HARNACK INEQUALITY FOR A CLASS OF DEGENERATE ELLIPTIC EQUATIONS IN NON-DIVERGENCE FORM

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ABSTRACT

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Professor Cristian E. Gutiérrez, Chair

We provide two proofs of an invariant Harnack inequality in small balls for a class of second order elliptic operators in non-divergence form, structured on Heisenberg vector fields. We assume that the coefficient matrix is uniformly positive definite, continuous, and symplectic. The first proof emulates a method of E. M. Landis [27], and is based on the so-called growth lemma, which establishes a quantitative decay of oscillation for subsolutions. The second proof consists in establishing a critical density property for non-negative supersolutions, and then invoking the axiomatic approach developed by Di Fazio, Gutiérrez and Lanconelli [11] to obtain Harnack's inequality.

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Farhan Abedin

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INTRODUCTION

A second order elliptic operator of non-divergence form on an open set $\Omega \subset \mathbb{R}^n$ can be written as

$$L = \mathrm{tr}(M(x)D^2\cdot) = \sum_{i,j=1}^{n} m_{ij}(x)\partial_{x_i}\partial_{x_j} \tag{1.1}$$

where $M(x) = (m_{ij}(x))$ is a symmetric, non-negative definite matrix for each $x \in \Omega$. If $\lambda(x)$ and $\Lambda(x)$ are, respectively, the smallest and largest eigenvalues of the matrix $M(x)$, then we have

$$0 \le \lambda(x)|\xi|^2 \le \sum_{i,j=1}^{n} m_{ij}(x)\xi_i\xi_j \le \Lambda(x)|\xi|^2 \quad \text{at each } x \in \Omega \text{ for all } \xi \in \mathbb{R}^n. \tag{1.2}$$

We will assume throughout that the matrix $M(\cdot)$ has uniformly bounded entries on Ω; hence, there exists a constant $\Lambda > 0$ such that $\sup_{\Omega} \Lambda(x) \le \Lambda$. The operator L is said to be *uniformly elliptic* on Ω if, in addition, $\inf_{\Omega} \lambda(x) > 0$. In this case, there exists a constant $\lambda > 0$ such that $\lambda(x) \ge \lambda$ for all $x \in \Omega$. Therefore,

$$0 < \lambda|\xi|^2 \le \sum_{i,j=1}^{n} m_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \quad \text{at each } x \in \Omega \text{ for all } \xi \in \mathbb{R}^n. \tag{1.3}$$

We say the operator L is *degenerate elliptic* on Ω if $\inf_{\Omega} \lambda(x) = 0$.

A quintessential property of elliptic operators is the weak maximum principle, which states that if L is a second order elliptic operator on a bounded domain Ω, and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies $Lu \geq 0$ (resp. $Lu \leq 0$), then

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u \qquad \left(\text{resp. } \min_{\overline{\Omega}} u = \min_{\partial\Omega} u\right). \tag{1.4}$$

Uniformly elliptic operators satisfy, in addition, the following Harnack inequality, which can be viewed as a quantitative form of the maximum principle.

Theorem 1.0.1. *Suppose L is a uniformly elliptic operator in non-divergence form, with coefficients $M(\cdot)$ satisfying* (1.3)*. There exists a constant $C = C(n, \lambda, \Lambda) > 0$ such that for all $u \in C^2(\Omega)$ non-negative and satisfying $Lu = 0$, we have*

$$\sup_{B_r} u \leq C \inf_{B_r} u \qquad \textit{for all balls } B_r \textit{ such that } B_{2r} \Subset \Omega. \tag{1.5}$$

Theorem 1.0.1 is a seminal contribution of N. V. Krylov and M. V. Safonov to the theory of second order elliptic equations (cf. [24], [25]). Perhaps the most significant attribute of their work is that the constant C appearing in (1.5) is independent of the smoothness of the coefficient matrix $M(x)$. This is of paramount importance in the regularity theory for fully nonlinear uniformly elliptic equations; see [6], [18], [21] and [23] for the many applications of Theorem 1.0.1.

A key ingredient in the proof of Theorem 1.0.1 is the so-called Aleksandrov-Bakelman-Pucci (ABP) Maximum Principle (cf. [16, Lemma 9.3]).

Theorem 1.0.2. *Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain and L is a uniformly elliptic operator in non-divergence form, with coefficients $M(\cdot)$ satisfying* (1.3)*. There exists* *a constant* $C = C\left(diam(\Omega), n, \frac{\Lambda}{\lambda}\right)$ *such that for all* $u \in C^2(\Omega) \cap C(\overline{\Omega})$, *we have*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u + C||Lu||_{L^n(\Gamma_u^+)}, \tag{1.6}$$

where $\Gamma_u^+ = \{y \in \Omega : \exists p = p(y) \in \mathbb{R}^n \ s.t.\ u(x) \leq u(y) + p \cdot (x - y)\ \forall x \in \Omega\}$ *is the upper contact set of* u.

Once the ABP maximum principle is at hand, Theorem 1.0.1 can be established by following a sequence of well-understood steps; see, for instance, [6, Chapter 4], [16, Chapter 9], [18, Chapter 2] and [22, Chapter 5]. There have been subsequent efforts to generalize Theorem 1.0.1 to the setting of degenerate elliptic operators. A pioneering result in this direction was obtained by Caffarelli and Gutiérrez in [7], where the authors prove Harnack's inequality for positive solutions of the linearized Monge-Ampère equation; see [18, Chapter 7] for a detailed exposition of this approach.

1.1 Harnack Inequality in the Heisenberg Group

An important class of degenerate elliptic operators is formed by the analogue of non-divergence form operators in the setting of homogeneous Lie groups. The most basic example of such an operator is the sub-Laplacian in the Heisenberg group, which arises naturally in the theory of several complex variables and was studied extensively by Folland and Stein [13], [15]; see also [29] for a general discussion of analysis and geometry in the Heisenberg group. Let us describe these operators in more detail.

Denote points in $\mathbb{R}^{2n+1}$ by $z = (x,t) = (x_1, \ldots, x_{2n}, t) \in \mathbb{R}^{2n} \times \mathbb{R}$. Let $\mathbb{I}_n$ denote the $n \times n$ identity matrix, and define the $2n \times 2n$ matrix

$$\mathcal{J} := \begin{pmatrix} 0 & -\mathbb{I}_n \\ \mathbb{I}_n & 0 \end{pmatrix}. \tag{1.7}$$

The Heisenberg group $\mathbb{H}^n$ is the homogeneous Lie group $(\mathbb{R}^{2n+1}, \circ, \delta_r)$ equipped with the composition law

$$(x,t) \circ (\xi, \tau) := (x + \xi, t + \tau + 2\langle \mathcal{J}x, \xi \rangle), \tag{1.8}$$

and the family of dilations

$$\delta_r : \mathbb{H}^n \to \mathbb{H}^n, \ \delta_r(x,t) = (rx, r^2 t), \ r > 0. \tag{1.9}$$

Here $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^{2n}$. The identity element is $0 = (0,0)$, and the inverse is $(x,t)^{-1} := (-x,-t)$. We can define a δ_r-homogeneous symmetric norm on $\mathbb{H}^n$ using the function

$$\rho(x,t) := (|x|^4 + t^2)^{\frac{1}{4}}. \tag{1.10}$$

The associated metric is

$$d((x,t),(\xi,\tau)) := \rho((x,t)^{-1} \circ (\xi,\tau)). \tag{1.11}$$

The balls defined by this metric will be denoted

$$B_R((x,t)) := \left\{ (\xi,\tau) \in \mathbb{R}^{2n+1} : d((x,t),(\xi,\tau)) < R \right\}. \tag{1.12}$$

The Haar measure on $\mathbb{H}^n$ is $(2n+1)$-dimensional Lebesgue measure, which we will denote by $|\cdot|$. For any $(x,t) \in \mathbb{H}^n$ and any $R > 0$, we have

$$|B_R((x,t))| = |B_R(0)| = R^{2n+2}|B_1(0)|.$$

The number $Q := 2n + 2$ is thus called the homogeneous dimension of $\mathbb{H}^n$.

Consider the vector fields

$$X_i := \partial_{x_i} + 2(\mathcal{J}x)_i \partial_t, \quad i = 1, \ldots, 2n. \tag{1.13}$$

These span the first (horizontal) layer of the Lie algebra of $\mathbb{H}^n$, and the only non-trivial commutation relations they satisfy are

$$[X_i, X_{i+n}] = 4\partial_t, \qquad i \in \{1, \ldots, n\}. \tag{1.14}$$

The linear second order differential operators we will consider are of the form

$$\mathcal{L}_A := \sum_{i,j=1}^{2n} a_{ij}(x,t) X_i X_j, \tag{1.15}$$

where $A(x,t) = (a_{ij}(x,t))_{i,j=1,\ldots,2n} \in \mathbb{R}^{2n\times 2n}$ is symmetric and satisfies the uniform ellipticity condition

$$0 < \lambda \mathbb{I}_{2n} \leq A(x,t) \leq \Lambda \mathbb{I}_{2n} \quad \text{for all } (x,t) \in \Omega \subset \mathbb{H}^n. \tag{1.16}$$

We refer to these as *horizontally uniformly elliptic* operators in non-divergence form. Even though the coefficient matrix A satisfies (1.16), the corresponding operator $\mathcal{L}_A$ is, in general, degenerate elliptic, as illustrated by the following simple example.

Example 1.1. *Consider the sub-Laplacian in $\mathbb{H}^1$, which corresponds to the case $A(x,t) = \mathbb{I}_2$. Relabeling the t coordinate x_3, we see that the operator $\mathcal{L} = X_1^2 + X_2^2$ is of the form* (1.1) *with coefficients $m_{ij}(x_1, x_2, x_3)$ given by the 3×3 matrix*

$$M(x_1, x_2, x_3) := \begin{pmatrix} 1 & 0 & -2x_2 \\ 0 & 1 & 2x_1 \\ -2x_2 & 2x_1 & 4(x_1^2 + x_2^2) \end{pmatrix}. \tag{1.17}$$

It is easy to see that M is non-negative definite, and that det$(M) = 0$ at any point $(x_1, x_2, x_3) \in \mathbb{R}^3$. Consequently, the operator $\mathcal{L}$ is degenerate elliptic.

The book [4, Chapter 5] presents in detail the potential theory associated to sub-Laplacians formed by left-invariant vector fields in homogeneous Lie groups; see also [8] for a discussion of the isoperimetric problem in $\mathbb{H}^n$. More details of the analytic and geometric properties of the Heisenberg group $\mathbb{H}^n$ and the operators $\mathcal{L}_A$ are provided in Section 2.1.

A central problem in the study of horizontally elliptic operators in non-divergence form is to establish the analogue of Theorem 1.0.1 for non-negative solutions to $\mathcal{L}_A u = 0$. Such a result would bound the supremum of u on a metric ball B_R by a constant times the infimum of u on the same ball, while assuming no smoothness of the coefficient matrix A. As a first step toward solving this problem, there have been efforts to prove an ABP-type maximum principle corresponding to the operators $\mathcal{L}_A$ (cf. [19], [9], [3]). However, two obstructions to the development of such an estimate are the strong degeneracy of $\mathcal{L}_A$, and the difficulty in defining an appropriate "convex envelope" in this setting.

1.2 Review of Existing Literature

Inspired by the techniques introduced by Caffarelli and Gutiérrez in [7] for the linearized Monge-Ampère operator, an axiomatic approach to Harnack's inequality for linear elliptic operators in the setting of doubling Hölder quasi-metric spaces of homogeneous type was developed by DiFazio, Gutiérrez and Lanconelli in [11]. The Heisenberg group $\mathbb{H}^n$ and the operators $\mathcal{L}_A$ can be treated using this approach, and so by the results of [11], a scale-invariant Harnack's inequality for $\mathcal{L}_A$ holds once the following two properties are verified:

(i) *Critical Density Property*: There exist constants $\epsilon > 0$ and $M > 1$, such that for all balls $B_{2r} \subset \Omega$, and any $u \geq 0$ on B_{2r} satisfying $\mathcal{L}_A u \leq 0$ in B_{2r}, we have

$$\inf_{B_{\frac{r}{2}}} u \leq 1 \;\Rightarrow\; |\{u < M\} \cap B_r| \geq \epsilon \, |B_r| \, .$$

(ii) *Double Ball Property*: There exists a constant $\gamma > 0$ such that for all balls $B_{2r} \subset \Omega$ and any $u \geq 0$ on B_{2r} satisfying $\mathcal{L}_A u \leq 0$ on B_{2r}, we have

$$\inf_{B_{\frac{r}{2}}} u \geq 1 \;\Rightarrow\; \inf_{B_r} u \geq \gamma.$$

The Double Ball Property for $\mathcal{L}_A$ has been established by Gutiérrez and Tournier in [20] assuming no smoothness on the coefficient matrix $A(\cdot)$. Thus, the only remaining obstruction to proving Harnack's inequality for $\mathcal{L}_A$ is the Critical Density Property. It is worth mentioning here that, for uniformly elliptic operators, the proof of the Critical Density Property requires the use of the ABP maximum principle, Theorem 1.0.2. The lack of an analogous principle for the operator $\mathcal{L}_A$ is the key obstruction to proving the most general form of the Krylov-Safonov Harnack inequality in the Heisenberg group.

Nevertheless, some partial results have been obtained in recent years. In [20], the authors prove the existence of a dimensional constant $C_n > 1$ such that if the constants λ, Λ in (1.16) satisfy $1 \leq \frac{\Lambda}{\lambda} \leq C_n$ (the so-called Cordes-Landis condition), then the Critical Density Property holds for the operator $\mathcal{L}_A$. No assumptions are made on the smoothness of the entries of A; however, the constant C_n degenerates to 1 as the dimension n goes to infinity, and so this greatly restricts the class of operators $\mathcal{L}_A$ for which a scale-invariant Harnack inequality can be shown to hold. When the coefficients are assumed to be Hölder continuous, Bonfiglioli and Uguzzoni [5] show via parametrix methods that Harnack's inequality holds with no restriction on the eigenvalue ratio $\frac{\Lambda}{\lambda}$; however, the constant in Harnack's inequality now depends on the Hölder semi-norm of the coefficients. For more general Carnot groups and associated horizontally elliptic operators, the Double Ball and Critical Density properties have been established by Tralli assuming the Cordes-Landis condition (cf. [30], [31]). As of the writing of this document, the problem of establishing the Critical Density Property for $\mathcal{L}_A$ without any restrictive assumptions on the smoothness or on the eigenvalue ratio of the coefficient matrix A remains a challenging open problem.

1.3 Description of Results

In our recent work [1], we establish the critical density property for non-negative supersolutions of the operator $\mathcal{L}_A$ defined in (1.15) in small balls in $\mathbb{H}^1$, assuming the coefficient matrix A only satisfies the assumption of uniform equicontinuity in Ω. In the following, ω denotes the modulus of continuity of $A(\cdot)$.

Theorem 1.3.1. *There exist constants* $0 < \epsilon = \epsilon(\lambda, \Lambda) < 1$*, and* $\delta_0 = \delta_0(\lambda, \Lambda, \omega) > 0$ *such that for all* $z_0 \in \mathbb{H}^1$*,* $0 < r \leq \delta_0$*, and* $u \in C^2$ *satisfying* *(i)* $u \geq 0$ *in* $B_{2r}(z_0)$

(ii) $\mathcal{L}_A u \leq 0$ *in* $B_{2r}(z_0)$

(iii) $\inf\limits_{B_{\frac{r}{2}}(z_0)} u < \frac{1}{2}$,

we have $|\{z \in B_r(z_0) : u(z) < 1\}| \geq \epsilon \, |B_r(z_0)|$.

For $\mathbb{H}^n$, $n > 1$, the analogue of Theorem 1.3.1 holds if the matrix A satisfies an additional algebraic condition, namely that it is symplectic at every point, once it is normalized to have unit determinant. We recall below the definition of a symplectic matrix.

Definition 1.3.2. *A symmetric positive definite matrix* $M \in \mathbb{R}^{2n \times 2n}$ *is symplectic if it satisfies* $\mathcal{J}^t M \mathcal{J} = M^{-1}$*, where* $\mathcal{J}$ *is defined in* (1.7).

The following Harnack inequality follows by combining Theorem 1.3.1 with the Double Ball Property established in [20], and invoking the results of [11].

Theorem 1.3.3. *Assume that* $A(\cdot)$ *is uniformly elliptic, continuous and symplectic, with modulus of continuous* ω*. There exist constants* $C \geq 1$ *and* $\eta > 0$ *depending only on* Λ, λ, Q *and a constant* δ_0 *depending in addition on* ω*, such that for any* $u \in C^2(\Omega)$ *satisfying* $u \geq 0$*,* $\mathcal{L}_A u = 0$ *in* $B_{\eta r}(z_0) \subset \Omega$ *for some* $r \leq \delta_0$*, we have*

$$\sup_{B_r(z_0)} u \leq C \inf_{B_r(z_0)} u. \tag{1.18}$$

In $\mathbb{H}^1$, the symplectic condition is unnecessary, as every symmetric positive definite 2×2 matrix of unit determinant is symplectic. Consequently, Theorem 1.3.3 subsumes the case of Hölder continuous coefficients considered in [5] when $n = 1$.

The approach to proving Theorem 1.3.1 is a perturbative one: we construct barriers in small balls by using the fundamental solution of the frozen operator, and then use the barriers to obtain pointwise-to-measure estimates for supersolutions of $\mathcal{L}_A$. It is in the construction of these barriers where the assumption that A is continuous and symplectic is used. More precisely, the fact that A is symplectic allows us to explicitly identify the fundamental solution for a class of constant coefficient operators; see Remark 2.1.5. Once the fundamental solution is at hand, it can be used to construct barriers for $\mathcal{L}_A$ on arbitrary open sets of small enough diameter determined by the modulus of continuity of the coefficients $A(\cdot)$. After the construction of the barriers, one can use maximum principle arguments to obtain the desired pointwise-to-measure estimate. See Chapter 3 for all the details of this argument.

Another goal of this thesis is to furnish an alternate proof of Theorem 1.3.3 by following an approach of E. M. Landis [27]. The main ingredient of Landis' method is the so-called Growth Lemma (see Theorem 2.3.1), which shows in a quantitative manner how the oscillation of subsolutions of $\mathcal{L}_A$ reduces from a larger ball to a smaller one. Although the Landis approach does not yield a stronger result than Theorem 1.3.3, the method of proof is more straightforward compared to that in [1]. One only needs to establish the growth lemma and then perform an elementary but ingenious iteration argument to obtain Harnack's inequality (see Theorem 2.4.3). This bypasses the use of sophisticated tools such as the Besicovitch covering theorem and the Calderón-Zygmund set decomposition, which are needed in the axiomatic approach of [11]. See Chapter 2 for details of Landis' technique adapted to the setting of horizontally elliptic operators. For more instances in which Landis-type growth theorems are established, we refer to [12], [17] for parabolic equations, and [28] for elliptic equations with unbounded drift.

We believe that the Landis approach is quite versatile and has the potential to be applied to other degenerate PDEs. As an example, we mention our latest work with G. Tralli [2], where we establish regularity properties of solutions to a class of parabolic equations in non-divergence form of Kolmogorov-Fokker-Planck type. The prototypical operator in this class is

$$\mathcal{K}_A := \operatorname{tr}\left(A(x,y,t)D_x^2\right) + \langle x, \nabla_y\rangle - \partial_t, \quad (x,y,t) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}, \tag{1.19}$$

where the $d \times d$ matrix $A(x,y,t)$ satisfies

$$0 < \lambda \mathbb{I}_d \leq A(x,y,t) \leq \Lambda \mathbb{I}_d, \text{ for all } (x,y,t) \in \Omega \subset \mathbb{R}^{2n+1}. \tag{1.20}$$

Note that $\mathcal{K}_A$ is highly degenerate, as the highest order derivatives occur only in the x variable. However, like $\mathcal{L}_A$, the operator $\mathcal{K}_A$ possesses an underlying homogeneous Lie group structure, and moreover, the corresponding constant coefficient operator has an explicit fundamental solution. Assuming that the coefficients $A(x,y,t)$ are either uniformly continuous or satisfy a smallness assumption on the eigenvalue ratio $\frac{\Lambda}{\lambda}$, we were able to construct barriers for $\mathcal{K}_A$ using the fundamental solution for constant coefficient operators. This allowed us to implement the Landis approach and establish a parabolic growth lemma and Harnack inequality on cylindrical sets adapted to the homogeneous Lie group structure of the operator $\mathcal{K}_A$. We refer the interested reader to [2] for details.

CHAPTER 2

A LANDIS-TYPE APPROACH FOR HORIZONTALLY ELLIPTIC EQUATIONS

Our goal in this chapter is to describe a Landis-type approach to proving Theorem 1.3.3. While we follow closely the presentation in Landis' book [27], we deviate in one significant manner; namely, we do not use the notion of s-capacity. In this regard, we choose to follow the more direct approach for the construction of barriers, as described in [26].

2.1 Basic Notions

We recall and expand upon some of the notions introduced in Section 1.1. Denote coordinates in $\mathbb{R}^{2n+1}$ as $(x,t) = (x_1, \dots, x_{2n}, t) \in \mathbb{R}^{2n} \times \mathbb{R}$. Recall the matrix

$\mathcal{J}$ from (1.7), and the composition and dilation laws (1.8), (1.9).

$$\mathcal{J} := \begin{pmatrix} 0 & -\mathbb{I}_n \\ \\ \mathbb{I}_n & 0 \end{pmatrix};$$

$$(x,t) \circ (\xi, \tau) := (x + \xi, t + \tau + 2 \langle \mathcal{J}x, \xi \rangle);$$

$$\delta_r(x,t) := (rx, r^2 t), \ r > 0.$$

The identity element of $\mathbb{H}^n$ is $0 = (0,0)$ and the inverse is $(x,t)^{-1} := (-x,-t)$. To simplify notation, we will from here onwards denote the points in $\mathbb{H}^n$ as $z = (x,t)$ and $\zeta = (\xi, \tau)$.

The following lemma shows how to construct a δ_r-homogeneous norm on $\mathbb{H}^n$. This will naturally yield a metric compatible with group translations.

Lemma 2.1.1. *The function*

$$\rho(z) = \rho(x,t) := (|x|^4 + t^2)^{\frac{1}{4}}, \qquad z \in \mathbb{R}^{2n+1} \tag{2.1}$$

defines a δ_r-homogeneous norm on $\mathbb{H}^n$.

Proof. We modify the proof given in [8, pg. 18] for $\mathbb{H}^1$. Clearly, $\rho(z) = 0$ if and only if $z = 0$ and $\rho(\delta_r(z)) = r\rho(z)$ for all $r > 0$. To verify the triangle inequality, we must show that for all $z = (x,t)$ and $\zeta = (\xi, \tau)$, we have $\rho(z \circ \zeta) \leq \rho(z) + \rho(\zeta)$. In the following, we will denote by $\mathbf{i}$ the imaginary number $\sqrt{-1}$. The notation $|\cdot|$ will perform double duty and denote both the modulus of a complex number and the length of a vector in real Euclidean space. We also write $x = (x', x'')$ and $\xi = (\xi', \xi'')$, where $x', x'', \xi', \xi'' \in \mathbb{R}^n$. Then the complex vectors $v, w \in \mathbb{C}^n$ can be defined as $v := x' + \mathbf{i}x''$ and $w := \xi' + \mathbf{i}\xi''$. A direct calculation shows that

$$\langle x, \xi \rangle + \mathbf{i} \langle \mathcal{J}x, \xi \rangle = H(v, w),$$

where $H(v, w) = \overline{v} \cdot w$ is the standard Hermitian inner product on $\mathbb{C}^n$. Notice that by the Cauchy-Schwarz inequality,

$$|H(v, w)| \leq H(v, v)^{1/2} H(w, w)^{1/2} = |x||\xi| \leq \rho(z)\rho(\zeta).$$

We can thus write

$$\begin{aligned}
(\rho(z \circ \zeta))^4 &= |x + \xi|^4 + (t + \tau + 2 \langle \mathcal{J}x, \xi \rangle)^2 \\
&= \left| |x + \xi|^2 + \mathbf{i}(t + \tau + 2 \langle \mathcal{J}x, \xi \rangle) \right|^2 \\
&= \left| |x|^2 + 2 \langle x, \xi \rangle + |\xi|^2 + \mathbf{i}(t + \tau + 2 \langle \mathcal{J}x, \xi \rangle) \right|^2 \\
&= \left| \left(|x|^2 + \mathbf{i}t\right) + \left(|\xi|^2 + \mathbf{i}\tau\right) + 2 \left(\langle x, \xi \rangle + \mathbf{i} \langle \mathcal{J}x, \xi \rangle\right) \right|^2 \\
&\leq \left(\left||x|^2 + \mathbf{i}t\right| + \left||\xi|^2 + \mathbf{i}\tau\right| + 2 \left|\langle x, \xi \rangle + \mathbf{i} \langle \mathcal{J}x, \xi \rangle\right| \right)^2 \\
&= \left(\rho(z)^2 + \rho(\zeta)^2 + 2|H(v, w)|\right)^2 \\
&\leq \left(\rho(z)^2 + \rho(\zeta)^2 + 2\rho(z)\rho(\xi)\right)^2 \\
&= (\rho(z) + \rho(\zeta))^4 .
\end{aligned}$$

□

We claim that the function

$$d(z, \zeta) := \rho(z^{-1} \circ \zeta) \tag{2.2}$$

defines a metric on $\mathbb{H}^n$. Indeed, by Lemma 2.1.1, d is positive-definite and satisfies the triangle inequality. Symmetry follows from the identity $z^{-1} \circ \zeta = -(\zeta^{-1} \circ z)$.

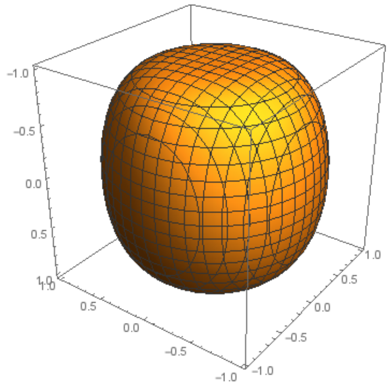

Figure 2.1: The unit d-ball in $\mathbb{H}^1$.

The balls defined by the metric (2.2) (see Figure 2.1 above) are

$$B_R(z) := \left\{\zeta \in \mathbb{R}^{2n+1} : d(z,\zeta) < R\right\}.$$

Equivalently, $B_R(z) = z \circ B_R(0) = z \circ (\delta_R(B_1(0)))$. Since the Jacobian of δ_R is easily verified to be R^{2n+2}, and Lebesgue measure is invariant under translations in the group, we have $|B_R(z)| = |B_R(0)| = R^Q|B_1(0)|$ for all $z \in \mathbb{H}^n$ and $R > 0$, where $Q = 2n + 2$.

The Lie algebra of $\mathbb{H}^n$ is generated by the horizontal vector fields

$$X_i := \partial_{x_i} + 2(\mathcal{J}x)_i\partial_t, \qquad i = 1, \dots, 2n. \tag{2.3}$$

The following left-invariance and homogeneity properties of the differential operators X_i follow from basic multivariable calculus and will be used frequently.

Lemma 2.1.2. *For all $f \in C^1(\mathbb{R}^{2n+1})$, we have*

(i) $X_i[f(\zeta^{-1} \circ z)] = (X_i f)(\zeta^{-1} \circ z) \quad \forall\ z, \zeta \in \mathbb{R}^{2n+1}$;

(ii) $X_i[f(\delta_r z)] = r(X_i f)(\delta_r z) \quad \forall\ z \in \mathbb{R}^{2n+1},\ r > 0.$

The horizontal gradient of a function $\psi \in C^1(\mathbb{R}^{2n+1})$ is the vector

$$\nabla_{\mathbb{H}}\psi := (X_1\psi, \ldots, X_{2n}\psi).$$

The horizontal Hessian of a function $\psi \in C^2(\mathbb{R}^{2n+1})$ is the matrix

$$D^2_{\mathbb{H}}\psi := (X_{i,j}\psi)_{i,j=1,\ldots,2n}\,, \qquad \text{where } X_{i,j}\psi := \frac{1}{2}\left(X_iX_j\psi + X_jX_i\psi\right).$$

Direct calculation shows that

$$X_{i,j}\psi(x,t) = \partial_{x_i}\partial_{x_j}\psi + 2(\mathcal{J}x)_j\partial_{x_i}\partial_t\psi + 2(\mathcal{J}x)_i\partial_{x_j}\partial_t\psi + 4(\mathcal{J}x)_i(\mathcal{J}x)_j\partial_t\partial_t\psi, \tag{2.4}$$

and so $D^2_{\mathbb{H}}$ is a pure second order differential operator like the standard Hessian D^2.

Let $\Omega \subset \mathbb{H}^n \equiv \mathbb{R}^{2n+1}$ be an open set. As stated in Section 1.1, we will be concerned with the second order differential operators

$$\mathcal{L}_A := \operatorname{tr}\left(A(z)D^2_{\mathbb{H}}\,\cdot\right) = \sum_{i,j=1}^{2n} a_{ij}(z)X_{i,j} = \sum_{i,j=1}^{2n} a_{ij}(z)X_iX_j, \tag{2.5}$$

where $A(z) = (a_{ij}(z))_{i,j=1,\ldots,2n} \in \mathbb{R}^{2n\times 2n}$ is symmetric and uniformly elliptic for each $z \in \Omega$; i.e., there exist positive constants λ, Λ such that

$$\lambda\mathbb{I}_{2n} \leq A(z) = A^T(z) \leq \Lambda\mathbb{I}_{2n}, \text{ for all } z \in \Omega \subset \mathbb{H}^n. \tag{2.6}$$

We claim $\mathcal{L}_A$ is an elliptic operator in the sense of (1.1). Indeed, using (2.4), we find that for any $\psi \in C^2(\mathbb{R}^{2n+1})$, we have $\mathcal{L}_A\psi = \operatorname{tr}\left(M(x,t)D^2\psi(x,t)\right)$, where

$$M(x,t) = \begin{pmatrix} A(x,t) & 2A(x,t)\mathcal{J}x \\ 2(A(x,t)\mathcal{J}x)^T & 4\left\langle A(x,t)\mathcal{J}x, \mathcal{J}x\right\rangle \end{pmatrix} \in \mathbb{R}^{(2n+1)\times(2n+1)}. \tag{2.7}$$

If $\zeta = (\xi, \tau) \in \mathbb{R}^{2n} \times \mathbb{R}$, it is easy to check that

$$\zeta^T M(x,t)\zeta = |A^{\frac{1}{2}}(x,t)(\xi + 2\tau\mathcal{J}x)|^2 \geq 0. \tag{2.8}$$

In fact, $\mathcal{L}_A$ is degenerate elliptic; the vector $\zeta = (2\mathcal{J}x, -1)$ belongs to the kernel of $M(x,t)$, and so $\det(M(x,t)) = 0$ for any $(x,t) \in \mathbb{R}^{2n+1}$. The weak maximum principle (1.4) for $\mathcal{L}_A$ follows from the properties (2.6) and (2.8) (see [16, Remark after Theorem 3.1]).

Since we will only consider solutions u to the equation $\mathcal{L}_A u = 0$, we may assume, without loss of generality, that $\det(A(z)) = 1$ for all $z \in \Omega$. This implies $\lambda \leq 1 \leq \Lambda$. The class of symmetric matrices with unit determinant satisfying (2.6) will be denoted by $M_n(\lambda, \Lambda, \Omega)$. From here onwards, any constant that depends solely on n, λ, Λ will be referred to as a *structural constant*.

The following algebraic condition on the coefficient matrices will be needed to establish our results. Recall the definition of a symplectic matrix, Definition 1.3.2.

Definition 2.1.3. *$A \in M_n(\lambda, \Lambda, \Omega)$ is said to be symplectic if $A(z)$ is symplectic at each point $z \in \Omega$ in the sense of Definition 1.3.2.*

If $A \in M_n(\lambda, \Lambda, \Omega)$ is symplectic, we then have the identity

$$A^{-1}(z) = \mathcal{J}^t A(z) \mathcal{J} \qquad \text{for all } z \in \Omega. \tag{2.9}$$

Notice that every symmetric, positive definite 2×2 matrix with unit determinant is symplectic. As a result, this condition will be satisfied automatically in $\mathbb{H}^1$. Let us also remark that symplectic transformations arise naturally in the study of the automorphisms of $\mathbb{H}^n$, see [14, Theorem 1.22].

Example 2.1. *Suppose A is given in block form*

$$A(z) = \begin{pmatrix} A_{11}(z) & A_{12}(z) \\ A_{12}^t(z) & A_{22}(z) \end{pmatrix},$$

where $A_{11}, A_{22}, A_{12} \in \mathbb{R}^{n\times n}$ and A_{11}, A_{22} are symmetric. Then A satisfies (2.9) *if and only if the blocks satisfy the identities*

$$\begin{aligned} A_{11}(z)A_{22}(z) - A_{12}^2(z) = \mathbb{I}_n, \\ A_{11}(z)A_{12}^t(z) = A_{12}(z)A_{11}(z), \\ A_{22}(z)A_{12}(z) = A_{12}^t(z)A_{22}(z). \end{aligned}$$

In particular the matrix

$$A(z) = \begin{pmatrix} A_{11}(z) & 0 \\ 0 & A_{11}^{-1}(z) \end{pmatrix}$$

is symplectic, for any A_{11} symmetric and positive definite.

For any constant matrix $M \in M_n(\lambda, \Lambda, \Omega)$ we let

$$\phi_M(x,t) := \left\langle M^{-1}x, x\right\rangle^2 + t^2. \tag{2.10}$$

The following lemma (cf. [1, Lemma 3.2]) establishes some useful identities satisfied by the function ϕ_M when M is a symplectic matrix with constant entries.

Lemma 2.1.4. *Suppose M is a symmetric, positive definite and symplectic constant matrix. Then for all $(x,t) \in \mathbb{H}^n$, we have*

$$\frac{Q+2}{4}\left\langle M\nabla_{\mathbb{H}}\phi_M, \nabla_{\mathbb{H}}\phi_M\right\rangle = \phi_M \mathcal{L}_M \phi_M = 4(Q+2)\left\langle M^{-1}x, x\right\rangle \phi_M. \tag{2.11}$$

Conversely, if the first identity in (2.11) *holds for ϕ_M in* (2.10)*, then the matrix M must be symplectic.*

Proof. By direct computation, we have

$$X_j\phi_M = 4(M^{-1}x)_j\left\langle M^{-1}x, x\right\rangle + 4t(\mathcal{J}x)_j, \tag{2.12}$$

$$X_iX_j\phi_M = 4(M^{-1})_{ji}\left\langle M^{-1}x, x\right\rangle + 8(M^{-1}x)_i(M^{-1}x)_j + 4t\mathcal{J}_{ji} + 8(\mathcal{J}x)_i(\mathcal{J}x)_j. \tag{2.13}$$

Using the antisymmetry of $\mathcal{J}$, we thus have

$$X_{i,j}\phi_M = 4(M^{-1})_{ji}\left\langle M^{-1}x, x\right\rangle + 8(M^{-1}x)_i(M^{-1}x)_j + 8(\mathcal{J}x)_i(\mathcal{J}x)_j. \tag{2.14}$$

By (2.9) we obtain

$$\begin{aligned}
\langle M\nabla_{\mathbb{H}}\phi_M, \nabla_{\mathbb{H}}\phi_M\rangle &= \left\langle 4\left\langle M^{-1}x, x\right\rangle x + 4tM\mathcal{J}x, 4\left\langle M^{-1}x, x\right\rangle M^{-1}x + 4t\mathcal{J}x\right\rangle \\
&= 16\left\langle M^{-1}x, x\right\rangle^3 + 16t^2\left\langle M\mathcal{J}x, \mathcal{J}x\right\rangle \\
&= 16\left\langle M^{-1}x, x\right\rangle^3 + 16t^2\left\langle M^{-1}x, x\right\rangle \\
&= 16\left\langle M^{-1}x, x\right\rangle \phi_M(x,t), \text{ and}
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_M\phi_M &= \sum_{i,j=1}^{2n} M_{ij}X_{i,j}\phi_M \\
&= \operatorname{tr}\left(M\left(4\left\langle M^{-1}x, x\right\rangle M^{-1} + 8(M^{-1}x)\otimes(M^{-1}x) + 8(\mathcal{J}x)\otimes(\mathcal{J}x)\right)\right) \\
&= 4(Q-2)\left\langle M^{-1}x, x\right\rangle + 8\left\langle M^{-1}x, x\right\rangle + 8\left\langle M\mathcal{J}x, \mathcal{J}x\right\rangle \\
&= 4(Q+2)\left\langle M^{-1}x, x\right\rangle,
\end{aligned}$$

which proves (2.11).

The converse follows by setting $t = 0$ in the previous identities for $\phi_M\mathcal{L}_M\phi_M$ and $\langle M\nabla_{\mathbb{H}}\phi_M, \nabla_{\mathbb{H}}\phi_M\rangle$. □

Remark 2.1.5. *From* (2.11), (2.12) *and* (2.13), *it follows that the function*

$$\Gamma_M := \phi_M^{-\frac{Q-2}{4}} \tag{2.15}$$

is, up to a multiplicative constant, the fundamental solution of $\mathcal{L}_M$ with pole at 0. *In fact, away from the origin we have*

$$\mathcal{L}_M \Gamma_M = \frac{Q-2}{4} \phi_M^{-\frac{Q+6}{4}} \left[\frac{Q+2}{4} \langle M \nabla_{\mathbb{H}} \phi_M, \nabla_{\mathbb{H}} \phi_M \rangle - \phi_M \mathcal{L}_M \phi_M \right] = 0. \tag{2.16}$$

2.2 Construction of Barriers

We now proceed to state precisely the continuity assumptions that are needed on the coefficients $A(z)$ in (2.5). In the following, $C(\Omega)$ denotes the set of continuous matrix-valued functions on Ω, and $||\cdot||$ denotes the operator norm of a matrix.

Definition 2.2.1. *Let $\omega : [0,1) \to [0,1)$ be a non-decreasing function satisfying $\lim_{s\to 0^+} \omega(s) = \omega(0) = 0$. The class $C(\Omega, \omega)$ is the set of matrices $A \in C(\Omega)$ for which*

$$\omega_A(z_0, \epsilon) := \sup_{z \in B_\epsilon(z_0)\cap\Omega} ||A(z) - A(z_0)|| \leq \omega(\epsilon) \qquad \textit{for all } z_0 \in \Omega \textit{ and } 0 < \epsilon < 1.$$

Example 2.2. *(i) The class of d-Hölder continuous matrices with exponent $\alpha \in (0,1)$ satisfies $\omega_A(z_0;\epsilon) \leq C\epsilon^\alpha$ for some constant $C > 0$. In this setting an invariant Harnack inequality for $\mathcal{L}_A$ is proved in [5].*

(ii) The class of d-Dini continuous matrices consists of matrices $A(\cdot)$ satisfying

$$\int_0^1 \frac{\omega_A(z_0;s)}{s}\, ds \leq D_0 < +\infty \qquad \textit{for all } z_0 \in \Omega.$$

We claim this class belongs to $C(\Omega,\omega)$ with $\omega(\epsilon) = \frac{D_0}{\log(\frac{1}{\epsilon})}$. Indeed, for all $z_0 \in \Omega$ and any $0 < \epsilon < 1$, we have

$$\int_0^1 \frac{\omega_A(z_0;s)}{s}\, ds \geq \int_\epsilon^1 \frac{\omega_A(z_0;s)}{s}\, ds \geq \omega_A(z_0;\epsilon) \int_\epsilon^1 \frac{1}{s}\, ds = \omega_A(z_0;\epsilon) \log\left(\frac{1}{\epsilon}\right).$$

We are now ready to prove the following important lemma (cf. [1, Lemma 3.6]), which is a key player in both the Landis-type approach outlined in this chapter, and the proof of the critical density estimates in Chapter 3. The main ingredients here are the continuity of $A(\cdot)$, the property (2.9), and Lemma 2.1.4.

Lemma 2.2.2. *Suppose $A \in M_n(\lambda, \Lambda) \cap C(\Omega, \omega)$ is symplectic. Fix $0 < s < \frac{1}{2}$ and $z_0 \in \Omega$. Let $\alpha = \dfrac{Q-2}{4} + s$ and $M := A(z_0)$. Consider the function*

$$\phi_{z_0,s}(\zeta) := \phi_M^{-\alpha}(\zeta), \qquad \text{for } \ \zeta \neq 0. \tag{2.17}$$

There exists $\delta_0 > 0$ depending only on λ, Λ, Q, s and ω such that

$$\operatorname{tr}\left(A(z)(D^2_{\mathbb{H}}\phi_{z_0,s})(\zeta)\right) \geq 0, \quad \text{for all } z \in B_{\delta_0}(z_0) \cap \Omega \text{ and } \zeta \neq 0. \tag{2.18}$$

Proof. For any $z \in \Omega$ and for all $\zeta \neq 0$, we have

$$\operatorname{tr}\left(A(z)(D^2_{\mathbb{H}}\phi_{z_0,s})(\zeta)\right) = \sum_{i,j=1}^{2n} a_{ij}(z) X_i X_j \phi_{z_0,s}(\zeta)$$

$$= \alpha\phi_M^{-\alpha-2}(\zeta)\left\{(\alpha+1)\left\langle A(z)\nabla_{\mathbb{H}}\phi_M(\zeta), \nabla_{\mathbb{H}}\phi_M(\zeta)\right\rangle - \phi_M(\zeta)\operatorname{tr}\left(A(z)D^2_{\mathbb{H}}\phi_M(\zeta)\right)\right\}$$

$$= \alpha\phi_M^{-\alpha-2}(\zeta)\left\{\mathrm{I} + \mathrm{II}\right\},$$

where

$$\mathrm{I} := s\left\langle A(z)\nabla_{\mathbb{H}}\phi_M(\zeta), \nabla_{\mathbb{H}}\phi_M(\zeta)\right\rangle,$$

$$\mathrm{II} := \frac{Q+2}{4}\left\langle A(z)\nabla_{\mathbb{H}}\phi_M(\zeta), \nabla_{\mathbb{H}}\phi_M(\zeta)\right\rangle - \phi_M\operatorname{tr}\left(A(z)D^2_{\mathbb{H}}\phi_M(\zeta)\right).$$

We first estimate I. Since $A(\cdot), M \in M_n(\lambda, \Lambda)$, we have

$$\mathrm{I} = s\left\langle A(z)\nabla_{\mathbb{H}}\phi_M(\zeta), \nabla_{\mathbb{H}}\phi_M(\zeta)\right\rangle \geq s\left(\frac{\lambda}{\Lambda}\right)\left\langle M\nabla_{\mathbb{H}}\phi_M(\zeta), \nabla_{\mathbb{H}}\phi_M(\zeta)\right\rangle. \tag{2.19}$$

To estimate II, we write $A(z) = A(z_0) + [A(z) - A(z_0)] = M + R(z)$. Using (2.11), we thus obtain

$$\begin{aligned}
\mathrm{II} &= \frac{Q+2}{4}\langle A(z)\nabla_{\mathbb{H}}\phi_M(\zeta), \nabla_{\mathbb{H}}\phi_M(\zeta)\rangle - \phi_M(\zeta)\mathrm{tr}\left(A(z)D^2_{\mathbb{H}}\phi_M(\zeta)\right) \\
&= \frac{Q+2}{4}\langle (M+R(z))\nabla_{\mathbb{H}}\phi_M(\zeta), \nabla_{\mathbb{H}}\phi_M(\zeta)\rangle - \phi_M(\zeta)\mathrm{tr}\left((M+R(z))D^2_{\mathbb{H}}\phi_M(\zeta)\right) \\
&= \frac{Q+2}{4}\langle R(z)\nabla_{\mathbb{H}}\phi_M(\zeta), \nabla_{\mathbb{H}}\phi_M(\zeta)\rangle - \phi_M(\zeta)\mathrm{tr}\left(R(z)D^2_{\mathbb{H}}\phi_M(\zeta)\right).
\end{aligned}$$

By uniform elliptiicty of $A(\cdot)$, there exists a positive constant $C_1 = C_1(\lambda, \Lambda, Q)$ such that

$$\left|\frac{Q+2}{4}\langle R(z)\nabla_{\mathbb{H}}\phi_M(\zeta), \nabla_{\mathbb{H}}\phi_M(\zeta)\rangle\right| \leq C_1\|R(z)\|\langle M\nabla_{\mathbb{H}}\phi_M(\zeta), \nabla_{\mathbb{H}}\phi_M(\zeta)\rangle. \quad (2.20)$$

Also, by (2.14), we have

$$\begin{aligned}
\mathrm{tr}\left(R(z)D^2_{\mathbb{H}}\phi_M(\zeta)\right) &= 4\left\langle M^{-1}\xi, \xi\right\rangle \mathrm{tr}(M^{-1}R(z)) \\
&\quad + 8\left\langle R(z)M^{-1}\xi, M^{-1}\xi\right\rangle + 8\langle R(z)\mathcal{J}\xi, \mathcal{J}\xi\rangle.
\end{aligned}$$

Since $\Lambda^{-1} \leq M^{-1} \leq \lambda^{-1}$, we conclude that there exists a constant $C_2 = C_2(\lambda, \Lambda)$ such that

$$\left|\mathrm{tr}\left(R(z)D^2_{\mathbb{H}}\phi_M(\zeta)\right)\right| \leq C_2\|R(z)\||\xi|^2.$$

Multiplying by ϕ_M, we have

$$\left|\phi_M(\zeta)\mathrm{tr}\left(R(z)D^2_{\mathbb{H}}\phi_M(\zeta)\right)\right| \leq C_2\|R(z)\||\xi|^2\phi_M(\zeta). \quad (2.21)$$

Now, by (2.11), we obtain

$$\langle M\nabla_{\mathbb{H}}\phi_M(\zeta), \nabla_{\mathbb{H}}\phi_M(\zeta)\rangle = 16\left\langle M^{-1}\xi, \xi\right\rangle \phi_M(\zeta) \geq \frac{16}{\Lambda}|\xi|^2\phi_M(\zeta).$$

In conjunction with (2.21), this implies the existence of a constant $C_3 = C_3(\lambda, \Lambda)$ such that

$$\left|\phi_M(\zeta)\mathrm{tr}\left(R(z)D^2_{\mathbb{H}}\phi_M(\zeta)\right)\right| \leq C_3\|R(z)\| \left\langle M\nabla_{\mathbb{H}}\phi_M(\zeta), \nabla_{\mathbb{H}}\phi_M(\zeta)\right\rangle.$$

With the bounds (2.20) and (2.21), we thus conclude there exists some constant $C = C(\lambda, \Lambda, Q)$ such that

$$|\mathrm{II}| \leq C\|R(z)\| \left\langle M\nabla_{\mathbb{H}}\phi_M(\zeta), \nabla_{\mathbb{H}}\phi_M(\zeta)\right\rangle. \tag{2.22}$$

Combining our estimates (2.19) and (2.22) for I and II respectively, and by noticing that $\phi_M \geq 0$, we obtain

$$\mathrm{tr}\left(A(z)(D^2_{\mathbb{H}}g)(\zeta)\right) \geq \alpha\phi_M^{-\alpha-2}(\zeta)\left\{s\left(\frac{\lambda}{\Lambda}\right) - C\|R(z)\|\right\}\left\langle M\nabla_{\mathbb{H}}\phi_M(\zeta), \nabla_{\mathbb{H}}\phi_M(\zeta)\right\rangle.$$

We now choose $\delta_0 > 0$ such that

$$\omega(\delta_0) \leq \frac{s\lambda}{C\Lambda}. \tag{2.23}$$

Since $||R(z)|| \leq \omega(\delta)$ if $|z| \leq \delta$, this implies (2.18). $\square$

Fix $0 < s < \frac{1}{2}$ and $z_0 \in \Omega$. For any bounded, Borel set $E \subset \mathbb{R}^{2n+1}$, consider the function

$$U_E(z) := \int_E \phi_{z_0,s}(z^{-1}\circ\zeta)\ d\zeta. \tag{2.24}$$

This function is well-defined due to the restriction on the values of s (cf. [4, Corollary 5.4.5]). The following is immediate from (2.18).

Corollary 2.2.3. *We have $\mathcal{L}_A U_E(z) \geq 0$ for all $z \in B_{\delta_0}(z_0)\backslash E$, where $\delta_0 > 0$ is the constant determined by Lemma 2.2.2.*

We now proceed to obtain pointwise bounds on U_E. These will be required for maximum principle arguments in the proof of the Landis-type growth lemma for horizontally elliptic operators, Theorem 2.3.1.

Lemma 2.2.4. *Fix $z_0 \in \mathbb{R}^{2n+1}$, $\tau > 1$ and $0 < s < \frac{1}{2}$. Let $\alpha = \dfrac{Q-2}{4} + s$. Then there exist positive constants $C_1 = C_1(\Lambda, \alpha, \tau), C_2 = C_2(\Lambda, \alpha, \tau), C_3 = C_3(\lambda, \alpha)$ such that for any Borel set $E \subset B_R(z_0)$, the function U_E satisfies the pointwise bounds*

(i) $U_E(z) \leq C_1 R^{-4\alpha}|E|$ *for all* $z \in \partial B_{\tau R}(z_0)$;

(ii) $U_E(z) \leq C_2 R^{-4\alpha}|B_R(z_0)|$ *for all* $z \in B_{\tau R}(z_0)$;

(iii) $U_E(z) \geq C_3 R^{-4\alpha}|E|$ *for all* $z \in B_R(z_0)$.

Proof. Recall the δ_r-homogeneous norm ρ, defined in (2.1), and the corresponding metric d, defined in (2.2). It suffices to establish pointwise estimates for the function

$$\psi_{E,\alpha}(z) := \int_E \frac{1}{d(z,\zeta)^{4\alpha}}\, d\zeta = \int_E \frac{1}{\rho(z^{-1}\circ\zeta)^{4\alpha}}\, d\zeta, \qquad z \in \mathbb{R}^{2n+1},\ E \subset B_R(z_0).$$

Indeed, by uniform ellipticity (2.6), we have

$$\lambda^{2\alpha}\psi_{E,\alpha}(z) \leq U_E(z) \leq \Lambda^{2\alpha}\psi_{E,\alpha}(z) \qquad \text{for all } z \in \Omega. \tag{2.25}$$

(i) Let $z \in \partial B_{\tau R}(z_0)$. Then $d(z,\zeta) \geq (\tau - 1)R$ for all $\zeta \in E$ and so

$$\psi_{E,\alpha}(z) \leq ((\tau-1)R)^{-4\alpha}|E|.$$

Therefore, by (2.25), we have

$$U_E(z) \leq \Lambda^{2\alpha}((\tau-1)R)^{-4\alpha}|E| = C_1 R^{-4\alpha}|E| \qquad \text{for all } z \in \partial B_{\tau R}(z_0).$$

(ii) Let $z \in B_{\tau R}(z_0)$. Then $E \subset B_{(\tau+1)R}(z)$ and so

$$\begin{aligned}\psi_{E,\alpha}(z) &\leq \int_{B_{(\tau+1)R}(z)} d(z,\zeta)^{-4\alpha}\, d\zeta \\ &= ((\tau+1)R)^{Q-4\alpha} \int_{B_1(0)} \rho(\zeta)^{-4\alpha}\, d\zeta = \sigma |B_R(z_0)| R^{-4\alpha},\end{aligned}$$

where

$$\sigma = \sigma(\alpha,\tau) := \frac{(\tau+1)^{Q-4\alpha}}{|B_1(0)|} \int_{B_1(0)} \rho(\zeta)^{-4\alpha}\, d\zeta < +\infty \qquad \text{since } 4\alpha < Q.$$

Therefore, by (2.25), we have

$$U_E(z) \leq \Lambda^{2\alpha} \sigma |B_R(z_0)| R^{-4\alpha} = C_2 R^{-4\alpha} |B_R(z_0)| \qquad \text{for all } z \in B_{\tau R}(z_0).$$

(iii) Let $z \in B_R(z_0)$. Then $d(z,\zeta) \leq 2R$ for all $\zeta \in E$ and so

$$\psi_{E,\alpha}(z) \geq (2R)^{-4\alpha} |E|.$$

Therefore, by (2.25), we have

$$U_E(z) \geq \lambda^{2\alpha} (2R)^{-4\alpha} |E| = C_3 R^{-4\alpha} |E| \qquad \text{for all } z \in B_R(z_0).$$

$\square$

2.3 Growth Lemma

We are now ready to prove the Landis-type growth lemma for subsolutions. Fix $0 < s < 2$ and let $\alpha = \frac{Q-2}{4} + s$. Let C_1, C_2, C_3 be the constants appearing in Lemma 2.2.4. We choose $\tau = \tau(\alpha, \lambda, \Lambda) > 1$ so that

$$C_1(\Lambda, \alpha, \tau) = \frac{1}{2} C_3(\lambda, \alpha). \tag{2.26}$$

This is possible because, from the proof of Lemma 2.2.4,

$$C_1(\Lambda, \alpha, \tau) = \Lambda^{2\alpha}((\tau - 1))^{-4\alpha} \quad \text{and} \quad C_3(\lambda, \alpha) = \lambda^{2\alpha} 2^{-4\alpha}.$$

Hence, we need τ to solve the equation

$$(\tau - 1)^{4\alpha} = \left(\frac{\Lambda}{\lambda}\right)^{2\alpha} 2^{4\alpha+1}.$$

Theorem 2.3.1. *Let $R > 0$ and $\tau R < \delta_0$, where $\delta_0 > 0$ is defined through* (2.23) *and τ is defined according to* (2.26). *Consider an open set $D \subset B_{\tau R}(z_0) \subset \Omega$ such that $D \cap B_R(z_0) \neq \emptyset$ (see Figure 2.2). Suppose $u \in C^2(D) \cap C(\overline{D})$ is non-negative in D, vanishes on $\partial D \cap B_{\tau R}(z_0)$ and satisfies $\mathcal{L}_A u \geq 0$ in D. Then there exists a structural constant $\eta > 0$ such that*

$$\sup_D u \geq \left(1 + \eta \frac{|B_R(z_0) \backslash D|}{|B_R(z_0)|}\right) \sup_{D \cap B_R(z_0)} u.$$

Proof. Define $E := B_R(z_0) \backslash D$. Consider the auxiliary function

$$v(z) := \left(\sup_D u\right) \left[1 - \frac{R^{4\alpha}}{C_2 |B_R(z_0)|} \left(U_E(z) - C_1 R^{-4\alpha} |E|\right)\right]$$

where U_E is defined as in (2.24), and the constants C_1, C_2 are from Lemma 2.2.4. Since $\tau R < \delta_0$, we may apply Lemma 2.2.2 and Corollary 2.2.3 to conclude that $\left(\mathcal{L}_A U_E\right)(z) \geq 0$ for all $z \in B_{\tau R}(z_0) \backslash E$. Hence, by the hypothesis $\mathcal{L}_A u \geq 0$, we have $\mathcal{L}_A v \leq 0 \leq \mathcal{L}_A u$ on D. By Lemma 2.2.4 (i), we have $v \geq \sup_D u \geq u$ on $\partial B_{\tau R}(z_0) \cap \overline{D}$. By Lemma 2.2.4 ($ii$), we have $v \geq 0$ on $B_{\tau R}(z_0)$. Since $u = 0$ on $\partial D \cap B_{\tau R}(z_0)$, we conclude that $v \geq u$ on $\partial D \cap B_{\tau R}(z_0)$. Therefore, $v \geq u$ on ∂D, and so by the maximum principle, $v \geq u$ on D. In particular, $u \leq v$ on $D \cap B_R(z_0)$.

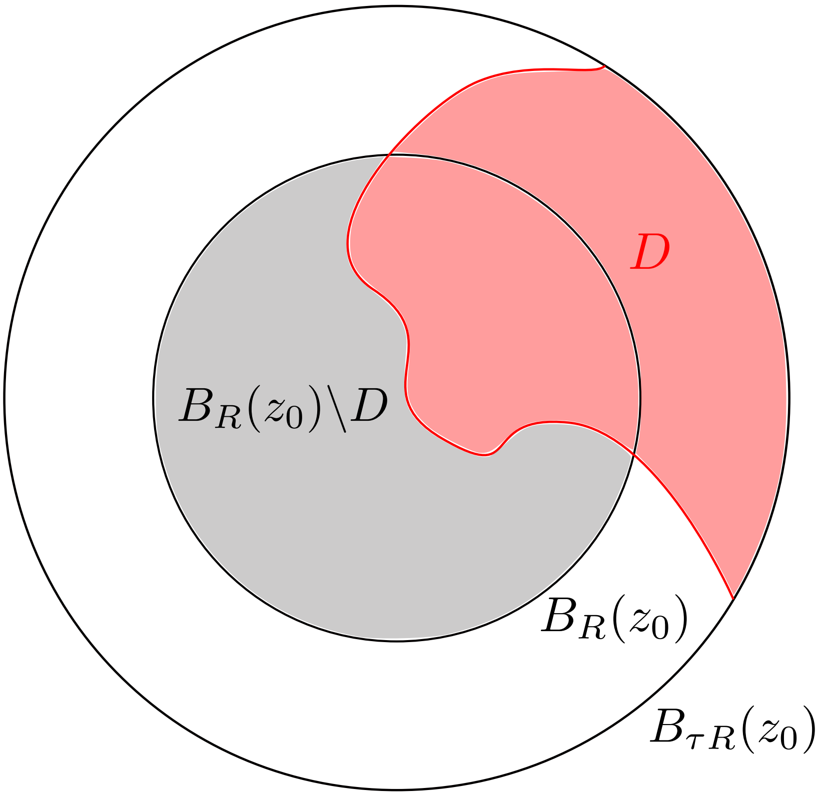

Figure 2.2: Growth Lemma

On the other hand, by Lemma 2.2.4 (*iii*) and (2.26), we have for all $z \in B_R(z_0)$

$$
\begin{aligned}
v(z) &\leq \left(\sup_D u\right)\left[1 - \frac{R^{4\alpha}}{C_2|B_R(z_0)|}\left(C_3 R^{-4\alpha}|E| - C_1 R^{-4\alpha}|E|\right)\right] \\
&= \left(\sup_D u\right)\left[1 - \frac{|E|}{C_2|B_R(z_0)|}\left(C_3 - C_1\right)\right] \\
&= \left(\sup_D u\right)\left[1 - \frac{|E|}{|B_R(z_0)|}\left(\frac{C_3}{2C_2}\right)\right].
\end{aligned}
$$

Letting $\eta := \dfrac{C_3}{2C_2}$, and using the fact that $u \leq v$ on $D \cap B_R(z_0)$, we conclude

$$
\sup_{D \cap B_R(z_0)} u \leq \left(1 - \eta \frac{|E|}{|B_R(z_0)|}\right) \sup_D u,
$$

which implies, after rearrangement,

$$
\sup_D u \geq \left(1 + \eta \frac{|E|}{|B_R(z_0)|}\right) \sup_{D \cap B_R(z_0)} u.
$$

Finally, since $E = B_R(z_0) \backslash D$, we obtain the desired result. $\square$

An immediate corollary of the growth lemma is the oscillation decay and local Hölder continuity of solutions to $\mathcal{L}_A u = 0$. For this, we will require the following definition.

Definition 2.3.2. *(d-Hölder Continuity) A function $u : \mathbb{R}^{2n+1} \to \mathbb{R}$ is said to be locally d-Hölder continuous at $z_0 \in \mathbb{R}^{2n+1}$ if there exist constants $C, r > 0$ and $\alpha \in (0,1)$ such that*

$$
|u(z) - u(z_0)| \leq C\ d(z, z_0)^\alpha \qquad \textit{for all } \ z \in B_r(z_0).
$$

It follows from the definition of the metric d in (1.11) that d-Hölder continuity is equivalent to Hölder continuity in the usual sense, possibly with different constants $C' > 0$ and $\alpha' \in (0,1)$ (cf. [4, Proposition 5.1.6]).

Corollary 2.3.3. *Suppose $u \in C^2(B_{\tau R}(z_0)) \cap C(\overline{B_{\tau R}(z_0)})$ solves $\mathcal{L}_A u = 0$ in $B_{\tau R}(z_0)$ with τ satisfying* (2.26) *and $\tau R < \delta_0$, where $\delta_0 > 0$ is defined through* (2.23)*. Then there exists a structural constant $\mu > 1$ such that*

$$\mathop{osc}_{B_{\tau R}(z_0)} u \geq \mu \mathop{osc}_{B_R(z_0)} u \qquad \text{for all } 0 < R < \frac{\delta_0}{\tau}.$$

Consequently, u is d-Hölder continuous at z_0.

Proof. Define the auxiliary function

$$v(z) := u(z) - \frac{1}{2}\left(\sup_{B_R(z_0)} u + \inf_{B_R(z_0)} u\right), \qquad z \in B_{\tau R}(z_0).$$

Let $D = \{v > 0\} \cap B_{\tau R}(z_0)$. Note that $D \cap B_R(z_0) \neq \emptyset$ if u is non-constant. We may also assume $|D \cap B_R(z_0)| \leq \frac{1}{2}|B_R(z_0)|$; otherwise, apply this argument to $w = -v$. Since $\mathcal{L}_A v = 0$, we can use Theorem 2.3.1 to obtain

$$\begin{aligned}
\sup_{B_{\tau R}(z_0)} v &\geq \sup_D v \\
&\geq \left(1 + \frac{\eta}{2}\right) \sup_{D \cap B_R(z_0)} v \qquad \text{(by Theorem 2.3.1)} \\
&= \left(1 + \frac{\eta}{2}\right) \sup_{B_R(z_0)} v \qquad \text{(because } v \leq 0 \text{ on } B_R(z_0) \backslash D) \\
&= \frac{1}{2}\left(1 + \frac{\eta}{2}\right) \mathop{\mathrm{osc}}_{B_R(z_0)} u.
\end{aligned}$$

Now

$$\begin{aligned}
\sup_{B_{\tau R}(z_0)} v &= \sup_{B_{\tau R}(z_0)} u - \frac{1}{2}\left(\sup_{B_R(z_0)} u + \inf_{B_R(z_0)} u\right) \\
&= \mathop{\mathrm{osc}}_{B_{\tau R}(z_0)} u + \inf_{B_{\tau R}(z_0)} u - \frac{1}{2}\left(\mathop{\mathrm{osc}}_{B_R(z_0)} u + \inf_{B_R(z_0)} u + \inf_{B_R(z_0)} u\right) \\
&= \mathop{\mathrm{osc}}_{B_{\tau R}(z_0)} u + \inf_{B_{\tau R}(z_0)} u - \frac{1}{2} \mathop{\mathrm{osc}}_{B_R(z_0)} u - \inf_{B_R(z_0)} u \\
&\leq \mathop{\mathrm{osc}}_{B_{\tau R}(z_0)} u - \frac{1}{2} \mathop{\mathrm{osc}}_{B_R(z_0)} u.
\end{aligned}$$

Therefore,

$$\operatorname*{osc}_{B_{\tau R}(z_0)} u - \frac{1}{2}\operatorname*{osc}_{B_R(z_0)} u \geq \frac{1}{2}\left(1+\frac{\eta}{2}\right)\operatorname*{osc}_{B_R(z_0)} u.$$

Rearranging, we obtain

$$\operatorname*{osc}_{B_{\tau R}(z_0)} u \geq \left(1+\frac{\eta}{4}\right)\operatorname*{osc}_{B_R(z_0)} u.$$

The oscillation decay thus follows with $\mu := 1 + \frac{\eta}{4}$.

The proof of Hölder continuity now follows by a standard argument, which we reproduce here. Let $z \in B_R(z_0)$ be arbitrary, $\tau R < \delta_0$, and denote $d(z, z_0) = \rho$. Then there exists a non-negative integer N such that $\tau^{-(N+1)}R \leq \rho < \tau^{-N}R$. By applying the oscillation decay inequality N times, we obtain

$$\operatorname*{osc}_{B_\rho(z_0)} u \leq \mu^{-N}\operatorname*{osc}_{B_R(z_0)} u \leq \frac{2\mu||u||_{L^\infty(B_R(z_0))}}{\mu^{N+1}}.$$

Since $\rho \geq \tau^{-(N+1)}R$ and $|u(z) - u(z_0)| \leq \operatorname*{osc}_{B_\rho(z_0)} u$, we have for all $\beta > 0$ that

$$\begin{aligned}
\frac{|u(z)-u(z_0)|}{d(z,z_0)^\beta} &\leq \frac{\tau^{(N+1)\beta}}{R^\beta}\operatorname*{osc}_{B_\rho(z_0)} u \\
&\leq \frac{\tau^{(N+1)\beta}}{R^\beta}\frac{2\mu||u||_{L^\infty(B_R(z_0))}}{\mu^{N+1}} \\
&= \frac{2\mu||u||_{L^\infty(B_R(z_0))}}{R^\beta}\left(\frac{\tau^\beta}{\mu}\right)^{N+1}.
\end{aligned}$$

Choose $\beta = \log_\tau \mu$ to obtain

$$|u(z) - u(z_0)| \leq \left(\frac{2\mu||u||_{L^\infty(B_R(z_0))}}{R^\beta}\right) d(z,z_0)^\beta \qquad \text{for all } z \in B_R(z_0).$$

$\square$

2.4 Harnack Inequality

In this section, we adapt an ingenious iteration argument of E. M. Landis to obtain Harnack's inequality for non-negative solutions to $\mathcal{L}_A u = 0$ as a direct consequence of the growth lemma, Theorem 2.3.1. We begin by establishing the following result.

Lemma 2.4.1. *Suppose τ satisfies* (2.26) *and $\tau r < \delta_0$, where $\delta_0 > 0$ is defined through* (2.23). *Consider an open set $D \subset B_{\tau r}(z_0) \subset \Omega$ such that $D \cap B_r(z_0) \neq \emptyset$. Suppose $u \in C^2(D) \cap C(\overline{D})$ is non-negative in D, vanishes on $\partial D \cap B_{\tau r}(z_0)$ and satisfies $\mathcal{L}_A u \geq 0$ in D. Then for any $M > 1$, there exists $\gamma > 0$ (depending on M and structural constants) such that*

$$|D| \leq \gamma |B_r| \quad \Rightarrow \quad \sup_D u \geq M \sup_{D \cap B_r(z_0)} u.$$

Proof. Let $m = m(\eta, M)$ be the smallest natural number such that $\left(1 + \frac{\eta}{2}\right)^m \geq M$, where η is the constant from Theorem 2.3.1. Consider the concentric balls

$$B^i := B_{(1+(\tau-1)\frac{i}{m})r}(z_0), \qquad i \in \{0, \dots, m\}.$$

Note that $B^0 = B_r(z_0)$ and $B^m = B_{\tau r}(z_0)$. For each $i \in \{0, \dots, m-1\}$, let $\xi_i \in \overline{D \cap B^i}$ be such that $u(\xi_i) = M_i := \sup_{D \cap B^i} u$. We claim that $\xi_i \in \partial B^i \cap D$. Indeed, by the maximum principle, we have $\sup_{D \cap B^i} u = \sup_{\partial(D \cap B^i)} u$. Now we may write $\partial(D \cap B^i) = (\partial D \cap B^i) \cup (\partial B^i \cap D)$. If $\xi_i \in \partial D \cap B^i$, then $u(\xi_i) = 0$, since u vanishes on $\partial D \cap B_{\tau R}(z_0)$ by assumption. This would imply u is identically zero on $D \cap B^i$, which in turn implies u vanishes on $D \cap B_r(z_0)$. The lemma then holds trivially, and so we conclude that $\xi_i \in \partial B^i \cap D$.

Define the sets

$$D_i := D \cap B_{(\frac{\tau-1}{m})r}(\xi_i) \text{ and } \tilde{B}^i := B_{(\frac{\tau-1}{\tau m})r}(\xi_i).$$

Given $M > 1$, we let

$$\gamma := \frac{1}{2}\left(\frac{\tau-1}{\tau m}\right)^Q. \tag{2.27}$$

Therefore, if $|D| \leq \gamma |B_r|$, then

$$|D_i \cap \tilde{B}^i| \leq |D| \leq \gamma |B_r| = \frac{1}{2}\left(\frac{\tau-1}{\tau m}\right)^Q |B_1(0)| r^Q = \frac{1}{2}|\tilde{B}^i|.$$

Hence, $|\tilde{B}^i \backslash D_i| \geq \frac{1}{2}|\tilde{B}^i|$. We now wish to apply Theorem 2.3.1 to u in $D_i \subset B_{(\frac{\tau-1}{m})r}(\xi_i)$, so we check all the hypotheses. Clearly, u is non-negative and satisfies $\mathcal{L}_A u \geq 0$ in D_i. Also, u vanishes on $\partial D_i \cap B_{(\frac{\tau-1}{m})r}(\xi_i)$ because $\partial D_i \cap B_{(\frac{\tau-1}{m})r}(\xi_i) \subset \partial D \cap B_{\tau r}(z_0)$, and u vanishes on $\partial D \cap B_{\tau r}(z_0)$ by hypothesis. Finally, $D_i \cap \tilde{B}^i \neq \emptyset$ because $\xi_i \in D_i \cap \tilde{B}^i$. Therefore, all the hypotheses of Theorem 2.3.1 are verified in $D_i \subset B_{(\frac{\tau-1}{m})r}(\xi_i)$, and so we obtain

$$\begin{aligned}
\sup_{D_i} u &\geq \left(1 + \eta \frac{|\tilde{B}^i \backslash D_i|}{|\tilde{B}^i|}\right) \sup_{D_i \cap \tilde{B}^i} u \\
&\geq \left(1 + \frac{\eta}{2}\right) \sup_{D_i \cap \tilde{B}^i} u \\
&\geq \left(1 + \frac{\eta}{2}\right) u(\xi_i) \\
&= \left(1 + \frac{\eta}{2}\right) M_i.
\end{aligned}$$

Since $B_{(\frac{\tau-1}{m})r}(\xi_i) \subset B^{i+1}$, it follows that $D_i \subset D \cap B^{i+1}$. We thus obtain the recursive inequality

$$M_{i+1} \geq \left(1 + \frac{\eta}{2}\right) M_i, \qquad i \in \{0, \ldots, m-1\}.$$

Iterating the above, we conclude

$$\sup_D u \geq \left(1 + \frac{\eta}{2}\right)^m \sup_{D \cap B_r(z_0)} u \geq M \sup_{D \cap B_r(z_0)} u.$$

$\square$

The following technical lemma will also be used in the proof of Harnack's inequality.

Lemma 2.4.2. *Let $0 \leq \frac{R}{2} \leq r < R$, and let B_r denote the d-ball of radius r centered at the origin. Let $U \subset \mathbb{R}^{2n+1}$ be an open ball centered at a point $\zeta \in \partial B_r$. There exists a constant $\hat{C} > 0$ depending only on U and R such that*

$$|(B_{r+s} \backslash B_r) \cap U| \geq \hat{C} s \qquad \textit{for all } s > 0 \textit{ sufficiently small.} \tag{2.28}$$

Proof. We begin by recalling the co-area formula (cf. [10, Section 3.4.2, Theorem 1]). Let $\Omega \subset \mathbb{R}^{2n+1}$ be a bounded domain, $u : \Omega \to \mathbb{R}$ a Lipschitz function and $g \in L^1(\Omega)$. Then we have

$$\int_\Omega g(z)|Du(z)|\, dz = \int_{-\infty}^{\infty} \left(\int_{u^{-1}(\tau)} g(\sigma)\, d\mathcal{H}_{2n}(\sigma) \right) d\tau,$$

where $\mathcal{H}_{2n}$ denotes $2n$-dimensional Hausdorff measure. Recall the definition of the norm (1.10)

$$\rho(z) = \rho(x,t) = (|x|^4 + t^2)^{1/4}.$$

The set ∂B_r thus corresponds to the level set $\{\rho = r\}$. We calculate $D\rho(z)$:

$$\partial_{x_i} \rho(x,t) = \frac{|x|^2 x_i}{(|x|^4 + t^2)^{3/4}} \qquad i \in \{1, \ldots, 2n\};$$

$$\partial_t \rho(x,t) = \frac{t/2}{(|x|^4 + t^2)^{3/4}}.$$

If $z = (x,t) \in B_R$, then $|x| \le R$. This implies

$$\begin{aligned}|D\rho(z)|^2 &= \frac{|x|^4}{(|x|^4+t^2)^{3/2}}|x|^2 + \frac{t^2/4}{(|x|^4+t^2)^{3/2}} \\ &\le \frac{R^2|x|^4 + \frac{1}{4}t^2}{(|x|^4+t^2)^{3/2}} \\ &\le \max\left\{R^2, \frac{1}{4}\right\} \frac{|x|^4+t^2}{(|x|^4+t^2)^{3/2}} \\ &= \max\left\{R^2, \frac{1}{4}\right\} \frac{1}{(|x|^4+t^2)^{1/2}}\end{aligned}$$

Therefore,

$$|D\rho(z)| \le \max\left\{R, \frac{1}{2}\right\} \frac{1}{\rho(z)} \qquad \text{for all } z \in B_R. \tag{2.29}$$

Now let U be as in the statement of the lemma. By the gradient bound (2.29), we have for all $s > 0$ sufficiently small

$$\int_{B_{r+s}\setminus B_r} \chi_U(z)\rho(z)|D\rho(z)|\, dz \le \max\left\{R, \frac{1}{2}\right\} |(B_{r+s}\setminus B_r)\cap U|,$$

where χ_U is the characteristic function of U. On the other hand, by the co-area formula,

$$\begin{aligned}\int_{B_{r+s}\setminus B_r} \chi_U(z)\rho(z)|D\rho(z)|\, dz &= \int_r^{r+s} \left(\int_{\rho^{-1}(\tau)} \chi_U(\sigma)\rho(\sigma)\, d\mathcal{H}_{2n}(\sigma)\right) d\tau \\ &= \int_r^{r+s} \left(\int_{\partial B_\tau \cap U} \tau\, d\mathcal{H}_{2n}(\sigma)\right) d\tau \\ &= \int_r^{r+s} \tau\, \mathcal{H}_{2n}(\partial B_\tau \cap U)\, d\tau.\end{aligned}$$

Therefore,

$$\int_r^{r+s} \tau\, \mathcal{H}_{2n}(\partial B_\tau \cap U)\, d\tau \le \max\left\{R, \frac{1}{2}\right\} |(B_{r+s}\setminus B_r)\cap U|.$$

Since U is an open ball centered at a point $\zeta \in \partial B_r$, there exists a constant $\sigma > 0$ (depending on U and r) such that $\mathcal{H}_{2n}(\partial B_r \cap U) \ge 2\sigma$. The fact that U is open also implies $(B_{r+s}\backslash B_r)\cap U \neq \emptyset$ for all $s>0$ sufficiently small. Hence, by continuity of the function $\tau \mapsto \mathcal{H}_{2n}(\partial B_\tau \cap U)$ at $\tau = r$, we have $\mathcal{H}_{2n}(\partial B_\tau \cap U) \geq \sigma$ for all $\tau \in (r, r+s)$ with $s>0$ sufficiently small. Therefore,

$$
\begin{aligned}
\int_r^{r+s} \tau\ \mathcal{H}_{2n}(\partial B_\tau \cap U)\ d\tau &\geq \sigma \int_r^{r+s} \tau\ d\tau \\
&= \frac{\sigma}{2}\left((r+s)^2 - r^2\right) \\
&= \frac{\sigma}{2}(2rs + s^2) \\
&\geq \sigma r s \geq \frac{\sigma R}{2} s \qquad \left(\text{since } r \geq \frac{R}{2}\right).
\end{aligned}
$$

The claim (2.28) now follows with $\hat{C} := \dfrac{\sigma R}{\max\{2R, 1\}}$. □

We are finally ready to prove Harnack's inequality in small balls.

Theorem 2.4.3. *Suppose τ satisfies (2.26) and $\tau R < \delta_0$, where $\delta_0 > 0$ is defined through (2.23). There exists a structural constant $C > 0$ such that for all $B_{\tau R}(z_0) \Subset \Omega$ and for all $u \in C^2(B_{\tau R}(z_0)) \cap C(\overline{B_{\tau R}(z_0)})$ non-negative satisfying $\mathcal{L}_A u = 0$ in $B_{\tau R}(z_0)$, we have*

$$
\sup_{B_{R/2}(z_0)} u \leq C \inf_{B_R(z_0)} u. \tag{2.30}
$$

Proof. By translation invariance and scaling, we may assume without loss of generality that $z_0 = 0$ and $\sup\limits_{B_{R/2}(z_0)} u = 2$; from here onward, we denote by B_ρ the ball of radius ρ centered at the origin. We aim to find a structural lower bound for u on B_R. To this end, let

$$
G := \{u > 1\} \cap B_R.
$$

Let $M := 2^{Q+1}$ and let $\gamma > 0$ be the constant appearing in Lemma 2.4.1 corresponding to this choice of M. Define

$$\epsilon_0 := \gamma \left(\frac{1}{8\tau}\right)^Q. \tag{2.31}$$

We are confronted with two possible scenarios: either $|G| \geq \epsilon_0 |B_R|$, or $|G| < \epsilon_0 |B_R|$.

Case 1: $|G| \geq \epsilon_0 |B_R|$.

Consider the function $w = 1 - u$, which satisfies $\mathcal{L}_A w = 0$ in $B_{\tau R}$. Let $D = \{w > 0\} \cap B_{\tau R}$. We may assume $D \cap B_R \neq \emptyset$; otherwise, $u \geq 1$ on B_R and we are done. Since u is non-negative, we have $w \leq 1$ on $B_{\tau R}$. Furthermore, $G \subset B_R \backslash D$, and so $|B_R \backslash D| \geq |G| \geq \epsilon_0 |B_R|$. Consequently, by applying Theorem 2.3.1 to w, we obtain

$$1 \geq \sup_D w \geq \left(1 + \eta \frac{|B_R \backslash D|}{|B_R|}\right) \sup_{D \cap B_R} w \geq (1 + \eta\epsilon_0)\left(1 - \inf_{D \cap B_R} u\right).$$

It follows that $\inf\limits_{D \cap B_R} u \geq \dfrac{\eta\epsilon_0}{1 + \eta\epsilon_0}$. Since $\inf\limits_{B_R \backslash D} u \geq 1$, we conclude that $\inf\limits_{B_R} u \geq \dfrac{\eta\epsilon_0}{1 + \eta\epsilon_0}$, and (2.30) follows.

Case 2: $|G| < \epsilon_0 |B_R|$.

The argument for this case is more delicate, and so we break it into a number of steps. The idea is to construct a sequence of concentric balls $\{\mathbb{B}^{s_\ell}, \ell \geq 1\}$ of radius $(1 + s_\ell)\frac{R}{2} \in \left(\frac{R}{2}, R\right)$ centered at the origin such that

$$\sup_{\mathbb{B}^{s_\ell}} u \geq 2\left(\frac{M}{2}\right)^\ell, \tag{2.32}$$

where we recall $M = 2^{Q+1}$. Associated to each ball $\mathbb{B}^{s_\ell}$ is the set

$$G^\ell := \left\{u > \left(\frac{M}{2}\right)^\ell\right\} \cap B_R.$$

The boundedness of u on B_R, coupled with the geometric growth of $\sup\limits_{\mathbb{B}^{s_\ell}} u$ suggested by (2.32), implies there exists a smallest index k for which the radius of $\mathbb{B}^{s_k}$ exceeds $\frac{3R}{4}$. This will allow us to show that for some index $i_0 \in \{1, \ldots, k\}$, the set G^{i_0} satisfies a measure estimate from below. We can then apply an argument similar to that for Case 1 to extract a lower bound on $\inf\limits_{B_R} u$.

Step 1: Construction of the first ball, $\mathbb{B}^{s_1}$.

Define

$$\mathbb{B}^s := B_{(1+s)\frac{R}{2}}, \qquad 0 < s < 1; \tag{2.33}$$

$$G^0 := G, \qquad G_s^0 := G^0 \cap \left(\mathbb{B}^s \backslash \mathbb{B}^0\right). \tag{2.34}$$

Observe that $\mathbb{B}^0 = B_{R/2}$ and $\mathbb{B}^1 = B_R$. Recalling the definition of ϵ_0 from (2.31), we have

$$|G_{1/2}^0| \leq |G^0| < \epsilon_0 |B_R| = \gamma \left(\frac{1/2}{4\tau}\right)^Q |B_1(0)| R^Q = \gamma |B_{\frac{1}{2}\frac{R}{4\tau}}|. \tag{2.35}$$

On the other hand, we claim

$$|G_s^0| \geq \gamma \left(\frac{s}{4\tau}\right)^Q |B_1(0)| R^Q \quad \text{for all } s > 0 \text{ sufficiently small.} \tag{2.36}$$

To see this, choose $\xi_0 \in \overline{B_{R/2}}$ such that $u(\xi_0) = 2 = \sup\limits_{B_{R/2}} u$. The weak maximum principle allows us to assume $\xi_0 \in \partial B_{R/2} = \partial \mathbb{B}^0$. By continuity of u, there exists an open ball U^0 centered at ξ_0 such that $u > 1$ on U^0 (see Figure 2.3). Applying Lemma 2.4.2 to U^0 with $r = \frac{R}{2}$, we find that there exists a constant $\sigma_0 > 0$ (depending on U^0 and R) such that

$$\left|U^0 \cap \left(\mathbb{B}^s \backslash \mathbb{B}^0\right)\right| \geq \sigma_0 s \qquad \text{for all } s > 0 \text{ sufficiently small.}$$

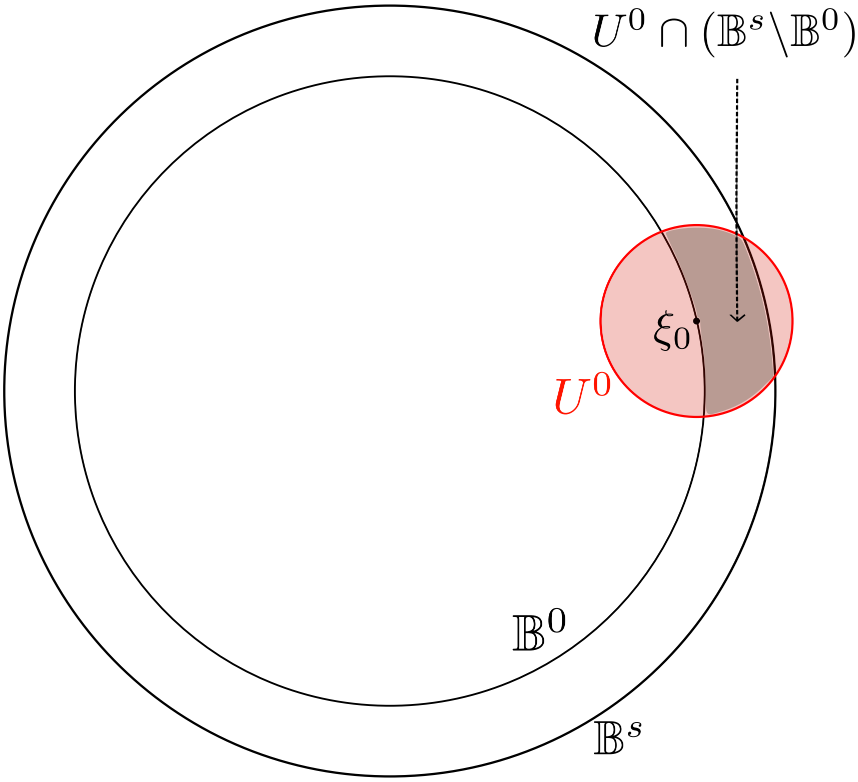

Figure 2.3: Construction of $\mathbb{B}^{s_1}$

Since $U^0 \subset G^0$, it follows from the definition of G_s^0 (2.34) that, for all $s > 0$ sufficiently small

$$|G_s^0| \geq \left|U^0 \cap \left(\mathbb{B}^s \backslash \mathbb{B}^0\right)\right| \geq \sigma_0 s.$$

Since $Q > 2$, we have

$$\lim_{s \to 0^+} \frac{\sigma_0 s}{\gamma \left(\frac{s}{4\tau}\right)^Q |B_1(0)| R^Q} = +\infty.$$

Therefore,

$$\sigma_0 s \geq \gamma \left(\frac{s}{4\tau}\right)^Q |B_1(0)| R^Q \quad \text{for all } s > 0 \text{ sufficiently small.}$$

This finishes the proof of (2.36).

It follows from (2.35), (2.36) and the intermediate value theorem that there exists $s_1 \in (0, 1/2)$ (depending on u and structural constants) for which

$$|G_{s_1}^0| = \gamma \left(\frac{s_1}{4\tau}\right)^Q |B_1(0)| R^Q = \gamma |B_{\frac{s_1 R}{4\tau}}|. \tag{2.37}$$

Since $\sup_{\mathbb{B}^{\frac{s_1}{2}}} u \geq \sup_{\mathbb{B}^0} u = \sup_{B_{R/2}} u = 2$, the weak maximum principle allows us to choose $\zeta_0 \in \partial \mathbb{B}^{\frac{s_1}{2}}$ such that $u(\zeta_0) \geq 2$. By basic metric arguments, we find that $B_{\frac{s_1 R}{4}}(\zeta_0) \subset \mathbb{B}^{s_1} \backslash \mathbb{B}^0$ and so, by (2.34),

$$G^0 \cap B_{\frac{s_1 R}{4}}(\zeta_0) \subset G^0 \cap \left(\mathbb{B}^{s_1} \backslash \mathbb{B}^0\right) = G_{s_1}^0. \tag{2.38}$$

Define $D^0 := G^0 \cap B_{\frac{s_1 R}{4}}(\zeta_0)$. Observe that $\zeta_0 \in D^0$, since $\zeta_0 \in G^0$. Consider the function $v = u - 1$, which is non-negative on D^0, satisfies $\mathcal{L}_A v = 0$, and vanishes on $\partial D^0 \cap B_{\frac{s_1 R}{4}}(\zeta_0)$. By (2.37) and (2.38), it follows that

$$|D^0| \leq |G_{s_1}^0| = \gamma |B_{\frac{s_1 R}{4\tau}}|.$$

Applying Lemma 2.4.1 to v in $D^0 \subset B_{\frac{s_1 R}{4}}(\zeta_0)$, and recalling that $\zeta_0 \in D^0$, we obtain

$$\sup_{D^0} v \geq M \sup_{D^0 \cap B_{\frac{s_1 R}{4\tau}}(\zeta_0)} v \geq M v(\zeta_0) \geq M,$$

where we have used that $v(\zeta_0) = u(\zeta_0) - 1 \geq 2 - 1 = 1$. Since $B_{\frac{s_1 R}{4}}(\zeta_0) \subset \mathbb{B}^{s_1}$ and $u \geq v$, it follows that

$$\sup_{\mathbb{B}^{s_1}} u \geq M.$$

Step 2: Construction of the sequence $\{\mathbb{B}^{s_\ell}, \ell \geq 1\}$.

We now perform an iteration based on the process outlined in Step 1. Suppose that for some $\ell \geq 1$, the ball $\mathbb{B}^{s_\ell}$ has been constructed so that

$$\sup_{\mathbb{B}^{s_\ell}} u \geq 2 \left(\frac{M}{2} \right)^{\ell}.$$

We show how to construct the ball $\mathbb{B}^{s_{\ell+1}}$ with $s_{\ell+1} \geq s_\ell$ and $\sup_{\mathbb{B}^{s_{\ell+1}}} u \geq 2 \left(\frac{M}{2}\right)^{\ell+1}$. If $s_\ell \geq \frac{1}{2}$, then proceed to Step 3. Otherwise, we must have $1 - s_\ell > \frac{1}{2}$. Define

$$G^\ell := \left\{ u > \left(\frac{M}{2} \right)^{\ell} \right\} \cap B_R; \tag{2.39}$$

$$G^\ell_s := G^\ell \cap \left(\mathbb{B}^{s_\ell + s} \backslash \mathbb{B}^{s_\ell} \right), \qquad 0 < s < 1 - s_\ell. \tag{2.40}$$

Since $G^\ell \subset G$, we have as in (2.35)

$$|G^\ell_{1/2}| \leq |G| < \epsilon_0 |B_R| = \gamma \left(\frac{1/2}{4\tau} \right)^{Q} |B_1(0)| R^Q. \tag{2.41}$$

Arguing as in the proof of (2.36), we also have

$$|G^\ell_s| \geq \gamma \left(\frac{s}{4\tau} \right)^{Q} |B_1(0)| R^Q \quad \text{for all } s > 0 \text{ sufficiently small.} \tag{2.42}$$

Indeed, choose $\xi_\ell \in \overline{\mathbb{B}^{s_\ell}}$ such that $u(\xi_\ell) = \sup_{\mathbb{B}^{s_\ell}} u \geq 2 \left(\frac{M}{2}\right)^{\ell}$. The weak maximum principle allows us to assume $\xi_\ell \in \partial \mathbb{B}^{s_\ell}$. By continuity of u, there exists an open ball U^ℓ centered at ξ_ℓ such that $u > \left(\frac{M}{2}\right)^\ell$ on U^ℓ. Applying Lemma 2.4.2 to U^ℓ with $r = (1+s_\ell)\frac{R}{2} < R$, we find that there exists a constant $\sigma_\ell > 0$ (depending on U^ℓ and R) such that

$$\left|U^\ell \cap \left(\mathbb{B}^{s_\ell+s} \backslash \mathbb{B}^{s_\ell}\right)\right| \geq \sigma_\ell s \qquad \text{for all } s > 0 \text{ sufficiently small.}$$

Since $U^\ell \subset G^\ell$, it follows from the definition of G^ℓ_s (2.40) that, for all $s > 0$ sufficiently small,

$$|G^\ell_s| \geq \left|U^\ell \cap \left(\mathbb{B}^{s_\ell+s} \backslash \mathbb{B}^{s_\ell}\right)\right| \geq \sigma_\ell s.$$

Since $Q > 2$, we have

$$\lim_{s \to 0^+} \frac{\sigma_\ell s}{\gamma \left(\frac{s}{4\tau}\right)^Q |B_1(0)| R^Q} = +\infty.$$

Therefore,

$$\sigma_\ell s \geq \gamma \left(\frac{s}{4\tau}\right)^Q |B_1(0)| R^Q \quad \text{for all } s > 0 \text{ sufficiently small.}$$

This finishes the proof of (2.42).

It follows from (2.41), (2.42) and the intermediate value theorem that there exists $\rho_\ell \in (0, 1/2)$ (depending on u) for which

$$|G^\ell_{\rho_\ell}| = \gamma \left(\frac{\rho_\ell}{4\tau}\right)^Q |B_1(0)| R^Q = \gamma |B_{\frac{\rho_\ell R}{4\tau}}|. \tag{2.43}$$

Since $\sup\limits_{\mathbb{B}^{s_\ell + \frac{\rho_\ell}{2}}} u \geq \sup\limits_{\mathbb{B}^{s_\ell}} u \geq 2\left(\frac{M}{2}\right)^\ell$, the weak maximum principle allows us to choose $\zeta_\ell \in \partial \mathbb{B}^{s_\ell + \frac{\rho_\ell}{2}}$ such that $u(\zeta_\ell) \geq 2\left(\frac{M}{2}\right)^\ell$. Define $s_{\ell+1} := s_\ell + \rho_\ell$. Then $B_{\frac{\rho_\ell R}{4}}(\zeta_\ell) \subset \mathbb{B}^{s_{\ell+1}} \backslash \mathbb{B}^{s_\ell}$ and so, by (2.40),

$$G^\ell \cap B_{\frac{\rho_\ell R}{4}}(\zeta_\ell) \subset G^\ell \cap (\mathbb{B}^{s_{\ell+1}} \backslash \mathbb{B}^{s_\ell}) = G^\ell_{\rho_\ell}. \tag{2.44}$$

Define $D^\ell := G^\ell \cap B_{\frac{\rho_\ell R}{4}}(\zeta_\ell)$. Observe that $\zeta_\ell \in D^\ell$, since $\zeta_\ell \in G^\ell$. Consider the function $v = u - \left(\frac{M}{2}\right)^\ell$, which is non-negative on D^ℓ, satisfies $\mathcal{L}_A v = 0$, and vanishes on $\partial D^\ell \cap B_{\frac{\rho_\ell R}{4}}(\zeta_\ell)$. By (2.43) and (2.44), it follows that

$$|D^\ell| \leq |G^\ell_{\rho_\ell}| = \gamma |B_{\frac{\rho_\ell R}{4\tau}}|.$$

Applying Lemma 2.4.1 to v in $D^\ell \subset B_{\frac{\rho_\ell R}{4}}(\zeta_\ell)$, and recalling that $\zeta_\ell \in D^\ell$, we obtain

$$\sup_{D^\ell} v \geq M \sup_{D^\ell \cap B_{\frac{\rho_\ell R}{4\tau}}(\zeta_\ell)} v \geq M v(\zeta_\ell) \geq 2\left(\frac{M}{2}\right)^{\ell+1},$$

where we have used that $v(\zeta_\ell) = u(\zeta_\ell) - \left(\frac{M}{2}\right)^\ell \geq 2\left(\frac{M}{2}\right)^\ell - \left(\frac{M}{2}\right)^\ell = \left(\frac{M}{2}\right)^\ell$. Since $B_{\frac{\rho_\ell R}{4}}(\zeta_\ell) \subset \mathbb{B}^{s_{\ell+1}}$ and $u \geq v$, it follows that

$$\sup_{\mathbb{B}^{s_{\ell+1}}} u \geq 2\left(\frac{M}{2}\right)^{\ell+1}.$$

Step 3: Obtaining a lower bound for $\inf\limits_{B_R} u$.

We claim that the above process must terminate after finitely many steps. Indeed, notice that if $s < \frac{1}{2}$, then $\mathbb{B}^s \subset B_{3R/4}$. Hence, if there are infinitely many balls $\mathbb{B}^{s_\ell}$ with $s_\ell < \frac{1}{2}$, then by (2.32), the function u is unbounded on $B_{3R/4}$, which is a contradiction. Therefore, there must exist a smallest integer $k \geq 1$ (depending on u and structural constants) such that $s_{k+1} \geq \frac{1}{2}$. Denoting $\rho_0 := s_1$ and recalling that $s_{\ell+1} := s_\ell + \rho_\ell$ for $\ell \in \{0, \ldots, k\}$, we thus have

$$s_{k+1} = \rho_0 + \rho_1 + \cdots + \rho_k \geq \frac{1}{2} \quad \text{and} \quad s_k = \rho_0 + \rho_1 + \cdots + \rho_{k-1} < \frac{1}{2}.$$

For each $\ell \in \{0, \ldots, k\}$, we know that the corresponding set $G^\ell_{\rho_\ell}$ defined through (2.40) satisfies, by (2.43),

$$|G^\ell_{\rho_\ell}| = \gamma \left(\frac{\rho_\ell}{4\tau}\right)^Q |B_1(0)| R^Q = \gamma \left(\frac{\rho_\ell}{4\tau}\right)^Q |B_R|,$$

and, by (2.39),

$$u > \left(\frac{M}{2}\right)^{\ell} \quad \text{on } G^{\ell}_{\rho_\ell}.$$

Now since $\rho_0+\rho_1+\cdots+\rho_k \geq 1/2$, there must exist at least one index $i_0 \in \{0,\ldots,k\}$ such that

$$\rho_{i_0} \geq \left(\frac{1}{2}\right)^{i_0+2}.$$

Thus,

$$|G^{i_0}_{\rho_{i_0}}| = \gamma\left(\frac{\rho_{i_0}}{4\tau}\right)^{Q} |B_R| \geq \gamma\left(\frac{2^{-(i_0+2)}}{4\tau}\right)^{Q} |B_R|, \tag{2.45}$$

and

$$u > \left(\frac{M}{2}\right)^{i_0} \quad \text{on } G^{i_0}_{\rho_{i_0}}. \tag{2.46}$$

Consider the function $v = \left(\frac{M}{2}\right)^{i_0} - u$. Then $\mathcal{L}_A v = 0$, and $v \leq \left(\frac{M}{2}\right)^{i_0}$ on $B_{\tau R}$ since u is non-negative on $B_{\tau R}$. Let

$$D = \{v > 0\} \cap B_{\tau R} = \left\{u < \left(\frac{M}{2}\right)^{i_0}\right\} \cap B_{\tau R}.$$

Notice that if $x \notin D$, then $u(x) \geq \left(\frac{M}{2}\right)^{i_0} \geq 1$. Therefore, we may assume $D \cap B_R \neq \emptyset$, for otherwise $\inf\limits_{B_R} u \geq 1$ and (2.30) follows immediately. By (2.40) and (2.46), $G^{i_0}_{\rho_{i_0}} \subset B_R \backslash D$. Hence, by (2.45), we have the measure estimate

$$|B_R \backslash D| \geq |G^{i_0}_{\rho_{i_0}}| \geq \gamma\left(\frac{2^{-(i_0+2)}}{4\tau}\right)^{Q} |B_R|.$$

Consequently,

$$\frac{|B_R \backslash D|}{|B_R|} \geq \gamma\left(\frac{2^{-(i_0+2)}}{4\tau}\right)^{Q}.$$

Applying Theorem 2.3.1 to v in $D \subset B_{\tau R}$, we obtain

$$\begin{aligned}\left(\frac{M}{2}\right)^{i_0} &\geq \sup_D v \\ &\geq \left(1+\eta\frac{|B_R\backslash D|}{|B_R|}\right)\sup_{D\cap B_R} v \\ &\geq \left[1+\eta\gamma\left(\frac{2^{-(i_0+2)}}{4\tau}\right)^Q\right]\left[\left(\frac{M}{2}\right)^{i_0}-\inf_{D\cap B_R} u\right].\end{aligned}$$

This implies

$$\begin{aligned}\left(\inf_{D\cap B_R} u\right)\left[1+\eta\gamma\left(\frac{2^{-(i_0+2)}}{4\tau}\right)^Q\right] &\geq \eta\gamma\left(\frac{2^{-(i_0+2)}}{4\tau}\right)^Q\left(\frac{M}{2}\right)^{i_0} \\ &= \eta\gamma\left(\frac{1}{16\tau}\right)^Q\left(\frac{M}{2^{Q+1}}\right)^{i_0}.\end{aligned}$$

Recalling that $M := 2^{Q+1}$ and noticing that $2^{-(i_0+2)} \leq \frac{1}{4}$, we obtain

$$\inf_{D\cap B_R} u \geq \frac{\hat{c}}{1+\hat{c}} \qquad \text{where } \hat{c} = \eta\gamma\left(\frac{1}{16\tau}\right)^Q.$$

Since $u(x) \geq 1$ if $x \notin D$, we have $\inf_{B_R\backslash D} u \geq 1$. Therefore, $\inf_{B_R} u \geq \dfrac{\hat{c}}{1+\hat{c}}$. Noticing that $\hat{c}$ is a structural constant, we conclude the proof of (2.30). $\square$

We conclude this chapter with a few remarks.

Remark 2.4.4. *Note that* (2.30) *easily implies*

$$\sup_{B_{R/2}(z_0)} u \leq C \inf_{B_{R/2}(z_0)} u. \tag{2.47}$$

On the other hand, (2.47) *implies*

$$C \inf_{B_{R/2}(z_0)} u \geq \sup_{B_{R/2}(z_0)} u \geq \sup_{B_{R/4}(z_0)} u.$$

Consequently, (2.30) *and* (2.47) *are equivalent.*

Remark 2.4.5. *For the argument in Case 1 to work, it is only necessary to assume* $u \geq 0$ *and* $\mathcal{L}_A u \leq 0$ *on* B_R*, as this is enough to invoke Theorem 2.3.1. Thus, Case 1 can be interpreted as a critical-density estimate; i.e. there exist structural constants* $0 < \epsilon_0 < 1$ *and* $\mathcal{M} > 1$ *such that for all non-negative supersolutions* u*,*

$$\inf_{B_{R/2}} u \leq 1 \quad \Rightarrow \quad |\{u \leq \mathcal{M}\} \cap B_R| \geq \epsilon_0 |B_R|.$$

Indeed, the argument above shows $\mathcal{M} = \dfrac{1 + \eta\epsilon_0}{\eta\epsilon_0}$*. See the appendix for a discussion on the connections between the growth lemma and the critical density property.*

CHAPTER 3

CRITICAL DENSITY PROPERTY FOR HORIZONTALLY ELLIPTIC OPERATORS

In this chapter, we provide the proof of the critical density estimate, Theorem 1.3.1. This was established in our work [1]. As mentioned in Chapter 1, Theorem 1.3.1 combined with the double ball property proved in [20] allows us to invoke the axiomatic approach from [11] to obtain the Harnack inequality, Theorem 1.3.3.

For any constant matrix $M \in M_n(\lambda, \Lambda, \Omega)$ we recall the definition of the function ϕ_M from (2.10).

$$\phi_M(x,t) := \langle M^{-1}x, x\rangle^2 + t^2.$$

At times, it will be convenient for us to work with the quasi-norms $\rho_M := \phi_M^{1/4}$ and the corresponding quasi-metrics $d_M(z,\zeta) := \rho_M(\zeta^{-1} \circ z)$, which are both δ_r-homogeneous. It is easy to show that ρ and ρ_M are equivalent; in fact, recalling that $\lambda \leq 1 \leq \Lambda$, we have

$$\sqrt{\frac{1}{\Lambda}}\rho \leq \rho_M \leq \sqrt{\frac{1}{\lambda}}\rho. \tag{3.1}$$

This implies, in particular, that d_M satisfies the quasi-triangular inequality with constant $\sqrt{\Lambda/\lambda}$. Moreover, if we denote by B^M the balls with respect to d_M, we have

$$B_{\lambda^{1/2}r}(z) \subseteq B_r^M(z) \subseteq B_{\Lambda^{1/2}r}(z) \quad \forall r > 0,\, z \in \mathbb{H}^n. \tag{3.2}$$

Throughout this chapter, we will denote by $\mathrm{dist}(\cdot,\cdot)$ and $\mathrm{diam}(\cdot)$ the distance and the diameter of sets with respect to d, whereas we will denote by $\mathrm{dist_M}(\cdot,\cdot)$ the distance with respect to the modified quasi-metric d_M.

3.1 Construction of Barriers

We proceed to construct barriers similar to those used in the proof of Theorem 2.3.1. Let us provide an informal description of our approach, which is based on the techniques employed in [20]. Recall the function U_E defined in (2.24), where $E \subset B_R$ for some $R > 0$ sufficiently small. In the proof of Theorem 2.4.1, U_E was used as a barrier *outside* the set E. It is evident from the definition of U_E that it is not twice differentiable on E, and hence cannot be used for maximum principle arguments on the set E itself. The strategy is to appropriately modify U_E so that it can act as a barrier on compact subsets of E. To do this, we use a cutoff function inside the integral defining $U_E(z)$, which smooths out the pole at z and allows us to differentiate twice at points inside E. We then break the integral $\mathcal{L}_A U_E$ into two pieces: one over B_R and another over the set $B_R \backslash E$. Lemma 2.2.2 takes care of the second piece, as long as $R \leq \delta_0$. To control the first piece, we integrate by parts to obtain an integral over ∂B_R. It can be shown that this boundary term exhibits some uniform behavior as a function of R, and this allows us to perform comparisons with appropriately scaled distance paraboloids. All these properties are then used when carrying out maximum principle arguments in the proof of Theorem 1.3.1.

Fix $0 < s < \frac{1}{2}$, a point $z_0 \in \Omega$, and for $A \in M_n(\lambda, \Lambda) \cap C(\Omega, \omega)$ symplectic, we let $M = A(z_0)$. Let $\alpha = \dfrac{Q-2}{4} + s$ and recall the function $\phi_{z_0,s}$ defined in (2.17). For any bounded open set $O \subset \mathbb{H}^n$, consider the function

$$h(z) := -\frac{1}{\alpha} \int\limits_{O} \phi_{z_0,s}(z^{-1} \circ \zeta)\, d\zeta. \tag{3.3}$$

(Compare this to the function U_E from (2.24)). Let ψ be a smooth, non-decreasing function of one variable such that $\psi(t) = 1$ for $t \geq 2$ and $\psi(t) = 0$ for $t < 1$. For $\mu > 0$, define

$$h_\mu(z) := -\frac{1}{\alpha} \int\limits_{O} \psi_\mu\left(d_M(z,\zeta)\right) \phi_{z_0,s}(z^{-1} \circ \zeta)\, d\zeta, \tag{3.4}$$

where $\psi_\mu(r) := \psi\left(r/\mu\right)$. The function h_μ is smooth, and converges uniformly to h as $\mu \to 0^+$. In the following we denote

$$\eta := 2\sqrt{\Lambda/\lambda} + 1. \tag{3.5}$$

Lemma 3.1.1. *Let δ_0 be the constant given in Lemma 2.2.2. There exists a positive constant C depending only on λ, Λ, Q, s such that for all $0 < r \le \delta_0$ and $z_0 \in \Omega$ with $B_{\eta r}(z_0) \subseteq \Omega$, and for all open sets $O' \Subset O \subseteq B_r(z_0)$, we have*

$$\mathcal{L}_A h_\mu(z) \ge C r^{-4s} \qquad \forall\, z \in O', \tag{3.6}$$

for all $0 < \mu < \min\left\{\frac{r}{\sqrt{\lambda}}, \frac{dist(O', \partial O)}{2\sqrt{\Lambda}}\right\}$, and for all $A \in M_n(\lambda, \Lambda) \cap C(\Omega, \omega)$ symplectic.

Proof. Define $\phi_\mu(\zeta) := \psi_\mu(\rho_M(\zeta))\phi_{z_0,s}(\zeta)$. Since d_M is symmetric,

$$h_\mu(z) := \int_O \phi_\mu(z^{-1} \circ \zeta)\, d\zeta = \int_O \phi_\mu(\zeta^{-1} \circ z)\, d\zeta.$$

By (3.2) and the hypotheses of the lemma we have for all $z \in B_r(z_0)$

$$B_r(z_0) \subset B_{2r}(z) \subseteq B^M_{\frac{2r}{\sqrt{\lambda}}}(z) \subseteq B_{2\sqrt{\frac{\Lambda}{\lambda}}r}(z) \subset B_{\eta r}(z_0) \subset \Omega.$$

In particular, $O \subset B^M_{2r/\sqrt{\lambda}}(z) \subset \Omega$ for any $z \in O$. By the smoothness of ϕ_μ and the left-invariance of the vector fields and the Lebesgue measure, we have

$$\begin{aligned}
X_iX_jh_\mu(z) &= -\frac{1}{\alpha}\int_O (X_iX_j\phi_\mu)(\zeta^{-1} \circ z)\, d\zeta \\
&= -\frac{1}{\alpha}\int_{B^M_{2r/\sqrt{\lambda}}(z)} (X_iX_j\phi_\mu)(\zeta^{-1} \circ z)\, d\zeta + \frac{1}{\alpha}\int_{B^M_{2r/\sqrt{\lambda}}(z)\setminus O} (X_iX_j\phi_\mu)(\zeta^{-1} \circ z)\, d\zeta \\
&= -\frac{1}{\alpha}\int_{B^M_{2r/\sqrt{\lambda}}(0)} (X_iX_j\phi_\mu)(\zeta)\, d\zeta + \frac{1}{\alpha}\int_{B^M_{2r/\sqrt{\lambda}}(z)\setminus O} (X_iX_j\phi_\mu)(\zeta^{-1} \circ z)\, d\zeta \\
&= -\frac{1}{\alpha}\int_{\partial B^M_{2r/\sqrt{\lambda}}(0)} (X_j\phi_\mu)(\zeta)\frac{X_i\rho_M(\zeta)}{|D\rho_M(\zeta)|}\, d\sigma(\zeta) + \frac{1}{\alpha}\int_{B^M_{2r/\sqrt{\lambda}}(z)\setminus O} (X_iX_j\phi_\mu)(\zeta^{-1} \circ z)\, d\zeta,
\end{aligned}$$

where $|D\rho_M|$ stands for the Euclidean length of the standard gradient in $\mathbb{R}^{2n+1}$ and $d\sigma$ is the standard $2n$-dimensional Hausdorff measure in $\mathbb{R}^{2n+1}$. The final equality follows from the divergence theorem, since the vector fields X_j in (2.3) are divergence-free.

We wish to replace ϕ_μ by $\phi_{z_0,s}$ in the final expression above. If $0 < \mu < r/\sqrt{\lambda}$, then for $\zeta \in \partial B^M_{2r/\sqrt{\lambda}}(0)$, $\rho_M(\zeta) > 2\mu$ and so $\phi_\mu = \phi_{z_0,s}$ in the first integral. In the second integral, $\phi_\mu(\zeta^{-1} \circ z) = \phi_{z_0,s}(\zeta^{-1} \circ z)$ if and only if $d_M(\zeta, z) > 2\mu$. If O' is a compactly contained subset of O, then for $z \in O'$ and $\zeta \notin O$ we have $d_M(\zeta, z) \geq \mathrm{dist}_M(O', \partial O) \geq \frac{1}{\sqrt{\Lambda}}\mathrm{dist}(O', \partial O)$ by (3.1). So if μ satisfies $0 < 2\sqrt{\Lambda}\mu < \mathrm{dist}(O', \partial O)$, then we can eliminate μ in the second integral. Therefore, for any $z \in O'$, we obtain

$$
\begin{aligned}
X_i X_j h_\mu(z) = &-\frac{1}{\alpha} \int\limits_{\partial B^M_{2r/\sqrt{\lambda}}(0)} (X_j \phi_{z_0,s})(\zeta) \frac{X_i \rho_M(\zeta)}{|D\rho_M(\zeta)|}\, d\sigma(\zeta) \\
&+ \frac{1}{\alpha} \int\limits_{B^M_{2r/\sqrt{\lambda}}(z) \setminus O} (X_i X_j \phi_{z_0,s})(\zeta^{-1} \circ z)\, d\zeta.
\end{aligned}
$$

Multiplying the previous identity by $a_{ij}(z)$ and adding over i, j yields

$$
\begin{aligned}
(\mathcal{L}_A h_\mu)(z) = &-\frac{1}{\alpha} \int\limits_{\partial B^M_{2r/\sqrt{\lambda}}(0)} \frac{\langle A(z)\nabla_{\mathbb{H}} \phi_{z_0,s}(\zeta), \nabla_{\mathbb{H}} \rho_M(\zeta)\rangle}{|D\rho_M(\zeta)|}\, d\sigma(\zeta) \\
&+ \frac{1}{\alpha} \int\limits_{B^M_{2r/\sqrt{\lambda}}(z) \setminus O} \mathrm{tr}\left(A(z)(D^2_{\mathbb{H}} \phi_{z_0,s})(\zeta^{-1} \circ z)\right)\, d\zeta \qquad \forall\, z \in O'.
\end{aligned}
$$

Since $z \in O' \subset O \subseteq B_r(z_0) \subseteq B_{\delta_0}(z_0)$ and $A(\cdot)$ satisfies the assumptions of Lemma 2.2.2, we conclude that

$$
\mathrm{tr}\left(A(z)(D^2_{\mathbb{H}} \phi_{z_0,s})(\zeta^{-1} \circ z)\right) \geq 0 \text{ for all } \zeta \in B^M_{2r/\sqrt{\lambda}}(z) \setminus O.
$$

Thus, for all $z \in O'$,

$$(\mathcal{L}_A h_\mu)(z) \geq -\frac{1}{\alpha} \int\limits_{\partial B^M_{2r/\sqrt{\lambda}}(0)} \frac{\langle A(z)\nabla_{\mathbb{H}}\phi_{z_0,s}(\zeta), \nabla_{\mathbb{H}}\rho_M(\zeta)\rangle}{|D\rho_M(\zeta)|}\, d\sigma(\zeta).$$

We now estimate the boundary integral from below. Recall that $\phi_{z_0,s} = \phi_M^{-\alpha} = \rho_M^{-4\alpha}$. Therefore,

$$\begin{aligned}
(\mathcal{L}_A h_\mu)(z) &\geq -\frac{1}{\alpha} \int\limits_{\partial B^M_{2r/\sqrt{\lambda}}(0)} \frac{\langle A(z)\nabla_{\mathbb{H}}\phi_{z_0,s}(\zeta), \nabla_{\mathbb{H}}\rho_M(\zeta)\rangle}{|D\rho_M(\zeta)|}\, d\sigma(\zeta) \\
&= 4 \int\limits_{\partial B^M_{2r/\sqrt{\lambda}}(0)} \frac{\langle A(z)\nabla_{\mathbb{H}}\rho_M(\zeta), \nabla_{\mathbb{H}}\rho_M(\zeta)\rangle}{\rho_M^{4\alpha+1}(\zeta)|D\rho_M(\zeta)|}\, d\sigma(\zeta) \\
&= 4\left(\frac{\sqrt{\lambda}}{2r}\right)^{4\alpha+1} \int\limits_{\partial B^M_{2r/\sqrt{\lambda}}(0)} \frac{\langle A(z)\nabla_{\mathbb{H}}\rho_M(\zeta), \nabla_{\mathbb{H}}\rho_M(\zeta)\rangle}{|D\rho_M(\zeta)|}\, d\sigma(\zeta) \\
&= 4\left(\frac{\sqrt{\lambda}}{2r}\right)^{Q-1+4s} \int\limits_{\partial B^M_{2r/\sqrt{\lambda}}(0)} \frac{\langle A(z)\nabla_{\mathbb{H}}\rho_M(\zeta), \nabla_{\mathbb{H}}\rho_M(\zeta)\rangle}{|D\rho_M(\zeta)|}\, d\sigma(\zeta) \\
&\geq \left(\frac{4^{1-2s}\lambda^{2s}}{r^{4s}}\frac{\lambda}{\Lambda}\right)\mathcal{B}(M, 2r/\sqrt{\lambda}), \qquad (3.7)
\end{aligned}$$

where

$$\mathcal{B}(M,r) := \frac{1}{r^{Q-1}} \int\limits_{\partial B^M_r(0)} \frac{\langle M\nabla_{\mathbb{H}}\rho_M(\zeta), \nabla_{\mathbb{H}}\rho_M(\zeta)\rangle}{|D\rho_M(\zeta)|}\, d\sigma(\zeta).$$

We now adapt an argument from [4, Section 5.5] to show that $\mathcal{B}(M,r)$ can be bounded below by a positive constant independent of r and M. First notice that, for all $r > 0$,

$$\mathcal{B}(M,r) = \frac{1}{2-Q} \int\limits_{\partial B^M_r(0)} \frac{\langle M\nabla_{\mathbb{H}}\Gamma_M(z), \nabla_{\mathbb{H}}\rho_M(z)\rangle}{|D\rho_M(z)|}\, d\sigma(z), \qquad (3.8)$$

where Γ_M is as in (2.15). Recalling that $\mathcal{L}_M\Gamma_M = 0$ outside the origin, we conclude from the divergence theorem and (3.8) that for any $r_2 > r_1 > 0$,

$$0 = \frac{1}{2-Q} \int_{B^M_{r_2}(0) \setminus B^M_{r_1}(0)} \mathcal{L}_M \Gamma_M(z)\ dz = \mathcal{B}(M, r_2) - \mathcal{B}(M, r_1).$$

Multiplying the above by r^{Q-1}, integrating from 0 to 1, and using the coarea formula yields

$$\begin{aligned}
\frac{1}{Q}\mathcal{B}(M,1) &= \int_0^1 r^{Q-1}\mathcal{B}(M,r)\ dr \\
&= \int_0^1 \int_{\partial B^M_r(0)} \frac{\langle M\nabla_{\mathbb{H}}\rho_M(z), \nabla_{\mathbb{H}}\rho_M(z)\rangle}{|D\rho_M(z)|} d\sigma(z)\ dr \\
&= \int_{B^M_r(0)} \langle M\nabla_{\mathbb{H}}\rho_M(z), \nabla_{\mathbb{H}}\rho_M(z)\rangle\, dz \\
&= \frac{1}{16} \int_{B^M_1(0)} \frac{\langle M\nabla_{\mathbb{H}}\phi_M(z), \nabla_{\mathbb{H}}\phi_M(z)\rangle}{\phi_M^{\frac{3}{2}}(z)}\ dz \\
&= \int_{B^M_1(0)} \frac{\left\langle M^{-1}x, x\right\rangle}{\phi_M^{\frac{1}{2}}(x,t)}\ dxdt \\
&\geq \frac{\lambda}{\Lambda} \int_{B_{\sqrt{\lambda}}(0)} \frac{|x|^2}{\sqrt{|x|^4 + t^2}}\ dxdt =: \tilde{C}.
\end{aligned}$$

Therefore we obtain that $\mathcal{B}(M, r)$ is bounded below uniformly in r and M, and so from (3.7)

$$(\mathcal{L}_A h_\mu)(z) \geq \frac{4^{1-2s}\lambda^{2s}}{r^{4s}} \frac{\lambda}{\Lambda}\tilde{C} =: \frac{C}{r^{4s}}$$

for all $z \in O'$ and for all μ satisfying $0 < \mu < \min\left\{\frac{r}{\sqrt{\lambda}}, \frac{\operatorname{dist}(O', \partial O)}{2\sqrt{\Lambda}}\right\}$. □

Remark 3.1.2. *Among all the possible sets O of fixed measure, the set that maximizes the quantity*

$$\int_O \frac{1}{d^{4\alpha}(z,\zeta)}\ d\zeta$$

is the ball centered at z satisfying $|B_1|\, R^Q = |B_R(z)| = |O|$ (see [20, pg. 2112]). We thus have

$$\begin{aligned}\int_O \frac{1}{d^{4\alpha}(z,\zeta)}\, d\zeta &\leq \int_{B_R(z)} \frac{1}{d^{4\alpha}(z,\zeta)}\, d\zeta = R^{Q-4\alpha} \int_{B_1(0)} \frac{1}{\rho^{4\alpha}(\zeta)}\, d\zeta = \\ &= |O|^{1-\frac{4\alpha}{Q}}\, |B_1|^{\frac{4\alpha}{Q}-1} \int_{B_1(0)} \frac{1}{\rho^{4\alpha}(\zeta)}\, d\zeta.\end{aligned}$$

Hence, by (3.1)*, we get*

$$\begin{aligned}0 \geq h(z) &= -\frac{1}{\alpha} \int_O \frac{1}{d_M^{4\alpha}(z,\zeta)}\, d\zeta \geq -\frac{\Lambda^{2\alpha}}{\alpha} \int_O \frac{1}{d^{4\alpha}(z,\zeta)}\, d\zeta \\ &\geq -\frac{\Lambda^{2\alpha}}{\alpha}\, |O|^{1-\frac{4\alpha}{Q}}\, |B_1|^{\frac{4\alpha}{Q}-1} \int_{B_1(0)} \frac{1}{\rho^{4\alpha}(\zeta)}\, d\zeta.\end{aligned}$$

Therefore there exists a positive constant γ depending only on Q, Λ, α such that

$$0 \geq h(z) \geq -\gamma\, |O|^{1-\frac{4\alpha}{Q}}\,. \tag{3.9}$$

3.2 Critical Density Estimate

We now use the barriers constructed in Lemma 3.1.1 to obtain critical density estimates on balls of radius less than δ_0, where δ_0 is as in Lemma 2.2.2. Recall that η is as in (3.5), and s is a fixed number in $(0, \frac{1}{2})$.

Theorem 3.2.1. *There exists $0 < \epsilon = \epsilon(Q, \Lambda, \lambda) < 1$ such that for all $z_0 \in \Omega$ and $0 < r \leq \delta_0$ with $B_{\eta r}(z_0) \subseteq \Omega$, for any $A \in M_n(\lambda, \Lambda) \cap C(\Omega, \omega)$ symplectic and for any $u \in C^2(B_{\eta r}(z_0))$ satisfying*

(i) $u \geq 0$ *in* $B_{\eta r}(z_0)$,

(ii) $\mathcal{L}_A u \leq 0$ *in* $B_{\eta r}(z_0)$,

(iii) $\inf_{B_{r/2}(z_0)} u < \frac{1}{2}$,

we have

$$|\{z \in B_r(z_0) \,:\, u(z) < 1\}| \geq \epsilon \, |B_r(z_0)| \,. \tag{3.10}$$

Proof. Let $\varphi(z) := d(z, z_0)^4$. We have

$$\begin{aligned} \mathcal{L}_A \varphi(z) &= 4\mathrm{tr}(A(z))|x - x_0|^2 + 8\,\langle A(z)(x - x_0), x - x_0\rangle + 8\,\langle A(z)\mathcal{J}(x - x_0), \mathcal{J}(x - x_0)\rangle \\ &\leq 4\Lambda(Q + 2)|x - x_0|^2 \leq 4\Lambda(Q + 2)r^2 \qquad \text{for any } z \in B_r(z_0). \end{aligned}$$

Let C be the constant in (3.6) and consider

$$w(z) := \frac{Cr^{2-4s}}{4\Lambda(Q + 2)}\left(u(z) + \frac{1}{r^4}\varphi(z) - 1\right).$$

By (i), w is nonnegative on $\partial B_r(z_0)$. By (iii), there exists a point $\overline{z} \in B_{\frac{r}{2}}(z_0)$ such that $u(\overline{z}) < 1/2$. Therefore,

$$w(\overline{z}) \leq \frac{Cr^{2-4s}}{4\Lambda(Q + 2)}\left(\frac{1}{2} + \frac{1}{16} - 1\right) = -\frac{7Cr^{2-4s}}{64\Lambda(Q + 2)}. \tag{3.11}$$

Let $O := \{z \in B_r(z_0) \,:\, w(z) < 0\}$. Notice that O is open, $\overline{z} \in O$, and

$$O \subseteq \{z \in B_r(z_0) \,:\, u(z) < 1\}.$$

With this choice of O and by defining $M := A(z_0)$, we consider the barriers h, h_μ in (3.3), (3.4) respectively. We claim that

$$h - w \leq 0 \quad \text{in } O.$$

By definition, h is non-positive. Since $w = 0$ on ∂O, it follows that $h - w \leq 0$ on ∂O. Suppose, for contradiction, that there exists $\zeta_0 \in O$ such that $h(\zeta_0) -$

$w(\zeta_0) = 2\sigma > 0$. Since h_μ converges uniformly to h as μ goes to 0, there exists $\mu_0 > 0$ such that $h_\mu(\zeta_0) - w(\zeta_0) \geq \sigma$ for $\mu \leq \mu_0$. Let $O' \Subset O$ containing ζ_0 and $0 < \mu < \min\left\{\frac{r}{\sqrt{\lambda}}, \frac{\operatorname{dist}(O', \partial O)}{2\sqrt{\Lambda}}, \mu_0\right\}$. By (ii), $\mathcal{L}_A u \leq 0$ in $B_{\eta r}(z_0)$, and so $\mathcal{L}_A w \leq Cr^{-4s}$ in $B_r(z_0)$. Therefore, by Lemma 3.1.1, $\mathcal{L}_A(h_\mu - w) \geq 0$ in O'. From the weak maximum principle for $\mathcal{L}_A$ we then infer that $\max_{\partial O'}(h_\mu - w) \geq \sigma$. Letting $\mu \to 0^+$, we conclude that $\max_{\partial O'}(h - w) \geq \sigma$ for any O' containing ζ_0. This is a contradiction, as $h - w \leq 0$ on ∂O.

Therefore, by (3.9), (3.11) and recalling that $4\alpha = Q - 2 + 4s$, we obtain

$$-\frac{7Cr^{2-4s}}{64\Lambda(Q+2)} \geq w(\overline{z}) \geq h(\overline{z}) \geq -\gamma\,|O|^{1-\frac{4\alpha}{Q}} = -\gamma\,|O|^{\frac{2}{Q}(1-2s)}.$$

This, of course, implies

$$\begin{aligned}
|O|^{\frac{2}{Q}(1-2s)} &\geq \frac{C}{\gamma\Lambda}\frac{7}{64(Q+2)}r^{2-4s} \\
&= \frac{C}{\gamma\Lambda}\frac{7}{64(Q+2)}|B_1(0)|^{-\frac{2}{Q}(1-2s)}\,|B_r(z_0)|^{\frac{2}{Q}(1-2s)} \\
&=: C_0\,|B_r(z_0)|^{\frac{2}{Q}(1-2s)}.
\end{aligned}$$

Choosing $\epsilon = C_0^{\frac{Q}{2(1-2s)}}$ therefore gives us

$$|\{z \in B_r(z_0) \,:\, u(z) < 1\}| \geq |O| \geq \epsilon\,|B_r(z_0)|.$$

Notice that ϵ depends only on Q, λ, Λ. $\square$

Theorem 1.3.3 now follows from the double ball property (see [20, 30]) and the results in [11].

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APPENDIX

APPENDIX: ON THE RELATION BETWEEN THE GROWTH LEMMA AND THE CRITICAL DENSITY ESTIMATE

The purpose of this appendix is to show how the growth lemma and the critical density estimate are related [1]. In the following, we let L be any second order linear elliptic operator of the form (1.1), where $M(\cdot)$ is assumed to be non-negative

[1] The contents of this appendix were discovered over several illuminating conversations with Giulio Tralli. We warmly thank him for allowing us to reproduce his notes here.

definite on some domain $\Omega \subset \mathbb{R}^n$; in particular, L could possibly be degenerate elliptic. The sets $B_R(x_0)$ below will denote metric balls centered at x_0 corresponding to the operator L. The number $\tau > 1$ will be a fixed structural constant.

Definition .0.1. *($\epsilon - M$ Critical Density) We say that the $\epsilon - M$ Critical Density is satisfied in $B_{\tau R}(x_0) \Subset \Omega$ if there exist constants $0 < \epsilon < 1$ and $M > 1$ such that for all $u \in C^2(B_{\tau R}(x_0)) \cap C(\overline{B_{\tau R}(x_0)})$ non-negative and satisfying $Lu \leq 0$ in $B_{\tau R}(x_0)$, we have*

$$|\{u < M\} \cap B_{\tau R}(x_0)| \leq \epsilon |B_{\tau R}(x_0)| \quad \Rightarrow \quad \inf_{B_R(x_0)} u \geq 1.$$

Definition .0.2. *($\delta - N$ Growth Lemma) We say that the $\delta - N$ Growth Lemma is satisfied in $B_{\tau R}(x_0) \Subset \Omega$ if there exist constants $0 < \delta < 1$ and $N > 1$ such that for all $v \in C^2(B_{\tau R}(x_0)) \cap C(\overline{B_{\tau R}(x_0)})$ satisfying $\{v > 0\} \cap B_R(x_0) \neq \emptyset$ and $Lv \geq 0$ in $B_{\tau R}(x_0)$, we have*

$$|\{v > 0\} \cap B_R(x_0)| \leq \delta |B_R(x_0)| \quad \Rightarrow \quad \sup_{B_{\tau R}(x_0)} v \geq N \sup_{B_R(x_0)} v.$$

Proposition .0.3. *(i) If the $\epsilon - M$ critical density property holds, then the $\delta - N$ growth lemma holds with $\delta = \epsilon$ and $N = \frac{M}{M-1}$.*

(ii) If the $\delta - N$ growth lemma holds, then the $\epsilon - M$ critical density property holds with $\epsilon = \delta$ and $M = \frac{N}{N-1}$.

Proof. (i) Assume $\epsilon - M$ critical density holds, and suppose $v \in C^2(B_{\tau R}(x_0)) \cap C(\overline{B_{\tau R}(x_0)})$ satisfies $\{v > 0\} \cap B_R(x_0) \neq \emptyset$, $Lv \geq 0$ in $B_{\tau R}(x_0)$ and $|\{v > 0\} \cap B_R(x_0)| \leq \epsilon |B_R(x_0)|$. Notice that since $\{v > 0\} \cap B_R(x_0) \neq \emptyset$, we may assume

$\sup_{B_{\tau R}(x_0)} v > 0$. Consider the function

$$u(x) = M\left(1 - \frac{v(x)}{\sup_{B_{\tau R}(x_0)} v}\right).$$

Then u is non-negative in $B_{\tau R}(x_0)$ and satisfies $Lu \le 0$ in $B_{\tau R}(x_0)$. Moreover, $\{u < M\} = \{v > 0\}$, and so $|\{u < M\} \cap B_R(x_0)| \le \epsilon |B_R(x_0)|$. Consequently, by $\epsilon - M$ critical density, we have $\inf_{B_R(x_0)} u \ge 1$. This implies

$$M\left(1 - \frac{\sup_{B_R(x_0)} v}{\sup_{B_{\tau R}(x_0)} v}\right) \ge 1.$$

Rearranging the above, we get

$$\sup_{B_{\tau R}(x_0)} v \ge \left(\frac{M}{M-1}\right) \sup_{B_R(x_0)} v.$$

(ii) Assume $\delta - N$ growth lemma holds, and suppose $u \in C^2(B_{\tau R}(x_0)) \cap C(\overline{B_{\tau R}(x_0)})$ is non-negative, $Lu \le 0$ in $B_{\tau R}(x_0)$ and

$$\left|\left\{u < \frac{N}{N-1}\right\} \cap B_{\tau R}(x_0)\right| \le \delta |B_{\tau R}(x_0)|.$$

We may asume $\left\{u < \frac{N}{N-1}\right\} \cap B_R(x_0) \ne \emptyset$, for otherwise $u \ge \frac{N}{N-1} > 1$ on $B_R(x_0)$ and so $\inf_{B_R(x_0)} u \ge 1$. Consider the function

$$v(x) = 1 - \left(\frac{N-1}{N}\right) u(x).$$

Then $Lv \ge 0$ and $\left\{u < \frac{N}{N-1}\right\} = \{v > 0\}$. Consequently, $\{v > 0\} \cap B_R(x_0) \ne \emptyset$ and $|\{v > 0\} \cap B_{\tau R}(x_0)| \le \delta |B_{\tau R}(x_0)|$. By the $\delta - N$ growth lemma, we have

$$\sup_{B_{\tau R}(x_0)} v \ge N \sup_{B_R(x_0)} v.$$

Now since u is non-negative, we have $v \leq 1$ on $B_{\tau R}(x_0)$. Therefore,

$$1 \geq N \sup_{B_R(x_0)} v = N \left(1 - \left(\frac{N-1}{N} \right) \inf_{B_R(x_0)} u \right).$$

Rearranging, we obtain $\inf_{B_R(x_0)} u \geq 1$. $\square$