dedekind-Rademacher Sums as Building Blocks

We will now take a closer look at the finite sums over roots of unity appearing in Theorem 2.9, namely,

$$\frac{1}{cp} \sum_{\lambda^{cp}=1\neq\lambda^c} \frac{\lambda^w}{(1-\lambda^{cq})(1-\lambda)}$$

for some integers c, p, q, w, where p and q are relatively prime. Viewing this as a finite Fourier series in w suggests the use of the well-known convolution theorem for finite Fourier series (see, for example, [Te]):

Theorem 2.11 *Let* $f(t) = \frac{1}{N}\sum_{\lambda^N=1} a_\lambda \lambda^t$ *and* $g(t) = \frac{1}{N}\sum_{\lambda^N=1} b_\lambda \lambda^t$*. Then*

$$\frac{1}{N} \sum_{\lambda^N=1} a_\lambda b_\lambda \lambda^t = \sum_{m=0}^{N-1} f(t-m) g(m) \ .$$

□

We first define the sawtooth function

$$((x)) := x - [x] - 1/2 \ ,$$

which differs from the one appearing in the classical Dedekind sum only at the integers. The reason for introducing this slightly modified sawtooth function is its natural appearance in our formulas.

The key ingredient to be able to apply the convolution theorem to our case is

Lemma 2.12 *For* $p \in \mathbb{N}, t \in \mathbb{Z}$*,*

$$\frac{1}{p} \sum_{\lambda^p=1\neq\lambda} \frac{\lambda^t}{\lambda - 1} = \left(\left(\frac{-t}{p}\right)\right) + \frac{1}{2p} \ .$$

This lemma is well-known (see, for example, [RG]), however, for sake of completeness we give a short proof based on the residue theorem method of section 2.1.1.

Proof. Consider the interval $\mathcal{I} := [0, \frac{1}{p}]$, viewed as a one-dimensional polytope. Then the lattice point count in the dilated interval is clearly

$$L\left(\overline{\mathcal{I}}, t\right) = \left[\frac{t}{p}\right] + 1 \ . \tag{2.14}$$

On the other hand, we can write this number, by applying the ideas in section 2.1.1, as

$$L\left(\overline{\mathcal{I}}, t\right) = \operatorname{Res}\left(\frac{z^{-t-1}}{(1-z^p)(1-z)}, z = 0\right) \ .$$

Equivalently, we could expand this generating function into partial fractions. Using the residue theorem, this can be rewritten as

$$L\left(\overline{\mathcal{I}},t\right) = \frac{t}{p} + \frac{1}{2p} + \frac{1}{2} - \frac{1}{p} \sum_{\lambda^p = 1 \neq \lambda} \frac{\lambda^{-t}}{\lambda - 1} \ . \tag{2.15}$$

Comparing (2.14) with (2.15) yields the statement. □

Corollary 2.13 *For $c, p, q, t \in \mathbb{Z}, (p,q) = 1$,*

$$\frac{1}{cp} \sum_{\lambda^{cp} = 1 \neq \lambda^c} \frac{\lambda^t}{1 - \lambda^{cq}} = \begin{cases} -\left(\left(\frac{-q^{-1}t}{cp}\right)\right) - \frac{1}{2p} & \text{if } c|t \\ 0 & \text{else.} \end{cases}$$

Here, $qq^{-1} \equiv 1 \mod p$.

Proof. If $c|t$, write $t = cw$ to obtain

$$\frac{1}{cp} \sum_{\lambda^{cp} = 1 \neq \lambda^c} \frac{\lambda^t}{1 - \lambda^{cq}} = \frac{1}{cp} \sum_{\lambda^{cp} = 1 \neq \lambda^c} \frac{\lambda^{cw}}{1 - \lambda^{cq}} = \frac{1}{p} \sum_{\lambda^p = 1 \neq \lambda} \frac{\lambda^w}{1 - \lambda^q} = \frac{1}{p} \sum_{\lambda^p = 1 \neq \lambda} \frac{\lambda^{q^{-1}w}}{1 - \lambda}$$
$$\stackrel{(\star)}{=} -\left(\left(\frac{-q^{-1}w}{p}\right)\right) - \frac{1}{2p} = -\left(\left(\frac{-q^{-1}t}{cp}\right)\right) - \frac{1}{2p} \ .$$

Here, $(\star)$ follows from Lemma 2.12. If c does not divide t, let $\xi = e^{2\pi i/cp}$. Then

$$\frac{1}{cp} \sum_{\lambda^{cp} = 1 \neq \lambda^c} \frac{\lambda^t}{1 - \lambda^{cq}} = \frac{1}{cp} \sum_{m=1}^{p-1} \sum_{n=0}^{c-1} \frac{\xi^{(mc+np)t}}{1 - \xi^{(mc+np)cq}} = \frac{1}{cp} \sum_{n=0}^{c-1} \xi^{npt} \sum_{m=1}^{p-1} \frac{\xi^{mct}}{1 - \xi^{mc^2q}} = 0 \ .$$

□

Corollary 2.14 *For $c, p, q, t \in \mathbb{Z}, (p,q) = 1$,*

$$\frac{1}{cp} \sum_{\lambda^{cp} = 1 \neq \lambda^c} \frac{\lambda^{-t}}{(1 - \lambda^{cq})(1 - \lambda)} = -\mathfrak{s}\left(q, p; \frac{t}{cp}, 0\right) - \frac{1}{2}\left(\left(\frac{t}{cp}\right)\right) + \frac{1}{2p}\left(\left(\frac{t}{c}\right)\right) \ .$$

Proof. We will repeatedly use the periodicity of the sawtooth function. One consequence is, for $p \in \mathbb{Z}, x \in \mathbb{R}$,

$$\sum_{m=0}^{p-1} \left(\left(\frac{m+x}{p}\right)\right) = ((x)) \ , \tag{2.16}$$

the proof of which is left as an exercise ([RG]). Now by Lemma 2.12,

$$\frac{1}{cp} \sum_{\lambda^{cp} = 1 \neq \lambda^c} \frac{\lambda^t}{(1 - \lambda)} = \frac{1}{cp} \sum_{\lambda^{cp} = 1 \neq \lambda} \frac{\lambda^t}{(1 - \lambda)} - \frac{1}{cp} \sum_{\lambda^c = 1 \neq \lambda} \frac{\lambda^t}{(1 - \lambda)}$$
$$= -\left(\left(\frac{-t}{cp}\right)\right) - \frac{1}{2cp} - \frac{1}{p}\left(-\left(\left(\frac{-t}{c}\right)\right) - \frac{1}{2c}\right)$$
$$= -\left(\left(\frac{-t}{cp}\right)\right) + \frac{1}{p}\left(\left(\frac{-t}{c}\right)\right) \ .$$

Finally we use the Convolution Theorem 2.11 and Corollary 2.13 to obtain

$$\begin{aligned}
&\frac{1}{cp} \sum_{\lambda^{cp}=1\neq\lambda^{c}} \frac{\lambda^{t}}{(1-\lambda^{cq})(1-\lambda)} = \\
&= \sum_{\substack{m=0 \\ c|m}}^{cp-1} \left(- \left(\left(\frac{-q^{-1}m}{cp} \right) \right) - \frac{1}{2p} \right) \left(- \left(\left(\frac{-(t-m)}{cp} \right) \right) + \frac{1}{p} \left(\left(\frac{-(t-m)}{c} \right) \right) \right) \\
&= \sum_{k=0}^{p-1} \left(\left(\frac{-q^{-1}k}{p} \right) \right) \left(\left(\frac{k}{p} - \frac{t}{cp} \right) \right) - \frac{1}{p} \sum_{k=0}^{p-1} \left(\left(\frac{-q^{-1}k}{p} \right) \right) \left(\left(\frac{-t}{c} \right) \right) \\
&\quad + \frac{1}{2p} \sum_{k=0}^{p-1} \left(\left(\frac{k}{p} - \frac{t}{cp} \right) \right) - \frac{1}{2p^2} \sum_{k=0}^{p-1} \left(\left(\frac{-t}{c} \right) \right) \\
&\stackrel{(2.16)}{=} \sum_{k=0}^{p-1} \left(\left(\frac{-k}{p} \right) \right) \left(\left(\frac{qk}{p} - \frac{t}{cp} \right) \right) + \frac{1}{2p} \left(\left(\frac{-t}{c} \right) \right) + \frac{1}{2p} \left(\left(\frac{-t}{c} \right) \right) - \frac{1}{2p} \left(\left(\frac{-t}{c} \right) \right) \\
&\stackrel{(2.16)}{=} -\mathfrak{s}\left(q, p; \frac{-t}{cp}, 0\right) - \frac{1}{2} \left(\left(\frac{-t}{cp} \right) \right) + \frac{1}{2p} \left(\left(\frac{-t}{c} \right) \right) .
\end{aligned}$$

In the last step, we used

$$\mathfrak{s}(a, b; x, 0) = \sum_{k=0}^{p-1} \left(\left(\frac{ka}{b} + x \right) \right) \left(\left(\frac{k}{b} \right) \right) + \frac{1}{2}((x)) . \tag{2.17}$$

□

One of the Dedekind-Rademacher sums appearing in Theorem 2.9 actually turns out to be of an even simpler form. To show this, we first need to rewrite Theorem 2.9 for the special case where $\mathcal{T}$ has the origin as a vertex:

Theorem 2.15 *For the rectangular rational triangle $\mathcal{T}$ given by* (2.9) *with $a = b = 0$, $c = r = 1$, and p and q relatively prime,*

$$\begin{aligned}
L\left(\overline{\mathcal{T}}, t\right) = &\frac{t^2}{2pq} + \frac{t}{2}\left(\frac{1}{p} + \frac{1}{q} + \frac{1}{pq}\right) + \frac{1}{4} + \frac{1}{12}\left(\frac{p}{q} + \frac{q}{p} + \frac{1}{pq}\right) \\
&-\mathfrak{s}\left(q, p; \frac{t}{p}, 0\right) - \mathfrak{s}\left(p, q; \frac{t}{q}, 0\right) - \frac{1}{2}\left(\left(\frac{t}{p}\right)\right) - \frac{1}{2}\left(\left(\frac{t}{q}\right)\right) .
\end{aligned}$$

Proof. Theorem 2.9 gives for this special case

$$\begin{aligned}
L\left(\overline{\mathcal{T}}, t\right) = &\frac{t^2}{2pq} + \frac{t}{2}\left(\frac{1}{p} + \frac{1}{q} + \frac{1}{pq}\right) + \frac{1}{4}\left(1 + \frac{1}{p} + \frac{1}{q}\right) + \frac{1}{12}\left(\frac{p}{q} + \frac{q}{p} + \frac{1}{pq}\right) \\
&+\frac{1}{p} \sum_{\lambda^{p}=1\neq\lambda} \frac{\lambda^{-t}}{(1-\lambda^{q})(1-\lambda)} + \frac{1}{q} \sum_{\mu^{q}=1\neq\mu} \frac{\mu^{-t}}{(1-\mu^{p})(1-\mu)} .
\end{aligned}$$

The statement now follows from Corollary 2.14. □

We use this Theorem to show

Lemma 2.16 *For $p, t \in \mathbb{Z}$,*

$$\mathfrak{s}\left(1, p; \frac{t}{p}, 0\right) = \sum_{k=0}^{p-1} \left(\left(\frac{k+t}{p}\right)\right) \left(\left(\frac{k}{p}\right)\right) = -\frac{p}{24} + \frac{1}{6p} + \frac{p}{2}\left(\left(\frac{t}{p}\right)\right)^2 + \frac{1}{2}\left(\left(\frac{t}{p}\right)\right) .$$

Proof. Consider the triangle $\Delta := \left\{(x, y) \in \mathbb{R}^2 \ : \ x + py \le 1\right\}$ and its integer dilates. By summing over vertical line segments in the triangle, we obtain

$$\begin{aligned} L(\Delta, t) &= \sum_{m=0}^{\left[\frac{t}{p}\right]} (t - pm + 1) = (t+1)\left(\left[\frac{t}{p}\right] + 1\right) - \frac{p}{2}\left[\frac{t}{p}\right]\left(\left[\frac{t}{p}\right] + 1\right) \\ &= \frac{t^2}{2p} + \left(\frac{1}{p} + \frac{1}{2}\right) t + \frac{1}{2} + \frac{p}{8} - \left(\left(\frac{t}{p}\right)\right) - \frac{p}{2}\left(\left(\frac{t}{p}\right)\right)^2 . \end{aligned} \tag{2.18}$$

On the other hand, we can compute the same number via Theorem 2.15:

$$\begin{aligned} L(\Delta, t) &= \frac{t^2}{2p} + \frac{t}{2}\left(\frac{2}{p} + 1\right) + \frac{1}{4} + \frac{1}{12}\left(p + \frac{2}{p}\right) - \mathfrak{s}\left(1, p; \frac{t}{p}, 0\right) \\ &\quad -\frac{1}{4} - \frac{1}{2}\left(\left(\frac{t}{p}\right)\right) + \frac{1}{2} . \end{aligned} \tag{2.19}$$

Again we used (2.16). Equating (2.18) with (2.19) yields the statement. □

Using these ingredients, we can finally restate Theorem 2.9 as the main theorem of this section:

Theorem 2.17 *For the rectangular rational triangle $\mathcal{T}$ given by* (2.9)*,*

$$\begin{aligned} L\left(\overline{\mathcal{T}}, t\right) &= \frac{1}{2c^2pq}(tr - u - v)^2 + (tr - u - v)\left(\frac{1}{2cp} + \frac{1}{2cq} + \frac{1}{c^2pq} + \frac{1}{cpq}\left(\left(\frac{tr}{c}\right)\right)\right) \\ &\quad + \frac{1}{4} + \frac{1}{12}\left(\frac{p}{q} + \frac{q}{p}\right) - \frac{1}{24pq} + \frac{1}{c^2pq} - \frac{1}{2}\left(\left(\frac{tr - v}{cp}\right)\right) - \frac{1}{2}\left(\left(\frac{tr - u}{cq}\right)\right) \\ &\quad + \frac{1}{cpq}\left(\left(\frac{tr}{c}\right)\right) + \frac{1}{cpq}\left(\left(\frac{tr - 1}{c}\right)\right) + \frac{1}{2pq}\left(\left(\frac{tr - 1}{c}\right)\right)^2 \\ &\quad - \mathfrak{s}\left(q, p; \frac{tr - v}{cp}, 0\right) - \mathfrak{s}\left(p, q; \frac{tr - u}{cq}, 0\right) . \end{aligned}$$

Here u and v are given by (2.11)*.*

Proof. By Lemma 2.12,

$$\frac{1}{c}\sum_{\lambda^c=1\neq\lambda}\frac{\lambda^w}{1-\lambda}=-\left(\left(\frac{-w}{c}\right)\right)-\frac{1}{2c}\,. \tag{2.20}$$

By Corollary 2.14 and Lemma 2.16,

$$\begin{aligned}\frac{1}{c}\sum_{\lambda^c=1\neq\lambda}\frac{\lambda^w}{(1-\lambda)^2}&=-\mathfrak{s}\big(1,c;\frac{-w}{c},0\big)-\frac{1}{2}\left(\left(\frac{-w}{c}\right)\right)-\frac{1}{4c}\\&=\frac{c}{24}-\frac{5}{12c}-\left(\left(\frac{-w}{c}\right)\right)-\frac{c}{2}\left(\left(\frac{-w}{c}\right)\right)^2\,.\end{aligned} \tag{2.21}$$

Now simplify the identity in Theorem 2.9 by means of (2.20), (2.21), and Corollary 2.14.
□

2.2.3 Remarks and Consequences

An important property of $\mathfrak{s}(a,b;x,y)$ is the reciprocity law Theorem 1.5. From this reciprocity law it follows immediately that the function $\mathfrak{s}(a,b;x,0)$, the nontrivial part of our lattice point count formulas, is polynomial-time computable. It is amusing to note that $\mathfrak{s}(a,b;x,0)$ appears in the multiplier system of a weight-0 modular form ([Rob]).

To complete the picture for an *arbitrary* two-dimensional rational polytope $\mathcal{P}$, we return to the statements in the introduction of this section. After triangulating $\mathcal{P}$, the problem reduces to rational rectangles and the rectangular triangles which were treated above. A lattice point count formula for a rational rectangle $\mathcal{R}$ is easy to obtain. Suppose $\mathcal{R}$ has vertices

$$\left(\frac{a_1}{d},\frac{a_2}{d}\right),\left(\frac{b_1}{d},\frac{a_2}{d}\right),\left(\frac{b_1}{d},\frac{b_2}{d}\right),\left(\frac{a_1}{d},\frac{b_2}{d}\right)\,,$$

with $a_j<b_j$. Then it is not hard to see that

$$L\left(\overline{\mathcal{R}},t\right)=\left(\left[\frac{tb_1}{d}\right]-\left[\frac{ta_1-1}{d}\right]\right)\left(\left[\frac{tb_2}{d}\right]-\left[\frac{ta_2-1}{d}\right]\right)\,.$$

We summarize in

Theorem 2.18 *Let $\mathcal{P}$ be a two-dimensional rational polytope. The coefficients of $L\left(\overline{\mathcal{P}},t\right)$ can be written in terms of the sawtooth function $((x))$ and the Dedekind-Rademacher sum $\mathfrak{s}(a,b;x,0)$. Consequently, the formula given by Theorem* 2.17 *for the lattice point count operator can be computed in polynomial time.* □

Barvinok ([Ba]) showed that for any fixed dimension the lattice point enumerator of a rational polytope can be computed in polynomial time. The distinction here is that we get a simple *formula*, which happens to be also polynomial-time computable.

As another remark, we can deduce the reciprocity law Theorem 1.4 for the classical Dedekind sum ([De], [RG]) from our formulas:

Proof of Theorem 1.4. Ehrhart's Theorem 1.2 says that the constant term of a lattice polytope equals the Euler characteristic of the polytope. Consider the simplest case of our triangle mentioned in Theorem 2.15. If we dilate this polytope by $t = pqw$, that is, only by multiples of pq, we obtain the dilates of a lattice polytope $\mathcal{P}$. Theorem 2.15 simplifies for these t to

$$L\left(\overline{\mathcal{P}}, w\right) = \frac{pqw^2}{2} + \frac{w}{2}(p+q+1) + \frac{1}{4} + \frac{1}{12}\left(\frac{p}{q} + \frac{q}{p} + \frac{1}{pq}\right)$$
$$-\mathfrak{s}\big(q,p;0,0\big) - \mathfrak{s}\big(p,q;0,0\big) + \frac{1}{2} \ .$$

On the other hand, we know that the constant term is the Euler characteristic of $\mathcal{P}$ and hence equals 1, which yields the identity

$$-\frac{1}{4} + \frac{1}{12}\left(\frac{p}{q} + \frac{q}{p} + \frac{1}{pq}\right) - \mathfrak{s}\big(q,p;0,0\big) - \mathfrak{s}\big(p,q;0,0\big) = 0 \ .$$

□

As a concluding consequence of our formulas, we rederive a reciprocity law due to Gessel ([Ge]), at the same time interpreting it geometrically.

Corollary 2.19 (Gessel) *Let p and q be relatively prime and suppose that $1 \le t \le p+q$. Then*

$$\frac{1}{p}\sum_{\lambda^p=1\neq\lambda} \frac{\lambda^t}{(1-\lambda^q)(1-\lambda)} + \frac{1}{q}\sum_{\lambda^q=1\neq\lambda} \frac{\lambda^t}{(1-\lambda^p)(1-\lambda)}$$
$$= -\frac{t^2}{2pq} + \frac{t}{2}\left(\frac{1}{p} + \frac{1}{q} + \frac{1}{pq}\right) - \frac{1}{4}\left(\frac{1}{p} + \frac{1}{q} + 1\right) - \frac{1}{12}\left(\frac{p}{q} + \frac{1}{pq} + \frac{q}{p}\right) \ .$$

It is easy to see that the reciprocity law for classical Dedekind sums (Theorem 1.4) is a special case of Gessel's theorem. We prove Gessel's theorem below, after rephrasing it in terms of Dedekind-Rademacher sums by means of Corollary 2.14:

Corollary 2.20 *Let p and q be relatively prime and suppose that $1 \le t \le p+q$. Then*

$$\mathfrak{s}\big(q,p;\frac{-t}{p},0\big) + \mathfrak{s}\big(p,q;\frac{-t}{q},0\big) =$$
$$\frac{t^2}{2pq} - \frac{t}{2}\left(\frac{1}{p} + \frac{1}{q} + \frac{1}{pq}\right) + \frac{1}{4} + \frac{1}{12}\left(\frac{p}{q} + \frac{1}{pq} + \frac{q}{p}\right) - \frac{1}{2}\left(\left(\frac{-t}{p}\right)\right) - \frac{1}{2}\left(\left(\frac{-t}{q}\right)\right) \ .$$

□

We first need to rewrite Theorem 2.9 for the *interior* of our triangle. This can be done either from scratch or by using the Ehrhart-Macdonald reciprocity law (Theorem 1.3), which will be proved in chapter 4.

Corollary 2.21 *For the rectangular rational triangle $\mathcal{T}$ given by* (2.9) *with $a = b = 0$, $c = r = 1$, and p and q relatively prime,*

$$L\left(\mathcal{T}^\circ, t\right) = \frac{t^2}{2pq} - \frac{t}{2}\left(\frac{1}{p} + \frac{1}{q} + \frac{1}{pq}\right) + \frac{1}{4}\left(1 + \frac{1}{p} + \frac{1}{q}\right) + \frac{1}{12}\left(\frac{p}{q} + \frac{q}{p} + \frac{1}{pq}\right)$$
$$+ \frac{1}{p} \sum_{\lambda^p = 1 \neq \lambda} \frac{\lambda^t}{(1-\lambda^q)(1-\lambda)} + \frac{1}{q} \sum_{\lambda^q = 1 \neq \lambda} \frac{\lambda^t}{(1-\lambda^p)(1-\lambda)} \ .$$

□

Note that this allows us to conclude a computability statment for the *interior* of a two-dimensional rational polytope similar to Theorem 2.18.

Proof of Corollary 2.19. Consider dilates of the triangle given in Corollary 2.21, that is,

$$t\mathcal{T}^\circ = \left\{(x, y) \in \mathbb{R}^2 : \ x, y > 0, \ px + qy < t\right\} \ .$$

By the very definition, $t\mathcal{T}^\circ$ does not contain any integer points for $1 \leq t \leq p + q$, in other words, $L\left(\mathcal{T}^\circ, t\right) = 0$. Hence Corollary 2.21 yields an identiy for these values of t:

$$0 = \frac{t^2}{2pq} - \frac{t}{2}\left(\frac{1}{p} + \frac{1}{q} + \frac{1}{pq}\right) + \frac{1}{4}\left(1 + \frac{1}{p} + \frac{1}{q}\right) + \frac{1}{12}\left(\frac{p}{q} + \frac{q}{p} + \frac{1}{pq}\right)$$
$$+ \frac{1}{p} \sum_{\lambda^p = 1 \neq \lambda} \frac{\lambda^t}{(1-\lambda^q)(1-\lambda)} + \frac{1}{q} \sum_{\lambda^q = 1 \neq \lambda} \frac{\lambda^t}{(1-\lambda^p)(1-\lambda)} \ .$$

□

These two methods (proofs of Theorem 1.4 and Corollary 2.19) of obtaining reciprocity laws from lattice point enumeration formulas extend easily to higher dimensions. We will make use of this fact in section 3.1.2.

2.3 General Rational Polytopes

In the last section we set up a complete machinery for the computation of lattice point enumeration formulas for any 2-dimensional rational polytope. We will extend this in the next chapter to a certain class of rational polytopes in arbitrary dimension. Before doing so, we conclude this chapter with a remark on the general case.

It certainly suffices to look at *convex* rational polytopes. These can be described by a finite number of inequalities with integer coefficients. In other words, a convex lattice polytope

$\mathcal{P}$ is an intersection of finitely many half-spaces. Translation by a lattice vector does not change the lattice point count, so we can assume that the points in the polytope have nonnegative coordinates and apply the ideas of the previous sections to $\mathcal{P}$. Suppose $\mathcal{P}$ is given by the $n+q$ inequalities

$$\begin{aligned} & x_1, \ldots, x_n \geq 0 \\ \\ & a_{11}x_1 + \cdots + a_{1n}x_n \leq b_1 \\ & \qquad \vdots \\ & a_{q1}x_1 + \cdots + a_{qn}x_n \leq b_q \ , \end{aligned} \tag{2.22}$$

with $a_{jk}, b_j \in \mathbb{Z}$. Define a matrix

$$M = \left(a_{jk}\right)_{\substack{j=1..q \\ k=1..n}},$$

and let $\mathbf{C}_j$ denote the j'th column and $\mathbf{R}_k$ the k'th row of M. Then we can rewrite the nontrivial inequalities determining $t\mathcal{P}$ as

$$\begin{aligned} \mathbf{R}_1 \cdot \mathbf{x} &\leq tb_1 \\ &\vdots \\ \mathbf{R}_q \cdot \mathbf{x} &\leq tb_q, \end{aligned} \tag{2.23}$$

where $\mathbf{x} = (x_1, \ldots, x_n)$ and $\cdot$ denotes the usual scalar product. Now consider the function

$$f\left(z_1, \ldots, z_q\right) = \frac{z_1^{-tb_1-1} \cdots z_q^{-tb_q-1}}{\left(1-\mathbf{z}^{\mathbf{C}_1}\right) \cdots \left(1-\mathbf{z}^{\mathbf{C}_q}\right)\left(1-\mathbf{z}_1\right) \cdots \left(1-\mathbf{z}_q\right)}.$$

Here we use the standard multinomial notation $\mathbf{z}^v := z_1^{v_1} \cdots z_q^{v_q}$. We will integrate f with respect to each variable over a circle with small radius:

$$\int_{|z_1|=\epsilon_1} \cdots \int_{|z_q|=\epsilon_q} f\left(z_1, \ldots, z_q\right)\, dz_q \cdots dz_1 \ . \tag{2.24}$$

Here, $0 < \epsilon_0, \ldots, \epsilon_q < 1$ are chosen such that we can expand all the $\frac{1}{1-\mathbf{z}^{\mathbf{C}_k}}$ into power series about 0. To ensure the existence of $\epsilon_0, \ldots, \epsilon_q$, we may, if necessary, add an additional inequality $x_1 + \cdots + x_n \leq tb_0$ for a suitable large b_0. This is always possible, since $\mathcal{P}$ is bounded.

Since the integral over one variable will give the respective residue at 0, we can integrate with respect to one variable at a time. When f is expanded into its Laurent series about 0, each term has the form

$$z_1^{\mathbf{m}\cdot\mathbf{R}_1+r_1-tb_1-1} \cdots z_q^{\mathbf{m}\cdot\mathbf{R}_q+r_q-tb_q-1},$$

where $\mathbf{m} := (m_1, \ldots, m_n)$, and $m_1, \ldots, m_n, r_1, \ldots, r_q$ are nonnegative integers. Thus, in the integral (2.24), this term will give a contribution precisely if $\mathbf{m}$ satisfies the inequalities (2.23). In other words, we have proved

Theorem 2.22

$$L(\overline{\mathcal{P}}, t) = \frac{1}{(2\pi i)^q} \int_{|z_1|=\epsilon_1} \cdots \int_{|z_q|=\epsilon_q} f(z_1, \ldots, z_q) \; dz_q \cdots dz_1 \; .$$

□

However, in several complex variables, we do not have a maschinery equivalent to the residue theorem. The methods introduced above therefore do not easily extend to the most general rational polytopes.

Chapter 3

The Frobenius Problem

If you think it's simple, then you have misunderstood the problem.
Bjarne Strustrup (lecture at Temple University, 11/25/97)

Given relatively prime positive integers $a_1, \ldots, a_n$, we call a positive integer t **representable** if there exist nonnegative integers $m_1, \ldots, m_n$ such that

$$t = \sum_{j=1}^{n} m_j a_j \ .$$

In this chapter, we discuss the *linear diophantine problem of Frobenius*: namely, find the largest integer t which is not representable. We call this largest integer the **Frobenius number** $g(a_1, \ldots, a_n)$. We study a more general problem: namely, we consider $N_t(a_1, \ldots, a_n)$, the number of nonnegative integer solutions $(m_1, \ldots, m_n)$ to $\sum_{j=1}^{n} m_j a_j = t$ for any positive integral t. Geometrically, $N_t(a_1, \ldots, a_n)$ enumerates the lattice points on the dilates of a rational polytope. Finding $g(a_1, \ldots, a_n)$ simply means finding the largest integer zero of $N_t(a_1, \ldots, a_n)$. We can also interpret $N_t(a_1, \ldots, a_n)$ as a partition function: $N_t(a_1, \ldots, a_n)$ enumerates the number of partitions of t into parts which come from the set $\{a_1, \ldots, a_n\}$ ([EL], [Na]).

Frobenius inaugurated the study of $g(a_1, \ldots, a_n)$ in the 19th century. For $n = 2$, Sylvester ([Sy]) proved that $g(a_1, a_2) = a_1 a_2 - a_1 - a_2$. For $n > 2$, all attempts to obtain explicit formulas have proved elusive. Here we focus on the study of $N_t(a_1, \ldots, a_n)$, and show that it has an explicit representation as a quasipolynomial. Through the discussion of $N_t(a_1, \ldots, a_n)$, we gain new insights into Frobenius's problem.

Within our formulas there appears a generalized Dedekind sum, which shares some properties with its classical siblings. In particular, we prove two reciprocity laws for these sums: a rederivation of the reciprocity law for Zagier's higher-dimensional Dedekind sums (Theorem 1.6), and a new reciprocity law that generalizes Gessel's reciprocity law Corollary 2.19.

Another motivation to study $N_t(a_1,\ldots,a_n)$ is the following trivial reduction formula to lower dimensions:

$$N_t(a_1,\ldots,a_n) = \sum_{m\geq 0} N_{t-ma_n}(a_1,\ldots,a_{n-1}) \ . \tag{3.1}$$

Here we use the convention that $N_t(a_1,\ldots,a_n)=0$ if $t\leq 0$; in particular, the sum in (3.1) is finite. This identity can be easily verified by viewing $N_t(a_1,\ldots,a_n)$ as

$$N_t(a_1,\ldots,a_n) = \#\left\{(m_1,\ldots,m_n)\in\mathbb{Z}^n_{\geq 0} : \sum_{k=1}^{n-1} m_k a_k = t - m_n a_n\right\}.$$

Hence, precise knowledge of the values of t for which $N_t(a_1,\ldots,a_n)=0$ in lower dimensions sheds additional light on the Frobenius number in higher dimensions.

Finally, we extend the Frobenius problem in a way that is naturally motivated by studying $N_t(a_1,\ldots,a_n)=0$.

The literature on the Frobenius problem is vast, see for example [BS], [Da], [EG], [Ka], [NW], [Rod1], [Rod2], [Se], [Sy], [Vi].

3.1 A Related Polytope

$N_t(a_1,\ldots,a_n)$ enumerates the lattice points on the t-dilate of the rational polytope

$$\mathcal{P} = \left\{(x_1,\ldots,x_n)\in\mathbb{R}^n :\ x_k\geq 0, \sum_{k=1}^{n} x_k a_k = 1\right\} \ .$$

Our computation of the quantity $N_t(a_1,\ldots,a_n)$ is similar to that of the lattice point count formulas in chapter 2. We note that one does not have to think of $N_t(a_1,\ldots,a_n)$ as the lattice point count of a polytope to understand how to compute its formula; however, this geometric interpretation was the motivation for our proof and offers guidance for our intuition.

3.1.1 Computation of $N_t(a_1,\ldots,a_n)$

We first need to introduce the generalized Dedekind sum that appears in this context.

Definition 3.1 *Let $c_1,\ldots,c_n\in\mathbb{Z}$ be relatively prime to $c\in\mathbb{Z}$, and $t\in\mathbb{Z}$. Define the* **Fourier-Dedekind sum** *as*

$$\sigma_t\left(c_1,\ldots,c_n;c\right) = \frac{1}{c}\sum_{\lambda^c=1\neq\lambda} \frac{\lambda^t}{(1-\lambda^{c_1})\cdots(1-\lambda^{c_n})} \ .$$

Some properties of σ_t are discussed in section 3.1.2. With this notation, we are ready to state the central theorem of this chapter:

Theorem 3.1 *Suppose $a_1, \ldots, a_n$ are pairwise relatively prime, and t is a positive integer. Then*

$$N_t(a_1, \ldots, a_n) = R_{-t}(a_1, \ldots, a_n) + \sum_{j=1}^{n} \sigma_{-t}(a_1, \ldots, \hat{a}_j, \ldots, a_n; a_j) ,$$

where $R_t(a_1, \ldots, a_n) = -\mathrm{Res}(G_{-t}(z), z = 1)$, and

$$G_t(z) = \frac{z^{t-1}}{(1 - z^{a_1}) \cdots (1 - z^{a_n})} .$$

Remarks. 1. R_t can be computed in precisely the same way as the rational-function part in Theorem 2.1. The first values are

$$\begin{aligned}
R_t(a_1, a_2) &= \frac{t}{a_1 a_2} + \frac{1}{2} \left(\frac{1}{a_1} + \frac{1}{a_2} \right) \\
R_t(a_1, a_2, a_3) &= \frac{t^2}{2 a_1 a_2 a_3} + \frac{t}{2} \left(\frac{1}{a_1 a_2} + \frac{1}{a_1 a_3} + \frac{1}{a_2 a_3} \right) \\
&+ \frac{1}{12} \left(\frac{3}{a_1} + \frac{3}{a_2} + \frac{3}{a_3} + \frac{a_1}{a_2 a_3} + \frac{a_2}{a_1 a_3} + \frac{a_3}{a_1 a_2} \right) \\
R_t(a_1, a_2, a_3, a_4) &= \frac{t^3}{6 a_1 a_2 a_3 a_4} + \frac{t^2}{4} \left(\frac{1}{a_1 a_2 a_3} + \frac{1}{a_1 a_2 a_4} + \frac{1}{a_1 a_3 a_4} + \frac{1}{a_2 a_3 a_4} \right) \\
&+ \frac{t}{4} \left(\frac{1}{a_1 a_2} + \frac{1}{a_1 a_3} + \frac{1}{a_1 a_4} + \frac{1}{a_2 a_3} + \frac{1}{a_2 a_4} + \frac{1}{a_3 a_4} \right) \\
&+ \frac{t}{12} \left(\frac{a_1}{a_2 a_3 a_4} + \frac{a_2}{a_1 a_3 a_4} + \frac{a_3}{a_1 a_2 a_4} + \frac{a_4}{a_1 a_2 a_3} \right) \\
&+ \frac{1}{24} \left(\frac{a_1}{a_2 a_3} + \frac{a_1}{a_2 a_4} + \frac{a_1}{a_3 a_4} + \frac{a_2}{a_1 a_4} + \frac{a_2}{a_1 a_3} + \frac{a_2}{a_3 a_4} \right. \\
&\quad \left. + \frac{a_3}{a_1 a_2} + \frac{a_3}{a_1 a_4} + \frac{a_3}{a_2 a_4} + \frac{a_4}{a_1 a_2} + \frac{a_4}{a_1 a_3} + \frac{a_4}{a_2 a_3} \right) \\
&+ \frac{1}{8} \left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} \right) .
\end{aligned}$$

2. If $a_1, \ldots, a_n$ are *not* pairwise relatively prime, we can get similar, slightly more complicated formulas for $N_t(a_1, \ldots, a_n)$. This remark will become more transparent in the proof of the theorem.

Proof. As in the last chapter, we interpret

$$N_t(a_1, \ldots, a_n) = \# \left\{ (m_1, \ldots, m_n) \in \mathbb{Z}_{\geq 0}^n : \sum_{k=1}^{n} m_k a_k = t \right\}$$

as the Taylor coefficient of z^t of the function

$$\left(1 + z^{a_1} + z^{2a_1} + \dots\right) \cdots \left(1 + z^{a_n} + z^{2a_n} + \dots\right) \\ = \frac{1}{1 - z^{a_1}} \cdots \frac{1}{1 - z^{a_n}} \ .$$

Shifting this coefficient to the coefficient of z^{-1}, we obtain a residue

$$N_t(a_1, \dots, a_n) = \operatorname{Res}\left(\frac{z^{-t-1}}{(1 - z^{a_1}) \cdots (1 - z^{a_n})}, z = 0\right) = \operatorname{Res}\left(G_{-t}(z), z = 0\right) \ . \tag{3.2}$$

Thus, we have to find the other residues of $G_{-t}(z)$. The other poles of $G_{-t}(z)$ are at all a_1'th, $\dots, a_n$'th roots of unity. These poles are *simple* by the pairwise-coprime condition (which is why we imposed this condition). Let λ be a nontrivial a_1'th root of unity. Then

$$\operatorname{Res}\left(G_{-t}(z), z = \lambda\right) = \frac{\lambda^{-t-1}}{(1 - \lambda^{a_2}) \cdots (1 - \lambda^{a_n})} \operatorname{Res}\left(\frac{1}{1 - z^{a_1}}, z = \lambda\right) \\ = -\frac{\lambda^{-t}}{a_1 \left(1 - \lambda^{a_2}\right) \cdots \left(1 - \lambda^{a_n}\right)} .$$

Adding up all the nontrivial a_1'th roots of unity, we obtain

$$\sum_{\lambda^{a_1} = 1 \neq \lambda} \operatorname{Res}\left(G_{-t}(z), z = \lambda\right) = -\frac{1}{a_1} \sum_{\lambda^{a_1} = 1 \neq \lambda} \frac{\lambda^{-t}}{(1 - \lambda^{a_2}) \cdots (1 - \lambda^{a_n})} \\ = -\sigma_{-t}\left(a_2, \dots, a_n; a_1\right) .$$

Together with the other similar residues at the other roots of unity and the residue at $z = 1$, we can restate (3.2) by means of the residue theorem. □

3.1.2 The Fourier-Dedekind Sum

In the derivation of the previous lattice point count formula (Theorem 3.1), we naturally arrived at the Fourier-Dedekind sum

$$\sigma_t\left(c_1, \dots, c_n; c\right) = \frac{1}{c} \sum_{\lambda^c = 1 \neq \lambda} \frac{\lambda^t}{(1 - \lambda^{c_1}) \cdots (1 - \lambda^{c_n})} \ .$$

This expression is a generalization of the classical Dedekind sum $\mathfrak{s}(h, k)$ and its various generalizations mentioned in the introductory chapter. In fact, we came across various special cases of $\sigma_t\left(c_1, \dots, c_n; c\right)$ before: in section 2.1.3, we obtained

$$\sigma_0\left(a, 1; c\right) = \frac{1}{c} \sum_{\lambda^c = 1 \neq \lambda} \frac{1}{\left(1 - \lambda^a\right)\left(1 - \lambda\right)} = \frac{1}{4} - \frac{1}{4c} - \mathfrak{s}(a, c) \ .$$

In section 2.2.2, we got another easy special case: Corollary 2.13 gives

$$\sigma_t\left(q; p\right) = \frac{1}{p} \sum_{\lambda^p = 1 \neq \lambda} \frac{\lambda^t}{1 - \lambda^q} = -\left(\left(\frac{-q^{-1} t}{p}\right)\right) - \frac{1}{2p} \ .$$

The Dedekind-Rademacher sums appearing in the lattice point formulas in the same section 2.2.2 are another specialization: Corollary 2.14 says that

$$\sigma_t\left(q,1;p\right) = \frac{1}{p} \sum_{\lambda^p=1\neq\lambda} \frac{\lambda^t}{\left(1-\lambda^q\right)\left(1-\lambda\right)} = -\mathfrak{s}\left(q,p;\frac{-t}{p},0\right) - \frac{1}{2}\left(\left(\frac{-t}{p}\right)\right) - \frac{1}{4p} \ .$$

Finally, it is not hard to see that Zagier's higher-dimensional Dedekind sum can be expressed as a sum of Fourier-Dedekind sums with trivial numerator, that is, $\sigma_0\left(c_1,\ldots,c_n;c\right)$.

In general, note that $\sigma_t\left(c_1,\ldots,c_n;c\right)$ is a rational number: it is an element of the cyclotomic field of c'th roots of unity, and invariant under all Galois transformations of this field. Some other obvious properties are

$$\begin{aligned} \sigma_t\left(c_1,\ldots,c_n;c\right) &= \sigma_t\left(c_{\pi(1)},\ldots,c_{\pi(n)};c\right) \qquad \text{for any } \pi\in S_n \\ \sigma_t\left(c_1,\ldots,c_n;c\right) &= \sigma_{(t \bmod c)}\left(c_1 \bmod c,\ldots,c_n \bmod c;c\right) \\ \sigma_t\left(c_1,\ldots,c_n;c\right) &= \sigma_{bt}\left(bc_1,\ldots,bc_n;c\right) \qquad \text{for any } b\in\mathbb{Z} \text{ with } (b,c)=1 \end{aligned} \tag{3.3}$$

We conclude this section by proving two reciprocity laws for Fourier-Dedekind sums; these are the proposed generalizations of the two reciprocity laws proved in section 2.2.3. The first one is equivalent to Zagier's reciprocity law for his higher dimensional Dedekind sums, Theorem 1.6.

Theorem 3.2 *For pairwise relatively prime positive integers $a_1,\ldots,a_n$,*

$$\sum_{j=1}^{n} \sigma_0(a_1,\ldots,\hat{a}_j,\ldots,a_n;a_j) = 1 - R_0\left(a_1,\ldots,a_n\right) \ ,$$

where R_t is the rational function given in Theorem 3.1.

Proof. Recall once more that Ehrhart's Theorem 1.2 states that the constant term of the Ehrhart polynomial of a lattice polytope equals the Euler characteristic of the polytope. Consider the polytope

$$\mathcal{P} = \left\{ (x_1,\ldots,x_n) \in \mathbb{R}^n_{\geq 0} : \sum_{k=1}^{n} x_k a_k = 1 \right\} ,$$

whose dilates correspond to the enumerator $N_t(a_1,\ldots,a_n)$ of Theorem 3.1. If we dilate this polytope only by multiples of $a_1\cdots a_n$, say $t = a_1\cdots a_n w$, we obtain the dilates of a lattice polytope. Theorem 3.1 simplifies for these t to

$$N_{a_1\cdots a_n w}(a_1,\ldots,a_n) = R_{a_1\cdots a_n w}\left(a_1,\ldots,a_n\right) + \sum_{j=1}^{n} \sigma_0(a_1,\ldots,\hat{a}_j,\ldots,a_n;a_j) \ ,$$

using the periodicity of σ_t (3.3). On the other hand, we know that the constant term (in terms of w) is the Euler characteristic of the polytope and hence equals 1, which yields the identity

$$1 = R_0\left(a_1,\ldots,a_n\right) + \sum_{j=1}^{n} \sigma_0(a_1,\ldots,\hat{a}_j,\ldots,a_n;a_j) \ .$$

□

The second one is a new reciprocity law, which generalizes Gessel's reciprocity law Corollary 2.19. It is not hard to see that Gessel's theorem is the two-dimensional case of

Theorem 3.3 *Let $a_1, \ldots, a_n$ be pairwise relatively prime positive integers and $0 < t < a_1 + \cdots + a_n$. Then*

$$\sum_{j=1}^{n} \sigma_t(a_1, \ldots, \hat{a}_j, \ldots, a_n; a_j) = -R_t(a_1, \ldots, a_n) \ ,$$

where R_t is the rational function given in Theorem 3.1*.*

Proof. Consider the interior of our polytope,

$$\mathcal{P}^\circ = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n : \ x_k > 0, \sum_{k=1}^{n} x_k a_k = 1 \right\} .$$

By the Ehrhart-Macdonald reciprocity law Theorem 1.3,

$$L(\mathcal{P}^\circ, t) = (-1)^{n-1} N_{-t}(a_1, \ldots, a_n) \ .$$

By definition, $t\mathcal{P}^\circ$ does not contain any integer points for $0 < t < a_1 + \cdots + a_n$. Hence Theorem 3.1 yields an identiy for these values of t:

$$0 = (-1)^{n-1} \left(R_t(a_1, \ldots, a_n) + \sum_{j=1}^{n} \sigma_t(a_1, \ldots, \hat{a}_j, \ldots, a_n; a_j) \right) \ .$$

□

3.2 Classical Results

We finally get to apply our formulas to the Frobenius problem. In this section we prove two classical results, both due to Sylvester ([Sy]). Both follow immediately from the formula

$$N_t(a_1, a_2) = \frac{t}{a_1 a_2} - \left(\left(\frac{a_2^{-1} t}{a_1} \right) \right) - \left(\left(\frac{a_1^{-1} t}{a_2} \right) \right) \ , \qquad (3.4)$$

which is the two-dimensional version of Theorem 3.1, using Corollary 2.13.

Corollary 3.4 (Sylvester) *$g(a, b) = ab - a - b$.*

Proof. We have to show that $N_{ab-a-b}(a,b) = 0$ and that $N_t(a,b) > 0$ for every $t > ab-a-b$. First, by the periodicity of the sawtooth function,

$$\begin{aligned} N_{ab-a-b}(a,b) &= \frac{ab-a-b}{ab} - \left(\left(\frac{b^{-1}(ab-a-b)}{a}\right)\right) - \left(\left(\frac{a^{-1}(ab-a-b)}{b}\right)\right) \\ &= 1 - \frac{1}{b} - \frac{1}{a} - \left(\left(\frac{-1}{a}\right)\right) - \left(\left(\frac{-1}{b}\right)\right) \\ &= 1 - \frac{1}{b} - \frac{1}{a} - \left(\frac{1}{2} - \frac{1}{a}\right) - \left(\frac{1}{2} - \frac{1}{b}\right) = 0 \ . \end{aligned}$$

For any integer m, $\left(\left(\frac{m}{a}\right)\right) \le \frac{1}{2} - \frac{1}{a}$. Hence for any positive integer n,

$$N_{ab-a-b+n}(a,b) \ge \frac{ab-a-b+n}{ab} - \left(\frac{1}{2} - \frac{1}{a}\right) - \left(\frac{1}{2} - \frac{1}{b}\right) = \frac{n}{ab} > 0.$$

□

Corollary 3.5 (Sylvester) *Exactly half of the integers between* 1 *and* $(a-1)(b-1)$ *are representable.*

Proof. We first claim that, if $t \in [1, ab-1]$ is not a multiple of a or b,

$$N_t(a,b) + N_{ab-t}(a,b) = 1 \ . \tag{3.5}$$

This identity follows directly from (3.4):

$$\begin{aligned} N_{ab-t}(a,b) &= \frac{ab-t}{ab} - \left(\left(\frac{b^{-1}(ab-t)}{a}\right)\right) - \left(\left(\frac{a^{-1}(ab-t)}{b}\right)\right) \\ &= 1 - \frac{t}{ab} - \left(\left(\frac{-b^{-1}t}{a}\right)\right) - \left(\left(\frac{-a^{-1}t}{b}\right)\right) \\ &\overset{(\star)}{=} 1 - \frac{t}{ab} + \left(\left(\frac{b^{-1}t}{a}\right)\right) + \left(\left(\frac{a^{-1}t}{b}\right)\right) \\ &= 1 - N_t(a,b) \ . \end{aligned}$$

Here, $(\star)$ follows from the fact that $((-x)) = -((x))$ if $x \notin \mathbb{Z}$. This shows that, for t between 1 and $ab-1$ and not divisible by a or b, exactly one of t and $ab-t$ is not representable. There are

$$ab - a - b + 1 = (a-1)(b-1)$$

integers between 1 and $ab-1$ which are not divisible by a or b. Finally, we note that $N_t(a,b) > 0$ if t is a multiple of a or b, by the very definition of $N_t(a,b)$. Hence the number of non-representable integers is $\frac{1}{2}(a-1)(b-1)$. □

Note that we proved even more. By (3.5), every positive integer less than ab has at most one representation. Hence, the representable integers in the above corollary are *uniquely* representable.

These two results can now be generalized in two ways: we can stay in dimension two and generalize the notion of a nonrepresentable integer (section 3.4), or we can move to higher dimensions (section 3.3).

3.3 New Bounds on the Frobenius Number

We will now use our formulas to give new results on the first nontrivial case of the Frobenius problem, dimension 3. Known bounds on the Frobenius number include

$$g(a_1,\dots,a_n) \le \left[\tfrac{1}{2}(a_2-1)(a_n-2)\right] - 1$$

([Vi]) and

$$g(a_1,\dots,a_n) \le 2a_n\left[\tfrac{a_1}{n}\right] - a_1$$

([Se]); for both we assume that $a_1 < \cdots < a_n$.

We note that a bound in three dimensions yields a bound for the general case: directly from the definition of g, we conclude that for $n \ge 3$

$$g(a_1,\dots,a_n) \le g(a_1,a_2,a_3) \ . \tag{3.6}$$

Furthermore, in dimension 3 it suffices to assume that a_1, a_2, a_3 are pairwise coprime, because of Johnson's formula ([Jo]): If $c = (a_1, a_2)$, then

$$g(a_1,a_2,a_3) = c \cdot g\left(\frac{a_1}{c}, \frac{a_2}{c}, a_3\right) + (c-1)a_3 \ . \tag{3.7}$$

Hence, we may assume a, b, c pairwise relatively prime. We need a handy expression for $\sigma_t(a,b;c)$, which can be achieved through the application of Corollary 2.14:

$$\begin{aligned}
\sigma_t(a,b;c) &= \frac{1}{c}\sum_{\lambda^c=1\neq\lambda} \frac{\lambda^t}{(1-\lambda^a)(1-\lambda^b)} = \frac{1}{c}\sum_{\lambda^c=1\neq\lambda} \frac{\lambda^{b^{-1}t}}{\left(1-\lambda^{b^{-1}a}\right)(1-\lambda)} \\
&\overset{\text{Cor. 2.14}}{=} -\sum_{k=0}^{c-1}\left(\left(\frac{k}{c}\right)\right)\left(\left(\frac{b^{-1}(ak-t)}{c}\right)\right) - \left(\left(\frac{-b^{-1}t}{c}\right)\right) - \frac{1}{4c} \\
&= \sum_{k=1}^{c-1}\left(\left(\frac{k}{c}\right)\right)\left(\left(\frac{-b^{-1}(ak+t)}{c}\right)\right) - \frac{1}{2}\left(\left(\frac{-b^{-1}t}{c}\right)\right) - \frac{1}{4c} \\
&= \sum_{k=0}^{c-1}\left(\left(\frac{k}{c}\right)\right)\left(\left(\frac{-b^{-1}(ak+t)}{c}\right)\right) - \frac{1}{4c} \ .
\end{aligned}$$

We will use the Cauchy-Schwartz inequality

$$\left|\sum_{k=1}^{n} a_k a_{\pi(k)}\right| \le \sum_{k=1}^{n} a_k^2 \ . \tag{3.8}$$

Here $a_k \in \mathbb{R}$, and $\pi \in S_n$ is a permutation. Since $(b^{-1}a, c) = 1$, we can use (3.8) to obtain

$$\begin{aligned}
\sigma_t(a,b;c) &\ge -\sum_{m=0}^{c-1}\left(\left(\frac{m}{c}\right)\right)^2 - \frac{1}{4c} = -\sum_{m=0}^{c-1}\left(\frac{m}{c}-\frac{1}{2}\right)^2 - \frac{1}{4c} \\
&= -\frac{1}{c^2}\frac{(2c-1)(c-1)c}{6} + \frac{1}{c}\frac{c(c-1)}{2} - \frac{c}{4} - \frac{1}{4c} = -\frac{c}{12} - \frac{1}{12c} \ .
\end{aligned}$$

This also restates Rademacher's bound on the classical Dedekind sums ([RG]). Using this in the formula for dimension 3 (remark after Theorem 3.1), we get

$$\begin{aligned} N_t(a,b,c) \geq& \frac{t^2}{2abc} + \frac{t}{2}\left(\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc}\right) + \frac{1}{12}\left(\frac{3}{a}+\frac{3}{b}+\frac{3}{c}+\frac{a}{bc}+\frac{b}{ac}+\frac{c}{ab}\right) \\ &-\frac{1}{12}(a+b+c) - \frac{1}{12}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \\ =& \frac{t^2}{2abc} + \frac{t}{2}\left(\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc}\right) + \frac{1}{12}\left(\frac{a}{bc}+\frac{b}{ac}+\frac{c}{ab}\right) \\ &-\frac{1}{12}(a+b+c) + \frac{1}{6}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) . \end{aligned}$$

The larger zero of the right-hand side is an upper bound for the solution of the Frobenius problem:

$$\begin{aligned} g(a,b,c) \leq& \, abc\left(-\frac{1}{2}\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac}\right) + \left[\frac{1}{4}\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac}\right)^2 - \frac{2}{abc}\right.\right. \\ &\left.\left.\cdot\left(\frac{1}{12}\left(\frac{a}{bc}+\frac{b}{ac}+\frac{c}{ab}\right) - \frac{1}{12}(a+b+c) + \frac{1}{6}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\right)\right]^{1/2}\right) \\ \leq& -\frac{1}{2}(a+b+c) + abc\sqrt{\frac{1}{4}\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac}\right)^2 + \frac{1}{6}\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac}\right)} \\ =& -\frac{1}{2}(a+b+c) + abc\sqrt{\frac{1}{2}\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac}\right)\left(\frac{1}{2}\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac}\right)+\frac{1}{3}\right)} \\ \leq& -\frac{1}{2}(a+b+c) + abc\sqrt{\frac{1}{4}\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac}\right)} . \end{aligned}$$

For the last inequality, we used the fact that $\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac} \leq \frac{1}{6}+\frac{1}{10}+\frac{1}{15} = \frac{1}{3}$. This proves

Theorem 3.6 *Let a_1, a_2, a_3 be pairwise relatively prime. Then*

$$g(a_1,\ldots,a_n) \leq \frac{1}{2}\left(\sqrt{a_1 a_2 a_3 \left(a_1+a_2+a_3\right)} - a_1 - a_2 - a_3\right) .$$

□

Remarks. 1. This inequality is useful only for certain ranges of a_1, a_2, a_3. For example, the inequality becomes meaningless if $a_3 > a_1 + a_2$, as a straightforward calculation shows.

2. More general results can be easily obtained from Theorem 3.6 by way of (3.6) and (3.7).

3.4 Frobenius's Problem Extended

We now turn to the proposed natural extension of the Frobenius problem: we define an integer t to be **k-representable** if $N_t(a_1, \ldots, a_n) = k$; that is, t can be represented in exactly k ways. Define $g_k = g_k(a_1, \ldots, a_n)$ to be the largest k-representable integer. It is fairly easy to see that for each k, eventually all integers are representable in at least k ways. Hence g_k is well-defined, and every integer greater than g_k is representable in at least $k+1$ ways. In particular $g_0(a_1, \ldots, a_n) = g(a_1, \ldots, a_n)$. We will prove statements about g_k similar in spirit to the two classical results by Sylvester mentioned in section 3.2. Our proofs are again based on the dimension-two formula (3.4).

The fundamental result connecting the classical notion of (0-)representable integers to our new notion of k-representable integers is

Theorem 3.7 $N_{t+ab}(a,b) = N_t(a,b) + 1.$

Proof. By the periodicity of the sawtooth function,

$$N_{t+ab}(a,b) = \frac{t+ab}{ab} - \left(\left(\frac{b^{-1}(t+ab)}{a}\right)\right) - \left(\left(\frac{a^{-1}(t+ab)}{b}\right)\right)$$
$$= \frac{t}{ab} + 1 - \left(\left(\frac{b^{-1}t}{a}\right)\right) - \left(\left(\frac{a^{-1}t}{b}\right)\right) = N_t(a,b) + 1 \ .$$

□

Corollary 3.8 $g_k(a,b) = (k+1)ab - a - b.$

Proof. By the preceeding theorem, $g_{k+1} = g_k + ab$. The statement now follows inductively, starting with Sylvester's theorem (Corollary 3.4). □

Corollary 3.9 *Given $k \geq 2$, the smallest k-representable integer is $ab(k-1)$.*

Proof. Let n be a nonnegative integer. Then

$$N_{ab(k-1)-n}(a,b) =$$
$$= \frac{ab(k-1)-n}{ab} - \left(\left(\frac{b^{-1}(ab(k-1)-n)}{a}\right)\right) - \left(\left(\frac{a^{-1}(ab(k-1)-n)}{b}\right)\right) \tag{3.9}$$
$$= k - 1 - \frac{n}{ab} - \left(\left(\frac{-b^{-1}n}{a}\right)\right) - \left(\left(\frac{-a^{-1}n}{b}\right)\right) \ . \tag{3.10}$$

If $n = 0$, (3.10) equals k. If n is positive, we use $((x)) \geq -\frac{1}{2}$ to see that

$$N_{ab(k-1)-n}(a,b) \leq k - \frac{n}{ab} < k \ .$$

□

All nonrepresentable positive integers lie, by definition, in the interval $[1, g(a,b)]$. It is easy to see that the smallest interval containing all uniquely representable integers is $[\min(a,b), g_1]$. For $k \geq 2$, the corresponding interval always has length $2ab - a - b + 1$, and the precise interval is given next.

Corollary 3.10 *Given $k \geq 2$, the smallest interval containing all k-representable integers is $[g_{k-2} + a + b, g_k]$.*

Proof. By Corollaries 3.8 and 3.9, the smallest integer in the interval is

$$ab(k-1) = g_{k-2} + a + b \ .$$

The upper bound of the interval follows by definition of g_k. □

Corollary 3.11 *There are exactly $ab - 1$ integers which are uniquely representable. Given $k \geq 2$, there are exactly ab k-representable integers.*

Proof. First, in the interval $[1, ab]$, there are, by Corollaries 3.5 and 3.9,

$$ab - \frac{(a-1)(b-1)}{2} - 1$$

1-representable integers. Using Theorem 3.7, we see that there are

$$\frac{(a-1)(b-1)}{2}$$

1-representable integers above ab. For $k \geq 2$, the statement follows by similar reasoning. □

Chapter 4

Multidimensional Ehrhart Reciprocity

If things are nice there is probably a good reason why they are nice: and if you do not know at least one reason for this good fortune, then you still have work to do.
Richard Askey ([As])

One of the exercises on the greatest integer function $[x]$ in an elementary course in number theory is to prove the statement

$$\left[\frac{t-1}{a}\right] = -\left[\frac{-t}{a}\right] - 1 \tag{4.1}$$

for $t \in \mathbb{Z}$, $a \in \mathbb{N}$. Geometrically, this is a special instance of a much more general theme. Consider the interval $\left[0, \frac{1}{a}\right]$, viewed as a 1-dimensional rational polytope. Now we dilate this polytope by an integer factor $t > 0$, and count the number of integer points in the dilated polytope. It is straightforward that this number in the open dilated polytope is $\left[\frac{t-1}{a}\right]$, whereas in the closure there are $\left[\frac{t}{a}\right] + 1$ integer points. Hence (4.1) marks the simplest, namely one-dimensional, case of the Ehrhart-Macdonald reciprocity law for rational polytopes (Theorem 1.3).

In this chapter, we generalize the notion of dilated polytopes: we use the description of a convex polytope as the intersection of halfspaces, which determine the facets of the polytope. Instead of dilating the polytope by a single factor, we allow different dilation factors for each facet, such that the combinatorial type of the polytope does not change. Recall that two polytopes are **combinatorially equivalent** if there exists a bijection between their faces that preserves the inclusion relation.

For the following definition, it is a crucial fact that rational polytopes can be described by inequalities with integer coefficients.

Definition 4.1 *Let the convex rational polytope $\mathcal{P}$ be given by*

$$\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^n : \ \mathbf{A}\ \mathbf{x} \le \mathbf{b}\} \ ,$$

with $\mathbf{A} \in M_{m\times n}(\mathbb{Z}), \mathbf{b} \in \mathbb{Z}^m$. Here the inequality is understood componentwise. For $\mathbf{t} \in \mathbb{Z}^m$, define the **vector-dilated polytope** *$\mathcal{P}^{(\mathbf{t})}$ as*

$$\mathcal{P}^{(\mathbf{t})} = \{\mathbf{x} \in \mathbb{R}^n : \ \mathbf{A}\ \mathbf{x} \le \mathbf{t}\} \ .$$

For those $\mathbf{t}$ for which $P^{(\mathbf{t})}$ is combinatorially equivalent to $\mathcal{P} = \mathcal{P}^{(\mathbf{b})}$, we define the number of lattice points in the interior and closure of $\mathcal{P}^{(\mathbf{t})}$ as

$$i_{\mathcal{P}}(\mathbf{t}) = \# \left(\mathcal{P}^{(\mathbf{t})^\circ} \cap \mathbb{Z}^n \right) \qquad \text{and} \qquad j_{\mathcal{P}}(\mathbf{t}) = \# \left(\mathcal{P}^{(\mathbf{t})} \cap \mathbb{Z}^n \right) \ ,$$

respectively.

Geometrically, for a given polytope we fix the normal vectors to its facets and consider all possible translations of the facets that do not change the face structure of the polytope. Note that the dimension of $\mathbf{t}$ is the number of facets of the polytope. The previously defined quantities $L(\mathcal{P}^\circ, t)$ and $L(\overline{\mathcal{P}}, t)$ can be recovered from this new definition by choosing $\mathbf{t} = t\mathbf{b}$.

We will prove that these new lattice point count operators have again a quasi-polynomial behavior satisfying an Ehrhart-type reciprocity law. In fact, the quasi-polynomial behavior is the more difficult part of the theory. We start by proving such a reciprocity law for simplices, which implies already the classical Ehrhart-Macdonald reciprocity law (Theorem 1.3). It is worth noticing that our proof is elementary in the sense that it does not use any sophisticated machinery. In fact, the original motivation for our reciprocity law was to construct an elementary proof of Theorem 1.3. In section 4.2, we extend this reciprocity theorem to general rational polytopes.

4.1 Vector-dilated Simplices

Here is the proposed generalization of Theorems 1.2 and 1.3 for simplices.

Theorem 4.1 *Let $\mathcal{S}$ be an n-dimensional rational simplex. Then $i_{\mathcal{S}}(\mathbf{t})$ and $j_{\mathcal{S}}(\mathbf{t})$ are quasipolynomials in $\mathbf{t} \in \mathbb{Z}^{n+1}$, satisfying*

$$i_{\mathcal{S}}(-\mathbf{t}) = (-1)^n j_{\mathcal{S}}(\mathbf{t}) \ .$$

A **quasipolynomial** in the d-dimensional variable $\mathbf{t} = (t_1, \ldots, t_d)$ is the natural generalization of a quasipolynomial in a 1-dimensional variable: namely, an expression of the form

$$\sum_{0 \le k_1, \ldots, k_d \le n} c_{(k_1,\ldots,k_d)} \ t_1^{k_1} \cdots t_d^{k_d} \ ,$$

where $c_{(k_1,\ldots,k_d)} = c_{(k_1,\ldots,k_d)}(t_1, \ldots, t_d)$ is periodic in $t_1, \ldots, t_d$. We first need to prove a lemma on such expressions.

4.1.1 A Lemma on Quasipolynomials

Lemma 4.2 *Let $q(t_1, \ldots, t_m)$ be a quasipolynomial, and fix $a_1, \ldots, a_m, c_0, \ldots, c_m, d \in \mathbb{Z}, d \neq 0$. Then*

$$Q_1(\mathbf{t}) = Q_1(t_0, t_1, \ldots, t_m) = \sum_{k=1}^{\left[\frac{c_0 t_0 + \cdots + c_m t_m - 1}{d}\right]} q\left(t_1 + a_1 k, \ldots, t_m + a_m k\right)$$

and

$$Q_2(\mathbf{t}) = \sum_{k=0}^{\left[\frac{c_0 t_0 + \cdots + c_m t_m}{d}\right]} q\left(t_1 + a_1 k, \ldots, t_m + a_m k\right)$$

are also quasipolynomials.

Remark. Here and in the following we define a finite series $\sum_{k=a}^{b} \cdots$ for *both* cases $a \leq b$ and $a > b$, in the usual way:

$$\sum_{k=a}^{b} \cdots = \begin{cases} \sum_{k=a}^{b} \cdots & \text{if } a \leq b \\ 0 & \text{if } a = b+1 \\ -\sum_{k=b+1}^{a-1} \cdots & \text{if } a \geq b+2\ . \end{cases} \tag{4.2}$$

Proof. We will prove the statement for Q_2; the proof for Q_1 follows in a similar fashion. After expanding out q in all its terms and multiplying out the binomial expressions, it suffices to prove that

$$Q_3(\mathbf{t}) = \sum_{k=0}^{\left[\frac{c_0 t_0 + \cdots + c_m t_m}{d}\right]} f\left(t_1 + a_1 k, \ldots, t_m + a_m k\right)\ k^j$$

is a quasipolynomial, where j is a fixed nonnegative integer and f is a periodic function in m variables. Consider a period p which is common to all the arguments of f, that is, $f\left(x_1 + p, \ldots, x_m + p\right) = f\left(x_1, \ldots, x_m\right)$. To see that Q_3 is a quasipolynomial, use the properties of f to write it as

$$\begin{aligned} Q_3(\mathbf{t}) = f\left(t_1, \ldots, t_m\right) & \sum_{k=0}^{\left[\frac{c_0 t_0 + \cdots + c_m t_m}{dp}\right]} (kp)^j \\ + f\left(t_1 + a_1, \ldots, t_m + a_m\right) & \sum_{k=0}^{\left[\frac{c_0 t_0 + \cdots + c_m t_m - d}{dp}\right]} (1 + kp)^j\ + \\ + f\left(t_1 + 2a_1, \ldots, t_m + 2a_m\right) & \sum_{k=0}^{\left[\frac{c_0 t_0 + \cdots + c_m t_m - 2d}{dp}\right]} (2 + kp)^j\ +\ \ldots\ + \\ + f\Big(t_1 + (p-1)a_1, \ldots, t_m + (p-1)a_m\Big) & \sum_{k=0}^{\left[\frac{c_0 t_0 + \cdots + c_m t_m - (p-1)d}{dp}\right]} (p - 1 + kp)^j\ . \end{aligned}$$

Upon expanding all of the binomials, putting the finite sums into closed forms, and writing $[x] = x - \{x\}$, the only dependency on $\mathbf{t}$ is periodic (with period dividing dp) or polynomial. □

4.1.2 Proofs of Theorems 4.1 and 1.3

Proof of Theorem 4.1. We use induction on the dimension n. First, a 1-dimensional rational simplex $\mathcal{S}$ is an interval with rational endpoints. Hence $\mathcal{S}^{(\mathbf{t})}$ is given by

$$\frac{t_1}{a_1} \le x \le \frac{t_2}{a_2} \ ,$$

so that we obtain

$$i_{\mathcal{S}}(\mathbf{t}) = \left[\frac{t_2 - 1}{a_2}\right] - \left[\frac{t_1}{a_1}\right] \quad \text{and} \quad j_{\mathcal{S}}(\mathbf{t}) = \left[\frac{t_2}{a_2}\right] - \left[\frac{t_1 - 1}{a_1}\right] \ .$$

These are quasipolynomials, as can be seen by writing $[x] = x - \{x\}$. Furthermore, by (4.1),

$$i_{\mathcal{S}}(-\mathbf{t}) = \left[\frac{-t_2 - 1}{a_2}\right] - \left[\frac{-t_1}{a_1}\right] = -\left[\frac{t_2}{a_2}\right] + \left[\frac{t_1 - 1}{a_1}\right] = -j_{\mathcal{S}}(\mathbf{t}) \ .$$

Now, let $\mathcal{S}$ be an n-dimensional rational simplex. After harmless unimodular transformations, which leave the lattice point count invariant, we may assume that the defining inequalities for $\mathcal{S}$ are

$$\begin{array}{rcccccl} a_{11}x_1 & & & & & \le & b_1 \\ a_{21}x_1 & + & \dots & + & a_{2n}x_n & \le & b_2 \\ \vdots & & & & & & \\ a_{n+1,1}x_1 & + & \dots & + & a_{n+1,n}x_n & \le & b_{n+1} \ . \end{array}$$

(Actually, we could obtain a lower triangular form for $\mathbf{A}$ by Hermite normal form; however, the above form suffices for our purposes.) Hence there exists a vertex $\mathbf{v} = (v_1, \dots, v_n)$ with $v_1 = \frac{b_1}{a_{11}}$ and another vertex $\mathbf{w} = (w_1, \dots, w_n)$ whose first component is not $\frac{b_1}{a_{11}}$. After switching x_1 to $-x_1$, if necessary, we may further assume that $v_1 < w_1$. Since $\mathbf{w}$ satisfies all equalities but the first one, it is not hard to see that $\mathbf{w}$ has first component $w_1 = r_2 b_2 + \cdots + r_n b_n$, for some rational numbers $r_2, \dots, r_n$; write this number as $w_1 = \frac{c_2 b_2 + \cdots + c_n b_n}{d}$ with $c_2, \dots, c_n, d \in \mathbb{Z}$. Viewing the defining inequalities of the vector-dilated simplex $\mathcal{S}^{(\mathbf{t})}$ as

$$\begin{array}{rcccccl} & & \frac{t_1}{a_{11}} & \le & x_1 & \le & \frac{c_2 t_2 + \cdots + c_n t_n}{d} \\ a_{22}x_2 & + & \dots & + & a_{2n}x_n & \le & t_2 - a_{21}x_1 \\ \vdots & & & & & & \\ a_{n+1,2}x_2 & + & \dots & + & a_{n+1,n}x_n & \le & t_{n+1} - a_{n+1,1}x_1 \ , \end{array}$$

we can compute the number of lattice points in the interior and closure of $\mathcal{S}^{(\mathbf{t})}$ as

$$i_{\mathcal{S}}(\mathbf{t}) = \sum_{m=\left[\frac{t_1}{a_{11}}\right]+1}^{\left[\frac{c_2t_2+\cdots+c_nt_n-1}{d}\right]} i_{\mathcal{Q}}\left(t_2 - a_{21}m, \ldots, t_{n+1} - a_{n+1,1}m\right) \tag{4.3}$$

and

$$j_{\mathcal{S}}(\mathbf{t}) = \sum_{m=\left[\frac{t_1-1}{a_{11}}\right]+1}^{\left[\frac{c_2t_2+\cdots+c_nt_n}{d}\right]} j_{\mathcal{Q}}\left(t_2 - a_{21}m, \ldots, t_{n+1} - a_{n+1,1}m\right) \ , \tag{4.4}$$

respectively, where the $(n-1)$-dimensional simplex $\mathcal{Q}^{(\mathbf{b})}$ is given by

$$\mathcal{Q}^{(\mathbf{b})} = \left\{\mathbf{x} \in \mathbb{R}^{n-1} : \ \mathbf{B}\,\mathbf{x} \leq \mathbf{b}\right\} \ ,$$

and

$$\mathbf{B} = \begin{pmatrix} a_{22} & \ldots & a_{2n} \\ & \vdots & \\ a_{n+1,2} & \ldots & a_{n+1,n} \end{pmatrix} \in M_{n\times(n-1)}(\mathbb{Z}) \ .$$

Note that if we start with some $\mathbf{t} \in \mathbb{Z}^{n+1}$ which satisfies Definition 4.1, then the dilation parameters for $\mathcal{Q}$ in (4.3) and (4.4) will ensure that the lattice point count operators are well defined. $i_{\mathcal{Q}}(\mathbf{t})$ and $j_{\mathcal{Q}}(\mathbf{t})$ are, by induction hypothesis, quasipolynomials satisfying the reciprocity law Theorem 4.1. Hence, by Lemma 4.2, $i_{\mathcal{S}}(\mathbf{t})$ and $j_{\mathcal{S}}(\mathbf{t})$ are also quasipolynomials. Note that we again use (4.2) to define these expressions for all $\mathbf{t} \in \mathbb{Z}^{n+1}$. Furthermore,

$$\begin{aligned}
i_{\mathcal{S}}(-\mathbf{t}) &= \sum_{m=\left[\frac{-t_1}{a_{11}}\right]+1}^{\left[\frac{-c_2t_2-\cdots-c_nt_n-1}{d}\right]} i_{\mathcal{Q}}\left(-t_2 - a_{21}m, \ldots, -t_{n+1} - a_{n+1,1}m\right) \\
&\overset{\text{Thm. 4.1}}{=} -\sum_{\left[\frac{-c_2t_2-\cdots-c_nt_n-1}{d}\right]+1}^{\left[\frac{-t_1}{a_{11}}\right]} (-1)^{n-1} j_{\mathcal{Q}}\left(t_2 + a_{21}m, \ldots, t_{n+1} + a_{n+1,1}m\right) \\
&\overset{(4.1)}{=} (-1)^n \sum_{m=-\left[\frac{c_2t_2+\cdots+c_nt_n}{d}\right]}^{-\left[\frac{t_1-1}{a_{11}}\right]-1} j_{\mathcal{Q}}\left(t_2 + a_{21}m, \ldots, t_{n+1} + a_{n+1,1}m\right) \\
&= (-1)^n \sum_{m=\left[\frac{t_1-1}{a_{11}}\right]+1}^{\left[\frac{c_2t_2+\cdots+c_nt_n}{d}\right]} j_{\mathcal{Q}}\left(t_2 - a_{21}m, \ldots, t_{n+1} - a_{n+1,1}m\right) = (-1)^n j_{\mathcal{S}}\left(\mathbf{t}\right) \ .
\end{aligned}$$

□

We are finally in a position that allows us to prove the classical Ehrhart-Macdonald reciprocity law.

Proof of Theorem 1.3. We use double induction on the dimension of the polytope $\mathcal{P}$ and on the number of n-dimensional simplices which triangulate $\mathcal{P}$. We saw already that Theorem 1.3 follows for 1-dimensional polytopes (that is, intervals) from (4.1). Also, Theorem 1.3 holds for simplices, as a special case of Theorem 4.1. For a general $\mathcal{P}$ satisfying the hypotheses of the statement, write

$$\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \ ,$$

where $\mathcal{P}_1$ is an n-dimensional simplex such that $\mathcal{P}_2 := \overline{\mathcal{P} - \mathcal{P}_1}$ is again a polytope homeomorphic to an n-manifold. Note that the conditions on $\mathcal{P}$ imply that $\mathcal{P}_1$ and $\mathcal{P}_2$ share an $(n-1)$-dimensional polytopal boundary, which we denote by $\mathcal{P}_3$. Hence

$$L(\overline{\mathcal{P}}, t) = L(\overline{\mathcal{P}_1}, t) + L(\overline{\mathcal{P}_2}, t) - L(\overline{\mathcal{P}_3}, t)$$

and

$$L(\mathcal{P}^\circ, t) = L(\mathcal{P}_1^\circ, t) + L(\mathcal{P}_2^\circ, t) + L(\mathcal{P}_3^\circ, t) \ .$$

By induction, we can apply Theorem 1.3 to $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$:

$$\begin{aligned} L(\overline{\mathcal{P}}, -t) &= L(\overline{\mathcal{P}_1}, -t) + L(\overline{\mathcal{P}_2}, -t) - L(\overline{\mathcal{P}_3}, -t) \\ &= (-1)^n L(\mathcal{P}_1^\circ, t) + (-1)^n L(\mathcal{P}_2^\circ, t) - (-1)^{n-1} L(\mathcal{P}_3^\circ, t) \\ &= (-1)^n L(\mathcal{P}^\circ, t) \ . \end{aligned}$$

□

4.1.3 Some Remarks and an Example

An obvious generalization of Theorem 4.1 is a similar statement for arbitrary rational polytopes (with any number of facets). This is the theme of section 4.2.

Another variation of the idea of vector-dilating a polytope is to dilate the *vertices* by certain factors, instead of the facets. This would most certainly require methods completely different from the ones used here.

It is, finally, of interest to compute precise formulas (that is, the coefficients of the quasipolynomials) for $i_{\mathcal{S}}(\mathbf{t})$ and $j_{\mathcal{S}}(\mathbf{t})$, corresponding to the various existing formulas for $L(\mathcal{P}^\circ, t)$ and $L\left(\overline{\mathcal{P}}, t\right)$.

To illustrate this, we will compute $j_{\mathcal{S}}(\mathbf{t})$ for a two-dimensional rectangular rational triangle, namely,

$$\mathcal{S} = \left\{ \mathbf{x} \in \mathbb{R}^2 : \begin{array}{ccccc} a_1 x_1 & & & \geq & 1 \\ & & a_2 x_2 & \geq & 1 \\ c_1 x_1 & + & c_2 x_2 & \leq & 1 \end{array} \right\} .$$

Here, a_1, a_2, c_1, c_2 are positive integers; we will also assume that c_1 and c_2 are relatively prime. To derive a formula for $j_{\mathcal{S}}(\mathbf{t})$ we can use once more the methods of chapter 2. A straightforward computation yields

$$
\begin{aligned}
j_{\mathcal{S}}(\mathbf{t}) = & \frac{1}{2c_1c_2}(e_1+e_2-t_3)^2 - \frac{1}{2}(e_1+e_2-t_3)\left(\frac{1}{c_1}+\frac{1}{c_2}+\frac{1}{c_1c_2}\right) \\
& + \frac{1}{4} + \frac{1}{12}\left(\frac{c_1}{c_2}+\frac{c_2}{c_1}+\frac{1}{c_1c_2}\right) + \sum_{k=0}^{c_1-1}\left(\left(\frac{t_3-e_2-c_2k}{c_1}\right)\right)\left(\left(\frac{k}{c_1}\right)\right) \\
& + \sum_{k=0}^{c_2-1}\left(\left(\frac{t_3-e_1-c_1k}{c_2}\right)\right)\left(\left(\frac{k}{c_2}\right)\right) .
\end{aligned}
$$

Here we introduced, for ease of notation, $e_j := \left(\left[\frac{t_j-1}{a_j}\right]+1\right)c_j$ for $j = 1, 2$. To see the quasipolynomial character better, we substitute back the expressions for e_1 and e_2, and write $[x] = x - ((x)) - 1/2$ for the greatest integer function. After a somewhat tedious calculation, we obtain

$$
\begin{aligned}
j_{\mathcal{S}}(\mathbf{t}) = & \frac{c_1}{2a_1^2c_2}t_1^2 + \frac{c_2}{2a_2^2c_1}t_2^2 + \frac{1}{2c_1c_2}t_3^2 + \frac{1}{a_1a_2}t_1t_2 - \frac{1}{a_1c_2}t_1t_3 - \frac{1}{a_2c_1}t_2t_3 \\
& + \nu_1(\mathbf{t})\ t_1 + \nu_2(\mathbf{t})\ t_2 + \nu_3(\mathbf{t})\ t_3 + \nu_0(\mathbf{t}) ,
\end{aligned}
$$

where

$$
\begin{aligned}
\nu_1(\mathbf{t}) = & -\frac{c_1}{a_1^2c_2}\left(1+\left(\left(\frac{t_1-1}{a_1}\right)\right)\right) - \frac{1}{a_1}\left(\left(\frac{t_2-1}{a_2}\right)\right) - \frac{1}{a_1a_2} - \frac{1}{2a_1c_2} \\
\nu_2(\mathbf{t}) = & -\frac{c_2}{a_2^2c_1}\left(1+\left(\left(\frac{t_2-1}{a_2}\right)\right)\right) - \frac{1}{a_2}\left(\left(\frac{t_1-1}{a_1}\right)\right) - \frac{1}{a_1a_2} - \frac{1}{2a_2c_1} \\
\nu_3(\mathbf{t}) = & \frac{1}{a_1c_2} + \frac{1}{a_2c_1} + \frac{1}{2c_1c_2} + \frac{1}{c_2}\left(\left(\frac{t_1-1}{a_1}\right)\right) + \frac{1}{c_1}\left(\left(\frac{t_2-1}{a_2}\right)\right) \\
\nu_0(\mathbf{t}) = & -\frac{1}{4c_1} - \frac{1}{4c_2} + \frac{1}{a_1a_2} + \frac{1}{2a_1c_2} + \frac{1}{2a_2c_1} + \frac{1}{12c_1c_2} - \frac{c_1}{24c_2} - \frac{c_2}{24c_1} \\
& + \frac{c_1}{2a_1^2c_2} + \frac{c_2}{2a_2^2c_1} + \left(\left(\frac{t_1-1}{a_1}\right)\right)\left(\frac{1}{a_2}+\frac{1}{2c_2}+\frac{c_1}{a_1c_2}\right) \\
& + \left(\left(\frac{t_2-1}{a_2}\right)\right)\left(\frac{1}{a_1}+\frac{1}{2c_1}+\frac{c_2}{a_2c_1}\right) + \frac{c_1}{2c_2}\left(\left(\frac{t_1-1}{a_1}\right)\right)^2 \\
& + \frac{c_2}{2c_1}\left(\left(\frac{t_2-1}{a_2}\right)\right)^2 + \left(\left(\frac{t_1-1}{a_1}\right)\right)\left(\left(\frac{t_2-1}{a_2}\right)\right) \\
& + \sum_{k=0}^{c_1-1}\left(\left(\frac{t_3}{c_1} - \frac{t_2-1}{a_2c_1} + \frac{1}{c_1}\left(\left(\frac{t_2-1}{a_2}\right)\right) - \frac{1}{2c_1} - \frac{c_2k}{c_1}\right)\right)\left(\left(\frac{k}{c_1}\right)\right) \\
& + \sum_{k=0}^{c_2-1}\left(\left(\frac{t_3}{c_2} - \frac{t_1-1}{a_1c_2} + \frac{1}{c_2}\left(\left(\frac{t_1-1}{a_1}\right)\right) - \frac{1}{2c_2} - \frac{c_1k}{c_2}\right)\right)\left(\left(\frac{k}{c_2}\right)\right) .
\end{aligned}
$$

4.2 Extension to General Polytopes

In this section, we finish the picture by extending Theorem 4.1 to general rational polytopes. We should start by extending Definition 4.1 to nonconvex polytopes. This can be done naturally in an additive way: write the polytope as the union of convex polytopes, and apply the above Definition 4.1 to these components. More thoroughly, we make the following

Definition 4.2 *Let $\mathcal{P}$ be a rational polytope. Write $\mathcal{P} = \bigcup_{k=1}^{r} \mathcal{P}_k$, where $\mathcal{P}_k$ are convex rational polytopes, say,*

$$\mathcal{P}_k = \{\mathbf{x} \in \mathbb{R}^n : \ \mathbf{A}_k \ \mathbf{x} \le \mathbf{b}_k\} \ ,$$

with $\mathbf{b}_k \in \mathbb{Z}^{m_k}$. Given $\mathbf{t} \in \mathbb{Z}^m$, where $m = m_1 + \cdots + m_r$, combine the first m_1 components of $\mathbf{t}$ in a vector $\mathbf{t}_1$, the next m_2 components in $\mathbf{t}_2$, etc. Define the **vector-dilated polytope** *$\mathcal{P}^{(\mathbf{t})}$ as*

$$\mathcal{P}^{(\mathbf{t})} = \bigcup_{k=1}^{r} \mathcal{P}_k^{(\mathbf{t}_k)} \ .$$

For those $\mathbf{t}$ for which $P^{(\mathbf{t})}$ is combinatorially equivalent to $\mathcal{P}$, we define as above

$$i_{\mathcal{P}}(\mathbf{t}) = \# \left(\mathcal{P}^{(\mathbf{t})^\circ} \cap \mathbb{Z}^n \right) \quad \text{and} \quad j_{\mathcal{P}}(\mathbf{t}) = \# \left(\mathcal{P}^{(\mathbf{t})} \cap \mathbb{Z}^n \right) \ .$$

4.2.1 Extending Ehrhart Reciprocity

From the Ehrhart-Macdonald reciprocity law we will now conclude a generalized version of Theorem 4.1:

Theorem 4.3 *Suppose the rational polytope $\mathcal{P}$ is homeomorphic to an n-manifold. Then $i_{\mathcal{P}}(\mathbf{t})$ and $j_{\mathcal{P}}(\mathbf{t})$ are quasipolynomials in $\mathbf{t} \in \mathbb{Z}^m$, satisfying*

$$i_{\mathcal{P}}(-\mathbf{t}) = (-1)^n j_{\mathcal{P}}(\mathbf{t}) \ .$$

Proof. It suffices to prove that $i_{\mathcal{P}}(\mathbf{t})$ and $j_{\mathcal{P}}(\mathbf{t})$ are quasipolynomials. In fact, *once we know this*, the statement follows from Theorem 1.3:

$$i_{\mathcal{P}}(-\mathbf{t}) = L\left(\mathcal{P}^{(\mathbf{t})^\circ}, -1\right) = (-1)^n L\left(\overline{\mathcal{P}^{(\mathbf{t})}}, 1\right) = (-1)^n j_{\mathcal{P}}(\mathbf{t}) \ .$$

To show that our lattice point count operators are quasipolynomials, it clearly suffices to prove that $i_{\mathcal{P}}(\mathbf{t})$ and $j_{\mathcal{P}}(\mathbf{t})$ are quasipolynomials in one of the components of $\mathbf{t}$, say t_1. Because we leave only this one component variable, we may also assume that $\mathcal{P}$ is convex. We make a unimodular transformation (which leaves the lattice invariant) similar to that in the last section: we may assume that the defining inequalities for $\mathcal{P}^{(\mathbf{t})}$ are

$$\begin{array}{ccccccc} a_{11}x_1 & & & & & \le & t_1 \\ a_{21}x_1 & + & \dots & + & a_{2n}x_n & \le & t_2 \\ & \vdots & & & & & \\ a_{m,1}x_1 & + & \dots & + & a_{m,n}x_n & \le & t_m \ . \end{array}$$

(Again, we could obtain a lower triangular form.) Viewing these inequalities as

$$\begin{array}{rcccccl} & & & & x_1 & \leq & \frac{t_1}{a_{11}} \\ a_{22}x_2 & + & \dots & + & a_{2n}x_n & \leq & t_2 - a_{21}x_1 \\ & \vdots & & & & & \\ a_{m,2}x_2 & + & \dots & + & a_{m,n}x_n & \leq & t_m - a_{m,1}x_1 \ , \end{array}$$

we can compute the number of lattice points in the interior and closure of $\mathcal{P}^{(\mathbf{t})}$ as

$$i_{\mathcal{P}}(\mathbf{t}) = \sum_{k=s_1}^{\left[\frac{t_1-1}{a_{11}}\right]} i_{\mathcal{Q}}\left(t_2 - a_{21}k, \dots, t_m - a_{m,1}k\right) \tag{4.5}$$

and

$$j_{\mathcal{P}}(\mathbf{t}) = \sum_{k=s_2}^{\left[\frac{t_1}{a_{11}}\right]} j_{\mathcal{Q}}\left(t_2 - a_{21}k, \dots, t_m - a_{m,1}k\right) \ , \tag{4.6}$$

respectively. Here s_1 and s_2 are rational numbers not depending on t_1, and the $(n-1)$-dimensional polytope $\mathcal{Q}^{(\mathbf{b})}$ is given by

$$\mathcal{Q}^{(\mathbf{b})} = \left\{\mathbf{x} \in \mathbb{R}^{n-1} : \ \mathbf{B}\,\mathbf{x} \leq \mathbf{b}\right\} \ ,$$

where

$$\mathbf{B} = \begin{pmatrix} a_{22} & \dots & a_{2n} \\ & \vdots & \\ a_{m,2} & \dots & a_{m,n} \end{pmatrix} \in M_{(m-1)\times(n-1)}(\mathbb{Z}) \ .$$

The functions $i_{\mathcal{Q}}$ and $j_{\mathcal{Q}}$, which are summed in (4.5) and (4.6), are constant in t_1. Thus we only need a weak form of Lemma 4.2 to deduce that $i_{\mathcal{P}}(\mathbf{t})$ and $j_{\mathcal{P}}(\mathbf{t})$ are quasipolynomials in t_1. □

At this point, we find it appropriate to remark why we did not simply start the notion of vector-dilated polytopes with this proof, assuming classical Ehrhart-Macdonald reciprocity. The point of section 4.1 (or at least half of it) was really to give an *elementary* proof of Theorem 1.3. It is for this reason that we chose to build our proof of Theorem 4.3 upon the work in section 4.1. The course of the proof looks like the following diagram:

$$(4.1) \implies \text{Theorem 4.1} \implies \text{Theorem 1.3} \implies \text{Theorem 4.3}$$

4.2.2 Extending Stanley's Theorem

We conclude by proving an appropriate generalization of the following theorem due to Stanley ([St1]). The Ehrhart-Macdonald reciprocity law compares the lattice point count of the polytope with that of the interior, that is, the polytope with all its facets removed. Stanley's theorem tells us what to expect if we only remove *some* of the facets.

Theorem 4.4 (Stanley) *Suppose the rational polytope $\mathcal{P}$ is homeomorphic to an n-manifold. Denote the set of all (closed) facets of $\mathcal{P}$ by F, and let T be a subset of F, such that $\bigcup_{\mathcal{F}\in T}\mathcal{F}$ is homeomorphic to an $(n-1)$-manifold. Let*

$$j_{\mathcal{P},T}(t) = \#\left(t\left(\mathcal{P} - \bigcup_{\mathcal{F}\in T}\mathcal{F}\right)\cap\mathbb{Z}^n\right)$$

and

$$i_{\mathcal{P},T}(t) = \#\left(t\left(\mathcal{P} - \bigcup_{\mathcal{F}\in F-T}\mathcal{F}\right)\cap\mathbb{Z}^n\right) .$$

Then

$$i_{\mathcal{P},T}(-t) = (-1)^n j_{\mathcal{P},T}(t) .$$

□

Note that Theorem 1.3 is the special case $T=\emptyset$ of Theorem 4.4. For an example which shows that this result does not hold in general, see [St1].

Our generalization will be proved essentially in the same way Stanley deduced Theorem 4.4 from Theorem 1.3.

Corollary 4.5 *Suppose the rational polytope $\mathcal{P}$ is homeomorphic to an n-manifold. Denote the set of all (closed) facets of $\mathcal{P}$ by F, and let T be a subset of F, such that $\bigcup_{\mathcal{F}\in T}\mathcal{F}$ is homeomorphic to an $(n-1)$-manifold. Let*

$$j_{\mathcal{P},T}(\mathbf{t}) = \#\left(\left(\mathcal{P}^{(\mathbf{t})} - \bigcup_{\mathcal{F}\in T}\mathcal{F}^{(\mathbf{t})}\right)\cap\mathbb{Z}^n\right)$$

and

$$i_{\mathcal{P},T}(\mathbf{t}) = \#\left(\left(\mathcal{P}^{(\mathbf{t})} - \bigcup_{\mathcal{F}\in F-T}\mathcal{F}^{(\mathbf{t})}\right)\cap\mathbb{Z}^n\right) .$$

Then

$$i_{\mathcal{P},T}(-\mathbf{t}) = (-1)^n j_{\mathcal{P},T}(\mathbf{t}) .$$

Again, note that Theorem 4.3 is the special case $T=\emptyset$ of this corollary.

Proof. By definition,

$$j_{\mathcal{P},T}(\mathbf{t}) = j_{\mathcal{P}}(\mathbf{t}) - \sum_{\mathcal{F}\in T} j_{\mathcal{F}}(\mathbf{t})$$

and

$$i_{\mathcal{P},T}(\mathbf{t}) = j_{\mathcal{P}}(\mathbf{t}) - \sum_{\mathcal{F}\in F-T} j_{\mathcal{F}}(\mathbf{t}) = i_{\mathcal{P}}(\mathbf{t}) + \sum_{\mathcal{F}\in T} i_{\mathcal{F}}(\mathbf{t}) .$$

Hence by Theorem 4.3,

$$i_{\mathcal{P},T}(-\mathbf{t}) = (-1)^n j_{\mathcal{P}}(\mathbf{t}) + \sum_{\mathcal{F}\in T}(-1)^{n-1} j_{\mathcal{F}}(\mathbf{t}) = (-1)^n j_{\mathcal{P},T}(\mathbf{t}) .$$

□

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