IDENTITIES BETWEEN HECKE EIGENFORMS

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ABSTRACT

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In this dissertation, we study solutions to certain low degree polynomials in terms of Hecke eigenforms. We show that the number of solutions to the equation $h = af^2 + bfg + g^2$ is finite for all N, where f, g, h are Hecke newforms with respect to $\Gamma_1(N)$ of weight $k > 2$ and $a, b \neq 0$. Using polynomial identities between Hecke eigenforms, we give another proof that the j-function is algebraic on zeros of Eisenstein series of weight $12k$. Assuming Maeda's conjecture, we prove that the Petersson inner product $\langle f^2, g \rangle$ is nonzero, where f and g are any nonzero cusp eigenforms for $SL_2(\mathbb{Z})$ of weight k and $2k$, respectively. As a corollary, we obtain that, assuming Maeda's conjecture, identities between cusp eigenforms for $SL_2(\mathbb{Z})$ of the form $X^2 + \sum_{i=1}^{n} \alpha_i Y_i = 0$ all are forced by

dimension considerations, i.e., a square of an eigenform for the full modular group is unbiased. We show by an example that this property does not hold in general for a congruence subgroup. Finally we attach our Sage code in the appendix.

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To my teachers

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CHAPTER 1

INTRODUCTION

Fourier coefficients of Hecke eigenforms often encode important arithmetic information. Given an identity between eigenforms, one obtains nontrivial relations between their Fourier coefficients and may further obtain solutions to certain related problems in number theory. For instance, let $\tau(n)$ be the nth Fourier coefficient of the weight 12 cusp form Δ for $SL_2(\mathbb{Z})$ given by

$$\Delta = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n)q^n$$

and define the weight 11 divisor sum function $\sigma_{11}(n) = \sum_{d|n} d^{11}$. Then the Ramanujan congruence

$$\tau(n) \equiv \sigma_{11}(n) \mod 691$$

can be deduced easily from the identity

$$E_{12} - E_6^2 = \frac{1008 \times 756}{691}\Delta,$$

where E_6 (respectively E_{12}) is the Eisenstein series of weight 6 (respectively 12) for $SL_2(\mathbb{Z})$.

Specific polynomial identities between Hecke eigenforms have been studied by many authors. Duke [9] and Ghate [11] independently investigated identities of the form $h = fg$ between eigenforms with respect to the full modular group $\Gamma = SL_2(\mathbb{Z})$ and proved that there are only 16 such identities. Emmons [10] extended the search to $\Gamma_0(p)$ and found 8 new cases. Later Johnson [13] studied the above equation for Hecke eigenforms with respect to $\Gamma_1(N)$ and obtained a complete list of 61 eigenform product identities. J. Beyerl and K. James and H. Xue [3] studied the problem of when an eigenform for $SL_2(\mathbb{Z})$ is divisible by another eigenform and proved that this can only occur in very special cases. Recently, Richey and Shutty [19] studied polynomial identities and showed that for a fixed polynomial (excluding trivial ones), there are only finitely many decompositions of normalized Hecke eigenforms for $SL_2(\mathbb{Z})$ described by a given polynomial.

Since the product of two eigenforms is rarely an eigenform, in this thesis we loosen the constraint and study solutions to the equation $h = P(f, g)$, where

h, f, g are Hecke eigenforms and P is a general degree two polynomial. The ring

of modular forms is graded by weight, therefore P is necessarily homogeneous,

i.e., $P(f, g) = af^2 + bfg + cg^2$. With proper normalization, we can assume

$c = 1$.

The first main result of this thesis is the following:

Theorem 1.0.1. *For all $N \in \mathbb{Z}_+$ and $a, b \in \mathbb{C}^\times$, there are at most finitely*

triples (f, g, h) of newforms with respect to $\Gamma_1(N)$ of weight $k > 2$ satisfying

the equation

$$h = af^2 + bfg + g^2. \tag{1.1}$$

Theorem 1.0.1 is obtained by estimating the Fourier coefficients of cusp eigen-

forms and its proof is presented in Chapter 3. In the same chapter, as further

motivation for studying identities between eigenforms, we also give an elemen-

tary proof via identities of the following theorem:

Theorem 1.0.2. *Let $E_{12k}(z)$ be the Eisenstein series of weight $12k$, and*

$$A := \{\rho \in \mathfrak{F} | E_{12k}(\rho) = 0 \text{ for some } k \in \mathbb{Z}_+\}$$

be the set of zeros for all Eisenstein series of weight $12k$ with $k \geq 1$ in the

fundamental domain $\mathfrak{F}$ for $SL_2(\mathbb{Z})$. If j is the modular j-function and $\rho \in A$,

then $j(\rho)$ is an algebraic number.

For a stronger result, which says that $j(\rho)$ is algebraic for any zero ρ of a

meromorphic modular form $f = \sum_{n=h}^{\infty} a_n q^n$ for $SL_2(\mathbb{Z})$ with $a_n(h) = 1$ for which all coefficients lie in a number field, see Corollary 2 of the paper by Bruinier, Kohnen and Ono [4].

Fourier coefficients of a normalized Hecke eigenform are all algebraic integers. As for Galois symmetry between Fourier coefficients of normalized eigenforms in the space $\mathcal{S}_k$ of cusp forms for $SL_2(\mathbb{Z})$ of weight $k \geq 12$, Maeda (see [12] Conjecture 1.2) conjectured that:

Conjecture 1.0.3. *The Hecke algebra over $\mathbb{Q}$ of $\mathcal{S}_k$ is simple (that is, a single number field) whose Galois closure over $\mathbb{Q}$ has Galois group isomorphic to the symmetric group $\mathfrak{S}_n$, where $n = \dim \mathcal{S}_k$.*

Maeda's conjecture can be used to study polynomial identities between Hecke eigenforms. See J.B. Conrey and D.W. Farmer [6] and J. Beyerl and K. James and H. Xue [3] for some previous work. In this paper, we prove the following:

Theorem 1.0.4. *Assuming Maeda's conjecture for $\mathcal{S}_k$ and $\mathcal{S}_{2k}$, if $f \in \mathcal{S}_k$, $g \in \mathcal{S}_{2k}$ are any nonzero cusp eigenforms, then the Petersson inner product $\langle f^2, g \rangle$ is nonzero.*

Theorem 1.0.4 says that, assuming Maeda's conjecture, the square of a cusp eigenform for $SL_2(\mathbb{Z})$ is "unbiased", i.e., it does not lie in the subspace spanned

by a proper subset of an eigenbasis. The proof of Theorem 1.0.4 is presented

in Section 4.

CHAPTER 2

PRELIMINARIES AND

CONVENTIONS

A standard reference for this section is [8]. Let f be a holomorphic function on the upper half plane $\mathcal{H} = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$. Then define the slash operator $[\gamma]_k$ of weight k as

$$f|_k[\gamma](z) := (cz + d)^{-k} f(\gamma z).$$

If Γ is a subgroup of $SL_2(\mathbb{Z})$, we say that f is a modular form of weight k with respect to Γ if $f|_k[\gamma](z) = f(z)$ for all $z \in \mathcal{H}$ and all $\gamma \in \Gamma$. Define

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod N \right\},$$

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \middle| c \equiv 0 \mod N \right\}.$$

Denote by $\mathcal{M}_k(\Gamma_1(N))$ the vector space of weight k modular forms with respect to $\Gamma_1(N)$.

Let χ be a Dirichlet character mod N, i.e., a homomorphism $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$. By convention one extends the definition of χ to $\mathbb{Z}$ by defining $\chi(m) = 0$ for $\gcd(m, N) > 1$. In particular the trivial character modulo N extends to the function

$$\mathbb{1}_N(n) = \begin{cases} 1 & \text{if } \gcd(n, N) = 1, \\ 0 & \text{if } \gcd(n, N) > 1. \end{cases}$$

Take $n \in (\mathbb{Z}/N\mathbb{Z})^\times$, and define the diamond operator

$$\langle n \rangle : \mathcal{M}_k(\Gamma_1(N)) \to \mathcal{M}_k(\Gamma_1(N))$$

as $\langle n \rangle f := f|_k[\gamma]$ for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ with $d \equiv n \mod N$. Define the χ eigenspace

$$\mathcal{M}_k(N, \chi) := \{ f \in \mathcal{M}_k(\Gamma_1(N)) | \langle n \rangle f = \chi(n)f, \forall n \in (\mathbb{Z}/N\mathbb{Z})^\times \}.$$

Then one has the following decomposion:

$$\mathcal{M}_k(\Gamma_1(N)) = \bigoplus_\chi \mathcal{M}_k(N, \chi),$$

where χ runs through all Dirichlet characters mod N such that $\chi(-1) = (-1)^k$.

See Section 5.2 of [8] for details.

Let $f \in \mathcal{M}_k(\Gamma_1(N))$, since $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma_1(N)$, one has a Fourier expansion:

$$f(z) = \sum_{n=0}^\infty a_n(f)q^n, \quad q = e^{2\pi i z}.$$

If $a_0 = 0$ in the Fourier expansion of $f|_k[\alpha]$ for all $\alpha \in SL_2(\mathbb{Z})$, then f is called

a cusp form. Denote the vector space of weight k cusp forms for Γ by $\mathcal{S}_k(\Gamma)$

For a modular form $f \in \mathcal{M}_k(\Gamma_1(N))$ the Petersson inner product of f with

$g \in \mathcal{S}_k(\Gamma_1(N))$ is defined by the formula

$$\langle f, g \rangle = \int_{\mathcal{H}/\Gamma_1(N)} \overline{f(z)}g(z)y^{k-2}dxdy,$$

where $z = x + iy$ and $\mathcal{H}/\Gamma_1(N)$ is the quotient space. See [8] Section 5.4.

We define the space $\mathcal{E}_k(\Gamma_1(N))$ of weight k Eisenstein series with respect to

$\Gamma_1(N)$ as the orthogonal complement of $\mathcal{S}_k(\Gamma_1(N))$ with respect to the Peters-

son inner product, i.e., one has the following orthogonal decomposition:

$$\mathcal{M}_k(\Gamma_1(N)) = \mathcal{S}_k(\Gamma_1(N)) \oplus \mathcal{E}_k(\Gamma_1(N)).$$

We also have decomposition of the cusp space and the space of Eisenstein

series in terms of χ eigenspaces:

$$\mathcal{S}_k(\Gamma_1(N)) = \bigoplus_\chi \mathcal{S}_k(N,\chi),$$

$$\mathcal{E}_k(\Gamma_1(N)) = \bigoplus_\chi \mathcal{E}_k(N,\chi).$$

See Section 5.11 of [8].

For two Dirichlet characters ψ modulo u and φ modulo v such that $uv|N$ and $(\psi\varphi)(-1) = (-1)^k$ define

$$E_k^{\psi,\varphi}(z) = \delta(\psi)L(1-k,\varphi) + 2\sum_{n=1}^\infty \sigma_{k-1}^{\psi,\varphi}(n)q^n, \qquad (2.1)$$

where $\delta(\psi)$ is 1 if $\psi = \mathbb{1}_1$ and 0 otherwise,

$$L(1-k,\varphi) = -\frac{B_{k,\varphi}}{k}$$

is the special value of the Dirichlet L function at $1-k$, $B_{k,\varphi}$ is the kth generalized Bernoulli number defined by the equality

$$\sum_{k\geq 0} B_{k,\varphi}\frac{t^k}{k!} := \sum_{a=0}^{N-1} \frac{\varphi(a)te^{at}}{e^{Nt} - 1},$$

and the generalized power sum in the Fourier coefficient is

$$\sigma_{k-1}^{\psi,\varphi}(n) = \sum_{\substack{m|n \\ m>0}} \psi(n/m)\varphi(m)m^{k-1}.$$

See page 129 in [8].

Let $A_{N,k}$ be the set of triples of (ψ,φ,t) such that ψ and φ are primitive Dirichlet characters modulo u and v with $(\psi\varphi)(-1) = (-1)^k$ and $t \in \mathbb{Z}_+$ such

that $tuv|N$. For any triple $(\psi, \varphi, t) \in A_{N,k}$, define

$$E_k^{\psi, \varphi, t}(z) = E_k^{\psi, \varphi}(tz). \tag{2.2}$$

Let $N \in \mathbb{Z}_+$ and let $k \geq 3$. The set

$$\{E_k^{\psi, \varphi, t} : (\psi, \varphi, t) \in A_{N,k}\}$$

gives a basis for $\mathcal{E}_k(\Gamma_1(N))$ and the set

$$\{E_k^{\psi, \varphi, t} : (\psi, \varphi, t) \in A_{N,k}, \psi\varphi = \chi\}$$

gives a basis for $\mathcal{E}_k(N, \chi)$ (see Theorem 4.5.2 in [8]).

For $k = 2$ and $k = 1$ an explicit basis can also be obtained but is more technical. Therefore we assume $k \geq 3$ when dealing with Eisenstein series. For details about the remaining two cases see Chapter 4 in [8].

Now we define Hecke operators. For a reference see page 305 in [16] . Define $\Delta_m(N)$ to be the set

$$\left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \middle| c \equiv 0 \mod N, \det \gamma = m \right\}.$$

Then $\Delta_m(N)$ is invariant under right multiplication by elements of $\Gamma_0(N)$. One checks that the set

$$\mathscr{S} = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in M_2(\mathbb{Z}) \middle| a, d > 0, ad = m, b = 0, \ldots, d - 1 \right\}$$

forms a complete system of representatives for the action of $\Gamma_0(N)$ on $\Delta_m(N)$.

One defines the Hecke operator $T_m \in \text{End}(\mathcal{M}_k(N, \chi))$ as:

$$f \mapsto f|_k T_m = m^{k/2-1} \sum_\sigma \chi(a_\sigma) f|_k \sigma,$$

where $\sigma = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix} \in \mathscr{S}$ and $\gcd(m, N) = 1$.

One can compute the action of T_m explicitly in terms of the Fourier expansion:

$$f|_k T_m = a_0 \sum_{m_1|m} \chi(m_1) m_1^{k-1} + \sum_{n=1}^{\infty} \sum_{m_1|(m,n)} \chi(m_1) m_1^{k-1} a_{mn/m_1^2} q^n.$$

The multiplication rule for weight k operators T_m is as follows:

$$T_m T_n = \sum_{m_1|(m,n)} \chi(m_1) m_1^{k-1} T_{mn/m_1^2}. \tag{2.3}$$

An eigenform $f \in \mathcal{M}_k(N, \chi)$ is defined as the simultaneous eigenfunction of all T_m with $\gcd(m, N) = 1$.

If $f \in \mathcal{M}_k(N, \chi)$ is an eigenform for the Hecke operators with character χ, i.e., if

$$f|_k T_m = \lambda_f(m) f \quad (\gcd(m, N) = 1), \tag{2.4}$$

then equation (2.3) implies that

$$\lambda_f(m) \lambda_f(n) = \sum_{m_1|(m,n)} \chi(m_1) m_1^{k-1} \lambda_f(mn/m_1^2).$$

Comparing the Fourier coefficients in equation (2.4), one sees that

$$a_0 \sum_{m_1 | m} \chi(m_1) m_1^{k-1} = \lambda_f(m) a_0, \qquad (2.5)$$

$$\sum_{m_1 | (m,n)} \chi(m_1) m_1^{k-1} a_{mn/m_1^2} = \lambda_f(m) a_n.$$

For $n = 1$ one then has

$$a_m = \lambda_f(m) a_1. \qquad (2.6)$$

Therefore if $a_1 \neq 0$ and we normalize f such that $a_1 = 1$, then $a_m = \lambda_m(f)$.

Eisenstein series are eigenforms. Let $E_k^{\psi,\varphi,t}$ be the Eisenstein series defined by equation (2.2). Then

$$T_p E_k^{\psi,\varphi,t} = (\psi(p) + \varphi(p) p^{k-1}) E_k^{\psi,\varphi,t}$$

if $uv = N$ or $p \nmid N$. See Proposition 5.2.3 in [8].

Let $M | N$ and $d | N/M$. Let $\alpha_d = \left(\begin{smallmatrix} d & 0 \\ 0 & 1 \end{smallmatrix} \right)$. Then for any $f : \mathcal{H} \to \mathbb{C}$

$$(f|_k[\alpha_d])(z) = d^{k-1} f(dz)$$

defines a linear map $[\alpha_d]$ taking $\mathcal{S}_k(\Gamma_1(M))$ to $\mathcal{S}_k(\Gamma_1(N))$.

Definition 2.0.1. *Let d be a divisor of N and let i_d be the map*

$$i_d : \quad (\mathcal{S}_k(\Gamma_1(Nd^{-1})))^2 \to \mathcal{S}_k(\Gamma_1(N))$$

given by

$$(f, g) \mapsto f + g|_k[\alpha_d].$$

The subspace of oldforms of level N is defined by

$$\mathcal{S}_k(\Gamma_1(N)) := \sum_{\substack{p \mid N \\ p \text{ prime}}} i_p((\mathcal{S}_k(\Gamma_1(Np^{-1})))^2)$$

and the subspace of newforms at level N is the orthogonal complement with respect to the Petersson inner product,

$$\mathcal{S}_k(\Gamma_1(N))^{new} = (\mathcal{S}_k(\Gamma_1(N))^{old})^{\perp}.$$

See Section 5.6 in [8].

Let ι_d be the map $(\iota_d f)(z) = f(dz)$. The main lemma in the theory of newforms, due to Atkin and Lehner [2], is the following:

Theorem 2.0.2. *(Thm. 5.7.1 in [8]) If $f \in S_k(\Gamma_1(N))$ has Fourier expansion $f(z) = \sum a_n(f)q^n$ with $a_n(f) = 0$ whenever $\gcd(n, N) = 1$, then f takes the form $f = \sum_{p \mid N} \iota_p f_p$ with each $f_p \in \mathcal{S}_k(\Gamma_1(N/p))$.*

Definition 2.0.3. *A newform is a normalized eigenform in $\mathcal{S}_k(\Gamma_1(N))^{new}$.*

By Theorem 2.0.2, if $f \in \mathcal{M}_k(N, \chi)$ is a Hecke eigenform with $a_1(f) = 0$, then f is not a newform.

CHAPTER 3

IDENTITIES BETWEEN

HECKE EIGENFORMS

In this section we study solutions to equation (1.1) with f, g, h being newforms with respect to $\Gamma_1(N)$. The first observation is that if $a_f(0) = a_g(0) = 0$ then $a_h(1) = 0$, which implies $h = 0$ by equation (2.6), hence without loss of generality we can assume that $a_g(0) \neq 0$. We normalize g such that $a_g(0) = 1$. In the following we talk about two cases according to whether or not $a_f(0) = 0$.

Lemma 3.0.1. *Assume that g, f, h satisfies equation (1.1) with f and g linearly independent over $\mathbb{C}$, and let χ_g, χ_f, χ_h be the Dirichlet characters associated with g, f, h respectively. Then $\chi_h = \chi_g^2 = \chi_f^2 = \chi_f \chi_g$.*

Proof. Take the diamond operator $\langle d \rangle$ and act on the identity $h = af^2 + bfg + g^2$ to obtain

$$\chi_h(d)h = a\chi_f^2(d)f^2 + b\chi_f(d)\chi_g(d)fg + \chi_g^2(d)g^2.$$

We then substitute (1.1) for h to obtain

$$a(\chi_h(d) - \chi_f^2(d))f^2 + b(\chi_h(d) - \chi_g(d)\chi_f(d))fg + (\chi_h(d) - \chi_g^2(d))g^2 = 0. \quad (3.1)$$

Note that since f and g are linearly independent over $\mathbb{C}$, so are f^2, fg and g^2, which one can easily prove by considering the Wronksian. Then equation (3.1) implies that

$$a(\chi_h(d) - \chi_f^2(d)) = b(\chi_h(d) - \chi_g(d)\chi_f(d)) = \chi_h(d) - \chi_g^2(d) = 0.$$

Since $a, b \neq 0$, one finds

$$\chi_h(d) = \chi_f^2(d) = \chi_f(d)\chi_g(d) = \chi_g^2(d).$$

Since d is arbitrary we are done. $\qquad\square$

Proposition 3.0.2. *Suppose that the triple of newforms (f, g, h) is a solution to (1.1), $a_f(0) \neq 0$ and $a_g(0) \neq 0$. Then $f = g$.*

Proof. By Lemma 3.0.1, f and g have the same character. Denote it by χ. Note that the space of weight k newforms with character χ with nonvanishing constant term is of dimension 1 with basis $E_k^{\mathbb{1}_1, \chi}$ by equation (2.5). Since f and g are normalized, we have $f = g$. $\qquad\square$

Remark. *If $f = g$, then equation (1.1) reduces to $h = fg$, which was classified in [13].*

Now we focus on the case where $a_f(0) = 0$. We have the following solutions forced by dimension considerations.

Lemma 3.0.3. *Let $f, g \in \mathcal{M}_k(N, \chi)$ be two eigenforms that are algebraically independent, and assume that $\dim_{\mathbb{C}} \mathcal{M}_{2k}(N, \chi^2) \leq 3$. Then every eigenform $h \in \mathcal{M}_{2k}(N, \chi^2)$ satisfies an identity of the form $h = af^2 + bfg + cg^2$.*

Recall that the dimension of $\mathcal{M}_k(\Gamma)$ can be computed by the Riemann-Roch Theorem [21]. If $\Gamma = SL_2(\mathbb{Z})$, it reads:

$$\dim_{\mathbb{C}} \mathcal{M}_k = \begin{cases} \left\lfloor \frac{k}{12} \right\rfloor & \text{if } k \equiv 2 \mod 12 \\ \left\lfloor \frac{k}{12} \right\rfloor + 1 & \text{otherwise} \end{cases}$$

Hence for $k = 12$ or 16 one obtains the following examples computed by Sage [7]:

$$E_{24} = a\Delta^2 + bE_{12}\Delta + E_{12}^2,$$

$$a = -\frac{2^{14} \cdot 3^8 \cdot 5^4 \cdot 7^4 \cdot 13^2 \cdot 1571}{103 \cdot 691^2 \cdot 2294797}, \qquad b = -\frac{2^8 \cdot 3^5 \cdot 5^3 \cdot 7^2 \cdot 13^3 \cdot 37}{103 \cdot 691 \cdot 2294797},$$

$$E_{32} = a(E_4\Delta)^2 + b(E_4\Delta)E_{16} + E_{16}^2,$$

$$a = \frac{-2^{18} \cdot 3^8 \cdot 5^5 \cdot 7^4 \cdot 11 \cdot 13^2 \cdot 17^2 \cdot 4273}{37 \cdot 683 \cdot 3617^2 \cdot 305065927},$$

$$b = \frac{-2^{12} \cdot 3^4 \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 17^2 \cdot 23 \cdot 1433}{37 \cdot 683 \cdot 3617 \cdot 305065927},$$

where $\Delta = \prod_{n=1}^{\infty} q(1-q^n)^{24}$ is the cusp form of weight 12 for $SL_2(\mathbb{Z})$ and

$$E_k = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n$$

is the Eisenstein series of weight k for $SL_2(\mathbb{Z})$.

The following two results follows from earlier work by Bruinier, Kohnen and Ono [4], which describes remarkable algebraic information contained in the zeros of Hecke eigenforms. Our independent proof starts from the point of view of polynomial identities.

Let

$$A := \{\rho \in \mathfrak{F} | E_{12k}(\rho) = 0 \text{ for some } k \in \mathbb{Z}_+\}$$

be the set of zeros of Eisenstein series of weight $12k$ that are contained in the fundamental domain $\mathfrak{F}$ of $SL_2(\mathbb{Z})$. By work of Rankin and Swinnerton-Dyer [18] we know that $A \subset \{e^{i\theta} | \frac{\pi}{3} \leq \theta \leq \frac{\pi}{2}\}$.

Theorem 3.0.4. *For all $\rho \in A$, $E_{12}(\rho)/\Delta(\rho)$ is an algebraic number.*

Proof. Note that the $n + 1$ monomials $E_{12}(z)^{n-l}\Delta(z)^l$ are linearly independent over $\mathbb{C}$ by considering the order of vanishing at infinity. One also has $\dim_{\mathbb{C}} M_{12n} = n + 1$, so the above monomials form a basis for M_{12n}. Therefore there exist $a_0, a_1, \cdots, a_n$ such that

$$E_{12n}(z) = \sum_{l=0}^{n} a_l E_{12}(z)^{n-l}\Delta(z)^l. \tag{3.2}$$

For $\rho \in A$ we have $E_{12n}(\rho) = 0$. Divide equation (3.2) by $\Delta(\rho) \neq 0$ to obtain

$$\sum_{l=0}^{n} a_l \left(E_{12}(\rho)/\Delta(\rho)\right)^{n-l} = 0. \tag{3.3}$$

To show that $E_{12}(\rho)/\Delta(\rho)$ is algebraic, it suffices to show that all the a_l are

rational. One sees this by the following algorithm to compute a_l:

First $a_0 = 1$ by considering the constant term.

Suppose $a_0, \cdots, a_l$ are all rational. Then the Fourier coefficients of the function

$E_{12n}(z) - \sum_{k=0}^{l} a_k E_{12}(z)^{n-k} \Delta(z)^k$ are all rational since each term has rational

Fourier coefficients. Further we know that the order of the function $E_{12n}(z) -$

$\sum_{k=0}^{l} a_k E_{12}(z)^{n-k} \Delta(z)^k$ at $i\infty$ is $O(q^{l+1})$. Now set

$$a_{l+1} = \left. \frac{E_{12n}(z) - \sum_{k=0}^{l} a_k E_{12}(z)^{n-k} \Delta(z)^k}{\Delta^{l+1}} \right|_{i\infty}$$

as the constant term. Then one sees that a_{l+1} is rational since the Fourier

coefficients of the function

$$\frac{E_{12n}(z) - \sum_{k=0}^{l} a_k E_{12}(z)^{n-k} \Delta(z)^k}{\Delta^{l+1}}$$

are all rational. $\qquad \square$

Corollary 3.0.5. *For all $\rho \in A$, the function value of the discriminant $j(\rho)$*

is algebraic.

Proof. By the following well known identities:

$$\Delta(z) = \frac{E_4^3 - E_6^2}{1728}$$

$$j(z) = \frac{E_4^3}{\Delta(z)}$$

$$E_{12}(z) = \frac{441 E_4^3 + 250 E_6^2}{691}$$

one then computes

$$\frac{E_{12}(z)}{\Delta(z)} - j(z) = -\frac{432000}{691}.$$

Therefore $j(\rho) = \dfrac{E_{12}(\rho)}{\Delta(\rho)} + \dfrac{432000}{691}$ is an algebraic number by Theorem 3.0.4.

$\square$

Now we prove the finiteness result stated in the introduction.

Theorem 3.0.6. *For all $N \in \mathbb{Z}_+$ and $a, b \in \mathbb{C}^\times$, there are at most finitely triples (f, g, h) of newforms with respect to $\Gamma_1(N)$ of weights $k > 2$ satisfying the equation $h = af^2 + bfg + g^2$.*

Proof. By Proposition 3.0.2 and the remark thereafter we only need to consider the case where $a_f(0) = 0$ and g is an Eisenstein series. Let $g = E_k^{\mathbb{1}, \varphi}$, $f \in \mathcal{M}_k(N, \varphi)$, $h = E_{2k}^{\mathbb{1}, \varphi^2}$. For a fixed N and k, the number of triples of such newforms (f, g, h) is finite. We first show that k is bounded. Consider the Fourier expansions

$$f = q + O(q^2)$$

$$g = 1 + \beta \left(\sum_{n=1}^\infty \sigma_{k-1}^\varphi(n) q^n \right)$$

$$h = 1 + \alpha \left(\sum_{n=1}^\infty \sigma_{2k-1}^{\varphi^2}(n) q^n \right)$$

where

$$\alpha = -\frac{4k}{B_{2k,\varphi^2}}, \tag{3.4}$$

$$\beta = -\frac{2k}{B_{k,\varphi}}, \tag{3.5}$$

$$\sigma^{\varphi}_{k-1}(n) = \sum_{d|n} \varphi(d)d^{k-1}.$$

Comparing the coefficients of q on both sides of the equation $h = af^2+bfg+g^2$,

one obtains

$$b + 2\beta = \alpha. \tag{3.6}$$

Recall that we have the following bound for the generalized Bernoulli number

$B_{k,\chi}$, where χ is a primitive character with conductor l_χ and $\chi(-1) = (-1)^k$:

$$2\zeta(2k)\zeta(k)^{-1}k!(2\pi)^{-k}l_\chi^{k-\frac{1}{2}} \le |B_{k,\chi}| \le 2\zeta(k)k!(2\pi)^{-k}l_\chi^{k-\frac{1}{2}}. \tag{3.7}$$

See [13]. Applying Stirling's formula in equation (3.7) and using the fact

that $\zeta(2k)/\zeta(k)$ is bounded one sees that $|B_{k,\chi}|$ increases quickly as $k \to \infty$.

Then by equations (3.4) and (3.5) one sees that $\alpha, \beta \to 0$ as $k \to \infty$. The

convergence is uniform for all l_χ, hence for a fixed $b \ne 0$, k is bounded for all

N.

For each fixed k, if $l_\chi \to \infty$, then $|B_{k,\chi}| \to \infty$ by equation (3.7). Then

equations (3.4)-(3.6) show that l_χ is bounded. Note that g, h are newforms,

φ and φ^2 are necessarily primitive, i.e., $l_\varphi = l_{\varphi^2} = N$. This shows that N

is bounded for each fixed k. Hence there can only be finitely many pairs

(N, k). Since for each pair there are only finitely many tuples of newforms, this completes the proof of the theorem. $\qquad \square$

CHAPTER 4

MAEDA'S CONJECTURE

AND ITS APPLICATIONS

In this section, following J.B. Conrey and D.W. Farmer [6], we will show how Maeda's conjecure provides information about Hecke eigenform identities. Throughout this section, all eigenforms are eigenforms for $SL_2(\mathbb{Z})$ and we will write $\mathcal{S}_k$ instead of $\mathcal{S}_k(SL_2(\mathbb{Z}))$.

Recall that Maeda's conjecture (see [12] Conjecture 1.2) says the following:

Conjecture 4.0.1. *The Hecke algebra over $\mathbb{Q}$ of $\mathcal{S}_k$ is simple (that is, a single number field) whose Galois closure over $\mathbb{Q}$ has Galois group isomorphic to the symmetric group $\mathfrak{S}_n$, where $n = \dim \mathcal{S}_k$.*

f	g_1	g_2	$a_1 = -a_2$
Δ	$E_{12}\Delta + (12\sqrt{144169} + \frac{32404}{691})\Delta^2$	$\sigma(g_1)$	$\frac{24}{\sqrt{144169}}$
$E_4\Delta$	$\Delta(xE_4^5 + (1-x)E_4^2 E_6^2)$	$\sigma(g_1)$	$\frac{24}{\sqrt{18295489}}$
$x = \frac{12\sqrt{18295489}+20532}{1728}$, σ is the nontrivial element of $\mathrm{Gal}(F/\mathbb{Q})$			

Table 1

The main result of this section is the following:

Theorem 4.0.2. *Assuming Maeda's conjecture for $\mathcal{S}_k$ and $\mathcal{S}_{2k}$, if $f \in \mathcal{S}_k$, $g \in \mathcal{S}_{2k}$ are any nonzero cusp eigenforms, then the Petersson inner product $\langle f^2, g \rangle$ is nonzero.*

An immediate corollary of Theorem 4.0.2 is:

Theorem 4.0.3. *Assuming Maeda's conjecture for all $\mathcal{S}_k$, then identities between cusp eigenforms for $SL_2(\mathbb{Z})$ of the form $f^2 = \sum_{i=1}^{d} c_i g_i$ are all forced by dimension considerations, where $f \in \mathcal{S}_k$, $g_i \in \mathcal{S}_{2k}$ and $c_i \neq 0$. In particular, there are only two identities of the form $f^2 = a_1 g_1 + a_2 g_2$ given by Table 1.*

Proof. By Theorem 4.0.2, we know that $\langle f^2, g_i \rangle \neq 0$ for all $i = 1, \cdots, \dim \mathcal{S}_{2k}$. Then we have $d = \dim \mathcal{S}_{2k}$, i.e., the given identity is forced by dimension considerations. In particular, if $f^2 = a_1 g_1 + a_2 g_2$, then $a_1 \neq 0$ and $a_2 \neq 0$ since f^2 is not itself an eigenform. Then, by Maeda's conjecture we have $\dim_{\mathbb{C}} \mathcal{S}_{2k} = 2$, and so $k = 12$ or 16. See Table 1 for the data on the two cases, which we computed with Sage [7]. $\qquad\square$

We set up notation following [6] before giving the proof of Theorem 4.0.2.

Recall that $\Delta = q\prod_{n=1}^{\infty}(1-q^n)$ is the discriminant and

$$E_4 = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n$$

is the Eisenstein series of weight 4. Then the set

$$B_{\mathbb{Q}} := \{\Delta^j E_4^{\frac{k}{4}-3j}\}$$

forms a basis of $\mathcal{S}_k$ with integer Fourier coefficients [6]. The matrix representation of the Hecke operator T_n in the basis $B_{\mathbb{Q}}$ has integer entries, hence the characteristic polynomial $T_{n,k}(x)$ of T_n acting on $\mathcal{S}_k$ has integer coefficients and the eigenvalues a_n of T_n are all algebraic integers [6]. Let $d = \dim\mathcal{S}_k$. One defines the Hecke field F associated with $\mathcal{S}_k$ by

$$F := \mathbb{Q}(a_j(n) : 1 \leq j \leq d, n \in \mathbb{Z}_+).$$

The following two lemmas proved in [6] play an important role in this section.

Lemma 4.0.4. *The Hecke field F equals $\mathbb{Q}(a_j(n) : 1 \leq j, n \leq d)$. In particular, F is a finite Galois extension of $\mathbb{Q}$.*

The Galois group $\mathrm{Gal}(F/\mathbb{Q})$ acts on functions with Fourier coefficients in F in the following way:

$$\sigma\left(\sum a_n q^n\right) = \sum \sigma(a_n)q^n. \tag{4.1}$$

Lemma 4.0.5. *The group* $\mathrm{Gal}(F/\mathbb{Q})$ *acts on the set of* $B = \{f_1, \cdots, f_d\}$ *of normalized cusp eigenforms. If* $T_{n,k}(x)$ *is irreducible for some* n, *then the action is transitive. Furthermore, if* $T_{n,k}(x)$ *is irreducible then* $F/\mathbb{Q}$ *is the splitting field of* $T_{n,k}(x)$ *and* $\mathrm{Gal}(F/\mathbb{Q}) = \mathrm{Gal}(T_{n,k})$.

Now we begin the proof of Theorem 4.0.2.

Proof. Let $f \in \mathcal{S}_k$ be a normalized Hecke eigenform. Since $f^2 \in \mathcal{S}_{2k}$ we expand f^2 in a normalized eigenbasis to obtain

$$f^2 = \sum_{i=1}^{d_2} c_i g_i \tag{4.2}$$

for some $c_i \in \mathbb{C}$ and $d_2 = \dim \mathcal{S}_{2k}$. Let F_1 (resp. F_2) be the Hecke field for $\mathcal{S}_k$ (resp. $\mathcal{S}_{2k}$) and $F_1 F_2$ be the composite field of F_1 and F_2. Then we have the following:

Lemma 4.0.6. $c_i \in F_1 F_2$ *for* $1 \leq i \leq d_2$.

Proof. Comparing the Fourier coefficients in equation (4.2), one obtains the linear equations

$$\sum_{i=1}^{d_2} c_i a_n(g_i) = a_n(f^2) \tag{4.3}$$

for $1 \leq n \leq d_2$. We claim that the coefficient matrix $(a_n(g_i))_{1 \leq n, i \leq d_2}$ is non-singular. Assume our claim for now. Solve for c_i by Cramer's rule and note that $a_n(g_i) \in F_2$ and $a_n(f^2) \in F_1$, then one sees $c_i \in F_1 F_2$.

Now we prove our claim. Suppose for a contradiction that the coefficient matrix $(a_n(g_i))_{1 \leq n, i \leq d_2}$ is singular, then there exit $c_1, \cdots, c_n \in \mathbb{C}$ such that

$$0 = \sum_i c_i a_n(g_i) = a_n\left(\sum_i c_i g_i\right)$$

for $1 \leq n \leq d_2$, i.e., the cusp form $g := \sum_i c_i g_i \in \mathcal{S}_{2k}$ satisfies $a_n(g) = 0$ for $1 \leq n \leq d_2$. The order of vanishing of g at ∞ satisfies $v_\infty(g) \geq d_2 + 1 = \lfloor \frac{2k}{12} \rfloor + 1$. Recall that the valence formula (Theorem VII. 3.(iii) in [20]) for g reads

$$v_\infty(g) + \frac{1}{2} v_i(g) + \frac{1}{3} v_\omega(g) + \sum_{\substack{\tau \in \mathcal{H}/SL_2(\mathbb{Z}) \\ \tau \neq i, \omega}} v_\tau(g) = \frac{2k}{12}.$$

Since all the terms on the left are positive, one obtains the contradiction $\lfloor \frac{2k}{12} \rfloor + 1 \leq \frac{2k}{12}$. This proves our claim, and hence the lemma. $\square$

Lemma 4.0.7. *If the stabilizer of f^2 in $\mathrm{Gal}(F_1 F_2/\mathbb{Q})$ acts transitively on the set $B_2 = \{g_1, \cdots, g_{d_2}\}$ of cusp eigenforms of weight $2k$, then we have $\langle f^2, g_i \rangle \neq 0$ for $1 \leq i \leq d_2$.*

Proof. Suppose by contradiction that $\langle f^2, g_i \rangle = 0$. Using equation (4.2) we take the inner product with g_i and use the orthogonality relation $\langle g_i, g_j \rangle = \delta_{ij}$ to find $c_i = 0$. Then transitivity gives $c_i = 0$ for $1 \leq i \leq d_2$, which implies that $f^2 = 0$, which is a contradiction. $\square$

Lemma 4.0.8. *Assuming Maeda's conjecture for $\mathcal{S}_k$ and $\mathcal{S}_{2k}$, the stabilizer of f^2 in $\mathrm{Gal}(F_1 F_2/\mathbb{Q})$ acts transitively on the set B_2 of cusp eigenforms of weight $2k$.*

Proof. We have

$$\mathrm{Gal}(F_1 F_2/\mathbb{Q}) \cong \{(\sigma_1, \sigma_2) \in \mathrm{Gal}(F_1/\mathbb{Q}) \times \mathrm{Gal}(F_2/\mathbb{Q}) \big| \sigma_1|_{F_1 \cap F_2} = \sigma_2|_{F_1 \cap F_2}\}.$$

Note that $F_1 \cap F_2$ is Galois over $\mathbb{Q}$, since F_1, F_2 are both Galois over $\mathbb{Q}$, and by elementry Galois theory, the subgroup H_2 of $\mathrm{Gal}(F_2/\mathbb{Q})$ that fixes $F_1 \cap F_2$ is normal. Assuming Maeda's conjecture, we have $\mathrm{Gal}(F_2/\mathbb{Q}) \cong \mathfrak{S}_{d_2}$, $\mathrm{Gal}(F_1/\mathbb{Q}) \cong \mathfrak{S}_{d_1}$, where $d_1 = \dim \mathcal{S}_k$ and $d_2 = \dim \mathcal{S}_{2k}$.

If $d_2 \geq 5$, then the only normal subgroups of $\mathfrak{S}_{d_2}$ are $\mathfrak{S}_{d_2}$, the alternating group A_{d_2} and $\{1\}$. Since $F_1 \cap F_2 \subsetneq F_2$ by degree considerations, we have $H_2 \neq \{1\}$, therefore $H_2 \cong A_{d_2}$ or $H_2 \cong \mathfrak{S}_{d_2}$.

If $H_2 \cong \mathfrak{S}_{d_2}$, then $F_1 \cap F_2 = \mathbb{Q}$ and

$$\mathrm{Gal}(F_1 F_2/\mathbb{Q}) \cong \mathrm{Gal}(F_1/\mathbb{Q}) \times \mathrm{Gal}(F_2/\mathbb{Q}) \cong \mathfrak{S}_{d_1} \times \mathfrak{S}_{d_2},$$

using Maeda's conjecture to obtain the second isomorphism. The stabilizer of f^2 in $\mathrm{Gal}(F_1 F_2/\mathbb{Q})$ contains a subgroup isomorphic to $\mathfrak{S}_{d_2}$, hence acts transitively on B_2.

If $H_2 \cong A_{d_2}$, then

$$[F_1 \cap F_2 : \mathbb{Q}] = |\mathrm{Gal}(F_1 \cap F_2/\mathbb{Q})| = |\mathrm{Gal}(F_2/\mathbb{Q})|/|H_2| = 2,$$

so $F_1 \cap F_2$ is a quadratic field. Moreover we know that the subgroup of $\mathfrak{S}_{d_1}$ that fixes $F_1 \cap F_2$ elementwise is a normal subgroup of index two, i.e., A_{d_1}. In

this case, $F_1 \cap F_2$ is generated by the square root of the discriminant of $T_{n,k}(x)$ or $T_{n,2k}(x)$. Then we have

$$\mathrm{Gal}(F_1 F_2 / \mathbb{Q}) \cong \{(\sigma_1, \sigma_2) \in \mathfrak{S}_{d_1} \times \mathfrak{S}_{d_2} \mid \mathrm{sgn}\,\sigma_1 = \mathrm{sgn}\,\sigma_2\}.$$

In this case, the stabilizer of f^2 in $\mathrm{Gal}(F_1 F_2 / \mathbb{Q})$ contains a subgroup isomorphic to A_{d_2}, which acts transitively on B_2.

If $d_2 \leq 4$, we have $\lfloor 2k/12 \rfloor \leq 4$, so $k \leq 28$. It suffices to show that for $12 \leq k \leq 28$, $F_1 \cap F_2 = \mathbb{Q}$. We only need to check the cases $(k, 2k) = (24, 48), (28, 56)$ since for all other cases $\dim \mathcal{S}_k = 1$ and $F_1 = \mathbb{Q}$. We check these two cases by showing that no prime ramifies in $F_1 \cap F_2$. Then we obtain that $F_1 \cap F_2 = \mathbb{Q}$ by Minkowski's theorem. Indeed, if p is a prime that ramifies in $F_1 \cap F_2$, then p ramifies in F_1 and F_2. Let $\alpha_1, \cdots, \alpha_{d_1}$ be the d_1 roots of the polynomial $T_{n,k}(x)$. Note that the splitting field F_1 of $T_{n,k}(x)$ can be written as the composite field of the d_1 isomorphic subfields $\mathbb{Q}(\alpha_i)$, then we see that p ramifies in each $\mathbb{Q}(\alpha_i)$ by known properties of ramification in towers. Thus $p \mid D_{\mathbb{Q}(\alpha_i)}$, where $D_{\mathbb{Q}(\alpha_i)}$ is the discriminant of the field $\mathbb{Q}(\alpha_i)$. Similarly $p \mid D_{\mathbb{Q}(\beta_j)}$, where $\beta_1, \cdots, \beta_{d_2}$ are the d_2 roots of the polynomial $T_{n,2k}(x)$. Therefore $p \mid \gcd(D_{\mathbb{Q}(\alpha_i)}, D_{\mathbb{Q}(\beta_j)})$. Thus it suffices to check that $\gcd(D_{\mathbb{Q}(\alpha_i)}, D_{\mathbb{Q}(\beta_j)}) = 1$ for the two cases. One verifies this directly by Table 2, where the first column is the weight and the second column is the discriminant $D_{\mathbb{Q}(\alpha_i)}$ or $D_{\mathbb{Q}(\beta_j)}$ computed by Sage [7]. $\quad\square$

Now Theorem 4.0.2 follows easily by combining Lemmas 4.0.6-4.0.8. $\quad\square$

k	$D_{\mathbb{Q}(\alpha_i)}$
24	144169
28	131 * 139
48	31 * 6093733 * 16756155243992707260468295662812838
56	41132621 * 48033296728783687292737439509259855449806941

Table 2

CHAPTER 5

CONGRUENCE

SUBGROUPS

In this section, we study the Petersson inner product $\langle f^2, g \rangle$ with f, g being eigenforms for the congruence group $\Gamma_0(p)$. The characteristic polynomial $T_n(x)$ of Hecke operator T_n acting on $S_k(\Gamma_0(p))$ can be explicitly computed using Modular symbols (see Chapter 8 of [22]). Since $S_k(\Gamma_0(p)) = S_k^{new} \oplus S_k^{old}$ is semisimple Hecke algebra module, the characteristic polynomial $T_n(x)$ is no longer irreducible. Moreover the characteristic polynomial of T_n acting on the new subspace may fail to be irreducible. Since the Galois action may be not transitive on the eigenbasis, one may expect that $\langle f^2, g \rangle = 0$ for certain cases.

Note that in the case $p = 2$, an explicit basis of $\mathcal{M}_{2k}(\Gamma_0(2))$ is given by

$$\{(E_2^*)^a E_4^b | 2a + 4b = 2k\},$$

where $E_2^*(z) = E_2(z) - 2E(2z)$ and $E_2(z), E_4(z)$ is Eisenstein series of weight 2 and 4 respectively (see page 56 of [14]).

One checks that $\mathcal{S}_{10}(\Gamma_0(2))$ is one dimensional and the Fourier expansion of its basis is given by

$$f = q + 16q^2 - 156q^3 + 256q^4 + 870q^5 + O(q^6),$$

which can be obtained by the following Sage command:

$$f = \text{Newforms}(2,10,\text{names}=\text{"a"}).$$

Similarly one obtains the eigenbasis of $\mathcal{S}_{20}^{new}(\Gamma_0(2))$:

$$g_1 = q - 512q^2 - 13092q^3 + 262144q^4 + 6546750q^5 + O(q^6)$$

$$g_2 = q + 512q^2 - 53028q^3 + 262144q^4 - 5556930q^5 + O(q^6).$$

The eigenbasis that spans the old subspace is given by

$$g_3 = \Delta(z)E_8(z) = q + 456q^2 + 50652q^3 - 316352q^4 - 2377410q^5 + O(q^6)$$

$$g_4 = \Delta(2z)E_8(2z) = q^2 + 456q^4 + O(q^6).$$

Since $f^2 \in \mathcal{S}_{20}$, there exits $a_i \in \mathbb{C}$ such that $f^2 = \sum_{i=1}^{4} a_i g_i$. Solving this linear

system with Sage we find

$$a_1 = -1/1992$$

$$a_2 = 0$$

$$a_3 = 1/1992$$

$$a_4 = 128/249.$$

Note that $a_2 = 0$, then one obtains $\langle f^2, g_2 \rangle = 0$. This example shows that the unbiased property in Theorem 4.0.2 does not hold for the congruence subgroup $\Gamma_0(N)$.

Here we list a few more examples computed with Sage.

$\Gamma = \Gamma_0(5)$, $f \in \mathcal{S}_4(\Gamma)$ is the unique normalized eigenform with the Fourier expansion

$$f = q - 4q^2 + 2q^3 + 8q^4 - 5q^5 + O(q^6),$$

the weight 8 eigenforms for Γ have the following Fourier expansions:

$$g_1 = q - 14q^2 - 48q^3 + 68q^4 + 125q^5 + O(q^6)$$

$$g_2 = q + aq^2 + (-8a + 90)\, q^3 + (20a - 152)\, q^4 - 125q^5 + O(q^6)$$

$$g_3 = \sigma(g_2),$$

where a is a root of the equation $x^2 - 20x + 24 = 0$ and $\sigma(a)$ is the Galois

conjugate of a.

Write $f^2 = \sum_{i=1}^{3} c_i g_i$, then we have $(c_1, c_2, c_3) = \left(0, \frac{1}{152}a - \frac{5}{76}, -\frac{1}{152}a + \frac{5}{76}\right)$.

$\Gamma = \Gamma_0(7)$, Let $f \in \mathcal{S}_4(\Gamma)$ be the unique normalized eigenform with Fourier expansion

$$f = q - q^2 - 2q^3 - 7q^4 + 16q^5 + O(q^6),$$

g_1, g_2, g_3 are weight 8 eigenforms with Fourier expansions

$$g_1 = q - 6q^2 - 42q^3 - 92q^4 - 84q^5 + O(q^6),$$

$$g_2 = q + a_1 q^2 + (-2a_1 + 44)\, q^3 + (-3a_1 + 86)\, q^4 + (-10a_1 + 150)\, q^5 + O(q^6)$$

$$g_3 = \sigma(g_2),$$

where a_1 is a root of the equation $x^2 + 3x - 214 = 0$ and $\sigma(a_1)$ is the Galois conjugate of a_1.

Write $f^2 = \sum_{i=1}^{3} c_i g_i$, then we have $(c_1, c_2, c_3) = \left(0, \frac{2}{865}a_1 + \frac{3}{865}, -\frac{2}{865}a_1 - \frac{3}{865}\right)$.

CHAPTER 6

SAGE CODE FOR

COMPUTING c_n

In this section, we provide Sage code for our computations. Our computations are based on William Stein's Pari readable tables of dimensions of modular forms, which are available on his website [1]. To do the computations, one needs to load Stein's table by Pari and then generate Sage readable dababase. The goal of the following code is to solve for the coefficient c_n in the equation $f^2 = \sum_j c_j g_j$, where f is weight 2 level l newform and l is a prime bounded by 100. The upper bound 100 is chosen according to William Stein's database. If we use the Fourier expansions $f^2 = \sum_n a_{f^2}(n)q^n$ and $g_j = \sum_n a_{g_i}(n)q^n$, we

obtain a linear system

$$M\left(c_1, \cdots, c_k\right)^T = \left(a_{f^2}(1), \cdots, a_{f^2}(n)\right) \tag{6.1}$$

where n is the dimension of the new subspace, i.e., the carnality of the basis $\{g_j\}$ and M is the n by n coefficient matrix of g_js. In the following code, we arrange the coefficients of g_j in a list named lst and then generate a square matrix with complex entries, which corresponds to lines 11,23,37,53 in the following code. One thing to note is that the coefficients of g_j are algebraic numbers, one need embeds them into $\mathbb{C}$ to do computations in practice. We denote the right side of equation 6.1 by a vector v in our code. Note that $a_{f^2}(l) = \sum_k a_f(l-k)a_f(k)$ and this is what line 10 in our code does. Let u be the vector $(c_1, \cdots, c_n)$, then we have $u = M^{-1}v$, which is what lines 12,24,38,54 in our code compute. Actually these lines do a little more, taking the Galois conjugates of f into consideration.

The g_js lie in several Galois orbits, which correspond to the irreducible factors of the characteristic polynomial of the a Hecke operator T_p with p prime to the level. The main if loop in the code is used to detect the number of irreducible factors or the number of Galois orbits in g_js. In the following code, E[i,0] is the minimal polynomial of the Hecke eigenvalues of weight 2 and level i newforms and E41[i,0] is the minimal polynomial of the Hecke eigenvalues of weight 4 newforms and the last digit labels different Galois orbits. For

instance if $E41[i,0] \neq 0$ and $E42[i,0] = 0$, then we have that there is only one Galois orbit in the g_js and the cardinality of the set $\{g_j\}$ is the degree of the polynomial defined by $E41[i,0]$. Then E41list[i][l] is the l-th Fourier coefficient of the weight 4 newforms in Galois orbit 1. The database used in our computation is too long to be listed here, therefore we only list the essential part of the program. To make it more readable, here is a table of notation.

i	prime level in $(1, 98)$
$E[i,0]$	minimal polynomial of weight 2 level i Hecke eigenvalue
$E41[i,0]$	minimal polynomial of weight 4 level i Hecke eigenvalue in Galois orbit 1
$E42[i,0]$	minimal polynomial of weight 4 level i Hecke eigenvalue in Galois orbit 2
K	splitting field of $E41[i,0]$
L	splitting field of $E[i,0]$
n	dimension of the new subspace cusp forms of weight 4 level i
v	$(a_{f^2}(1), \cdots, a_{f^2}(n))$, vector formed by the first n Fourier coefficients of f^2
M	matrix formed by the first n Fourier coefficients of the n g_js
u	the vector $(c_1, \cdots, c_n)$

```
1
2  for i in prime_range(1,98):
3          if E[i,0]<>0:
4                  if E41[i,0]<>0:
5                          if E42[i,0]==0:
6                          print(i)
7                          K.<a>=NumberField(E41[i,0])
```

```
8                            L.<b>=NumberField(E[i,0])
9                            n=(E41[i,0]).degree()
10                           v=[[sum((Elist[i][l-k-1]*Elist[i][k-1])
                                .subs(x=L.embeddings(CC)[j](b))
                                for k in range(1,l)) for l in range
                                (1,n+1)] for j in range(0,(E[i,0]).
                                degree())]
11                           M=matrix(CC,n,n, [(E41list[i][l]).subs(
                                x=K.embeddings(CC)[j](a)) for l in
                                range(0,n) for j in range(0,n)])
12                           u=[M.inverse()*vector(v[j]) for j in
                                range(0,(E[i,0]).degree())]
13                           print(i,u)
14                     if (E42[i,0]<>0 and E43[i,0]==0):
15                           K.<a>=NumberField(E41[i,0])
16                           L.<b>=NumberField(E[i,0])
17                           K2.<c>=NumberField(E42[i,0])
18                           n=(E41[i,0]).degree()+(E42[i,0]).degree
                                ()
19                           v=[[sum((E2list[i][l-k-1]*E2list[i][k
                                -1]).subs(x=L.embeddings(CC)[j](b))
                                 for k in range(1,l)) for l in
                                range(1,n+1)] for j in range(0,(E[i
                                ,0]).degree())]
20                           lst1 = [(E41list[i][l]).subs(x=K.
                                embeddings(CC)[j](a)) for l in
                                range(0,n) for j in range(0,E41[i
                                ,0].degree() )]
21                           lst2 =  [(E42list[i][l]).subs(x=K2.
                                embeddings(CC)[j](c)) for l in
```

```
                            range (0 ,n) for j in range (0 , E42 [i
                                ,0]. degree ())]
22                          lst = lst1 + lst2
23                          M= matrix (CC ,n ,n , lst )
24                          u =[ M. inverse ()* vector (v[j]) for j in
                                range (0 ,(E[i ,0]). degree ())]
25                          print (i ,u)
26                  if ( E43 [i ,0] <>0 and E44 [i ,0]==0):
27                          K.<a>= NumberField ( E41 [i ,0])
28                          L.<b>= NumberField (E[i ,0])
29                          K2.<c>= NumberField ( E42 [i ,0])
30                          K3.<d>= NumberField ( E43 [i ,0])
31                          n =( E41 [i ,0]). degree ()+( E42 [i ,0]). degree
                                ()+( E43 [i ,0]). degree ()
32                          v =[[ sum (( E2list [i ][ l-k -1]* E2list [i ][k
                                -1]). subs (x=L. embeddings (CC )[j ](b))
                                for k in range (1 ,l)) for l in
                                range (1 ,n +1)] for j in range (0 ,(E[i
                                ,0]). degree ())]
33                          lst1 = ( E41list [i ][l]). subs (x=K.
                                embeddings (CC )[j ](a)) for l in
                                range (0 ,n) for j in range (0 , E41 [i
                                ,0]. degree () )]
34                          lst2 = [( E42list [i ][l]). subs (x=K2.
                                embeddings (CC )[j ](c)) for l in
                                range (0 ,n) for j in range (0 , E42 [i
                                ,0]. degree ())]
35                          lst3 = [( E43list [i ][l]). subs (x=K3.
                                embeddings (CC )[j ](d)) for l in
                                range (0 ,n) for j in range (0 , E43 [i
```

```
                            ,0].degree())]
36                          lst =lst1+lst2+lst3
37                          M= matrix(CC,n,n, lst)
38                          u=[M.inverse()*vector(v[j]) for j in
                                range(0,(E[i,0]).degree())]
39                          print(i,u)
40                  if (E44[i,0]<>0 and E45[i,0]==0):
41                          K.<a>=NumberField(E41[i,0])
42                          L.<b>=NumberField(E[i,0])
43                          K2.<c>=NumberField(E42[i,0])
44                          K3.<d>=NumberField(E43[i,0])
45                          K4.<f>=NumberField(E44[i,0])
46                          n=(E41[i,0]).degree()+(E42[i,0]).degree
                                ()+(E43[i,0]).degree()+(E44[i,0]).
                                degree()
47                          v=[[sum((E2list[i][l-k-1]*E2list[i][k
                                -1]).subs(x=L.embeddings(CC)[j](b))
                                 for k in range(1,l)) for l in
                                range(1,n+1)] for j in range(0,(E[i
                                ,0]).degree())]
48                          lst1 = [(E41list[i][l]).subs(x=K.
                                embeddings(CC)[j](a)) for l in
                                range(0,n) for j in range(0,E41[i
                                ,0].degree() )]
49                          lst2 = [(E42list[i][l]).subs(x=K2.
                                embeddings(CC)[j](c)) for l in
                                range(0,n) for j in range(0,E42[i
                                ,0].degree())]
50                          lst3 = [(E43list[i][l]).subs(x=K3.
                                embeddings(CC)[j](d)) for l in
```

```
                              range (0 ,n) for j in range (0 , E43 [i
                              ,0]. degree ())]
51                            lst4 = [( E44list [i][l]). subs (x=K4.
                              embeddings (CC)[j](f)) for l in
                              range (0 ,n) for j in range (0 , E44 [i
                              ,0]. degree ())]
52                            lst = lst1 + lst2 + lst3 + lst4
53                            M= matrix (CC ,n ,n, lst )
54                            u=[ M. inverse ()* vector (v[j]) for j in
                              range (0 ,( E[i ,0]). degree ())]
55                            print (i ,u)
```

For example the space $S_2(\Gamma_0(11))$ is 1 dimensional, whose basis can be computed by Sage using

$$\text{sage: } f = \text{Newforms}(11,2,\text{names}="a")$$

İn terms of the Fourier expansion

$$f = q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 - 2q^9 + O(q^{10})$$

This is the Hecke eigenform coming from the elliptic curve $y^2 + y = x^3 - x^2$ studied by Eichler.

Let g_1, g_2 be the two normalized newforms of $S_4(\Gamma_0(11))$. Explicitly, let a be a root of the polynomial $x^2 - 2x - 2$, then

$$g_1 = q + aq^2 + (-4a + 3) q^3 + (2a - 6) q^4 + (8a - 7) q^5 + O(q^6)$$

and g_2 is the Galois conjugate of g_1.

Let $f^2 = c_1 g_1 + c_2 g_2$, then one easily solves $c_1 = -\frac{1}{2\sqrt{3}}$, $c_2 = -c_1$. For level 11, the program gives $c_1 = -0.288675134594813$ and $c_2 = 0.288675134594813$.

It is straightforward to check that c_1 and c_2 are decimal approximations of $\pm\frac{1}{2\sqrt{3}}$.

REFERENCES

[1] http://wstein.org/Tables/tables.html.

[2] A. O. L. Atkin and J. Lehner. Hecke operators on $\Gamma_0(m)$. *Math. Ann.*, 185:134–160, 1970.

[3] Jeffrey Beyerl, Kevin James, and Hui Xue. Divisibility of an eigenform by an eigenform. *Proc. Amer. Math. Soc.*, 142(1):29–38, 2014.

[4] Jan H Bruinier, Winfried Kohnen, and Ken Ono. The arithmetic of the values of modular functions and the divisors of modular forms. *Compositio Mathematica*, 140(03):552–566, 2004.

[5] Kevin Buzzard. On the eigenvalues of the Hecke operator T_2. *J. Number Theory*, 57(1):130–132, 1996.

[6] J. B. Conrey and D. W. Farmer. Hecke operators and the nonvanishing of L-functions. In *Topics in number theory (University Park, PA,*

1997), volume 467 of *Math. Appl.*, pages 143–150. Kluwer Acad. Publ., Dordrecht, 1999.

[7] The Sage Developers. *Sage Mathematics Software (Version 6.10)*, 2016. `http://www.sagemath.org`.

[8] Fred Diamond and Jerry Shurman. *A first course in modular forms*, volume 228 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2005.

[9] W. Duke. When is the product of two Hecke eigenforms an eigenform? In *Number theory in progress, Vol. 2 (Zakopane-Kościelisko, 1997)*, pages 737–741. de Gruyter, Berlin, 1999.

[10] Brad A. Emmons. Products of Hecke eigenforms. *J. Number Theory*, 115(2):381–393, 2005.

[11] Eknath Ghate. On monomial relations between Eisenstein series. *J. Ramanujan Math. Soc.*, 15(2):71–79, 2000.

[12] Haruzo Hida and Yoshitaka Maeda. Non-abelian base change for totally real fields. *Pacific J. Math.*, (Special Issue):189–217, 1997. Olga Taussky-Todd: in memoriam.

[13] Matthew Leander Johnson. Hecke eigenforms as products of eigenforms.

Journal of Number Theory, 133(7):2339–2362, 2013.

[14] L. J. P. Kilford. *Modular forms.* Imperial College Press, London, 2008. A classical and computational introduction.

[15] The LMFDB Collaboration. The l-functions and modular forms database, 2013. [Online; accessed 16 September 2013].

[16] Yuri Ivanovic Manin and Alexei A. Panchishkin. *Introduction to modern number theory*, volume 49 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, second edition, 2005. Fundamental problems, ideas and theories, Translated from the Russian.

[17] Ken Ono and Matthew A Papanikolas. p-adic properties of values of the modular j-function. In *Galois Theory and Modular Forms*, pages 357–365. Springer, 2004.

[18] FKC Rankin and HPF Swinnerton-Dyer. On the zeros of eisenstein series. *Bulletin of the London Mathematical Society*, 2(2):169–170, 1970.

[19] Joseph Richey and Noah Shutty. Polynomial identities on eigenforms. *Journal of Number Theory*, 158:333 – 347, 2016.

[20] J.-P. Serre. *A course in arithmetic.* Springer-Verlag, New York-Heidelberg, 1973. Translated from the French, Graduate Texts in Math-

ematics, No. 7.

[21] Goro Shimura. *Introduction to the arithmetic theory of automorphic functions*, volume 11 of *Publications of the Mathematical Society of Japan*. Princeton University Press, Princeton, NJ, 1994. Reprint of the 1971 original, Kanô Memorial Lectures, 1.

[22] William Stein. *Modular forms, a computational approach*, volume 79 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2007. With an appendix by Paul E. Gunnells.