ON THE NONLINEAR INTERACTION OF CHARGED PARTICLES WITH FLUIDS

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ABSTRACT

We consider three different phenomena governing the fluid flow in the presence of charged particles: electroconvection in fluids, electroconvection in porous media, and electrodiffusion. Electroconvecton in fluids is mathematically represented by a nonlinear drift-diffusion partial differential equation describing the time evolution of a surface charge density in a two-dimensional incompressible fluid. The velocity of the fluid evolves according to Navier-Stokes equations forced nonlinearly by the electrical forces due to the presence of the charge density. The resulting model is reminiscent of the quasi-geostrophic equation, where the main difference resides in the dependence of the drift velocity on the charge density. When the fluid flows through a porous medium, the velocity and the electrical forces are related according to Darcy's law, which yields a challenging doubly nonlinear and doubly nonlocal model describing electroconvection in porous media. A different type of particle-fluid interaction, called electrodiffusion, is also considered. This latter phenomenon is described by nonlinearly advected and nonlinearly forced continuity equations tracking the time evolution of the concentrations of many ionic species having different valences and diffusivities and interacting with an incompressible fluid. This work is based on [1, 2, 3] and addresses the global well-posedness, long-time dynamics, and other features associated with the aforementioned three models.

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TABLE OF CONTENTS

Page

CHAPTER 1

Navier-Stokes Equations on the Torus

We consider the incompressible Navier-Stokes equations, forced by time-independent body forces in the fluid, on two-dimensional periodic domains. We address the existence, uniqueness, smoothness, and long-time behavior of solutions.

1.1 Viscous Homogeneous Incompressible Fluids

Let $\Omega \subset \mathbb{R}^d$ be a domain occupied by a viscous incompressible fluid. The particle trajectories are described by the flow map

$$X(\cdot, t) : \Omega \to \Omega, \qquad a \mapsto X(a, t) \tag{1.1}$$

at positive times $t \geq 0$. The velocity field

$$u(x, t) = (u_1(x, t), \ldots, u_d(x, t)) \tag{1.2}$$

at $(X(a, t), t)$ is tangent to the curve $\{X(a, s) : 0 \leq s \leq t\}$ at $X(a, t)$ and obeys

$$\partial_t X(a, t) = u(X(a, t), t). \tag{1.3}$$

The density scalar field $\rho(\cdot, t)$ determines the mass of a volume element V in the fluid via

$$m(V, t) = \int_V \rho(x, t) dx. \tag{1.4}$$

Assuming the conservation of mass, we have

$$\frac{d}{dt}m(X(V,t),t) = 0 \Leftrightarrow \int_{X(V,t)} \left(\partial_t\rho + \nabla\cdot(u\rho)\right)(x,t)dx = 0 \tag{1.5}$$

for any smooth volume element V in Ω. Consequently, mass is conserved if and only if the the density ρ obeys the equation

$$\partial_t\rho + \nabla\cdot(\rho u) = 0. \tag{1.6}$$

Due to the incompressibility of the fluid, the velocity field satisfies the divergence-free condition

$$\nabla\cdot u = 0, \tag{1.7}$$

hence the equation (1.6) reduces to

$$\partial_t\left(\rho(X(a,t),t)\right) = \partial_t\rho(X(a,t),t) + (u\cdot\nabla\rho)(X(a,t),t) = 0. \tag{1.8}$$

Therefore, $\rho(X(a,t),t)$ is constant in time and amounts to the initial density ρ_0 at a. This gives the explicit expression

$$\rho(x,t) = \rho_0(X^{-1}(x,t)) \tag{1.9}$$

for all $x\in\Omega$ and all nonnegative times. If ρ_0 is homogeneous (constant in space), then we obtain

$$\rho(x,t) = \rho_0 \tag{1.10}$$

for all $x\in\Omega$ and all times $t\geq 0$. The pressure and viscosity of the fluid exerts a force on the surface of any volume element in the fluid. Due to the conservation of momentum (Newton's second law of motion), and in the presence of body forces in the fluid, the following relation

$$\frac{d}{dt}\int_{X(V,t)} \rho_0 u(x,t)dx = \int_{X(V,t)} \rho f(x,t)dx + \int_{\partial X(V,t)} \big(pI + \nu(\nabla u + \nabla u^T)\big) \cdot n(x)d\sigma(x) \tag{1.11}$$

holds, where p is the pressure of the fluid, $\nu > 0$ is a positive constant denoting the dynamic viscosity, $n(x)$ is the outward unit normal to the boundary $\partial X(V,t)$, and I is the identity operator. An application of Green's formula yields

$$\frac{d}{dt}\int_{X(V,t)} \rho_0 u(x,t)dx = \int_{X(V,t)} \left(-\nabla p(x) + \nu\Delta(x) + \rho_0 f(x)\right) dx \tag{1.12}$$

for all volume elements in Ω, from which we infer that

$$\rho_0 \left(\partial_t u + u \cdot \nabla u\right) + \nabla p - \nu\Delta u = \rho_0 f. \tag{1.13}$$

Dividing both sides by ρ_0, we obtain the Navier-Stokes system describing the time evolution of the velocity field u,

$$\partial_t u + u \cdot \nabla u + \frac{1}{\rho_0}\nabla p - \frac{\nu}{\rho}\Delta u = f. \tag{1.14}$$

The constant $\nu\rho^{-1}$ is called the kinematic viscosity. We refer the reader to [6, Chapter 1] for more details.

In the following sections of this chapter, the constants ρ_0 and ν will be taken to be 1 as they don't have any contribution to the analysis of the problems we are addressing.

1.2 Functional Spaces and Notations

Let $\mathbb{T}^2 = [0, 2\pi]^2$.

For $1 \leq p \leq \infty$, we denote by $L^p(\mathbb{T}^2)$ the Lebesgue spaces of measurable periodic functions f from $\mathbb{T}^2$ to $\mathbb{R}$ (or $\mathbb{R}^2$) such that

$$\|f\|_{L^p} = \left(\int_{\mathbb{T}^2} \|f\|^p dx\right)^{1/p} < \infty \tag{1.15}$$

if $p \in [1, \infty)$ and

$$\|f\|_{L^\infty} = \text{esssup}_{\mathbb{T}^2}|f| < \infty \tag{1.16}$$

if $p = \infty$. The $L^2(\mathbb{T}^2)$ inner product is denoted by $(\cdot, \cdot)_{L^2}$.

We denote by $\mathbb{P}$ the Leray-Hodge projection onto the space divergence free vector fields. For a mean-free periodic vector field $v = (v_1, v_2)$ with Fourier series

$$v = \sum_{j \in \mathbb{Z}^2 \setminus \{0\}} v_j e^{ij \cdot x}, \tag{1.17}$$

$\mathbb{P}v$ has the following Fourier representation

$$\mathbb{P}v = \sum_{j \in \mathbb{Z}^2 \setminus \{0\}} \left[v_j - (v_j \cdot j)\frac{j}{|j|^2}\right] e^{ij \cdot x}. \tag{1.18}$$

The operator $\mathbb{P}$ is bounded on L^p spaces for any $p \in (1, \infty)$.

For $s > 0$, we denote by $H^s(\mathbb{T}^2)$ the Sobolev spaces of measurable periodic mean-free functions f

$$f = \sum_{j \in \mathbb{Z}^2 \setminus \{0\}} f_j e^{ij \cdot x} \tag{1.19}$$

from $\mathbb{T}^2$ to $\mathbb{R}$ (or $\mathbb{R}^2$), obeying

$$\|f\|_{H^s}^2 = \sum_{k \in \mathbb{Z}^2} |k|^{2s} |f_k|^2 < \infty. \tag{1.20}$$

Let H and V be the Hilbert spaces of $L^2(\mathbb{T}^2)$ and $H^1(\mathbb{T}^2)$ respectively, consisting of periodic vector fields which are mean zero and divergence-free, with norms

$$\|u\|_H = \|u\|_{L^2} \tag{1.21}$$

and

$$\|u\|_V = \|\nabla u\|_{L^2} \tag{1.22}$$

respectively.

For a Banach space $(X, \|\cdot\|_X)$ and $p \in [1, \infty]$, we consider the Lebesgue spaces $L^p(0, T; X)$ of functions f from X to $\mathbb{R}$ (or $\mathbb{R}^2$) satisfying

$$\int_0^T \|f\|_X^p dt < \infty \tag{1.23}$$

with the usual convention when $p = \infty$.

Throughout this chapter, C denotes a positive universal constant, and it changes from line to line along the proofs.

1.3 Existence and Uniqueness of Solutions

We consider the forced incompressible Navier-Stokes equations

$$\partial_t u + u \cdot \nabla u + \nabla p - \Delta u = f, \tag{1.24}$$

$$\nabla \cdot u = 0 \tag{1.25}$$

on $\mathbb{T}^2 \times [0, \infty)$, with initial data $u_0(x)$. All unknowns are periodic in space. The body forces f are divergence-free, time independent and have mean zero.

The Stokes operator $A := \mathbb{P}\Delta$ is positive, self-adjoint, with compact inverse. By the spectral theorem for Hilbert spaces, there is an orthonormal basis of H consisting of eigenfunctions $\{\Phi_k\}_{k=1}^{\infty}$ of the Stokes operator

$$A\Phi_k = \mathbb{P}(-\Delta\Phi_k) = \mu_k \Phi_k \tag{1.26}$$

with periodic boundary condition on $\mathbb{T}^2$, where the sequence of eigenvalues μ_k is increasing and obeys $0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_k \to \infty$. The functions Φ_k's are C^∞, divergence-free and have mean zero. Since the operators $\mathbb{P}$ and Δ are Fourier multipliers, they commute, and consequently, it holds that

$$-\Delta\Phi_k = \mu_k \Phi_k \tag{1.27}$$

for all $k \in \mathbb{N}$. For a positive integer $n \geq 1$, we let

$$u_n = \mathbb{P}_n u = \sum_{k=1}^{n} (u, \Phi_k)_{L^2} \Phi_k \tag{1.28}$$

be the Galerkin approximations of u. For each $n \in \mathbb{N}$, u_n is C^∞, divergence-free and has mean zero. We consider the approximating equations

$$\frac{\partial}{\partial t} u_n - \Delta u_n + \mathbb{P}_n(u_n \cdot \nabla u_n) = \mathbb{P}_n f. \tag{1.29}$$

These are equivalent to a system of nonlinear ODE's for the coefficients of the Galerkin approximations $(u, \Phi_i)_{L^2}$, $1 \leq i \leq n$,

$$\begin{aligned} &\frac{d}{dt}(u, \Phi_i)_{L^2} + \mu_i (u, \Phi_i)_{L^2} \\ &\qquad + \sum_{k,l=1}^{n} (\Phi_k \cdot \nabla \Phi_l, \Phi_i)_{L^2} (u, \Phi_k)_{L^2} (u, \Phi_l)_{L^2} = (f, \Phi_i)_{L^2}, \end{aligned} \tag{1.30}$$

hence a solution of the approximating system would exist if it is bounded in L^2. Indeed, we take the L^2 inner product of (1.29) with u_n. In view of the self-adjointness of the projector $\mathbb{P}_n$ and the divergence-free condition obeyed by u_n, the following cancellation

$$(\mathbb{P}_n(u_n \cdot \nabla u_n), u_n)_{L^2} = (u_n \cdot \nabla u_n, \mathbb{P}_n u_n)_{L^2} = (u_n \cdot \nabla u_n, u_n)_{L^2} = 0 \tag{1.31}$$

holds, yielding the differential inequality

$$\frac{1}{2}\frac{d}{dt}\|u_n\|_{L^2}^2 + \|\nabla u_n\|_{L^2} \leq \|f\|_{L^2}^2 \|u_n\|_{L^2} \leq \frac{1}{2}\|f\|_{L^2}^2 + \frac{1}{2}\|u_n\|_{L^2}^2, \tag{1.32}$$

after making use of Cauchy-Schwarz and Young inequalities. Now we control the L^2 norm of u_n by the L^2 of its gradient via application of the Poincaré inequality and obtain the energy inequality

$$\frac{d}{dt}\|u_n\|_{L^2}^2 + \|u_n\|_{L^2}^2 \leq \|f\|_{L^2}^2, \tag{1.33}$$

from which we infer that

$$\|u_n(t)\|_{L^2}^2 \leq \|u_0\|_{L^2}^2 e^{-t} + \|f\|_{L^2}^2 \tag{1.34}$$

holds for every $t \geq 0$. Moreover, we have

$$\int_0^T \|\nabla u_n(t)\|_{L^2}^2 dt \leq \|u_0\|_{L^2}^2 + \|f\|_{L^2}^2 T \tag{1.35}$$

for every $T > 0$. Therefore, the Galerkin approximants u_n are uniformly bounded in the Lebesgue spaces

$$u_n \in L^\infty(0,\infty; L^2) \cap L^2(0,T; H^1). \tag{1.36}$$

for any $T > 0$. Moreover, the time-derivatives of the approximants u_n obey

$$\frac{\partial}{\partial_t} u_n \in L^{4/3}(0,T; H^{-1}) \tag{1.37}$$

uniformly in n. Indeed, if $\Phi \in H^1$, then

$$|(-\Delta u_n, \Phi)_{L^2}| = |(\nabla u_n, \nabla \Phi)_{L^2}| \leq \|\nabla u_n\|_{L^2} \|\Phi\|_{H^1} \tag{1.38}$$

and so

$$\|\Delta u_n\|_{H^{-1}} \leq \|\nabla u_n\|_{L^2} \tag{1.39}$$

which implies that

$$\Delta u_n \in L^2(0,T; H^{-1}) \subset L^{4/3}(0,T; H^{-1}). \tag{1.40}$$

Also, if $\Phi \in H^1$, then the nonlinear term in u_n can be bounded as

$$|(\mathbb{P}_n(u_n \cdot \nabla u_n), \Phi)_{L^2}| \le \|u_n\|_{L^4} \|\nabla u_n\|_{L^2} \|\mathbb{P}_n \Phi\|_{L^4}$$
$$\le C\|u_n\|_{L^2}^{\frac{1}{2}} \|\nabla u_n\|_{L^2}^{\frac{3}{2}} \|\Phi\|_{H^1} \tag{1.41}$$

in view of Ladyzhenskaya's interpolation inequality. By taking the supremum over all functions Φ in H^1 whose H^1 norm is bounded by 1, we infer that

$$\|\mathbb{P}_n(u_n \cdot \nabla u_n)\|_{H^{-1}} \le C\|u_n\|_{L^2}^{\frac{1}{2}} \|\nabla u_n\|_{L^2}^{\frac{3}{2}}, \tag{1.42}$$

from which the inclusion

$$\mathbb{P}_n(u_n \cdot \nabla u_n) \in L^{4/3}(0, T; H^{-1}) \tag{1.43}$$

follows. By making use of the PDE (1.29) obeyed by u_n, we obtain the desired control (1.37) of the time derivatives. Now we apply the Aubin-Lions lemma and conclude that the sequence $\{u_n\}_{n=1}^{\infty}$ has a subsequence that converges strongly in $L^2(0, T; L^2)$ to some function u. Testing (1.29) with $\Phi \in V$ and integrating in time from 0 to t, we have

$$(u_n(t), \Phi)_{L^2} - (u_0, \Phi)_{L^2} + \int_0^t (\nabla u_n, \nabla \Phi)_{L^2} ds$$
$$+ \int_0^t (u_n \cdot \nabla u_n, \mathbb{P}_n \Phi)_{L^2} ds = t(f, \mathbb{P}_n \Phi)_{L^2}. \tag{1.44}$$

Since $u_n \to u$ strongly in $L^2(0, T; L^2)$, we conclude, passing to subsequences, that $u_n(t) \to u(t)$ in L^2 for a.e. $t \in [0, T]$, and thus

$$|(u_n, \Phi)_{L^2} - (u, \Phi)_{L^2}| \le \|u_n - u\|_{L^2} \|\Phi\|_{L^2} \to 0. \tag{1.45}$$

Now, since u_n is bounded in $L^2(0,T;H^1)$, we deduce, passing to subsequences, that $\nabla u_n \rightharpoonup \nabla u$ in $L^2(0,T;L^2)$, and so

$$\int_0^t (\nabla u_n, \nabla\Phi)_{L^2} ds \to \int_0^t (\nabla u, \nabla\Phi)_{L^2} ds. \tag{1.46}$$

Moreover,

$$\begin{aligned}
&\left|\int_0^t \left\{(u_n\cdot\nabla u_n, \mathbb{P}_n\Phi)_{L^2} - (u\cdot\nabla u, \Phi)_{L^2}\right\} ds\right| \\
&\quad \le \left|\int_0^t ((u_n-u)\cdot\nabla u_n, \mathbb{P}_n\Phi)_{L^2} ds\right| \\
&\qquad + \left|\int_0^t (u\cdot\nabla(u_n-u), \mathbb{P}_n\Phi)_{L^2} ds\right| \\
&\qquad\quad + \left|\int_0^t (u\cdot\nabla u, \mathbb{P}_n\Phi-\Phi)_{L^2} ds\right| \\
&\quad := A_n + B_n + C_n.
\end{aligned} \tag{1.47}$$

We show that the quantities A_n, B_n and C_n converge to 0. Indeed, we have

$$\begin{aligned}
A_n &\le \int_0^t C\|u_n-u\|_{L^2}^{\frac{1}{2}}\|\nabla u_n-\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla u_n\|_{L^2}\|\mathbb{P}_n\Phi\|_{L^4} ds \\
&\le C\|\Phi\|_{H^1}\left(\int_0^t \|\nabla u_n\|_{L^2}^2\right)^{\frac{1}{2}}\left(\int_0^t \|\nabla u_n-\nabla u\|_{L^2}^2\right)^{\frac{1}{4}}\left(\int_0^t \|u_n-u\|_{L^2}^2\right)^{\frac{1}{4}}
\end{aligned}$$

which approaches 0 as n blows up due to the estimate (1.35). Since $u\mathbb{P}_n\Phi \in L^2(0,T;L^2)$, then by the weak convergence $\nabla u_n \rightharpoonup \nabla u$ in $L^2(0,T;L^2)$ we conclude that

$$B_n = \left|\int_0^t (\nabla(u_n-u), u\mathbb{P}_n\Phi)_{L^2} ds\right| \to 0. \tag{1.48}$$

Finally, we have

$$C_n \leq C\|\mathbb{P}_n\Phi - \Phi\|_{H^1} \int_0^t \|u\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{3}{2}} \to 0. \quad (1.49)$$

Therefore, u obeys the equation

$$(u(t),\Phi)_{L^2} - (u_0,\Phi)_{L^2} + \int_0^t (\nabla u, \nabla \Phi)_{L^2} ds + \int_0^t (u\cdot\nabla u, \Phi)_{L^2} ds = t(f,\Phi)_{L^2}. \quad (1.50)$$

for all $\Phi \in V$ and a.e. $t \in [0,T]$. This gives the following theorem:

Theorem 1.1. *Let $T > 0$ be arbitrary. Let $u_0 \in H$ and $f \in H$. Then, the system* (1.24)–(1.25) *has a unique solution $u \in L^\infty(0,T;H) \cap L^2(0,T;V)$ satisfying*

$$\|u(t)\|_{L^2}^2 \leq \|u_0\|_{L^2} e^{-t} + \|f\|_{L^2}^2 \quad (1.51)$$

for any $t \in [0,T]$. Moreover, we have

$$\int_0^T \|\nabla u(t)\|_{L^2}^2 \leq \|u_0\|_{L^2}^2 + T\|f\|_{L^2}^2 \quad (1.52)$$

for every $T > 0$.

Proof: In view of the bounds (1.34) and (1.35), and the lower semicontinuity of the norm, we obtain (1.51) and (1.52). For uniqueness, suppose u_1 and u_2 are two solutions of (1.24)–(1.25) with same initial conditions, such that $u_1, u_2 \in L^\infty(0,T;H) \cap L^2(0,T;V)$. Then the difference $u = u_1 - u_2$ obeys

$$\partial_t u + u\cdot\nabla u_1 + u_2\cdot\nabla u + \nabla(p_1 - p_2) - \Delta u = 0. \quad (1.53)$$

Taking the L^2 inner product of this latter equation with u gives the energy equality

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 = -\int (u\cdot\nabla u_1)\cdot u dx. \quad (1.54)$$

We estimate

$$\left|\int (u \cdot \nabla u_1) \cdot u dx\right| \leq C\|u\|_{L^2}\|\nabla u\|_{L^2}\|\nabla u_1\|_{L^2}$$
$$\leq \frac{1}{2}\|\nabla u\|_{L^2}^2 + C\|\nabla u_1\|_{L^2}^2\|u\|_{L^2}^2 \tag{1.55}$$

using Ladyzhenskaya's interpolation inequality followed by an application of Young's inequality. Consequently, u satisfies the differential inequality

$$\frac{d}{dt}\|u\|_{L^2}^2 \leq C\|\nabla u_1\|_{L^2}^2\|u\|_{L^2}^2, \tag{1.56}$$

from which we infer that $u_1 = u_2$ a.e. in $\mathbb{T}^2$ for all $t \in [0, T]$.

Theorem 1.2. *Suppose $u_0 \in V$ and $f \in H$. Then, for any $t \geq 0$, it holds that*

$$\|\nabla u(t)\|_{L^2}^2 \leq \|\nabla u_0\|_{L^2}^2 e^{-t} + \|f\|_{L^2}^2. \tag{1.57}$$

Moreover,

$$\int_0^T \|\Delta u(t)\|_{L^2}^2 ds \leq \|\nabla u_0\|_{L^2}^2 + T\|f\|_{L^2}^2 \tag{1.58}$$

holds for all $T > 0$.

Proof: We take the L^2 inner product of equation (1.29) obeyed by u_n with $-\Delta u_n$. In view of the the identity

$$Tr(M^T M^2) = 0 \tag{1.59}$$

that holds for the two-by-two traceless matrix M with entries $M_{ij} = \frac{\partial (u_n)_i}{\partial x_j}$, the nonlinear term in u_n vanishes. Moreover, the forcing term $(f, -\Delta u_n)_{L^2}$ is bounded by the product of $\|f\|_{L^2}$ and $\|\Delta u_n\|_{L^2}$ due the Cauchy-Schwarz inequality. A straightforward application of Young's inequality yields

$$\frac{d}{dt}\|\nabla u_n\|_{L^2}^2 + \|\Delta u_n\|_{L^2}^2 \leq \|f\|_{L^2}^2. \tag{1.60}$$

By the Poincaré inequality, the term $\|\Delta u_n\|_{L^2}$ is bounded from below by $\|\nabla u_n\|_{L^2}$. We multiply by the integrating factor, integrate in time from 0 to t, apply the Banach Alaoglu theorem, and use the lower semicontinuity of the norm in order to obtain (1.57). Integrating (1.60) in time from 0 to T and using again the Banach Alaoglu theorem and the lower semicontinuity of the norm, we obtain (1.58).

Remark 1.1. *It follows from the proof provided above that*

$$\int_t^{t+1} \|\Delta u(s)\|_{L^2}^2 ds \leq \|\nabla u_0\|_{L^2}^2 e^{-t} + 2\|f\|_{L^2}^2 \tag{1.61}$$

for any $t \geq 0$. These bounds will be used later to bootstrap the regularity and exponential decay to higher-order derivatives.

Theorem 1.3. *Let $u_0 \in H, f \in H$, and u be the solution of* (1.24)–(1.25) *with initial data u_0. There exists a radius $R > 0$ depending only on $\|f\|_{L^2}$, and a time $t_0 > 0$ depending only on $\|u_0\|_{L^2}$, such that the bound*

$$\|\nabla u(t)\|_{L^2} \leq R \tag{1.62}$$

holds for $t \geq t_0$.

Proof: From the local-in-time integral (1.52), we infer the existence of a small positive time $t_1 \in [0, 1]$ with the property that

$$\|\nabla u(t_1)\|_{L^2}^2 \leq 2\|u_0\|_{L^2}^2 + 2\|f\|_{L^2}^2 < \infty. \tag{1.63}$$

In view of the bound (1.57), there exists a time $t_0 \geq t_1$ such that

$$\|\nabla u(t)\|_{L^2}^2 \leq 2\|f\|_{L^2}^2 := R \tag{1.64}$$

for all $t \geq t_0$. This ends the proof of Theorem 1.3.

Now we address higher regularity properties of the solutions. We need the following Gronwall Lemma:

Lemma 1.1. *Let $y(t) \geq 0$ obey a differential inequality*

$$\frac{d}{dt}y + c_1 y \leq F_1 + F(t) \tag{1.65}$$

with initial datum $y(0) = y_0$, with F_1 a positive constant and $F(t) \geq 0$ obeying

$$\int_t^{t+1} F(s)ds \leq g_0 e^{-c_2 t} + F_2 \tag{1.66}$$

where c_1, c_2, g_0, F_2 are positive constants. Then

$$y(t) \leq y_0 e^{-c_1 t} + g_0 e^{c_1 + c}(t+1)e^{-ct} + \frac{1}{c_1}F_1 + \frac{e^{c_1}}{1 - e^{-c_1}}F_2 \tag{1.67}$$

holds with $c = \min\{c_1, c_2\}$.

The main point of the lemma is that the constants y_0 and g_0 are multiplied by exponentially decaying factors.

Proof: Integrating, we have

$$y(t) \leq y_0 e^{-c_1 t} + \frac{1}{c_1}F_1 + \int_0^t e^{-c_1(t-s)}F(s)ds, \tag{1.68}$$

and, taking N to be the integer part of t, i.e. $t \in [N, N+1)$, we have

$$\begin{aligned}
\int_0^t e^{-c_1(t-s)}F(s)ds &\leq \sum_{k=0}^{N} e^{-c_1(t-k-1)} \int_k^{k+1} F(s)ds \\
&\leq e^{c_1} \sum_{k=0}^{N} e^{-c_1(N-k)}(g_0 e^{-c_2 k} + F_2) \\
&\leq e^{c_1}(N+1)e^{-\min\{c_1,c_2\}N} g_0 + \frac{e^{c_1}}{1-e^{-c_1}}F_2. \qquad (1.69)
\end{aligned}$$

Note that

$$e^{c_1}(N+1)e^{-\min\{c_1,c_2\}N} \leq e^{c_1+c}(t+1)e^{-ct} \leq C_\gamma e^{-\gamma t}$$

for $\gamma < c = \min\{c_1, c_2\}$.

By making use of this latter lemma, we obtain the following theorem:

Theorem 1.4. *(Higher Regularity) Let $k \geq 1$ be an integer. Let $u_0 \in H^k$ and $f \in H^{k-1}$. Then we have*

$$\|u(t)\|_{H^k} + \int_t^{t+1} \|u(s)\|_{H^{k+1}}^2 ds \leq C_k \|u_0\|_{H^k} e^{-\frac{t}{2^{k-1}}} + C_k \|f\|_{k-1} \tag{1.70}$$

for all $t \in [0, \infty)$. Here C_k is a positive constant depending only on k and some universal constants.

Proof: We present a proof by induction. In view of the estimate (1.57) and Remark 1.1, we infer that Theorem 1.4 holds for $k = 1$. Suppose the theorem holds at the $(k-1)$-th level. Taking the L^2 inner product of the equation (1.29) obeyed by the velocity approximants u_n with $(-\Delta)^{\frac{k}{2}} u_n$, we obtain the energy equality

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|(-\Delta)^{\frac{k}{2}} u_n\|_{L^2}^2 + \|(-\Delta)^{\frac{k+1}{2}} u_n\|_{L^2}^2 \\ &\quad = (\mathbb{P}_n f, (-\Delta)^k u_n)_{L^2} - (\mathbb{P}_n(u_n \cdot \nabla u_n), (-\Delta)^k u_n)_{L^2}. \end{aligned} \tag{1.71}$$

Here $(-\Delta)^{\frac{1}{2}}$ is the square root of the Laplacian, defined as a Fourier multiplier with symbol $|j|^{\frac{1}{2}}$. In view of the self-adjointness of $\mathbb{P}_n$, the fact that $\mathbb{P}_n$ and $(-\Delta)^k$ commute as they are Fourier multipliers, and the identity $\mathbb{P}_n u_n = u_n$, we bound the forcing term as follows,

$$\begin{aligned} |(\mathbb{P}_n f, (-\Delta)^k u_n)_{L^2}| &\leq \|(-\Delta)^{\frac{k-1}{2}} f\|_{L^2} \|(-\Delta)^{\frac{k+1}{2}} u_n\|_{L^2} \\ &\leq \frac{1}{4}\|(-\Delta)^{\frac{k+1}{2}} u_n\|_{L^2}^2 + C\|(-\Delta)^{\frac{k-1}{2}} f\|_{L^2}^2. \end{aligned} \tag{1.72}$$

The first inequality follows from integrating by parts whereas the second inequality is a direct consequence of Young's inequality for products. As for the nonlinear term in u_n, we distinguish two cases: $k = 2$ and $k > 2$. If $k = 2$, then we have

$$
\begin{aligned}
&|(\mathbb{P}_n(u_n \cdot \nabla u_n), (-\Delta)^2 u_n)_{L^2}| = |(\nabla(u_n \cdot \nabla u_n), \nabla(-\Delta) u_n)_{L^2}| \\
&\quad \le \|\nabla u_n\|_{L^4}^2 \|\nabla \Delta u_n\|_{L^2} + \|u_n\|_{L^4} \|\Delta u_n\|_{L^4} \|\nabla \Delta u_n\|_{L^2} \\
&\quad \le C\|\nabla u_n\|_{L^2} \|\Delta u_n\|_{L^2} \|\nabla \Delta u_n\|_{L^2} + C\|\nabla u_n\|_{L^2} \|\Delta u_n\|_{L^2}^{\frac{1}{2}} \|\nabla \Delta u_n\|_{L^2}^{\frac{3}{2}} \\
&\quad \le \frac{1}{4}\|\nabla \Delta u_n\|_{L^2}^2 + C\left(\|\nabla u_n\|_{L^2}^4 + \|\nabla u_n\|_{L^2}^2\right)\|\Delta u_n\|_{L^2}^2
\end{aligned}
\tag{1.73}
$$

after several applications of Ladyzhenskaya's interpolation inequality. If $k > 2$, then H^{k-1} is a Banach Algebra, and consequently, we have

$$
\begin{aligned}
&|(\mathbb{P}_n(u_n \cdot \nabla u_n), (-\Delta)^k u_n)_{L^2}| = |((-\Delta)^{\frac{k-1}{2}}(u_n \cdot \nabla u_n), (-\Delta)^{\frac{k+1}{2}} u_n)_{L^2}| \\
&\quad \le \|(-\Delta)^{\frac{k-1}{2}}(u_n \cdot \nabla u_n)\|_{L^2} \|(-\Delta)^{\frac{k+1}{2}} u_n\|_{L^2} \\
&\quad \le C\|(-\Delta)^{\frac{k-1}{2}} u_n\|_{L^2} \|(-\Delta)^{\frac{k}{2}} u_n\|_{L^2} \|(-\Delta)^{\frac{k+1}{2}} u_n\|_{L^2} \\
&\quad \le \frac{1}{4}\|(-\Delta)^{\frac{k+1}{2}} u_n\|_{L^2}^2 + C\|(-\Delta)^{\frac{k-1}{2}} u_n\|_{L^2}^2 \|(-\Delta)^{\frac{k}{2}} u_n\|_{L^2}^2
\end{aligned}
\tag{1.74}
$$

via integration by parts and use of Hölder and Young inequalities. Therefore, we obtain the differential inequalities

$$
\begin{aligned}
&\frac{d}{dt}\|\Delta u_n\|_{L^2}^2 + \|\nabla \Delta u_n\|_{L^2}^2 \\
&\qquad \le C\|\nabla f\|_{L^2}^2 + C\left(\|\nabla u_n\|_{L^2}^4 + \|\nabla u_n\|_{L^2}^2\right)\|\Delta u_n\|_{L^2}^2
\end{aligned}
\tag{1.75}
$$

when $k = 2$, and

$$
\begin{aligned}
&\frac{d}{dt}\|(-\Delta)^{\frac{k}{2}} u_n\|_{L^2}^2 + \|(-\Delta)^{\frac{k+1}{2}} u_n\|_{L^2}^2 \\
&\qquad \le C\|(-\Delta)^{\frac{k-1}{2}} f\|_{L^2}^2 + C\|(-\Delta)^{\frac{k-1}{2}} u_n\|_{L^2}^2 \|(-\Delta)^{\frac{k}{2}} u_n\|_{L^2}^2
\end{aligned}
\tag{1.76}
$$

when $k > 2$. An application of Lemma 1.1 completes the proof of Theorem 1.4.

Remark 1.2. *The solutions to the forced Navier-Stokes system are infinitely differentiable in space and time provided that the initial data is C^∞. This follows from the Sobolev H^k regularity obtained in Theorem 1.4 for all $k \geq 0$ and standard Sobolev embeddings.*

Remark 1.3. *In the absence of body forces in the fluid (that is $f = 0$), the velocity u and all its spatial derivatives decay exponentially in time to zero, a fact that follows from Theorem 1.4.*

1.4 Existence of a Finite Dimensional Global Attractor

Let

$$\mathcal{S}(t) : H \to H \tag{1.77}$$

be the solution map

$$\mathcal{S}(t)u_0 = u(t) \tag{1.78}$$

corresponding to the forced incompressible Navier-Stokes system (1.24)–(1.25). Note that $\mathcal{S}(t)$ is well-defined on H for every $t \geq 0$. Moreover, the uniqueness of solutions implies that

$$\mathcal{S}(t+s)u_0 = \mathcal{S}(t)\mathcal{S}(s)u_0 \tag{1.79}$$

for all $t, s \geq 0$. In other words, $\mathcal{S}(t)$ is a semi-group. We proceed to investigate other properties of the map $\mathcal{S}(t)$.

Theorem 1.5. *(Continuity) Let $u_0^1, u_0^2 \in H$. Let $t > 0$. There exists a constant $C(t)$, locally uniformly bounded as a function of $t \geq 0$ and locally bounded as initial data u_0^1, u_0^2 are varied in H, such that $\mathcal{S}(t)$ is Lipschitz continuous in H obeying*

$$\|\mathcal{S}(t)u_0^1 - \mathcal{S}(t)u_0^2\|_H^2 \leq C(t)\|u_0^1 - u_0^2\|_H^2. \tag{1.80}$$

Proof: Let $u_1(t) = \mathcal{S}(t)u_0^1, u_2(t) = \mathcal{S}(t)u_0^2$. The difference $u = u_1 - u_2$ obeys the differential inequality

$$\frac{d}{dt}\|u\|_{L^2}^2 \leq \|\nabla u_1\|_{L^2}^2\|u\|_{L^2}^2, \tag{1.81}$$

see the proof of Theorem 1.1. By Gronwall's inequality, we infer that

$$\|u(t)\|_{L^2}^2 \leq C(t)\|u_0^1 - u_0^2\|_{L^2}^2 \tag{1.82}$$

where

$$C(t) = \exp\left\{\int_0^t \|\nabla u_1(s)\|_{L^2}^2 ds\right\}. \tag{1.83}$$

This completes the proof of Theorem 1.5.

Now we address the injectivity of the solution map $\mathcal{S}(t)$ on H:

Theorem 1.6. *(Backward Uniqueness) Let $u_0^1, u_0^2 \in V$. If there exists $T > 0$ such that $\mathcal{S}(T)u_0^1 = \mathcal{S}(T)u_0^2$, then $u_0^1 = u_0^2$.*

Proof: Let $u(t) = \mathcal{S}(t)u_0^1 - \mathcal{S}(t)u_0^2, \widetilde{u}(t) = \frac{1}{2}(\mathcal{S}(t)u_0^1 + \mathcal{S}(t)u_0^2)$. WLOG, assume $u \neq 0$ on $[0, T)$.

Let $E_0(t) = \|u(t)\|_{L^2}^2, E_1(t) = \|\nabla u(t)\|_{L^2}^2, Y(t) = \log(1/E_0(t))$. Then $Y(t)$ is bounded from below on $[0,T)$ and $\lim\limits_{t\to T^-} Y(t) = +\infty$. We show that the following differential inequalities

$$\frac{d}{dt}\frac{E_1}{E_0} \leq C_1(t)\frac{E_1}{E_0} \tag{1.84}$$

and

$$\frac{d}{dt}Y(t) \leq C_2(t)\frac{E_1}{E_0} \tag{1.85}$$

hold, with

$$\int_0^T (C_1(t) + C_2(t))dt < \infty. \tag{1.86}$$

This implies that $Y \in L^\infty(0,T)$, yielding consequently a contradiction.

The equation obeyed by u is given by

$$\partial_t u + Au + B(\widetilde{u}, u) + B(u, \widetilde{u}) = 0, \tag{1.87}$$

where $A = -\mathbb{P}\Delta$ is the Stokes operator and B is the operator defined by $B(v,\omega) = \mathbb{P}(v \cdot \nabla\omega)$. We take the L^2 inner product of (1.87) with u and obtain the energy equality

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + (B(\widetilde{u}, u) + B(u, \widetilde{u}), u)_{L^2} = 0, \tag{1.88}$$

from which we derive the equation desribing the time evolution of $Y(t)$,

$$\frac{1}{2}\frac{d}{dt}Y(t) = -\frac{1}{2}\frac{d}{dt}\log E_0 = \frac{E_1}{E_0} + \frac{(B(\widetilde{u}, u) + B(u, \widetilde{u}), u)_{L^2}}{E_0}. \tag{1.89}$$

Due to the cancellation law

$$(B(\widetilde{u}, u), u)_{L^2} = 0, \tag{1.90}$$

and the estimate

$$|(B(u,\widetilde{u}),u)_{L^2}| \leq \|u\|_{L^4}^2\|\nabla\widetilde{u}\|_{L^2} \leq C\|\nabla\widetilde{u}\|_{L^2}E_1, \tag{1.91}$$

we conclude that (1.85) holds. On the other hand, the time evolution of E_1/E_0 depends upon the time evolution of both E_1 and Y via

$$\frac{d}{dt}\frac{E_1}{E_0} = \frac{1}{E_0}\frac{d}{dt}E_1 + \frac{E_1}{E_0}\frac{d}{dt}Y. \tag{1.92}$$

We derive the differential equality obeyed by E_1 by taking the L^2 inner product of (1.87) with Au and obtain

$$\frac{1}{2}\frac{d}{dt}E_1 + \|Au\|_{L^2}^2 + (B(u,\widetilde{u}) + B(\widetilde{u},u), Au)_{L^2} = 0. \tag{1.93}$$

Inserting (1.93) in (1.92) gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\frac{E_1}{E_0} &= \frac{1}{E_0}(-\|Au\|_{L^2}^2 - (B(u,\widetilde{u}) + B(\widetilde{u},u), Au)_{L^2}) \\ &\quad + \frac{E_1}{E_0}\left(\frac{E_1}{E_0} + \frac{(B(u,\widetilde{u}) + B(\widetilde{u},u), u)_{L^2}}{E_0}\right)\end{aligned} \tag{1.94}$$

which, after making use of the identity

$$\frac{E_1^2}{E_0^2} - \frac{\|Au\|_{L^2}^2}{E_0} = -E_0^{-1}\|(A - E_1E_0^{-1})u\|_{L^2}^2, \tag{1.95}$$

reduces to

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\frac{E_1}{E_0} &= -E_0^{-1}\|(A - E_1E_0^{-1})u\|_{L^2}^2 \\ &\quad - E_0^{-1}(B(u,\widetilde{u}) + B(\widetilde{u},u), (A - E_1E_0^{-1})u)_{L^2}.\end{aligned} \tag{1.96}$$

We apply the Cauchy-Schwarz inequality to the second term on the right hand-side of (1.96) and split the resulting product using Young's inequality.

We estimate

$$\begin{aligned}\|B(u,\widetilde{u})+B(\widetilde{u},u)\|_{L^2}^2 &\le C(\|B(u,\widetilde{u})\|_{L^2}+\|B(\widetilde{u},u)\|_{L^2}^2)\\ &\le C(\|u\|_{L^4}^2\|\nabla\widetilde{u}\|_{L^4}^2+\|\widetilde{u}\|_{L^\infty}^2\|\nabla u\|_{L^2}^2)\\ &\le C(\|\nabla\widetilde{u}\|_{L^4}^2+\|\widetilde{u}\|_{L^\infty}^2)E_1, \qquad (1.97)\end{aligned}$$

using Hölder's and Ladyzhenskaya's inequalities. We obtain (1.84), completing the proof of Theorem 1.6.

As a consequence of Theorem 1.3, there exists a positive radius $R > 0$ depending only on the body forces f, such that for any initial velocity $u_0 \in H$, there exists a time $t_0 > 0$, depending only on $\|u_0\|_{L^2}$, such that

$$\mathcal{S}(t)u_0 \in \mathcal{B}_R = \{u \in H : \|\nabla u\|_{L^2} \le R\} \tag{1.98}$$

for all $t \ge t_0$. Due to the Poincaré inequality, there exists a time $T > 0$, depending only on the radius R, such that the inclusion

$$\mathcal{S}(t)\mathcal{B}_R \subset \mathcal{B}_R \tag{1.99}$$

holds for all times $t \ge T$.

The continuity and injectivity properties of the solution map $\mathcal{S}(t)$, together with the connectedness and compactness properties of the absorbing ball $\mathcal{B}_R$, imply the existence of a global attractor:

Theorem 1.7. *(Global Attractor) Let*

$$X = \bigcap_{t>0} \mathcal{S}(t)\mathcal{B}_R \tag{1.100}$$

Then:

(i) X is compact in H.

(ii) $\mathcal{S}(t)X = X$ *for all* $t \geq 0$.

(iii) *If* Z *is bounded in* H *in the norm of of* H*, and* $\mathcal{S}(t)Z = Z$ *for all* $t \geq 0$*, then* $Z \subset X$.

(iv) *For every* $u_0 \in H$, $\lim_{t\to\infty} dist_H(\mathcal{S}(t)u_0, X) = 0$.

(v) X *is connected.*

Proof:

(i) Since $\mathcal{B}_R$ is compact in H, then by continuity of the solution map $\mathcal{S}(t)$, we see that $\mathcal{S}(t)\mathcal{B}_R$ is compact in H for all $t \geq 0$, and so is their intersection.

(ii) Let $x \in X$, and $t > 0$. We show that $\mathcal{S}(t)x \in X$. Since $x \in X$, then for each $\sigma > 0$, there exists $y_\sigma \in \mathcal{B}_R$ such that $x = \mathcal{S}(\sigma)y_\sigma$, so $\mathcal{S}(t)x = \mathcal{S}(t+\sigma)y_\sigma$, and so $\mathcal{S}(t)x \in \bigcap_{s>t} \mathcal{S}(s)\mathcal{B}_R$. Now, if $s \leq t$, then $\mathcal{S}(t)x = \mathcal{S}(s)\mathcal{S}(t-s)x$. But $x = \mathcal{S}(T+s)y_s$ for some $y_s \in \mathcal{B}_R$, so $\mathcal{S}(t-s)x = \mathcal{S}(t-s)\mathcal{S}(T+s)y_s = \mathcal{S}(t+T)y_s \in \mathcal{B}_R$ since $\mathcal{S}(T+t)\mathcal{B}_R \subset \mathcal{B}_R$. Thus $\mathcal{S}(s)\mathcal{S}(t-s)x \in \mathcal{S}(s)\mathcal{B}_R$, so $\mathcal{S}(t)x \in \mathcal{S}(s)\mathcal{B}_R$, and so $\mathcal{S}(t)x \in \bigcap_{s\leq t} \mathcal{S}(s)\mathcal{B}_R$. Therefore, $\mathcal{S}(t)x \in X$.

Now, let $x \in X$ and fix $t > 0$. We show that $x \in \mathcal{S}(t)X$. Well, there exists $y_t \in \mathcal{B}_R$ such that $x = \mathcal{S}(t)y_t$. If $s > 0$, then there exists $z_s \in \mathcal{B}_R$ such that $x = \mathcal{S}(t+s)z_s$, and so $\mathcal{S}(t)\mathcal{S}(s)z_s = \mathcal{S}(t)y_t$. By injectivity of $\mathcal{S}(t)$, it follows that $\mathcal{S}(s)z_s = y_t$, so $y_t \in \mathcal{S}(s)\mathcal{B}_R$. This is true for any $s > 0$, so $y_t \in \bigcap_{s>0} \mathcal{S}(s)\mathcal{B}_R$, which implies that $y_t \in X$ and thus $x = \mathcal{S}(t)y_t \in \mathcal{S}(t)X$.

(iii) Since Z is bounded in H, there exists t_Z such that $\mathcal{S}(t)u_0 \in \mathcal{B}_R$ for all $t \geq t_Z$ and all $u_0 \in Z$.

Then, let $z \in Z$, and fix $s > 0$. We show that there exists $y_s \in \mathcal{B}_R$ such that $z = \mathcal{S}(s)y_s$. From the invariance property of Z, it follows that there exists $z_s \in Z$ such that $z = \mathcal{S}(s + t_Z)z_s$. Thus, $z = \mathcal{S}(s)\mathcal{S}(t_Z)z_s$, with $\mathcal{S}(t_Z)z_s \in \mathcal{B}_R$. Hence, $z \in \bigcap_{s>0} \mathcal{S}(s)\mathcal{B}_R$, and thus $z \in X$.

(iv) Let $u_0 \in H$. Define $\omega(u_0)$ by

$$\omega(u_0) = \left\{ u \in H : \exists s_j \to \infty, u = \lim_{j\to\infty} \mathcal{S}(s_j)u_0 \right\} \tag{1.101}$$

where the limit is taken in H. We recall that there exists $t_0 = t_0(\|u_0\|_{L^2}) > 0$ such that $\mathcal{S}(t)u_0 \in \mathcal{B}_R$ for all $t \geq t_0$, thus $\omega(u_0)$ is non empty and bounded in H.

Now, we claim that $\mathcal{S}(t)\omega(u_0) = \omega(u_0)$ for all $t \geq 0$. To show this claim, we note that if $u \in \omega(u_0)$, then $u = \lim_{j\to\infty} \mathcal{S}(s_j)u_0$ and so $\mathcal{S}(t)u = \lim_{j\to\infty} \mathcal{S}(t + s_j)u_0$, thus $\mathcal{S}(t)u \in \omega(u_0)$. On the other hand, if $u \in \omega(u_0)$ and $\mathcal{S}(s_j)u_0 \to u$, in H, we consider the sequence $\mathcal{S}(s_j - t)u_0$ for all j such that $s_j \geq t$. Since $\mathcal{B}_R$ is compact and $\mathcal{S}(s_j - t)u_0 \in \mathcal{B}_R$ for all but finitely many j's, it follows, passing to a subsequence, that $\mathcal{S}(s_{j_k} - t)u_0$ converges to some $v \in \mathcal{B}_R$. But $\mathcal{S}(t)\mathcal{S}(s_{j_k} - t)u_0 = \mathcal{S}(s_{j_k})u_0$ converges to u and $\mathcal{S}(t)v$ simultaneously. Thus, $u \in \mathcal{S}(t)\omega(u_0)$.

Finally, we apply (iii) to $Z = \omega(u_0)$. More precisely, suppose there exists $\epsilon > 0$ and a sequence $t_j \to \infty$ such that $dist(\mathcal{S}(t_j)u_0, X) \geq \epsilon > 0$. By compactness of $\mathcal{B}_R$, $\mathcal{S}(t_j)u_0$ has a subsequence converging to an element of $\omega(u_0)$, which by our assumption would lie outside X, contradicting the fact that $\omega(u_0) \subset X$.

(v) Assume that D_1 and D_2 are non-empty open (in H) disjoint sets such that $X \subset D_1 \cup D_2$. Assume $x_1 \in X \cap D_1, x_2 \in X \cap D_2$. Let $t > 0$. Then, there exists $y_1 = y_1(t), y_2 = y_2(t) \in \mathcal{B}_R$ such that $x_1 = \mathcal{S}(t)y_1, x_2 = \mathcal{S}(t)y_2$. Let γ be a straight line in $\mathcal{B}_R$ joining y_1 to y_2. Thus, $\mathcal{S}(t)\gamma$ is a continuous curve joining x_1 to x_2. Choose a point $x(t) = \mathcal{S}(t)y(t)$ on $\mathcal{S}(t)\gamma$ such that $x(t) \in F = H \setminus (D_1 \cup D_2)$. Note that F is closed in H and $F \cap X = \emptyset$. Since $\mathcal{S}(t)\mathcal{B}_R \subset \mathcal{B}_R$ for all $t \geq T$, we see that $x(t) \in \mathcal{B}_R$ for all $t \geq T$, and so there exists $t_j \to \infty$ such that $x(t_j)$ converges in H to some x. But $x(t_j) \in F$ and F is closed, so $x \in F$. We claim that $x \in X$, and this will contradict the fact that $F \cap X = \emptyset$. To prove our claim, let $s > 0$. Take the sequence $\mathcal{S}(t_j - s)y(t_j)$ for $t_j \geq s + T$. Since $y(t_j) \in \gamma \subset \mathcal{B}_R$, it follows that $\mathcal{S}(t_j - s)y(t_j) \in \mathcal{B}_R$ for $t_j \geq s + T$. Since $\mathcal{B}_R$ is compact, there exists $t_{j_k} \to \infty$ such that $\mathcal{S}(t_{j_k} - s)y(t_{j_k})$ converges to some $y \in \mathcal{B}_R$. Thus $\mathcal{S}(s)\mathcal{S}(t_{j_k} - s)y_{t_{j_k}}$ converges to $\mathcal{S}(s)y$ in H. But $\mathcal{S}(t_{j_k})y_{t_{j_k}}$ converges to x in H. Thus, $x = \mathcal{S}(s)y$ with $y \in \mathcal{B}_R$. Thus is true for any $s > 0$. Thus, we proved our claim.

Remark 1.4. *The attractor X is smooth, a fact that follows from the C^∞ regularity of the solutions to the forced Navier-Stokes system.*

Let Φ be a smooth function defined on an open set $\Omega \subset \mathbb{R}^N$, $N \geq 1$ and taking values in H. Let

$$\Sigma_t = \mathcal{S}(t)\Phi(\Omega). \tag{1.102}$$

The volume element in Σ_t is

$$\left| \frac{\partial}{\partial \alpha_1}(\mathcal{S}(t)\Phi(\alpha)) \wedge \cdots \wedge \frac{\partial}{\partial \alpha_N}(\mathcal{S}(t)\Phi(\alpha)) \right| d\alpha_1 \ldots d\alpha_N \tag{1.103}$$

where $d\alpha_1 \ldots d\alpha_N$ is the volume element in $\mathbb{R}^N$.

The functions

$$v_i(t) = \frac{\partial}{\partial \alpha_i}(\mathcal{S}(t)\Phi(\alpha)), i = 1, \ldots, N \tag{1.104}$$

satisfy the linearized equation

$$\partial_t v + B(v,u) + B(u,v) + Av = 0 \tag{1.105}$$

along $u(t) = \mathcal{S}(t)\Phi(\alpha)$. Thus, the time evolution of the volume element of an N-dimensional surface transported by $\mathcal{S}(t)$ is characterized by that of

$$\|v_1(t) \wedge \cdots \wedge v_N(t)\|_{\Lambda^N H} \tag{1.106}$$

where $\Lambda^N H$ is the N-th exterior product of H, and $v_1, \ldots, v_N$ satisfy (1.105) along some $u(t) = \mathcal{S}(t)u_0$.

Theorem 1.8. *(Decay of Volume Elements) There exists a time t_0 depending only on $\|u_0\|_{L^2}$ and an integer N_0 depending only on the body forces $\|f\|_{L^2}$ such that*

$$\|v_1(t) \wedge \cdots \wedge v_N(t)\|_H \leq \|v_1(0) \wedge \cdots \wedge v_N(0)\|_H e^{-CN^2 t} \tag{1.107}$$

for all $t \geq t_0$ and for all $N \geq N_0$.

Proof: We consider the operator

$$Tv := Av + B(u,v) + B(v,u). \tag{1.108}$$

defined on $H^2 \cap H$. The wedge product $v_1 \wedge \cdots \wedge v_N$ evolves according to

$$\partial_t(v_1 \wedge \cdots \wedge v_N) + T_N(v_1 \wedge \cdots \wedge v_N) = 0 \tag{1.109}$$

where T_N is the operator defined by

$$T_N = T \wedge I \wedge \cdots \wedge I + I \wedge T \wedge I \wedge \cdots \wedge I + I \wedge \cdots \wedge I \wedge T, \tag{1.110}$$

and I is the identity operator. Consequently, it holds that

$$\frac{1}{2}\frac{d}{dt}\|v_1 \wedge \cdots \wedge v_N\|_H^2 + Tr(TQ)\|v_1 \wedge \cdots \wedge v_N\|_H^2 = 0 \tag{1.111}$$

where $Q = Q(v_1, \ldots, v_N)$ is the orthogonal projector in H onto the space spanned by $v_1, \ldots, v_N$. An application of Gronwall's inequality yields the bound

$$\begin{aligned}&\|v_1(t) \wedge \cdots \wedge v_N(t)\|_H \\ &\qquad \le \|v_1(0) \wedge \cdots \wedge v_N(0)\|_H \exp\left\{-\int_0^t Tr(TQ(s))ds\right\}.\end{aligned} \tag{1.112}$$

For each $t > 0$, let $b_i, i = 1, \ldots, N$, be an orthonormal family of functions in H spanning the linear span of $v_1, \ldots, v_N$. Then, the trace of TQ is given by

$$Tr(TQ) = \sum_{i=1}^N (Tb_i, b_i)_{L^2} = \sum_{i=1}^N (Ab_i, b_i)_{L^2} + \sum_{i=1}^N (B(b_i, u), b_i)_{L^2}. \tag{1.113}$$

We note that

$$\sum_{i=1}^N (Ab_i, b_i)_{L^2} \ge \mu_1 + \cdots + \mu_N \tag{1.114}$$

where μ_i are the eigenvalues of A in H. Asymptotically, each eigenvalue μ_i satisfies $\mu_i \ge Ci$, and so the asymptotic behavior of their sum is described by

$$\mu_1 + \cdots + \mu_N \ge C(1 + \cdots + N) \ge CN^2. \tag{1.115}$$

On the other hand, we have

$$\left|\sum_{i=1}^{N}(B(b_i,u),b_i)_{L^2}\right| \le \sum_{i=1}^{N}\|b_i\|_{L^4}^2\|\nabla u\|_{L^2}$$
$$\le C\sum_{i=1}^{N}\|b_i\|_{L^2}\|\nabla b_i\|_{L^2}\|\nabla u\|_{L^2}$$
$$\le C\sum_{i=1}^{N}\|\nabla b_i\|_{L^2}\|\nabla u\|_{L^2}$$
$$\le C\|\nabla u\|_{L^2}N^{1/2}\left(\sum_{i=1}^{N}\|\nabla b_i\|_{L^2}^2\right)^{1/2}$$
$$\le C\|\nabla u\|_{L^2}^2 N + \frac{1}{4}\sum_{i=1}^{N}\|\nabla b_i\|_{L^2}^2$$
$$= C\|\nabla u\|_{L^2}^2 N + \frac{1}{4}\sum_{i=1}^{N}(Ab_i,b_i)_{L^2} \tag{1.116}$$

due to applications of Hölder, Ladyzhenskaya, and Young inequalities. Putting (1.113)–(1.116) together, and integrating in time from 0 to t, we infer that

$$\int_0^t Tr(TQ(s))ds \ge \frac{1}{2}\int_0^t\sum_{i=1}^{N}(Ab_i,b_i)_{L^2}ds - CN\int_0^t\|\nabla u\|_{L^2}^2 ds$$
$$\ge CNt\left(N - \frac{1}{t}\int_0^t\|\nabla u(s)\|_{L^2}^2 ds\right). \tag{1.117}$$

We choose a time $t_0 = t_0(\|u_0\|_{L^2}) > 0$ such that the estimate

$$\|\nabla u(t)\|_{L^2}^2 \le 2\|f\|_{L^2}^2 \tag{1.118}$$

holds for all $t \ge t_0$. Starting at t_0, the following bound

$$\int_0^t Tr(TQ(s))ds \ge CNt(N - 2\|f\|_{L^2}^2) \tag{1.119}$$

holds. Finally, we choose an integer $N_0 \ge 4\|f\|_{L^2}^2$ and conclude that

$$\int_0^t Tr(TQ(s))ds \ge CN^2 t \tag{1.120}$$

for all integers $N \ge N_0$ and all times $t \ge t_0$. This ends the proof of Theorem 1.8.

The exponential decay of volume elements carried by the flow yield the finiteness of the fractal dimension of the attractor:

Theorem 1.9. *The global attractor X has finite fractal dimension $D_H(X)$,*

$$D_H(X) = \limsup_{r \to 0} \frac{\log N_H(r)}{\log\left(\frac{1}{r}\right)} \tag{1.121}$$

where $N_H(r)$ is the minimal number of balls in H of radii r needed to cover X.

The proof is based on global Lyapunov exponents and the Kaplan-Yorke formula. We omit it here and refer the reader to [35, Chapter 14] for more details.

CHAPTER 2

Electroconvection in Fluids

We study a model of electroconvection in which a two dimensional viscous fluid caries electrical charges and interacts with them. The system has global solutions, but in general the solutions do not have bounded mean. Tracking the mean, we associate to each solution a mean zero frame and show that in the mean zero frame the system has a compact, finite dimensional global attractor. If the fluid is forced only by electrical forces and no other body forces are present, then the attractor reduces to one point.

2.1 Introduction

We consider an electroconvection model that describes the evolution of a surface charge density interacting with a two dimensional fluid. The model was used in theoretical and numerical studies related to experiments of electroconvection in thin smectic layers of liquid crystals [25, 41]. Analogies with Rayleigh-Bénard convection motivated the physical studies [42].

The surface charge density $q = q(x,t)$ is a real valued function of position x and time t. Its evolution is a continuity equation, with the current density J given by the sum of the Ohmic density σE, with E the electric field, and the advective current density qu, where u is the velocity of the fluid. Magnetic effects are neglected and the electric field E is the gradient of a potential. The restriction to a two dimensional region results in a nonlocal relation between the surface charge density and the divergence of the electrical field [13, 42, 41]. The evolution of the surface charge density is given by

$$\partial_t q + \nabla \cdot J = 0 \tag{2.1}$$

where the current density J is given by

$$J = \sigma E + qu \tag{2.2}$$

with σ a constant conductivity, and the electric field given by

$$E = -\nabla\Phi - \nabla\Lambda^{-1}q. \tag{2.3}$$

Here Φ is a given smooth function which represents the restriction to the surface of the potential due to the applied voltage, and $\Lambda^{-1}q$ (with Λ the square root of the two dimensional spatially periodic Laplacian) is the restriction to the surface of the potential due to the surface density charge q. The equation is coupled to the incompressible Navier-Stokes system

$$\partial_t u + u \cdot \nabla u + \nabla p - \Delta u = qE + f, \quad \nabla \cdot u = 0, \tag{2.4}$$

where f are body forces in the fluid. In this chapter, we consider two dimensional periodic boundary conditions. The potential Φ and forces f are time independent and smooth.

The global existence of regular solutions of this system with homogeneous Dirichlet boundary conditions was established in [13]. In this work we focus on long time dynamics. The long time dynamics of dissipative partial differential equations has been investigated by many authors. The two dimensional forced Navier-Stokes equations are known to possess global finite dimensional attractors ([14, 35], and references therein). The long time behavior of various types of dissipative PDE has been studied extensively [9, 21, 23, 32, 34]. Closer to the present system, the study of long time dynamics of the critical dissipative SQG system with fractional Laplacian dissipation and the existence of a finite dimensional global attractor were done in [19].

We investigate the system (2.1)–(2.4). This has weak solutions in L^2 (Theorem 2.1) which, however, are not known to be unique. After any positive time, weak solutions become strong, and strong solutions exist globally and are unique (Theorem 2.2). Our main result is the existence of a global attractor X which is compact in a natural phase space of strong solutions and has finite fractal dimension. In order to establish the existence of the attractor we need to account for the fact that spatial averages of the velocity are time dependent, and might grow in time, driven by the integral $\int q\nabla\Phi$. This integral does not vanish in general, nor is it time integrable. The remarkable property of the system is that the spatial average of velocity can be tracked, or "moded" out, and the resulting system has a compact global attractor. In this mean zero frame, the initial value problem for the system is solved by a nonlinear semigroup $S(t)$ which has a compact absorbing ball, is Lipschitz continuous in various norms, is injective, and high dimensional volume elements carried by its flow decay in phase space.

This chapter is organized as follows. In Section 2.2 we gather preliminaries concerning the dissipative operators. A lower bound, in the spirit of [19], Proposition 2.2, is proved in Section 2.2. Commutator estimates for positive and negative fractional powers of the Laplacian (Proposition 2.3) are also proved in this section. Section 2.3 is devoted to basic PDE results: existence of weak solutions, existence and uniqueness of strong solutions. Here we also prove uniform long time bounds for various norms of the solutions, which have the feature that the initial data contributions to them decay exponentially, leaving only contributions coming from the steady forces. The passage to the mean zero frame is described in Section 2.4. The absorbing ball for the nonlinear semigroup is described in Section 2.5. In Section 2.6 continuity properties of the semigroup are established, and Section 2.7 is devoted to the proof of backward uniqueness. Decay of volume elements is proved in Section 2.8. In Section 2.9 we prove the finite dimensionality of the attractor for general fluid body forces f. We also show that in the absence of body forces in the fluid, the system has a unique globally attracting steady solution in the mean zero frame. In this case, in the original variables, the fluid's spatial average velocity has a finite limit in infinite time.

2.2 Preliminaries

We denote functions spaces of spatially periodic functions on the torus without distinct notation for vector valued functions. We write the Fourier series for mean zero velocities u as

$$u = \sum_{j \in \mathbb{Z}^2 \setminus \{0\}} u_j e^{ij \cdot x} \tag{2.5}$$

with $u_j \in \mathbb{C}^2$. The reality condition for the series is $\overline{u}_j = u_{-j}$. The divergence-free condition is

$$j \cdot u_j = 0. \tag{2.6}$$

For $s \in \mathbb{R}$, the fractional Laplacian Λ^s applied to a mean zero scalar function q is defined as a Fourier multiplier with symbol $|k|^s$, that is, for q given by

$$q = \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} q_k e^{ik \cdot x}, \tag{2.7}$$

we have that

$$\Lambda^s q = \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} |k|^s q_k e^{ik \cdot x}. \tag{2.8}$$

We consider the Hilbert space $\mathcal{H}$

$$\mathcal{H} = H \oplus L^2 \tag{2.9}$$

where H is the Hilbert space of L^2 periodic vector fields which are mean zero and divergence-free, $H = \mathbb{P}(L^2)$. The scalar product in $\mathcal{H}$ is denoted $(\cdot\,;\,\cdot)$:

$$((u_1, q_1); (u_2, q_2)) = \int_{\mathbb{T}^2} (u_1 \cdot u_2 + q_1 q_2) dx. \tag{2.10}$$

As all spatial integrals are on $\mathbb{T}^2$, we denote them simply by $\int$. We consider the operator $\mathcal{A}$ defined on $\mathcal{H}$ by

$$\mathcal{A}w = (Au, \Lambda q) \tag{2.11}$$

where $w = (u, q)$ and $A = -\mathbb{P}\Delta$ is the Stokes operator. The domain of definition of $\mathcal{A}$ is

$$\mathcal{D}(\mathcal{A}) = (H^2 \cap H) \oplus H^1. \tag{2.12}$$

The operator

$$\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H} \tag{2.13}$$

is positive and selfadjoint. There is an orthonormal basis of the Hilbert space $\mathcal{H}$ formed by a sequence w_k of eigenvectors,

$$\mathcal{A}w_k = \mu_k w_k. \tag{2.14}$$

The set of eigenvalues is precisely the union of the eigenvalues of A and those of Λ, counted with their multiplicities. The multiplicity of an eigenvalue λ of A is the same as the multiplicity of the same eigenvalue λ considered as an eigenvalue of the scalar Laplacian with periodic boundary conditions on $[0, 2\pi] \times [0, 2\pi]$. This follows from the fact that in two dimensions we can uniquely associate a stream function to each eigenfunction of the Stokes operator A. It can be shown that the eigenvalues μ_k obey $0 < \mu_1 \leq \ldots \mu_k \leq \ldots$ and that there exists a constant C_0 such that

$$\mu_k \geq C_0 \mu_1 \sqrt{k} \tag{2.15}$$

holds for all $k \geq 1$. If we denote the eigenvalues of A counted with multiplicity by $0 < \lambda_1 \leq \lambda_2 \cdots \leq \lambda_j \leq \ldots$ and those of Λ, counted also with multiplicity as $0 < r_1 \leq r_2 \cdots \leq r_j \leq \ldots$ then we have $j \leq c_1 \lambda_j$ and $k \leq c_2 r_k^2$ with c_1, c_2 positive constants. Assuming that

$$\{\mu_i \mid \ i = 1, \ldots N\} = \{\lambda_i \mid \ i = 1, \ldots, j\} \cup \{r_i \mid \ i = 1, \ldots, k\}$$

if $\mu_N = \lambda_j$ it follows that $j \leq c_1 \mu_N$ and if $\mu_N = r_k$ it follows that $k \leq c_2 \mu_N^2$. Because $N = j + k$ it follows that $N \leq c_1 \mu_N + c_2 \mu_N^2 \leq (c_1 + c_2)\mu_N^2$ because $\mu_N \geq 1$, and thus (2.15) follows.

We recall that the Riesz transforms $R = (R_1, R_2)$ for periodic functions are defined as multipliers

$$(R_j q)_k = i \frac{k_j}{|k|} q_k, \quad k \in \mathbb{Z}^2 \setminus \{0\}, \quad j = 1, 2, \tag{2.16}$$

and they are bounded operators in L^p, $1 < p < \infty$.

The fractional Laplacian has certain lower bounds in L^p spaces which we are going to use. A Poincaré inequality in L^p spaces is given in [19] in the following proposition

Proposition 2.1. *Let* $p = 2m$, $m \geq 1$, $0 \leq \alpha \leq 2$, *and let* $q \in C^\infty$ *have zero mean on* $\mathbb{T}^2$. *Then*

$$\int_{\mathbb{T}^2} q^{p-1}(x) \Lambda^\alpha q(x) dx \geq \frac{1}{p} \|\Lambda^{\alpha/2}(q^{p/2})\|_{L^2}^2 + \lambda \|q\|_{L^p}^p \tag{2.17}$$

holds, with an explicit constant $\lambda > 0$, *which is independent of* p.

Proposition 2.2. *The inequality*

$$\int \nabla q \cdot \Lambda \nabla q dx \geq c \|q\|_{L^4}^{-\frac{2}{3}} \|\nabla q\|_{L^{\frac{8}{3}}}^{\frac{8}{3}} \tag{2.18}$$

holds for $q \in H^{\frac{3}{2}}$.

Proof: This inequality is based on [19]. We recall the pointwise identity ([19])

$$\nabla q(x) \cdot \Lambda \nabla q(x) = \frac{1}{2}\Lambda(|\nabla q|^2)(x) + \frac{1}{2}D[q](x) \tag{2.19}$$

where

$$D[q](x) = cP.V. \int_{\mathbb{R}^2} \frac{|\nabla q(x) - \nabla q(x+y)|^2}{|y|^3} dy, \tag{2.20}$$

with c a universal constant. We abused notation and wrote q for the periodic extension of q, as a function defined on all $\mathbb{R}^2$.

We consider a cutoff function $\Psi : [0,\infty) \to [0,\infty)$, which is smooth, non-decreasing, identically 1 on $[2,\infty)$, vanishes on $[0,1]$ and obeys $|\Psi'| \le 3$.

For $l > 0$ to be determined, we have

$$\begin{aligned}
D[q](x) &\ge c \int_{\mathbb{R}^2} \frac{|\nabla q(x) - \nabla q(x+y)|^2}{|y|^3} \Psi\left(\frac{|y|}{l}\right) dy \\
&\ge c \int_{\mathbb{R}^2} \frac{|\nabla q(x)|^2 - 2\nabla q(x) \cdot \nabla q(x+y)}{|y|^3} \Psi\left(\frac{|y|}{l}\right) dy \\
&\ge c|\nabla q(x)|^2 \int_{|y|\ge l} \frac{1}{|y|^3} dy - 2c \sum_{j=1}^{2} \left| \int_{\mathbb{R}^2} \frac{\partial_j q(x) \partial_j q(x+y)}{|y|^3} \Psi\left(\frac{|y|}{l}\right) dy \right| \\
&\ge c_1 \frac{|\nabla q(x)|^2}{l} - c_2 |\nabla q(x)| \sum_{j=1}^{2} \int_{\mathbb{R}^2} |q(x+y)| \left| \nabla\left(\frac{1}{|y|^3}\Psi\left(\frac{|y|}{l}\right)\right)\right| dy.
\end{aligned}$$

Now

$$\begin{aligned}
&\int_{\mathbb{R}^2} |q(x+y)| \left|\nabla\left(\tfrac{1}{|y|^3}\Psi\left(\tfrac{|y|}{l}\right)\right)\right| dy \\
&\quad = \textstyle\sum_{j\in\mathbb{Z}^2} \int_{Q_0+2\pi j} |q(x+y)| \left|\nabla\left(\tfrac{1}{|y|^3}\Psi\left(\tfrac{|y|}{l}\right)\right)\right| dy \\
&\quad = \textstyle\sum_{j\in\mathbb{Z}^2} \int_{Q_0} |q(x+y)| \left|\nabla\left(\tfrac{1}{|y-2\pi j|^3}\Psi\left(\tfrac{|y-2\pi j|}{l}\right)\right)\right| dy \le k(l)\|q\|_{L^4},
\end{aligned} \tag{2.21}$$

where $Q_0 = [-\pi, \pi] \times [-\pi, \pi]$ and

$$k(l) = \sum_{j \in \mathbb{Z}^2} \left[\int_{Q_0} \left| \nabla \left(\frac{1}{|y - 2\pi j|^3} \Psi \left(\frac{|y - 2\pi j|}{l} \right) \right) \right|^{\frac{4}{3}} dy \right]^{\frac{3}{4}}. \tag{2.22}$$

The contribution of the term corresponding to $j = 0$ in the sum is of the order $l^{-\frac{5}{2}}$ for small l and $\Psi = 1$ for $j \neq 0$ and $l < \frac{\pi}{4}$. We obtain

$$k(l) \leq C(l^{-\frac{5}{2}} + 1), \tag{2.23}$$

for all $0 < l < \frac{\pi}{4}$, and hence have from (2.21)

$$D[q](x) \geq |\nabla q(x)|(c_1 l^{-1} |\nabla q(x)| - c_2 k(l)) \|q\|_{L^4}). \tag{2.24}$$

We may choose

$$l = \min \left\{ \left(2 \frac{C c_2 \|q\|_{L^4}}{c_1 |\nabla q(x)|} \right)^{2/3}, \frac{\pi}{4} \right\} \tag{2.25}$$

and deduce the pointwise inequality

$$D[q](x) \geq \frac{C_1}{20} \|q\|_{L^4}^{-\frac{2}{3}} |\nabla q(x)|^{\frac{8}{3}} - c_3 \|q\|_{L^4}^2 \tag{2.26}$$

with c_3 a positive absolute constant. Indeed, if $\left(\frac{2C c_2 \|q\|_{L^4}}{c_1 |\nabla q(x)|} \right)^{2/3} \leq \frac{\pi}{4}$, then $l = \left(\frac{2C c_2 \|q\|_{L^4}}{c_1 |\nabla q(x)|} \right)^{2/3}$. In this case, (2.24) implies that

$$\begin{aligned} D[q](x) \geq c_1 |\nabla q(x)|^2 & \left[\frac{c_1^{2/3} |\nabla q(x)|^{2/3}}{2^{2/3} C^{2/3} c_2^{2/3} \|q\|_{L^4}^{2/3}} \right] \\ & - C c_2 |\nabla q(x)| \|q\|_{L^4} \left[\frac{c_1^{5/3} |\nabla q(x)|^{5/3}}{2^{5/3} C^{5/3} c_2^{5/3} \|q\|_{L^4}^{5/3}} \right] - C c_2 |\nabla q(x)| \|q\|_{L^4} \\ = & \left(\frac{1}{2^{2/3}} - \frac{1}{2^{5/3}} \right) \frac{c_1^{5/3}}{C^{2/3} c_2^{2/3}} |\nabla q(x)|^{8/3} \|q\|_{L^4}^{-2/3} - C c_2 |\nabla q(x)| \|q\|_{L^4} \\ \geq & \frac{1}{4} \frac{c_1^{5/3}}{C^{2/3} c_2^{2/3}} |\nabla q(x)|^{8/3} \|q\|_{L^4}^{-2/3} - C c_2 |\nabla q(x)| \|q\|_{L^4}. \end{aligned}$$

Let

$$C_1 = \frac{c_1^{5/3}}{C^{2/3} c_2^{2/3}}.$$

Since

$$\frac{2Cc_2\|q\|_{L^4}}{c_1|\nabla q(x)|} \le \left(\frac{\pi}{4}\right)^{3/2},$$

then

$$\begin{aligned} Cc_2|\nabla q(x)|\|q\|_{L^4} &= Cc_2|\nabla q(x)|\|q\|_{L^4}^{-2/3}\|q\|_{L^4}^{5/3} \\ &\le Cc_2|\nabla q(x)|\|q\|_{L^4}^{-2/3}\left(\frac{\pi^{\frac{3}{2}}c_1|\nabla q(x)|}{4^{\frac{3}{2}}2Cc_2}\right)^{5/3} \\ &= \frac{\pi^{\frac{5}{2}}}{2^{\frac{20}{3}}}\frac{c_1^{5/3}}{C^{2/3}c_2^{2/3}}|\nabla q(x)|^{8/3}\|q\|_{L^4}^{-2/3} \end{aligned}$$

and so

$$\begin{aligned} D[q](x) &\ge \left(\frac{1}{4} - \frac{\pi^{\frac{5}{2}}}{2^{\frac{20}{3}}}\right)C_1|\nabla q(x)|^{8/3}\|q\|_{L^4}^{-2/3} \\ &\ge \frac{1}{20}C_1|\nabla q(x)|^{8/3}\|q\|_{L^4}^{-2/3} \\ &\ge \frac{C_1}{20}|\nabla q(x)|^{8/3}\|q\|_{L^4}^{-2/3} - c_3\|q\|_{L^4}^2. \end{aligned}$$

On the other hand, if the opposite inequality $\left(\frac{2Cc_2\|q\|_{L^4}}{c_1|\nabla q(x)|}\right)^{\frac{2}{3}} \ge \frac{\pi}{4}$ holds, then $l = \frac{\pi}{4}$, and (2.24) implies that

$$\begin{aligned} D[q](x) &\ge c_1|\nabla q(x)|^2\left[\frac{c_1^{2/3}|\nabla q(x)|^{2/3}}{2^{2/3}C^{2/3}c_2^{2/3}\|q\|_{L^4}^{2/3}}\right] - Cc_2\left(\left(\frac{\pi}{4}\right)^{-5/2}+1\right)|\nabla q(x)|\|q\|_{L^4} \\ &= \frac{1}{2^{2/3}}\frac{c_1^{5/3}}{C^{2/3}c_2^{2/3}}|\nabla q(x)|^{8/3}\|q\|_{L^4}^{-2/3} - C'c_2|\nabla q(x)|\|q\|_{L^4} \\ &\ge \frac{C_1}{2}|\nabla q(x)|^{8/3}\|q\|_{L^4}^{-2/3} - C'c_2|\nabla q(x)|\|q\|_{L^4}. \end{aligned}$$

This gives the desired estimate (2.26).

Integrating (2.26) over $\mathbb{T}^2$, we obtain

$$c_4\|q\|_{L^4}^2 + \int \nabla q\Lambda\nabla q \ge \frac{C_1}{20}\|q\|_{L^4}^{-\frac{2}{3}}\|\nabla q\|_{L^{\frac{8}{3}}}^{\frac{8}{3}}. \tag{2.27}$$

We also know that

$$\int \nabla q \Lambda \nabla q \geq c_5 \|q\|_{L^4}^2 \tag{2.28}$$

and therefore

$$\int \nabla q \Lambda \nabla q \geq c_6 \|q\|_{L^4}^{-\frac{2}{3}} \|\nabla q\|_{L^{\frac{8}{3}}}^{\frac{8}{3}} \tag{2.29}$$

follows with $c_6 = \frac{C_1}{20(1+\frac{c_4}{c_5})}$, and thus (2.18) holds.

The following commutator estimates are needed in the sequel.

Proposition 2.3. *Let* $u \in H^2 \cap H$ *and* $q \in H^{s+\alpha}$. *Let* $s \in (-1, 1)$ *and let* $0 \leq \alpha \leq 1$ *with* $s + \alpha \leq 1$. *Then the commutator* $[\Lambda^s, u \cdot \nabla]$ *obeys the inequality*

$$\|[\Lambda^s, u \cdot \nabla]\, q\|_{L^2} \leq C_s [u]_{1-\alpha} \|\Lambda^{s+\alpha} q\|_{L^2} \tag{2.30}$$

where

$$[u]_{1-\alpha} = \sum_{j \in \mathbb{Z} \setminus \{0\}} |j|^{1-\alpha} |u_j|. \tag{2.31}$$

Proof: The function $\phi = [\Lambda^s, u \cdot \nabla]\, q$ has the Fourier expansion

$$\phi_l = i \sum_{j+k=l} (u_j \cdot k) q_k (|l|^s - |k|^s). \tag{2.32}$$

In view of the fact that $u_j \cdot j = 0$ we have $u_j \cdot k = -u_j \cdot l$ and therefore

$$|u_j \cdot k| \leq |u_j| \min\{|l|, |k|\}.$$

If s is negative then we write

$$|l|^{-r} - |k|^{-r} = \frac{|k|^r - |l|^r}{|l|^r |k|^r}$$

with $r = |s|$, and we estimate for positive numbers $m \leq M$ and exponent $0 \leq r \leq 1$ using the conjugate powers:

$$(M^r - m^r)(M^{1-r} + m^{1-r}) = M - m + M^r m^{1-r} - m^r M^{1-r} \leq 2(M - m).$$

We denote by $M = \max\{|l|, |k|\}$, $m = \min\{|l|, |k|\}$. For $s < 0$, using the triangle inequality $M - m \le |j| \le 2M$, we obtain that

$$\frac{2m|j|^{\alpha}}{M^r m^r (M^{1-r} + m^{1-r})} \le \frac{2^{1+\alpha} m^{1-r} M^{\alpha}}{M} = 2^{1+\alpha} m^{-r+\alpha} \left(\frac{m}{M}\right)^{1-\alpha}$$
$$= 2^{1+\alpha} M^{-r+\alpha} \left(\frac{m}{M}\right)^{(1-r)} \le 2^{1+\alpha} |k|^{-r+\alpha}$$

and therefore

$$|(u_j \cdot k) q_k (|l|^s - |k|^s)| \le \frac{2m|j|}{M^r m^r (M^{1-r} + m^{1-r})} |u_j||q_k| \le 2^{1+\alpha} |j|^{1-\alpha} |k|^{s+\alpha} |u_j||q_k|.$$

Similarly for $s > 0$ we obtain with $s = r$

$$\frac{2m|j|^{\alpha}}{M^{1-r} + m^{1-r}} \le 2^{1+\alpha} m M^{\alpha+r-1} = 2^{1+\alpha} m^{r+\alpha} \left(\frac{m}{M}\right)^{1-r-\alpha} \le 2^{1+\alpha} |k|^{s+\alpha}$$

and thus

$$|(u_j \cdot k) q_k (|l|^s - |k|^s)| \le \frac{2m|j|}{M^{1-r} + m^{1-r}} |u_j||q_k| \le 2^{1+\alpha} |j|^{1-\alpha} |k|^{s+\alpha} |u_j||q_k|.$$

The proof is concluded by noting that the $\ell_2(\mathbb{Z}^2)$ norm of the sequence ϕ_l is bounded by the product of the $\ell_1(\mathbb{Z}^2)$ norm of the sequence $|j|^{1-\alpha}|u_j|$ and the $\ell_2(\mathbb{Z}^2)$ norm of the sequence $|k|^{s+\alpha}|q_k|$.

2.3 Existence and Uniqueness of Solutions

We consider the system

$$\begin{cases} \partial_t q + u \cdot \nabla q + \Lambda q = \Delta \Phi \\ \partial_t u + u \cdot \nabla u - \Delta u + \nabla p = -qRq - q\nabla\Phi + f \\ \nabla \cdot u = 0. \end{cases} \tag{2.33}$$

The unknowns u, q are periodic in space. We consider smooth, mean zero, divergence-free body forces f, and smooth potential Φ. The body forces and the potential are time independent. We discuss first a class of weak solutions. The equations (2.33) are meant in distribution sense, assuming that $q \in L^\infty(0,T;L^2)$ and u is divergence-free and belongs to $L^\infty(0,T;L^2)$.

Theorem 2.1. Weak solutions. *Let $u_0 \in L^2$ be divergence-free, let $q_0 \in L^2$ with $\int q_0 = 0$, and let $T > 0$ be arbitrary. There exists a weak solution (u,q) of the system (2.33) satisfying $u \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1)$ and $q \in L^\infty(0,T;L^2) \cap L^2(0,T;H^{\frac{1}{2}})$. Moreover the following inequalities hold a.e. in $0 \le t \le T$,*

$$\|q(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{\frac{1}{2}} q\|_{L^2}^2 \le \|q_0\|_{L^2}^2 + \int_0^t \|\Lambda^{\frac{3}{2}}\Phi\|_{L^2}^2, \tag{2.34}$$

$$\|q(t)\|_{L^2} \le \|q_0\|_{L^2} e^{-\lambda t} + \frac{1}{\lambda}\|\Delta\Phi\|_{L^2}, \tag{2.35}$$

and

$$\begin{aligned} &\|u(t)\|_{L^2}^2 + \|\Lambda^{-\frac{1}{2}}(q(t)-Q)\|_{L^2}^2 + \int_0^t \left(\|q(s)-Q\|_{L^2}^2 + \|\nabla u(s)\|_{L^2}^2\right) ds \\ &\le \|u_0\|_{L^2}^2 + \|\Lambda^{-\frac{1}{2}}(q_0 - Q)\|_{L^2}^2 + \int_0^t \|\Lambda^{-1} f\|_{L^2}^2 dt, \end{aligned} \tag{2.36}$$

where Q is defined by

$$Q = -\Lambda\Phi. \tag{2.37}$$

Furthermore,

$$t\|q(t)\|_{L^4}^2 \le C\|q_0\|_{L^2}^2 + C\int_0^T \|\Lambda^{\frac{3}{2}}\Phi\|_{L^2}^2 ds + \frac{1}{\lambda}\int_0^T s\|\Delta\Phi\|_{L^4}^2 ds \tag{2.38}$$

and

$$
\begin{aligned}
&t\|\nabla u(t)\|_{L^2}^2 \le \|u_0\|_{L^2}^2 + \|\Lambda^{-\frac{1}{2}}(q_0 - Q)\|_{L^2}^2 + \int_0^T \|\Lambda^{-1} f\|_{L^2}^2 ds \\
&+ C\int_0^T s(\|f\|_{L^2}^2 + \|Q\|_{L^4}^4) ds + C\|q_0\|_{L^2}^2 \left(\|q_0\|_{L^2}^2 + \int_0^t \|\Lambda^{\frac{3}{2}}\Phi\|_{L^2}^2 ds\right) \\
&+ C\left(\int_0^T \|\Lambda^{\frac{3}{2}}\Phi\|_{L^2}^2 ds\right)\left(\|q_0\|_{L^2}^2 + \int_0^t \|\Lambda^{\frac{3}{2}}\Phi\|_{L^2}^2 ds\right) \\
&+ C\left(\frac{1}{\lambda}\int_0^T s\|\Delta\Phi\|_{L^4}^2 ds\right)\left(\|q_0\|_{L^2}^2 + \int_0^t \|\Lambda^{\frac{3}{2}}\Phi\|_{L^2}^2 ds\right) \qquad (2.39)
\end{aligned}
$$

hold t*-a.e. in* $[0, T]$.

Proof: We consider a viscous approximation of the system with smoothed out initial data. For $0 < \epsilon \le 1$, we let J_ϵ be a standard mollifier operator, and we consider the system

$$
\begin{cases}
\partial_t q^\epsilon + u^\epsilon \cdot \nabla q^\epsilon + \Lambda q^\epsilon - \epsilon\Delta q^\epsilon = \Delta\Phi \\
\partial_t u^\epsilon + u^\epsilon \cdot \nabla u^\epsilon - \Delta u^\epsilon + \nabla p^\epsilon = -q^\epsilon R q^\epsilon - q^\epsilon \nabla\Phi + f, \\
\nabla \cdot u^\epsilon = 0
\end{cases} \qquad (2.40)
$$

with $q_0^\epsilon = J_\epsilon q_0, u_0^\epsilon = J_\epsilon u_0$. For fixed positive ϵ this system has global smooth solutions for $t > 0$, a fact that can be proved using a number of different methods. We provide a priori bounds and pass to the limit $\epsilon \to 0$.

We note that the mean of q^ϵ is zero, and therefore we can use the Poincaré inequality (2.17). Multiplying the first equation of system (2.40) by $(q^\epsilon)^{p-1}$, with $p \ge 2$ even, and integrating, we obtain, by using u^ϵ is divergence-free, the non-negativity of the integral involving the Laplacian, (2.17), and a Hölder inequality that

$$\frac{1}{p}\partial_t \|q^\epsilon\|_{L^p}^p + \lambda \|q^\epsilon\|_{L^p}^p \le \left| \int \Delta\Phi (q^\epsilon)^{p-1} dx \right| \le \|q^\epsilon\|_{L^p}^{p-1} \|\Delta\Phi\|_{L^p}. \tag{2.41}$$

Thus the L^p norms of q^ϵ obey diffferential inequalities

$$\partial_t \|q^\epsilon\|_{L^p} + \lambda \|q^\epsilon\|_{L^p} \le \|\Delta\Phi\|_{L^p}. \tag{2.42}$$

The $L^2(0,T;H^{\frac{1}{2}})$ norm of q^ϵ is bounded using

$$\frac{1}{2}\frac{d}{dt}\|q^\epsilon\|_{L^2}^2 + \int q^\epsilon \Lambda q^\epsilon \le \|\Lambda^{\frac{3}{2}}\Phi\|_{L^2} \|\Lambda^{\frac{1}{2}} q^\epsilon\|_{L^2}$$

and integrating in time, leading to

$$\|q^\epsilon(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{\frac{1}{2}} q^\epsilon\|_{L^2}^2 \le \|q_0\|_{L^2}^2 + \int_0^t \|\Lambda^{\frac{3}{2}}\Phi\|_{L^2}^2. \tag{2.43}$$

A cancellation is used to obtain bounds for u^ϵ in L^2. We take the scalar product in L^2 with u^ϵ in the second equation, and in the first equation we mutiply by $\Lambda^{-1}(q^\epsilon - Q)$ and integrate. We obtain

$$\frac{1}{2}\frac{d}{dt}\|u^\epsilon\|_{L^2}^2 + \|\nabla u^\epsilon\|_{L^2}^2 \le \int f \cdot u^\epsilon - \int q^\epsilon u^\epsilon \cdot R(q^\epsilon - Q)$$

and

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^{-\frac{1}{2}}(q^\epsilon - Q)\|_{L^2}^2 + \|(q^\epsilon - Q)\|_{L^2}^2 \le \int q^\epsilon u^\epsilon \cdot R(q^\epsilon - Q) + \epsilon \int q^\epsilon \Lambda Q.$$

Adding we obtain

$$\frac{1}{2}\frac{d}{dt}\left[\|u^\epsilon\|_{L^2}^2 + \|\Lambda^{-\frac{1}{2}}(q^\epsilon - Q)\|_{L^2}^2\right] + \|\nabla u^\epsilon\|_{L^2}^2 + \|(q^\epsilon - Q)\|_{L^2}^2$$
$$\le \|\Lambda^{-1} f\|_{L^2} \|\nabla u^\epsilon\|_{L^2} + \epsilon \int q^\epsilon \Lambda Q \tag{2.44}$$

and consequently

$$\|u^\epsilon(t)\|_{L^2}^2 + \|\Lambda^{-\frac{1}{2}}(q^\epsilon(t) - Q)\|_{L^2}^2 + \int_0^t \left(\|\nabla u^\epsilon\|_{L^2}^2 + \|(q^\epsilon - Q)\|_{L^2}^2\right) ds \le \|u_0\|_{L^2}^2$$
$$+ \|\Lambda^{-\frac{1}{2}}(q_0 - Q)\|_{L^2}^2 + \int_0^t \left(\|\Lambda^{-1} f\|_{L^2}^2 + \epsilon^2 \|\Lambda Q\|_{L^2}^2 + 2\epsilon \|\Lambda^{\frac{1}{2}} Q\|_{L^2}^2\right) ds. \tag{2.45}$$

Now, from (2.42) we deduce

$$\frac{d}{dt}t\|q^\epsilon(t)\|_{L^4}^2 + \lambda t\|q^\epsilon\|_{L^4}^2 \le t\frac{1}{\lambda}\|\Delta\Phi\|_{L^4}^2 + \|q^\epsilon(t)\|_{L^4}^2, \tag{2.46}$$

and in view of the embedding $H^{\frac{1}{2}} \subset L^4$ and (2.43) we deduce

$$t\|q^\epsilon(t)\|_{L^4}^2 \le C\|q_0\|_{L^2}^2 + C\int_0^t \|\Lambda^{\frac{3}{2}}\Phi\|_{L^2}^2 ds + \frac{1}{\lambda}\int_0^t se^{-\lambda(t-s)}\|\Delta\Phi\|_{L^4}^2 ds. \tag{2.47}$$

We take the second equation of (2.40), multiply by $-\Delta u^\epsilon$ and integrate in space. We use the identity

$$Tr(M^T M^2) = 0,$$

valid for any two-by-two traceless matrix M, which follows because M^2 is a multiple of the identity matrix. We use this identity in our case for a matrix M with entries $M_{ij} = \frac{\partial u_i^\epsilon}{\partial x_j}$, and obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla u^\epsilon\|_{L^2}^2 + \|\Delta u^\epsilon\|_{L^2}^2 = \int [f - q^\epsilon R(q^\epsilon - Q)] \cdot (-\Delta u^\epsilon) \tag{2.48}$$

and thus

$$\frac{d}{dt}\|\nabla u^\epsilon\|_{L^2}^2 + \|\Delta u^\epsilon\|_{L^2}^2 \le \|f - q^\epsilon R(q^\epsilon - Q)\|_{L^2}^2. \tag{2.49}$$

We multiply by t and integrate in time

$$\begin{aligned} t\|\nabla u^\epsilon(t)\|_{L^2}^2 &+ \int_0^t s\|\Delta u^\epsilon\|^2 ds \\ &\le \int_0^t \|\nabla u^\epsilon(s)\|_{L^2}^2 ds + C\int_0^t s\left(\|f\|_{L^2}^2 + \|q^\epsilon(s)\|_{L^4}^4 + \|Q\|_{L^4}^4\right) ds. \end{aligned} \tag{2.50}$$

In view of (2.43), (2.45) and (2.47) we obtain

$$
\begin{aligned}
t\|\nabla u^\epsilon(t)\|_{L^2}^2 + \int_0^t s\|\Delta u^\epsilon\|^2 ds \le \|u_0\|_{L^2}^2 + \|\Lambda^{-\frac{1}{2}}(q_0 - Q)\|_{L^2}^2 \\
+ \int_0^t \left(\|\Lambda^{-1} f\|_{L^2}^2 + \epsilon^2\|\Lambda Q\|_{L^2}^2 + 2\epsilon\|\Lambda^{\frac{1}{2}} Q\|_{L^2}^2\right) ds \\
+ C\int_0^t s(\|f\|_{L^2}^2 + \|Q\|_{L^4}^4) ds \\
+ C\|q_0\|_{L^2}^2 \left(\|q_0\|_{L^2}^2 + \int_0^t \|\Lambda^{\frac{3}{2}}\Phi\|_{L^2}^2 ds\right) \\
+ C\left(\int_0^T \|\Lambda^{\frac{3}{2}}\Phi\|_{L^2}^2 ds\right)\left(\|q_0\|_{L^2}^2 + \int_0^t \|\Lambda^{\frac{3}{2}}\Phi\|_{L^2}^2 ds\right) \\
+ C\left(\int_0^T s e^{-\lambda(T-s)} \frac{1}{\lambda}\|\Delta\Phi\|_{L^4}^2 ds\right)\left(\|q_0\|_{L^2}^2 + \int_0^t \|\Lambda^{\frac{3}{2}}\Phi\|_{L^2}^2 ds\right). \quad (2.51)
\end{aligned}
$$

These inequalities are used to pass to the limit. From (2.43) and (2.45) it folllows that q^ϵ is bounded in $L^2(0,T;H^{\frac{1}{2}})$ and u^ϵ is bounded in $L^2(0,T;H^1)$ on any sequence $\epsilon \to 0$. The equation (2.40) and the Aubin-Lions lemma imply that there exist $q \in L^2(0,T;H^{\frac{1}{2}})$ and $u \in L^2(0,T;H^1)$ such that

$$
\lim_{\epsilon\to 0}\int_0^T \left(\|u^\epsilon(t) - u(t)\|_{L^2}^2 + \|q^\epsilon(t) - q(t)\|_{L^2}^2\right) dt = 0, \quad (2.52)
$$

and, without loss of generality,

$$
\lim_{\epsilon\to 0}\left(\|u^\epsilon(t) - u(t)\|_{L^2}^2 + \|q^\epsilon(t) - q(t)\|_{L^2}^2\right) = 0, \quad t-\text{a.e. in } [0,T]. \quad (2.53)
$$

At each t where $q^\epsilon(t) \to q(t)$ strongly in L^2 it follows that $q^\epsilon(t)$ converges weakly to $q(t)$ in L^4, and therefore, by the weak lower semicontinuity of the L^4 norm, we have

$$
\|q(t)\|_{L^4} \le \liminf_{\epsilon\to 0}\|q^\epsilon(t)\|_{L^4}, \quad t-\text{a.e. in } [0,T]. \quad (2.54)
$$

Similarly, at any t where $u^\epsilon(t)$ converges strongly in L^2 to $u^\epsilon(t)$, the gradient $\nabla u^\epsilon(t)$ converges weakly in L^2 to ∇u. Therefore, by the weak lower semicontinuity of the

L^2 norm

$$\|\nabla u(t)\|_{L^2}^2 \leq \liminf_{\epsilon\to 0} \|\nabla u^\epsilon(t)\|_{L^2}, \quad t-\text{a.e. in } [0,T]. \tag{2.55}$$

The inequalities (2.47) and (2.51) thus yield (2.38) and (2.39) in the limit $\epsilon \to 0$. The fact that q and u obtained in the limit solve weakly the system (2.33) follows by testing the system (2.33) by test functions and passing to the limit. The proof of Theorem 2.1 is complete.

Remark 2.1. *Weak solutions are not known to be unique. The inequalities (2.38) and (2.39) show that for any $t_0 > 0$ the weak solutions become more regular, $u(t_0) \in H^1$, $q(t_0) \in L^4$ with quantitative bounds. This level of regularity generates strong solutions which are unique, as shown in the next theorem.*

Theorem 2.2. Strong solutions. *Let $u_0 \in H^1$ be divergence-free, let $q_0 \in L^4$ have mean zero, and let T be arbitrary. There exists a unique solution (u,q) of the system (2.33) with initial data (u_0, q_0) such that $u \in L^\infty(0,T;H^1) \cap L^2(0,T;H^2)$ is divergence-free and $q \in L^\infty(0,T;L^4) \cap L^2(0,T;H^{\frac{1}{2}})$. Moreover,*

$$\|q(t)\|_{L^4} \leq \|q(0)\|_{L^4} e^{-\lambda t} + \frac{1}{\lambda}\|\Delta\Phi\|_{L^4}, \tag{2.56}$$

$$\|\nabla u(t)\|_{L^2}^2 \leq \|\nabla u_0\|_{L^2}^2 e^{-t} + C_\gamma \|q_0\|_{L^4}^4 e^{-\gamma t} + C_\lambda\left(\|f\|_{L^2}^2 + \|Q\|_{L^4}^4 + \|\Delta\Phi\|_{L^4}^4\right), \tag{2.57}$$

with $0 < \gamma < \min\{1,\ 4\lambda\}$, and

$$\int_0^T \|\Delta u\|_{L^2}^2 \leq \|\nabla u_0\|_{L^2}^2 + C_\gamma \|q_0\|_{L^4}^4 + C_\lambda T\left(\|f\|_{L^2}^2 + \|Q\|_{L^4}^4 + \|\Delta\Phi\|_{L^4}^4\right). \tag{2.58}$$

hold.

Proof: We provide a priori bounds directly on the equations of (2.33). Their justification can be done using a viscous approximation of the q equation. The differential inequality

$$\partial_t \|q(t)\|_{L^4} + \lambda \|q(t)\|_{L^4} \le \|\Delta\Phi\|_{L^4} \tag{2.59}$$

is obtained as (2.42) above, and yields

$$\|q(t)\|_{L^4} \le \|q(0)\|_{L^4} e^{-\lambda t} + \frac{1}{\lambda}\|\Delta\Phi\|_{L^4}. \tag{2.60}$$

The differential inequality

$$\begin{aligned} \frac{d}{dt}\|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 &\le \|f - qR(q-Q)\|_{L^2}^2 \\ &\le C\left(\|f\|_{L^2}^2 + \|Q\|_{L^4}^4 + \|q\|_{L^4}^4\right) \end{aligned} \tag{2.61}$$

is obtained like the inequality (2.49) above. Because the gradient has mean zero, we have a Poincaré inequality for the gradient

$$\|\Delta u\|_{L^2}^2 \ge \|\nabla u\|_{L^2}^2 \tag{2.62}$$

and, using it, we obtain

$$\begin{aligned} \|\nabla u(t)\|_{L^2}^2 &\le \|\nabla u_0\|_{L^2}^2 e^{-t} + C_\gamma \|q_0\|_{L^4}^4 e^{-\gamma t} \\ &\quad + C_\lambda\left(\|f\|_{L^2}^2 + \|Q\|_{L^4}^4 + \|\Delta\Phi\|_{L^4}^4\right), \end{aligned} \tag{2.63}$$

with $0 < \gamma < \min\{1, 4\lambda\}$. This follows from (2.60) because

$$\begin{aligned} &\int_0^t e^{-(t-s)}\left(e^{-4\lambda s}\|q_0\|_{L^4}^4 + \lambda^{-4}\|\Delta\Phi\|_{L^4}^4\right) ds \\ &\le \|q_0\|_{L^4}^4 e^{-t}\int_0^t e^{(1-4\lambda)s} ds + \lambda^{-4}\|\Delta\Phi\|_{L^4}^4. \end{aligned}$$

Returning to (2.61) we deduce

$$\int_0^T \|\Delta u\|_{L^2}^2 dt \le \|\nabla u_0\|_{L^2}^2 + C_\gamma \|q_0\|_{L^4}^4$$
$$+ CT\left(\|f\|_{L^2}^2 + \|Q\|_{L^4}^4 + \|\Delta\Phi\|_{L^4}^4\right). \tag{2.64}$$

For the proof of uniqueness we take two solutions (u_1, q_1) and (u_2, q_2) of (2.33) and we write $q = q_2 - q_1$, $u = u_2 - u_1$. The differences obey the equations

$$\partial_t q + \Lambda q + u_1 \cdot \nabla q + u \cdot \nabla q + u \cdot \nabla q_1 = 0, \tag{2.65}$$

and

$$\partial_t u + u_2 \cdot \nabla u + u \cdot \nabla u_1 + \nabla p - \Delta u + q_1 R q + qRq + qRq_1 - qRQ = 0. \tag{2.66}$$

We multiply (2.65) by $\Lambda^{-1}q$, (2.66) by u and integrate. The cubic terms cancel

$$\int (u \cdot \nabla q)\Lambda^{-1} q + \int qRq \cdot u = 0$$

and the q_1 terms cancel as well

$$\int (u \cdot \nabla q_1)\Lambda^{-1} q + \int q_1 Rq \cdot u = 0,$$

and we are left with

$$\frac{1}{2}\frac{d}{dt}\left(\|\Lambda^{-\frac{1}{2}} q\|_{L^2}^2 + \|u\|_{L^2}^2\right) + \|\nabla u\|_{L^2}^2 + \|q\|_{L^2}^2$$
$$= \int q u_1 \cdot Rq - \int u \cdot \nabla u_1 \cdot u + \int q(R(Q - q_1) \cdot u. \tag{2.67}$$

We estimate

$$\left|\int u \cdot \nabla u_1 \cdot u\right| \le C\left(\|u\|_{L^2}\|\nabla u\|_{L^2} + \|u\|_{L^2}^2\right)\|\nabla u_1\|_{L^2} \tag{2.68}$$

and

$$\left|\int q(R(Q - q_1) \cdot u\right| \le C\|q\|_{L^2}\|Q - q_1\|_{L^4}\left(\|u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}} + \|u\|_{L^2}\right) \tag{2.69}$$

using L^4 bounds for u and the Ladyzhenskaya interpolation inequality. The first term in the right hand side of (2.67) can be written adding and subtracting zero as

$$\left|\int qu_1 \cdot Rq\right| = \left|\int \left(\left[\Lambda^{-\frac{1}{2}}, u_1 \cdot \nabla\right] q\right) \Lambda^{-\frac{1}{2}} q\right| \tag{2.70}$$

and using Proposition 2.3 with $s = -\frac{1}{2}$ and $\alpha = \frac{1}{2}$ we obtain

$$\left|\int qu_1 \cdot Rq\right| \le C[u_1]_{\frac{1}{2}} \|q\|_{L^2} \|\Lambda^{-\frac{1}{2}} q\|_{L^2} \tag{2.71}$$

Using Young inequalities in (2.68), (2.69) and (2.71) we obtain from (2.67),

$$\begin{aligned} &\frac{d}{dt}\left[\|\Lambda^{-\frac{1}{2}} q\|_{L^2}^2 + \|u\|_{L^2}^2\right] \\ &\quad \le C\left(\|\nabla u_1\|_{L^2}^2 + \|Q - q_1\|_{L^4}^4 + \|\nabla u_1\|_{L^2} + \|Q - q_1\|_{L^4}^2\right) \|u\|_{L^2}^2 \\ &\qquad + C[u_1]_{\frac{1}{2}}^2 \|\Lambda^{-\frac{1}{2}} q\|_{L^2}^2. \end{aligned} \tag{2.72}$$

Using the bound

$$[u_1]_{\frac{1}{2}} \le C\|\Delta u_1\|_{L^2} \tag{2.73}$$

for u_1 we obtain uniqueness from the fact that

$$\int_0^T (\|\Delta u_1\|_{L^2}^2 + \|q_1\|_{L^4}^4) dt < \infty \tag{2.74}$$

This concludes the proof of Theorem 2.2.

Remark 2.2. *The proof of uniqueness shows that we have weak-strong uniqueness: Strong solutions are unique among the larger class of weak solutions.*

Remark 2.3. *We have*

$$\begin{aligned} \int_t^{t+T} \|\Delta u(s)\|_{L^2}^2 ds &\le \|\nabla u_0\|_{L^2}^2 e^{-t} + C_\gamma \|q_0\|_{L^4}^4 e^{-\gamma t} \\ &\quad + C(1+T)\left(\|f\|_{L^2}^2 + \|Q\|_{L^4}^4 + \|\Delta\Phi\|_{L^4}^4\right). \end{aligned} \tag{2.75}$$

This is obtained by applying (2.58) on the interval $[t, t+T]$ and using the bounds (2.56) and (2.57) for the terms involving the "initial" time t.

Proposition 2.4. *The $H^{\frac{1}{2}}$ norm of the q component of strong solutions is locally uniformly bounded and their H^1 norm is locally uniformly square integrable in time. Moreover, for any $2 \le p \le \infty$, p even,*

$$\|q(t)\|_{L^p} \le \|q_0\|_{L^p} e^{-\lambda t} + \frac{1}{\lambda}\|\Delta\Phi\|_{L^p} \tag{2.76}$$

holds for all t.

Proof: The bound (2.34) holds for strong solutions. In view of it, for $t \ge t_0 > 0$ we consider the evolution of $\|\Lambda^{\frac{1}{2}} q\|_{L^2}$. We have

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^{\frac{1}{2}} q\|_{L^2}^2 + \|\Lambda q\|_{L^2}^2 = \int \Lambda Q \Lambda q - \int \left(\left[\Lambda^{\frac{1}{2}}, u \cdot \nabla \right] q \right) \Lambda^{\frac{1}{2}} q. \tag{2.77}$$

We use Proposition 2.3 with $s = \frac{1}{2}$ and $\alpha = \frac{1}{2}$ and (2.73) for u, and deduce, after using a Young inequality that

$$\frac{d}{dt}\|\Lambda^{\frac{1}{2}} q\|_{L^2}^2 + \|\Lambda q\|_{L^2}^2 \le \|\Lambda Q\|_{L^2}^2 + C\|\Delta u\|_{L^2}^2 \|\Lambda^{\frac{1}{2}} q\|_{L^2}^2. \tag{2.78}$$

Therefore the bound (2.64) implies

$$\|\Lambda^{\frac{1}{2}} q\|_{L^2}^2 \le C\left[T\|\Lambda Q\|_{L^2}^2 + \|\Lambda^{\frac{1}{2}} q(t_0)\|_{L^2}^2 \right] \exp K \tag{2.79}$$

with K given by

$$K = \|\nabla u_0\|_{L^2}^2 + C_\lambda \|q_0\|_{L^4}^4 + CT\left(\|f\|_{L^2}^2 + \|Q\|_{L^4}^4 + \|\Delta\Phi\|_{L^4}^4 \right) \tag{2.80}$$

and consequently

$$\int_{t_0}^{t} \|\Lambda q\|_{L^2}^2 \le T\|\Lambda Q\|_{L^2}^2 + C\left[T\|\Lambda Q\|_{L^2}^2 + \|\Lambda^{\frac{1}{2}} q(t_0)\|_{L^2}^2 \right] K \exp K + \|\Lambda^{\frac{1}{2}} q(t_0)\|_{L^2}^2 \tag{2.81}$$

hold for $0 < t_0 \le t \le T$.

The L^p bound (2.76) follows from the uniform Poincaré inequality (2.17) and the fact that u is divergence-free.

Remark 2.4. *The quantitative bound (2.81) shows that there exists $t_1 \in [t_0, t_0 + T]$ such that $q(t_1) \in H^1$, with a quantitative bound on its H^1 norm.*

Proposition 2.5. *Let $u_0 \in H^1$ be divergence-free and $q_0 \in H^1$ have mean zero. Then $\|\nabla q(t)\|_{L^2}$ can be bounded as*

$$\|\nabla q(t)\|_{L^2} \le C\left[1 + \|\nabla q_0\|_{L^2} + \|q_0\|_{L^4} + \|\nabla u_0\|_{L^2}\right]^8 e^{-c_1 t} + R_1(\Phi, f) \quad (2.82)$$

where $c_1 > 0$ is an explicit positive number and $R_1(\Phi, f)$ is an explicit function of norms of Φ and f. Moreover

$$\int_t^{t+T} \|\Lambda^{\frac{3}{2}} q(s)\|_{L^2}^2 ds$$
$$\le C\left[1 + \|\nabla q_0\|_{L^2} + \|q_0\|_{L^4} + \|\nabla u_0\|_{L^2}\right]^{16} e^{-c_2 t} + R_2(\Phi, f, T) \quad (2.83)$$

with $c_2 > 0$ and $R_2(\Phi, f, T)$ an explicit function of the norms of Φ,f and T. Moreover, if $u_0 \in H^2$ we have

$$\|\Delta u(t)\|_{L^2} \le C\left[1 + \|\nabla q_0\|_{L^2} + \|q_0\|_{L^4} + \|\Delta u_0\|_{L^2}\right]^{16} e^{-c_3 t} + R_3(\Phi, f) \quad (2.84)$$

with $c_3 > 0$ and $R_3(\Phi, f)$ and explicit function of the norms of Φ and f.

Proof: We take the first equation of (2.33) obeyed by q, multiply by $-\Delta q$ and integrate. We obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla q(t)\|_{L^2}^2 + \int (\Lambda \nabla q)\nabla q = \int \Lambda Q(-\Delta q) - \int (\nabla u \nabla q)\nabla q \quad (2.85)$$

We bound

$$\left|\int \Lambda Q(-\Delta q)\right| \le \|\Delta Q\|_{L^2}\|\Lambda^{\frac{3}{2}}q\|_{L^2} \tag{2.86}$$

and we bound

$$\left|\int (\nabla u \nabla q)\nabla q\right| \le \|\nabla u\|_{L^4}\|\nabla q\|^2_{L^{\frac{8}{3}}}. \tag{2.87}$$

Using (2.18) and a Young inequality we deduce

$$\frac{d}{dt}\|\nabla q\|^2_{L^2} + c\|\Lambda^{\frac{3}{2}}q\|^2_{L^2} \le C\|\Delta Q\|^2_{L^2} + C\|q\|^2_{L^4}\|\nabla u\|^4_{L^4}. \tag{2.88}$$

In view of the Ladyzhenskaya inequality

$$\|\nabla u\|^4_{L^4} \le C\|\nabla u\|^2_{L^2}\|\Delta u\|^2_{L^2}, \tag{2.89}$$

and the inequalities (2.57), (2.75), (2.76) it follows that the function

$$F(t) = \|q(t)\|^2_{L^4}\|\nabla u(t)\|^4_{L^4}$$

obeys the assumptions of the uniform Gronwall lemma, Lemma 1.1. The result (2.82) then follows using Lemma 1.1 for $y(t) = \|\nabla q\|^2_{L^2}$. The inequality (2.83) follows then by integrating in time (2.88).

For the bound (2.84) we apply $-\Delta$ to the equation obeyed by u. We obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta u\|^2_{L^2} + \|\nabla\Delta u\|^2_{L^2} = -\int \Delta(u\cdot\nabla u)\Delta u + \int \nabla(q(R(q-Q)-f)\nabla\Delta u. \tag{2.90}$$

After a cancellation due to the divergence-free condition, we have

$$\left|\int \Delta(u\cdot\nabla u)\Delta u\right| \le C\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}\|\nabla\Delta u\|_{L^2}. \tag{2.91}$$

Here we also used L^4 norms of the second order derivatives of u and Ladyzhenskaya interpolation inequality. We have also

$$\left|\int \nabla(q(R(q-Q)-f)\nabla\Delta u\right|$$
$$\leq C\|\nabla q\|_{L^4}\left(\|q\|_{L^4}+\|Q\|_{L^4}\right)\|\nabla\Delta u\|_{L^2}$$
$$+C\left(\|\nabla Q\|_{L^4}\|q\|_{L^4}+\|\nabla f\|_{L^2}\right)\|\nabla\Delta u\|_{L^2}. \tag{2.92}$$

Using the embedding $H^{\frac{1}{2}}\subset L^4$ for ∇q, we obtain

$$\frac{d}{dt}\|\Delta u\|_{L^2}^2+\|\nabla\Delta u\|_{L^2}^2$$
$$\leq C\left[\|\nabla u\|_{L^2}^2\|\Delta u\|_{L^2}^2+\left(\|\Lambda^{\frac{3}{2}}q\|_{L^2}^2+\|\nabla Q\|_{L^4}^2\right)\|q\|_{L^4}^2\right]$$
$$+C\left[\|\Lambda^{\frac{3}{2}}q\|_{L^2}^2\|Q\|_{L^4}^2+\|\nabla f\|_{L^2}^2\right]. \tag{2.93}$$

In view of (2.57), (2.75), (2.76), (2.83) we have that the function

$$F(t)=\|\nabla u\|_{L^2}^2\|\Delta u\|_{L^2}^2+\|\Lambda^{\frac{3}{2}}q\|_{L^2}^2\left(\|q\|_{L^4}^2+\|Q\|_{L^4}^2\right)+\|\nabla Q\|_{L^4}^2\|q\|_{L^4}^2+\|\nabla f\|_{L^2}^2$$

obeys the assumptions of Lemma 1.1. The inequality (2.84) then follows from this lemma applied to $y(t)=\|\Delta u\|_{L^2}^2$.

2.4 The Mean-Zero Frame

The second equation in (2.33) does not maintain a bounded average velocity u. Decomposing

$$u=v+u'(t) \tag{2.94}$$

where $v = v(t) \in \mathbb{R}^2$ is the average of $u(t)$, i.e.

$$u = v + \sum_{j\in\mathbb{Z}^2\setminus\{0\}} u_j e^{ij\cdot x} \tag{2.95}$$

we can rewrite the system (2.33) as

$$\begin{cases} \frac{d}{dt}v = -(2\pi)^{-2}\int q\nabla\Phi, \\ \partial_t q + (v+u')\cdot\nabla q + \Lambda q = \Delta\Phi \\ \partial_t u' + (v+u')\cdot\nabla u' - \Delta u' + \nabla p \\ \qquad\qquad = -qRq - q\nabla\Phi + (2\pi)^{-2}\int(q\nabla\Phi) + f \\ \nabla\cdot u' = 0 \end{cases} \tag{2.96}$$

where we used the fact that R is antisymmetric and f has mean zero. Given a solution of (2.96), we compute the displacement

$$\ell(t) = \int_0^t v(s)ds \tag{2.97}$$

and define the change of variables

$$X(x,t) = x - \int_0^t v(s) = x - \ell(t) \tag{2.98}$$

with inverse

$$Y(y,t) = y + \ell(t) \tag{2.99}$$

and note that

$$\frac{d}{dt}F(y+\ell(t),t) = (\partial_t + v(t)\cdot\nabla)F\circ Y(t). \tag{2.100}$$

Introducing the variables

$$\widetilde{u}(y,t) = u'(Y(y,t),t) \tag{2.101}$$

and

$$\widetilde{q}(y,t) = q(Y(y,t),t) \tag{2.102}$$

i.e.

$$u'(x,t) = \widetilde{u}(x-\ell(t),t) = \widetilde{u}\circ X, \quad q(x,t) = \widetilde{q}(x-\ell(t),t) = \widetilde{q}\circ X \tag{2.103}$$

we obtain the equations

$$\partial_t \widetilde{q} + \widetilde{u}\cdot\nabla\widetilde{q} + \Lambda\widetilde{q} = \Delta\widetilde{\Phi} \tag{2.104}$$

and

$$\partial_t \widetilde{u} + \widetilde{u}\cdot\nabla\widetilde{u} - \Delta\widetilde{u} + \nabla\widetilde{p} = -\widetilde{q}Rq - \widetilde{q}\nabla\widetilde{\Phi} + (2\pi)^{-2}\int \widetilde{q}\nabla\widetilde{\Phi} + \widetilde{f} \tag{2.105}$$

together with the divergence-free condition $\nabla\cdot\widetilde{u} = 0$. We used the translation invariance of the operators involved, and we used the notation

$$\widetilde{F}(y,t) = F(Y(y,t),t) \tag{2.106}$$

The new variables are still periodic in space with period 2π in each direction. The average of $\widetilde{u}$ is zero.

We note also that we can recover the solution (u,q) from the solution $(\widetilde{u},\widetilde{q})$ with the same initial data by the change of variables (2.103) and (2.97) where $v(t)$ is computed as

$$\frac{d}{dt}v(t) = -(2\pi)^{-2}\int \widetilde{q}\nabla\widetilde{\Phi}. \tag{2.107}$$

The two systems are equivalent, solution by solution. Dropping tildes we consider the system

$$\begin{cases} \partial_t q + u\cdot\nabla q + \Lambda q = \Delta\Phi \\ \partial_t u + u\cdot\nabla u - \Delta u + \nabla p = -qRq - q\nabla\Phi + (2\pi)^{-2}\int(q\nabla\Phi) + f \\ \nabla\cdot u = 0 \end{cases} \tag{2.108}$$

in which both u and q have mean zero. This is the system for which we can show that solutions have a finite dimensional attractor.

2.5 Long Time Dynamics

We are concerned with the long time behavior of solutions of (2.33) in the mean zero frame (2.108). Summarizing the results of Section 2.3 we know that solutions $(u(x,t), q(x,t))$ of the system (2.108) with initial data in L^2 exist globally, and they become strong at positive times. Strong solutions are unique, and have additional properties. We consider the subset $\mathcal{V}\subset\mathcal{H}$ where $\mathcal{H}$ is defined in (2.9)

$$\mathcal{V} = H^1\cap H\oplus L^4 \tag{2.109}$$

and study the evolution of solutions $(u(t), q(t))$ of (2.108) with initial data $w_0 = (u_0, q_0)\in\mathcal{V}$. The solution map

$$S(t)(u_0, q_0) = (u(t), q(t)) \tag{2.110}$$

is a semigroup

$$S(t):\mathcal{V}\mapsto\mathcal{H}, \tag{2.111}$$

$$S(t+s)w_0 = S(t)(S(s)w_0) \tag{2.112}$$

for $t, s \geq 0$. The abstract formulation of the system (2.108) is

$$\begin{cases} \partial_t u + Au + B(u,u) + \mathbb{P}(qR(q-Q)) = f, \\ \partial_t q + \Lambda q + u \cdot \nabla q = \Lambda Q \end{cases} \tag{2.113}$$

where

$$B(u,v) = \mathbb{P}(u \cdot \nabla v), \tag{2.114}$$

and $Q = -\Lambda\Phi$, as before. Note that, in view of

$$u = \mathbb{P}u \tag{2.115}$$

and the fact that $-\Delta$ commutes with $\mathbb{P}$ in the periodic case, we have

$$Au = -\Delta u. \tag{2.116}$$

Theorem 2.1 implies that there exist weak solutions of (2.113) with initial data in $\mathcal{H}$. If the initial data are in $\mathcal{V}$ the solutions are strong, unique and have additional properties.

Proposition 2.6. *There exists a constant R_0 depending on Φ and f, such that for any $w_0 = (u_0, q_0) \in \mathcal{V}$, there exists t_0 depending only on $\|u_0\|_{H^1}$ and $\|q_0\|_{L^4}$ such that the strong solution $(u(t), q(t)) = S(t)w_0$ of (2.108) with initial data $w_0 = (u_0, q_0)$ satisfies*

$$\|u(t)\|_{H^1} + \|q(t)\|_{L^4} \leq R_0 \tag{2.117}$$

for all $t \geq t_0$

Proof: Because u has mean zero we have the Poincaré inequality

$$\|u(t)\|_{L^2}^2 \leq \|\nabla u(t)\|_{L^2}^2. \tag{2.118}$$

The result (2.117) follows from (2.57) and (2.76) because of the translation invariance of norms

$$\|\nabla u\|_{L^2} = \|\nabla(u \circ X)\|_{L^2}, \quad \|q\|_{L^p} = \|q \circ X\|_{L^p}. \tag{2.119}$$

Proposition 2.7. *There exists R_1 depending only on Φ and f, and $t_1 > 0$ depending only on R_0 and R_1 such that for any $w_0 = (u_0, q_0) \in \mathcal{V}$ satsfying*

$$\|u_0\|_{H^1} + \|q_0\|_{L^4} \leq R_0 \tag{2.120}$$

we have

$$\|\Lambda^{\frac{1}{2}} q(t_1)\|_{L^2} \leq R_1 \tag{2.121}$$

and

$$\frac{1}{T}\int_{t_1}^{t_1+T} \left(\|\Delta u\|_{L^2}^2 + \|\Lambda q\|_{L^2}^2\right) dt \leq R_1^2 \tag{2.122}$$

for any $T > 0$. There exists $t_2 > t_1$, depending on R_1 such that

$$\|\Delta u(t_2)\|_{L^2}^2 + \|\Lambda q(t_2)\|_{L^2}^2 \leq R_1 \tag{2.123}$$

holds.

Proof: The bound on $\|\Lambda^{\frac{1}{2}} q(t_1)\|_{L^2}$ follows from

$$\int_0^t \|\Lambda^{\frac{1}{2}} q(s)\|_{L^2}^2 ds \leq \|q_0\|_{L^2}^2 + t\|\Lambda^{\frac{3}{2}}\Phi\|_{L^2}^2 \tag{2.124}$$

(see (2.43)) and the Chebyshev inequality. The inequality (2.122) follows from (2.58) and (2.81). The existence of t_2 for which (2.123) is true follows from (2.122).

Theorem 2.3. Absorbing ball. *There exists R_2 depending only on Φ an f such that, for any initial data $w_0 = (u_0, q_0) \in \mathcal{V}$, there exists $t_3 > 0$ depending only on the norms $\|u_0\|_{H^1}$, $\|q_0\|_{L^4}$ and on R_2 such that, for any $t \geq t_3$*

$$\|u(t)\|_{H^2} + \|q\|_{H^1} \leq R_2 \tag{2.125}$$

holds for t $\geq t_3$, i.e.

$$S(t)w_0 \in K_{R_2} = \{w \in \mathcal{V} \mid \ \|u\|_{H^2} + \|q\|_{H^1} \leq R_2\}. \tag{2.126}$$

holds for t $\geq t_3$.

Proof: By Proposition 2.6 and Proposition 2.7 above there exists R_1 depending on f and Φ and $t_2 > 0$ depending on the norms $\|u_0\|_{H^1}$ and $\|q_0\|_{L^4}$ such that $\|u(t_2)\|_{H^2} + \|q(t_2)\|_{H^1} \leq R_1$. Then the result follows from Proposition 2.5

2.6 Continuity Properties of the Solution Map

In addition to the topology of $\mathcal{H}$ with norm

$$\|w\|_{\mathcal{H}}^2 = \|u\|_{L^2}^2 + \|q\|_{L^2}^2 \tag{2.127}$$

we consider the natural topology of $\mathcal{V}$ which is a Banach space on its own, with norm

$$\|w\|_{\mathcal{V}}^2 = \|u\|_{H^1}^2 + \|q\|_{L^4}^2 \tag{2.128}$$

We consider the space

$$\mathcal{V}' = H^1 \cap H \oplus H^1 \tag{2.129}$$

and note that the absorbing ball K_{R_2} of Theorem 2.3 is included in $\mathcal{V}'$.

Theorem 2.4. Continuity. *Let $w_1^0 = (u_1^0, q_1^0) \in \mathcal{V}'$ and $w_2^0 = (u_2^0, q_2^0) \in \mathcal{V}'$. Let $t > 0$. There exist constants $C(t)$, $C_1(t)$, and $C_2(t)$ locally uniformly bounded above as functions of $t \geq 0$ and locally bounded as initial data w_1^0, w_2^0 are varied in $\mathcal{V}'$, such that $S(t)$ is Lipschitz continuous in $\mathcal{H}$, obeying*

$$\|S(t)w_1^0 - S(t)w_2^0\|_{\mathcal{H}} \leq C(t)\|w_1^0 - w_2^0\|_{\mathcal{H}}, \tag{2.130}$$

$S(t)$ is Lipschitz continuous in $\mathcal{V}$, obeying

$$\|S(t)w_1^0 - S(t)w_2^0\|_{\mathcal{V}} \leq C_1(t)\|w_1^0 - w_2^0\|_{\mathcal{V}}, \tag{2.131}$$

and $S(t)$ is Lipschitz continuous for $t > 0$ from $\mathcal{H}$ to $\mathcal{V}$, obeying

$$\sqrt{t}\|S(t)w_1^0 - S(t)w_2^0\|_{\mathcal{V}} \leq C_2(t)\|w_1^0 - w_2^0\|_{\mathcal{H}}. \tag{2.132}$$

Proof: We take the two solutions of (2.113) $w_1 = S(t)w_1^0 = (u_1(t), q_1(t))$ and $w_2 = S(t)w_2^0 = (u_2(t), q_2(t))$ and denote $w(t) = S(t)w_2^0 - S(t)w_1^0 = (u_2(t) - u_1(t), q_2(t) - q_1(t))$ and $\overline{w} = (\overline{u}, \overline{q}) = \frac{1}{2}(S(t)w_1^0 + S(t)w_2^0)$. Then $w(t)$ satisfies the system

$$\begin{cases} \partial_t u + Au + B(u, \overline{u}) + B(\overline{u}, u) + \mathbb{P}(qR(\overline{q} - Q) + \overline{q}Rq) = 0, \\ \partial_t q + \Lambda q + \overline{u} \cdot \nabla q + u \cdot \nabla \overline{q} = 0. \end{cases} \tag{2.133}$$

We obtain

$$\begin{aligned} &\frac{d}{dt}\|w(t)\|_{\mathcal{H}}^2 + \|\nabla u\|_{L^2}^2 + \|\Lambda^{\frac{1}{2}} q\|_{L^2}^2 \\ &\qquad \leq C(\|\nabla \overline{u}\|_{L^2}^2 + \|\overline{q}\|_{L^4}^2 + \|Q\|_{L^4}^2 + \|\nabla \overline{q}\|_{L^2}^4)\|w(t)\|_{\mathcal{H}}^2 \end{aligned} \tag{2.134}$$

by using estimates

$$\left| \int u \cdot \nabla \overline{q} q \right| \leq \|\nabla \overline{q}\|_{L^2} \|q\|_{L^4} \|u\|_{L^4}$$

and interpolation. Thus (2.130) holds with

$$C(t) = \exp\left\{C\int_0^t \left(\|\nabla\overline{u}\|_{L^2}^2 + \|\overline{q}\|_{L^4}^2 + \|Q\|_{L^4}^2 + \|\nabla\overline{q}\|_{L^2}^4\right) ds\right\} \tag{2.135}$$

which is a locally uniformly bounded function of time and initial data $w_1^0, w_2^0 \in \mathcal{V}'$.

The evolution of the norm the H^1 norm of u is obtained from the identity ([35])

$$(B(\overline{u}, u) + B(u, \overline{u}), Au)_H = -(B(u,u), A\overline{u})_H \tag{2.136}$$

which yields

$$\frac{1}{2}\frac{d}{dt}\|A^{\frac{1}{2}}u\|_H^2 + \|Au\|_H^2 = (B(u,u), A\overline{u})_H - (\mathbb{P}(qR(\overline{q}-Q)+\overline{q}Rq), Au)_H \tag{2.137}$$

and results in

$$\begin{aligned}&\frac{d}{dt}\|A^{\frac{1}{2}}u\|_H^2 + \|Au\|_H^2 \\ &\qquad \le C\|A\overline{u}\|_H^{\frac{4}{3}}\|u\|_H^{\frac{2}{3}}\|A^{\frac{1}{2}}u\|_H^{\frac{4}{3}} + C\left[\|\overline{q}\|_{L^4}^2 + \|Q\|_{L^4}^2\right]\|q\|_{L^4}^2.\end{aligned} \tag{2.138}$$

The L^4 norm of q evolves according to

$$\frac{1}{4}\frac{d}{dt}\|q\|_{L^4}^4 + \int q^3\Lambda q + \int q^3(u\nabla\overline{q}) = 0 \tag{2.139}$$

The inequality (2.17) and the embedding $H^{\frac{1}{2}} \subset L^4$ results in

$$\int q^3\Lambda q \ge c\|q\|_{L^8}^4 \tag{2.140}$$

and using the embedding $H^1 \subset L^8$ we deduce

$$\left|\int q^3(u\nabla\overline{q})\right| \le \|q\|_{L^8}^3\|u\|_{L^8}\|\nabla\overline{q}\|_{L^2} \le C\|q\|_{L^8}^3\|A^{\frac{1}{2}}u\|_H\|\nabla\overline{q}\|_{L^2}, \tag{2.141}$$

and therefore,

$$\frac{d}{dt}\|q\|_{L^4}^4 \le C\|A^{\frac{1}{2}}u\|_H^4\|\nabla\overline{q}\|_{L^2}^4. \tag{2.142}$$

Putting these together we obtain

$$\frac{d}{dt}\left[\|A^{\frac{1}{2}}u\|_H^4 + \|q\|_{L^4}^4\right]$$
$$\leq C\left(\|A\overline{u}\|_H^{\frac{4}{3}} + \|\overline{q}\|_{L^4}^2 + \|Q\|_{L^4}^2 + \|\nabla\overline{q}\|_{L^2}^4\right)\left[\|A^{\frac{1}{2}}u\|_{L^2}^4 + \|q\|_{L^4}^4\right] \quad (2.143)$$

Thus (2.131) holds with

$$C_1(t) = \exp\left\{C\int_0^t \left(\|A\overline{u}\|_H^{\frac{4}{3}} + \|\overline{q}\|_{L^4}^2 + \|Q\|_{L^4}^2 + \|\nabla\overline{q}\|_{L^2}^4\right) ds\right\} \quad (2.144)$$

which is a locally uniformly bounded function of $t > 0$ and initial data w_1^0, w_2^0 in $\mathcal{V}'$.

For the Lipschitz continuity from $\mathcal{H}$ to $\mathcal{V}$, we estimate slightly differently in (2.137),

$$\frac{d}{dt}\|A^{\frac{1}{2}}u\|_H^2 + \|Au\|_H^2 \leq C\|A\overline{u}\|_H^{\frac{4}{3}}\|u\|_H^{\frac{2}{3}}\|A^{\frac{1}{2}}u\|_H^{\frac{4}{3}}$$
$$+ C\left[\|\overline{q}\|_{L^\infty}^2 + \|R\overline{q}\|_{L^\infty}^2 + \|RQ\|_{L^\infty}^2\right]\|q\|_{L^2}^2. \quad (2.145)$$

Using the inequality $\|Au\|_H\|u\|_H \geq \|A^{\frac{1}{2}}u\|_H^2$ and a Young inequality, we obtain

$$\frac{d}{dt}\|A^{\frac{1}{2}}u\|_H^2 + \frac{1}{2}\|Au\|_H^2$$
$$\leq C\|A\overline{u}\|_H^2\|u\|_H^2 + C\left[\|\overline{q}\|_{L^\infty}^2 + \|R\overline{q}\|_{L^\infty}^2 + \|RQ\|_{L^\infty}^2\right]\|q\|_{L^2}^2. \quad (2.146)$$

Integrating in time in (2.134) and using (2.130) we have

$$\int_0^t \left(\|A^{\frac{1}{2}}u(s)\|_H^2 + \|\Lambda^{\frac{1}{2}}q(s)\|_{L^2}^2\right) ds \leq \widetilde{C}(t)\|w_0\|_{\mathcal{H}}^2 \quad (2.147)$$

with

$$\widetilde{C} = 1 + C\int_0^t C(s)(\|\nabla\overline{u}\|_{L^2}^2 + \|\overline{q}\|_{L^4}^2 + \|Q\|_{L^4}^2 + \|\nabla\overline{q}\|_{L^2}^4)ds \quad (2.148)$$

Multiplying (2.146) by t, using (2.147) and (2.130) we obtain

$$t\|A^{\frac{1}{2}}u(t)\|_H^2 \leq C_3(t)\|w_0\|_{\mathcal{H}}^2 \quad (2.149)$$

with $C_3(t)$ an explicit function of time which is locally uniformly bounded for $t \geq 0$, and locally bounded as initial data w_1^0, w_2^0 vary in $\mathcal{V}'$. Returning to (2.141) but estimating differently, using the Hölder inequality with exponents $2, 4, 4$ and then interpolation, we obatin

$$\left| \int q^3(u\nabla\overline{q}) \right| \leq C\|q\|_{L^8}^2 \|q\|_{L^4} \|u\|_H^{\frac{1}{2}} \|A^{\frac{1}{2}}u\|_H^{\frac{1}{2}} \|\nabla\overline{q}\|_{L^4} \tag{2.150}$$

and therefore, from (2.139) we obtain after a Young inequality and use of (2.140),

$$\frac{d}{dt}\|q\|_{L^4}^2 \leq C\|u\|_H \|A^{\frac{1}{2}}u\|_H \|\nabla\overline{q}\|_{L^4}^2. \tag{2.151}$$

Multiplying (2.151) by t, integrating in time, and using (2.147), the embedding $H^{\frac{1}{2}} \subset L^4$ and (2.149) we obtain

$$t\|q(t)\|_{L^4}^2 \leq C_4(t)\|w_0\|_H^2 \tag{2.152}$$

with $C_4(t)$ an explicit function of time which locally uniformly bounded for $t \geq 0$, and locally bounded as initial data w_1^0, w_2^0 vary in $\mathcal{V}'$. From (2.149) and (2.152) we obtain (2.132).

2.7 Backward Uniqueness

Theorem 2.5. Backward uniqueness. *Let w_1^0, w_2^0 be two initial data in $\mathcal{V}'$. For any $T > 0$, if $S(T)w_1^0 = S(T)w_2^0$, then $w_1^0 = w_2^0$.*

Proof: We use the notation of the proof of Theorem 2.4. The difference $w(t)$ obeys (2.133). We can write this abstractly as

$$\partial_t w + \mathcal{A}w + L(\overline{w})w = 0 \tag{2.153}$$

where $w = (u,q)$, $\overline{w} = (\overline{u},\overline{q})$, and

$$\begin{aligned} &L(\overline{w})w = (L_1(\overline{w})w,\ L_2(\overline{w})w),\ \text{with} \\ &L_1(\overline{w})w = B(u,\overline{u}) + B(\overline{u},u) + \mathbb{P}(qR(\overline{q}-Q) + \overline{q}Rq),\ \text{and} \\ &L_2(\overline{w})w = \overline{u}\cdot\nabla q + u\cdot\nabla\overline{q} \end{aligned} \tag{2.154}$$

Let us consider the evolution of the norm

$$E_0 = \|u\|_{L^2}^2 + \|q\|_{H^{-\frac{1}{2}}}^2 \tag{2.155}$$

obtained by taking the scalar product in $\mathcal{H}$ of the equation (2.153) with $(u, \Lambda^{-1}q) = (\mathbb{I}\oplus\Lambda^{-1})w = \mathcal{B}w$. The operator

$$\mathcal{B} = \mathbb{I}\oplus\Lambda^{-1} \tag{2.156}$$

is selfadjoint and commutes with $\mathcal{A}$. We obtain

$$\frac{1}{2}\frac{d}{dt}E_0 + E_1 + (L(\overline{w})w, \mathcal{B}w)_{\mathcal{H}} = 0 \tag{2.157}$$

where

$$E_1 = \|A^{\frac{1}{2}}u\|_H^2 + \|q\|_{L^2}^2 = (w, \mathcal{A}\mathcal{B}w)_{\mathcal{H}}. \tag{2.158}$$

Now we denote by

$$\mu = \frac{E_1}{E_0} \tag{2.159}$$

and observe that

$$\frac{1}{2}\frac{d}{dt}\log\left(\frac{1}{E_0}\right) = \mu + (L(\overline{w})\phi, \mathcal{B}\phi)_H \tag{2.160}$$

where

$$\phi = E_0^{-\frac{1}{2}}w. \tag{2.161}$$

Let us consider the function

$$Y(t) = \log\left(\frac{1}{E_0}\right), \tag{2.162}$$

and so we have

$$\frac{1}{2}\frac{d}{dt}Y(t) = \mu + (L(\overline{w})\phi, \mathcal{B}\phi)_{\mathcal{H}}. \tag{2.163}$$

The aim is to show that $Y(t)$ cannot reach the value $+\infty$ in finite time. To this end we take the derivative of μ and note

$$\frac{d}{dt}\mu = E_0^{-1}\frac{d}{dt}E_1 - \mu\frac{d}{dt}\log E_0 = E_0^{-1}\frac{d}{dt}E_1 + \mu\frac{d}{dt}Y. \tag{2.164}$$

We have

$$\frac{1}{2}\frac{d}{dt}E_1 + (w, \mathcal{A}^2\mathcal{B}w)_{\mathcal{H}} + (L(\overline{w})w, \mathcal{A}\mathcal{B}w)_{\mathcal{H}} = 0. \tag{2.165}$$

which implies that

$$E_0^{-1}\frac{d}{dt}E_1 = -2(\phi, \mathcal{A}^2\mathcal{B}\phi)_{\mathcal{H}} - 2(L(\overline{w})\phi, \mathcal{A}\mathcal{B}\phi)_{\mathcal{H}} \tag{2.166}$$

and therefore

$$\frac{1}{2}\frac{d}{dt}\mu = -(\phi, \mathcal{A}^2\mathcal{B}\phi)_{\mathcal{H}} - (L(\overline{w})\phi, \mathcal{A}\mathcal{B}\phi)_{\mathcal{H}} + \mu\left(\mu + (L(\overline{w})\phi, \mathcal{B}\phi)_{H}\right). \tag{2.167}$$

Let us note that

$$\mu = (\mathcal{A}\phi, \mathcal{B}\phi)_{\mathcal{H}} \tag{2.168}$$

and if we introduce the scalar product in $\mathcal{H}$ defined by

$$(a, b)_{\mathcal{B}} = (a, \mathcal{B}b)_{\mathcal{H}} \tag{2.169}$$

then we see that

$$\|\phi\|_{\mathcal{B}}^2 = 1 \tag{2.170}$$

and

$$(\mathcal{A}^2\phi, \phi)_{\mathcal{B}} - \mu^2 = \|(\mathcal{A} - \mu)\phi\|_{\mathcal{B}}^2 \tag{2.171}$$

hold. The equation (2.167) becomes

$$\frac{1}{2}\frac{d}{dt}\mu = -\|(\mathcal{A} - \mu)\phi\|_{\mathcal{B}}^2 - (L(\overline{w})\phi, (\mathcal{A} - \mu)\phi)_{\mathcal{B}}. \tag{2.172}$$

Let us note also that (2.163) can be written as

$$\frac{1}{2}\frac{d}{dt}Y(t) = \mu + (L(\overline{w})\phi, \phi)_{\mathcal{B}}. \tag{2.173}$$

This is a general structure, we could have used any postive selfadjoint operator $\mathcal{B}$ which commutes with $\mathcal{A}$, and it did really not matter what $L(\overline{w})$ or $\mathcal{A}$ were. Our choice is of course motivated by the properties of the latter, but some general features already can be taken advantage of.

We compute in our case

$$(L(\overline{w})\phi, \phi)_{\mathcal{B}} = \frac{1}{E_0}\left[(B(u, \overline{u}), u)_H + \int (qR(\overline{q} - Q) \cdot u - q\overline{u} \cdot Rq)\, dx\right] \tag{2.174}$$

where we used the cancellation of the terms involving $\overline{q}u \cdot Rq$ and $(u \cdot \nabla\overline{q})\Lambda^{-1}q$. The estimate

$$(L(\overline{w})\phi, \phi)_{\mathcal{B}} \leq K_0(t)\mu \tag{2.175}$$

with

$$K_0(t) = C\left[\|A\overline{u}\|_H + \|R(\overline{q} - Q)\|_{L^\infty}\right] \tag{2.176}$$

holds, and

$$\int_0^T K_0(t)dt < \infty \tag{2.177}$$

holds as well (see (2.75) and (2.83)). If we decompose

$$L(\overline{w})\phi = T_1\phi + T_2\phi \tag{2.178}$$

where

$$\|T_1\phi\|_{\mathcal{B}}^2 \leq K^2(t)\|\mathcal{A}^{\frac{1}{2}}\phi\|_{\mathcal{B}}^2 \tag{2.179}$$

then the contribution coming from T_1 can be estimated using the Schwarz inequality in the term $(T_1\phi, (\mathcal{A}-\mu)\phi)_{\mathcal{B}}$, and we obtain that

$$\frac{d}{dt}\mu \leq -\|(\mathcal{A}-\mu)\phi\|_{\mathcal{B}}^2 + K^2(t)\mu - 2(T_2\phi, (\mathcal{A}-\mu)\phi)_{\mathcal{B}}. \tag{2.180}$$

The bound (2.179) means that the velocity component of $T_1 w$ is bounded from $H^1 \times L^2$ to L^2 and the second component is bounded from $H^1 \times L^2$ to $H^{-\frac{1}{2}}$. The requirement (2.179) is satisfied in our case by

$$T_1 w = (L_1(\overline{w})w, u \cdot \nabla\overline{q}). \tag{2.181}$$

Indeed, (2.179) holds, i.e.

$$\|L_1(\overline{w})w\|_{L^2}^2 + \|\Lambda^{-\frac{1}{2}}(u\cdot\nabla\overline{q})\|_{L^2}^2 \leq K^2(t)\left[\|A^{\frac{1}{2}}u\|_H^2 + \|q\|_{L^2}^2\right] \tag{2.182}$$

with

$$K(t) = C\left[\|A\overline{u}\|_H + \|R(\overline{q}-Q)\|_{L^\infty} + \|\overline{q}\|_{L^\infty} + \|\nabla\overline{q}\|_{L^2}^{\frac{1}{2}}\|\nabla\overline{q}\|_{L^\infty}^{\frac{1}{2}}\right]. \tag{2.183}$$

It remains to examine what happens to T_2,

$$T_2 w = (0, \overline{u}\cdot\nabla q) \tag{2.184}$$

which does not satisfy (2.179). Its contribution to the evolution of μ in (2.180) is

$$\begin{aligned} 2(T_2\phi, (\mathcal{A}-\mu)\phi)_{\mathcal{B}} &= 2E_0^{-1}\int(\overline{u}\cdot\nabla q)(\Lambda-\mu)\Lambda^{-1}q \\ &= -2E_0^{-1}\mu\int(\overline{u}\cdot\nabla q)\Lambda^{-1}q. \end{aligned} \tag{2.185}$$

In view of the fact that

$$\int (\overline{u}\cdot\nabla q)\Lambda^{-1}q = -\int \Lambda^{-\frac{1}{2}}q\left[\overline{u}\cdot\nabla,\Lambda^{-\frac{1}{2}}\right]q \tag{2.186}$$

and Proposition 2.3 with $s=-\frac{1}{2}$ and $\alpha=0$, we have

$$-2(T_2\phi,(\mathcal{A}-\mu)\phi)_{\mathcal{B}} \le C[\overline{u}]_1\mu. \tag{2.187}$$

Thus, putting together the bounds (2.180) and (2.187) we obtain

$$\frac{d}{dt}\mu \le C(K^2(t)+[\overline{u}]_1)\mu \tag{2.188}$$

and because

$$\int_0^T (K^2(t)+[\overline{u}]_1)dt < \infty \tag{2.189}$$

it follows that $\mu(t)$ is locally bounded in time. From the bounds (2.175) and (2.177) it follows that $Y(t)$ is locally bounded.

2.8 Decay of Volume Elements

We consider a solution $\overline{w}=S(t)\overline{w}_0$ of (2.113) with initial data in the absorbing ball $\overline{w}_0\in K_{R_2}=\{w\in\mathcal{V}\mid \|u\|_{H^2}+\|q\|_{H^1}\le R_2\}$. We consider the linearization of $S(t)$ along $\overline{w}(t)$,

$$w_0\mapsto w(t)=S'(t,\overline{w})w_0 \tag{2.190}$$

viewed as an operator in $\mathcal{H}$. The function $w(t)$ solves

$$\partial_t w+\mathcal{A}w+L(\overline{w})w=0 \tag{2.191}$$

with initial data w_0. We denote $w = (u, q)$, $\overline{w} = (\overline{u}, \overline{q})$, and

$$\begin{aligned} &L(\overline{w})w = (L_1(\overline{w})w,\ L_2(\overline{w})w), \text{ with} \\ &L_1(\overline{w})w = B(u, \overline{u}) + B(\overline{u}, u) + \mathbb{P}(qR(\overline{q} - Q) + \overline{q}Rq), \text{ and} \\ &L_2(\overline{w})w = \overline{u} \cdot \nabla q + u \cdot \nabla \overline{q}. \end{aligned} \tag{2.192}$$

The volume elements associated to it are the norms in $\bigwedge^N \mathcal{H}$. The scalar product in $\bigwedge^N \mathcal{H}$ is

$$(w_1 \wedge \cdots \wedge w_N; y_1 \wedge \cdots \wedge y_N)_{\bigwedge^N \mathcal{H}} = \det (w_i, y_j)_{\mathcal{H}} \tag{2.193}$$

and the volume elements are norms

$$V_N(t) = \|w_1(t) \wedge \dots w_N(t)\|_{\bigwedge^N \mathcal{H}} \tag{2.194}$$

where

$$w_i(t) = S'(t, \overline{w})w_i(0) \tag{2.195}$$

are the images under the linearization of N linearly independent vectors. The monomial $w_1(t) \wedge \cdots \wedge w_N(t)$ evolves according to

$$\partial_t (w_1(t) \wedge \cdots \wedge w_N(t)) + (\mathcal{A} + L(\overline{w}))_N (w_1(t) \wedge \cdots \wedge w_N(t)) = 0 \tag{2.196}$$

with

$$\begin{aligned} &(\mathcal{A} + L(\overline{w}))_N(w_1(t) \wedge \cdots \wedge w_N(t)) \\ &\quad = (\mathcal{A} + L(\overline{w}))w_1 \wedge \cdots \wedge w_N + \cdots + w_1 \wedge \cdots \wedge (\mathcal{A} + L(\overline{w}))w_N \end{aligned} \tag{2.197}$$

and, as a consequence, the volume element evolves according to

$$\frac{d}{dt}V_N(t) + \text{Trace}\left((\mathcal{A} + L(\overline{w}))Q_N\right) V_N(t) = 0 \tag{2.198}$$

where Q_N is orthogonal projection in $\mathcal{H}$ onto the linear subspace spanned by the vectors w_i, $1 \le i \le N$. These are calculations which parallel well known calculations for the Navier-Stokes equations ([14], [35]).

The volume element $V_N(t)$ decays if N is large enough, as specified in the following theorem.

Theorem 2.6. *There exists a constant M depending on R_2 and norms of Φ and of f such that, for any initial data $\overline{w}_0$ in the absorbing ball K_{R_2}, for any $N \ge M$, and any initial data $w_1(0)$, $w_2(0)$, $\dots w_N(0)$ in $\mathcal{H}$, we have that*

$$\|S'(t,\overline{w})w_1(0) \wedge \cdots \wedge S'(t,\overline{w})w_N(0)\|_{\bigwedge^N \mathcal{H}} \le V_N(0)e^{-cN^{\frac{3}{2}}t} \tag{2.199}$$

holds for $t \ge t_0$, with t_0 depending on R_2.

Proof: The trace in (2.198) is computed as follows. At each instant of time t we choose an orthonormal basis $\phi_i = (v_i, r_i)$ of the linear span of $w_1, \dots w_N$. Then

$$\text{Trace}\,((\mathcal{A} + L(\overline{w}))Q_N) = \sum_{i=1}^{N} (\mathcal{A}\phi_i, \phi_i)_{\mathcal{H}} + \sum_{i=1}^{N} (L(\overline{w})\phi_i, \phi_i)_{\mathcal{H}}. \tag{2.200}$$

Now

$$\begin{aligned} \text{Trace}(\mathcal{A}Q_N) &= \sum_{i=1}^{N} (\mathcal{A}\phi_i, \phi_i)_{\mathcal{H}} \\ &= \sum_{i=1}^{N} [(Av_i, v_i)_H + (\Lambda r_i, r_i)_{L^2}] \ge \mu_1 + \dots \mu_N, \end{aligned} \tag{2.201}$$

and

$$\begin{aligned} &\sum_{i=1}^{N} (L(\overline{w})\phi_i, \phi_i)_{\mathcal{H}} \\ &= \sum_{i=1}^{N} \left[(B(v_i, \overline{u}), v_i)_H + (\mathbb{P}(r_i R(\overline{q} - Q) + \overline{q} R r_i), v_i)_H + \int (v_i \cdot \nabla \overline{q}) r_i \right]. \end{aligned} \tag{2.202}$$

On one hand we have a lower bound

$$\sum_{i=1}^{N}(\mathcal{A}\phi_i,\phi_i)_{\mathcal{H}} \geq \sum_{i=1}^{N}\left[\|A^{\frac{1}{2}}v_i\|_H^2 + c\|r_i\|_{L^4}^2\right], \tag{2.203}$$

and on the other hand we have the upper bound

$$\begin{aligned}\left|\sum_{i=1}^{N}(L(\overline{w})\phi_i,\phi_i)_{\mathcal{H}}\right| \qquad\qquad (2.204)\\ \leq C\sum_{i=1}^{N}\left[\|\nabla\overline{u}\|_{L^2}\|v_i\|_H\|A^{\frac{1}{2}}v_i\|_H + (\|\overline{q}\|_{L^4}+\|Q\|_{L^4})\|r_i\|_{L^4}\|v_i\|_{L^2}\right]\\ + C\sum_{i=1}^{N}\left[\|\nabla\overline{q}\|_{L^2}\|v_i\|_H^{\frac{1}{2}}\|A^{\frac{1}{2}}v_i\|_H^{\frac{1}{2}}\|r_i\|_{L^4}\right].\end{aligned}$$

Applying Schwarz inequalities in the first two terms in the right hand side of (2.204), and a Hölder inequality in $\mathbb{R}^N$ with exponents $4, 4, 2$ in the last term, followed by Young inequalities, we deduce after taking advantage of (2.203) that

$$\begin{aligned}\left|\sum_{i=1}^{N}(L(\overline{w})\phi_i,\phi_i)_{\mathcal{H}}\right| \leq \frac{1}{2}\mathrm{Trace}(\mathcal{A}Q_N)\\ + C\left(\|\nabla\overline{u}\|_{L^2}^2 + \|\overline{q}\|_{L^4}^2 + \|Q\|_{L^4}^2 + \|\nabla\overline{q}\|_{L^2}^4\right)\sum_{i=1}^{N}\|v_i\|_H^2. \qquad (2.205)\end{aligned}$$

Because of the normalization $\|v_i\|_H^2 + \|r_i\|_{L^2}^2 = \|\phi_i\|_{\mathcal{H}}^2 = 1$ we obtain

$$\begin{aligned}\left|\sum_{i=1}^{N}(L(\overline{w})\phi_i,\phi_i)_{\mathcal{H}}\right| \leq \frac{1}{2}\mathrm{Trace}(\mathcal{A}Q_N)\\ + CN\left(\|\nabla\overline{u}\|_{L^2}^2 + \|\overline{q}\|_{L^4}^2 + \|Q\|_{L^4}^2 + \|\nabla\overline{q}\|_{L^2}^4\right). \qquad (2.206)\end{aligned}$$

Let us note that, in view of the fact that K_{R_2} is an absorbing ball, we have

$$\sup_{T\geq 0}\frac{1}{T}\int_0^T\left(\|\nabla\overline{u}\|_{L^2}^2 + \|\overline{q}\|_{L^4}^2 + \|Q\|_{L^4}^2 + \|\nabla\overline{q}\|_{L^2}^4\right)dt \leq C(R_2) \tag{2.207}$$

with $C(R_2)$ a nondecreasing function of R_2. From (2.15) we have

$$\mu_1 + \cdots + \mu_N \geq cN^{\frac{3}{2}}, \tag{2.208}$$

and, in view of (2.198), (2.200), (2.201) and (2.206) we see that if

$$N^{\frac{1}{2}} \geq 8c^{-1}CC(R_2) \tag{2.209}$$

then $V_N(t)$ decays exponentially,

$$V_N(t) \leq V_N(0)e^{-cN^{\frac{3}{2}}t} \tag{2.210}$$

for $t \geq t_0$ with t_0 depending on R_2. Therefore the proof is complete.

2.9 Global Attractor

The properties of $S(t)$ of existence of a compact absorbing ball K_{R_2} (Theorem 2.3), continuity in $\mathcal{H}$ (Theorem 2.4), backward uniqueness (Theorem 2.5) imply the existence of a global attractor.

Theorem 2.7. *Let*

$$X = \bigcap_{t>0} S(t)K_{R_2} \tag{2.211}$$

where $S(t)$ is the semigroup solving (2.113) and K_{R_2} is the absorbing ball (2.126). Then:

(i) X *is compact in* $\mathcal{H}$.

(ii) $S(t)X = X$ *for all* $t \geq 0$.

(iii) If Z *is bounded in* $\mathcal{V}$ *in the norm of of* $\mathcal{V}$*, and* $S(t)Z = Z$ *for all* $t \geq 0$*, then* $Z \subset X$.

(iv) For every $w_0 \in \mathcal{V}$*,* $\lim_{t\to\infty} dist_{\mathcal{H}}(S(t)w_0, X) = 0$.

(v) X is connected.

The proof of this result follows verbatim the proof of Theorem 1.7. If the body forces vanish, then the attractor is particularly simple, it is a singleton.

Theorem 2.8. *Let $f = 0$. Then the attractor is a singleton, formed with the unique, globally attracting steady solution $w_Q = (0, Q)$,*

$$X = \{w_Q\}. \tag{2.212}$$

Proof: We take the scalar product in H of the first equation of (2.113) with u, we take the scalar product in L^2 of the second equation with $\Lambda^{-1}(q - Q)$ and add. The terms

$$(\mathbb{P}(qR(q-Q)), u)_H + (u \cdot \nabla q, \Lambda^{-1}(q-Q))_{L^2} = 0 \tag{2.213}$$

cancel, and we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|_H^2 + \|\Lambda^{-\frac{1}{2}}(q-Q)\|_{L^2}^2\right) + \|A^{\frac{1}{2}}u\|_H^2 + \|q-Q\|_{L^2}^2 = 0. \tag{2.214}$$

Because of the Poincaré inequality we obtain exponential decay of the distance to w_Q, first in $H \times H^{-\frac{1}{2}}$ and then in $\mathcal{H}$. The latter follows because

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|q-Q\|_{L^2}^2 + \|\Lambda^{\frac{1}{2}}(q-Q)\|_{L^2}^2 \\ &\quad = -\int qu \cdot \nabla Q dx \le \|u\|_H \|q\|_{L^2} \|\nabla Q\|_{L^\infty} \end{aligned} \tag{2.215}$$

and $\|q\|_{L^2}$ is bounded in time, while $\|u\|_H$ decays exponentially by (2.214), and therefore, from (2.215) we obtain the exponential convergence of w to w_Q in $\mathcal{H}$. This concludes the proof.

Remark 2.5. *When $f = 0$, returning to the nonzero mean velocity frame we see that the average velocity converges in time. Indeed, its time derivative, given in (2.107), obeys*

$$\left|\frac{d}{dt}v(t)\right| = \left|-(2\pi)^{-2}\int(q-Q)\nabla\Phi dx\right| \tag{2.216}$$

because $\int Q\nabla\Phi dx = 0$. The right hand side of (2.216) *belongs to $L^1(0,\infty)$ by* (2.214).

Employing methods initiated in [14] and used in many subsequent works, Theorem 2.6 implies

Theorem 2.9. *The global attractor X has finite fractal dimension*

$$D_{\mathcal{H}}(X) \leq M \tag{2.217}$$

where M depends only on norms of f and Φ.

The fractal dimension is defined as

$$\limsup_{r\to 0}\frac{\log N_{\mathcal{H}}(r)}{\log\left(\frac{1}{r}\right)} \tag{2.218}$$

where $N_{\mathcal{H}}(r)$ is the minimal number of balls in $\mathcal{H}$ of radii r needed to cover X.

Theorem 2.10. *The global attractor X has finite fractal dimension*

$$D_{\mathcal{V}}(X) = D_{\mathcal{H}}(X). \tag{2.219}$$

Proof: If $B_i \subset \mathcal{H}$ are a family of balls in $\mathcal{H}$ of radii ρ and centers w_i that cover X, then, because of the invariance $S(t)X = X$, the sets $S(t)B_i$ cover X. Now because of the continuity (2.132), the sets $S(t)B_i$ are included in balls in $\mathcal{V}$ of radii

$t^{-\frac{1}{2}}C_2(t)\rho = r$. Therefore

$$N_{\mathcal{H}}(r) \leq N_{\mathcal{V}}(r) \leq N_{\mathcal{H}}(\sqrt{t}C_2(t)^{-1}r). \tag{2.220}$$

Fixing $t > 0$ we obtain

$$\limsup_{r\to 0} \frac{\log N_{\mathcal{V}}(r)}{\log\left(\frac{1}{r}\right)} = \limsup_{r\to 0} \frac{\log N_{\mathcal{H}}(r)}{\log\left(\frac{1}{r}\right)}. \tag{2.221}$$

CHAPTER 3

Electroconvection in Porous Media

We consider the evolution of a surface charge density interacting with a two dimensional fluid in a porous medium. In the momentum equation, Stokes' law is replaced by Darcy's law balanced by the electrical forces. This results in an active scalar equation, in which the transport velocity is computed from the scalar charge density via a nonlinear and nonlocal relation. We address the model in the whole space $\mathbb{R}^2$ and in the periodic setting on $\mathbb{T}^2$. We prove the global existence and uniqueness of solutions in Besov spaces $\dot{B}^{\frac{2}{p}}_{p,1}$ for small initial data.

3.1 Introduction

Electroconvection, the evolution of charge distributions in fluids, was investigated experimentally and numerically in situations in which the fluid and charges are confined to thin films [25, 42, 41]. The charge distribution is carried by the fluid and diffuses due to the parallel component of the electrical field. This results in a nonlocal transport equation for the charge density ρ,

$$\partial_t \rho + u \cdot \nabla \rho + \Lambda \rho = 0 \tag{3.1}$$

where $\Lambda = (-\Delta)^{\frac{1}{2}}$ is the square root of the two dimensional Laplacian and u is the fluid velocity. The fluid is incompressible and is forced by electrical forces

$$F = \rho E \tag{3.2}$$

where E is the parallel component of the electrical field,

$$E = -\nabla \Phi, \tag{3.3}$$

with ∇ the gradient in $\mathbb{R}^2$. The relationship between the electrical potential Φ and the charge distribution confined to a two dimensional region is

$$\Phi = \Lambda^{-1} \rho \tag{3.4}$$

and we thus have

$$F = -\rho R \rho \tag{3.5}$$

with $R = \nabla \Lambda^{-1}$ the Riesz transforms. In general, the fluid obeys Navier-Stokes or related equations driven by the forces F. The derivation of this system for the physical setup in bounded domains was obtained in [13], where global regularity and uniqueness of solutions were obtained for the coupling with Navier-Stokes equations.

In this chapter, we consider flow through a porous medium, in which the dominant dissipation mechanism is due not to the viscosity of the fluid, but rather to an effective damping caused by flow through pores. The Stokes operator is then replaced by $u + \nabla p$. We consider a system in which the fluid equilibrates rapidly and the Reynolds number is low, so that forces are balanced by damping,

$$u + \nabla p = F. \tag{3.6}$$

This balance, together with (3.5) and the requirement of incompressibility,

$$\nabla \cdot u = 0, \tag{3.7}$$

leads to

$$u = -\mathbb{P}(\rho R \rho) \tag{3.8}$$

where $\mathbb{P}$ is the Leray-Hodge projector on divergence-free vector fields. The electroconvection situation described above leads to the active scalar equation (3.1) with constitutive law (3.8), which is the equation we study in this work. In comparison to the work [13], the nonlinear advection is missing, but also there is no viscosity, and because of the nonlinearity in the electrical force, the velocity's dependence of the charge density is more singular. The equation is L^∞-critical, and resembles critical SQG [15, 17, 19, 28] except for the constitutive law (3.8) which in this case is nonlinear and doubly nonlocal. Global regularity of critical SQG was originally proved by different methods in [8, 31] and was subsequently extensively studied. In [29], the balance law (3.8) was used to describe the solvent in a Nernst-Planck-Darcy system of ionic diffusion in 2D and 3D. An active scalar equation describing flow through porous media with fractional dissipation and linear nonlocal constitutive law was studied in [10] and global regularity was obtained.

In this chapter, we show that the equation (3.1), (3.8) has global weak solutions. We describe local existence and uniqueness results for strong solutions. We also show that solutions with small initial data in Besov spaces slightly smaller than L^∞ exist globally.

This chapter is organized as follows. In section 3.2, we recall results about Besov spaces and Littlewood-Paley decomposition. In section 3.3, we prove existence of global in time weak solutions of (3.1), (3.8) for initial data in $L^{2+\delta}(\mathbb{R}^2)$ for some $\delta > 0$. If the initial data is in $L^p(\mathbb{R}^2)$ for $p \in (2, \infty]$, then the L^p norm of any solution of (3.1), (3.8) remains bounded in time. If the initial data is $H^2(\mathbb{R}^2)$ regular, then we obtain a unique local strong solution. In section 3.4, we show that a global in time solution exists provided that the initial data is sufficiently small in Besov spaces that are slightly smaller than $L^\infty(\mathbb{R}^2)$. In section 3.5, we show that Hölder continuity of the charge distribution is a sufficient condition for the smoothness of solutions for arbitrary initial data, a result that is similar to the situation for SQG [20]. In section 3.6, we treat the periodic case, and we prove that the solution of the problem (3.1), (3.8) posed on the two dimensional torus converges exponentially in time to zero. Finally, we consider in section 3.7 the subcritical Darcy's law electroconvection, and we show existence of global smooth solutions for arbitrary initial data.

3.2 Preliminaries

We denote the Fourier transform of f by

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}^2} f(x) e^{-i\xi \cdot x} dx \tag{3.9}$$

and its inverse by $\mathcal{F}^{-1}$.

Let Φ be a nonnegative, nonincreasing, infinitely differentiable, radial function such that $\Phi(r) = 1$ for $r \in \left[0, \frac{1}{2}\right]$ and $\Phi(r) = 0$ for $r \in \left[\frac{5}{8}, \infty\right]$. Let

$$\Psi(r) = \Phi\left(\frac{r}{2}\right) - \Phi(r). \tag{3.10}$$

For each $j \in \mathbb{Z}$, let

$$\Psi_j(r) = \Psi(2^{-j}r). \tag{3.11}$$

We have

$$\Phi(|\xi|) + \sum_{j=0}^{\infty} \Psi_j(|\xi|) = 1 \tag{3.12}$$

for all $\xi \in \mathbb{R}^2$ and

$$\sum_{j=-\infty}^{\infty} \Psi_j(|\xi|) = 1 \tag{3.13}$$

for all $\xi \in \mathbb{R}^2 \setminus \{0\}$. We define the homogeneous dyadic blocks

$$\Delta_j f(x) = \mathcal{F}^{-1}\left[\Psi_j(|\cdot|)\widehat{f}(\cdot)\right](x) \tag{3.14}$$

and the lower frequency cutoff functions

$$S_j f = \sum_{k \leq j-1} \Delta_k f. \tag{3.15}$$

We note that the Fourier transform of each dyadic block is compactly supported. More precisely, we have

$$\text{supp}\, \mathcal{F}(\Delta_j f) \subset 2^j \left[\frac{1}{2}, \frac{5}{4}\right] \tag{3.16}$$

for all $j \in \mathbb{Z}$.

Let $\mathcal{S}'_h(\mathbb{R}^2)$ be the set of all tempered distributions $u \in \mathcal{S}'(\mathbb{R}^2)$ such that

$$\lim_{j \to -\infty} S_j u = 0 \tag{3.17}$$

in $\mathcal{S}'(\mathbb{R}^2)$. For $f \in \mathcal{S}'_h(\mathbb{R}^2)$, we denote the homogeneous Littlewood-Paley decomposition of f by

$$f = \sum_{j\in\mathbb{Z}} \Delta_j f. \tag{3.18}$$

For $s \in \mathbb{R}, 1 \leq p, q \leq \infty$, we denote the homogeneous Besov space

$$\dot{B}^s_{p,q}(\mathbb{R}^2) = \left\{ f \in \mathcal{S}'_h(\mathbb{R}^2) : \|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^2)} < \infty \right\} \tag{3.19}$$

where

$$\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^2)} = \left(\sum_{j\in\mathbb{Z}} 2^{jsq} \|\Delta_j f\|^q_{L^p(\mathbb{R}^2)} \right)^{1/q} \tag{3.20}$$

and the inhomogeneous Besov space

$$B^s_{p,q}(\mathbb{R}^2) = \left\{ f \in \mathcal{S}'(\mathbb{R}^2) : \|f\|_{B^s_{p,q}(\mathbb{R}^2)} < \infty \right\} \tag{3.21}$$

where

$$\|f\|_{B^s_{p,q}(\mathbb{R}^2)} = \left(2^{-sq} \|\tilde{\Delta}_{-1} f\|^q_{L^p(\mathbb{R}^2)} + \sum_{j=0}^{\infty} 2^{jsq} \|\Delta_j f\|^q_{L^p(\mathbb{R}^2)} \right)^{1/q} \tag{3.22}$$

with the usual modification when $q = \infty$. Here

$$\tilde{\Delta}_{-1} f = \mathcal{F}^{-1} \left[\Phi(|\cdot|) \widehat{f}(\cdot) \right](x). \tag{3.23}$$

We note that the definition of the space $\dot{B}^s_{p,q}$ is independent of the function Φ which defines the dyadic blocks. Indeed, any other dyadic partition yields an equivalent norm.

If $s > 0, 1 \leq p, q \leq \infty$, then

$$B^s_{p,q}(\mathbb{R}^2) = \dot{B}^s_{p,q}(\mathbb{R}^2) \cap L^p(\mathbb{R}^2). \tag{3.24}$$

Moreover, the norms $\|f\|_{B^s_{p,q}(\mathbb{R}^2)}$ and $\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^2)} + \|f\|_{L^p(\mathbb{R}^2)}$ are equivalent.

We also consider the following time dependent homogeneous Besov spaces

$$L^r(0,T;\dot{B}^s_{p,q}(\mathbb{R}^2)) = \left\{ f(t) \in \mathcal{S}'_h(\mathbb{R}^2) : \|f\|_{L^r(0,T;\dot{B}^s_{p,q}(\mathbb{R}^2))} < \infty \right\} \tag{3.25}$$

and

$$\tilde{L}^r(0,T;\dot{B}^s_{p,q}(\mathbb{R}^2)) = \left\{ f(t) \in \mathcal{S}'_h(\mathbb{R}^2) : \|f\|_{\tilde{L}^r(0,T;\dot{B}^s_{p,q}(\mathbb{R}^2))} < \infty \right\}, \tag{3.26}$$

where

$$\|f\|_{\tilde{L}^r(0,T;\dot{B}^s_{p,q}(\mathbb{R}^2))} = \left(\sum_{j\in\mathbb{Z}} 2^{jsq} \|\Delta_j f\|^q_{L^r(0,T;L^p(\mathbb{R}^2))} \right)^{1/q}.$$

We recall inequalities that are used in the upcoming sections (see for instance [5, 27, 43]).

Proposition 3.1. *Let* $f \in \mathcal{S}'_h(\mathbb{R}^2)$.

1. *(Bernstein's inequality) Let* $1 \le p \le \infty$. *Let* k *be a nonnegative integer. Then*

$$\sup_{|\alpha|=k} \|\partial^\alpha \Delta_j f\|_{L^p(\mathbb{R}^2)} \le C_k 2^{jk} \|\Delta_j f\|_{L^p(\mathbb{R}^2)} \tag{3.27}$$

holds for all $j \in \mathbb{Z}$.

2. *Let* $1 \le p \le q \le \infty$. *Then*

$$\|\Delta_j f\|_{L^q(\mathbb{R}^2)} \le C 2^{2j\left(\frac{1}{p}-\frac{1}{q}\right)} \|\Delta_j f\|_{L^p(\mathbb{R}^2)} \tag{3.28}$$

holds for all $j \in \mathbb{Z}$. *Moreover, the continuous Besov embedding*

$$\dot{B}^s_{p_1,q_1}(\mathbb{R}^2) \hookrightarrow \dot{B}^{s-2\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}_{p_2,q_2}(\mathbb{R}^2) \tag{3.29}$$

holds for $1 \le p_1 \le p_2 \le \infty, 1 \le q_1 \le q_2 \le \infty$ *and* $s \in \mathbb{R}$.

3. *Let* $1 \le p \le \infty, t \ge 0, \alpha > 0$. *Then*

$$\|e^{-t\Lambda^\alpha} \Delta_j f\|_{L^p(\mathbb{R}^2)} \le C e^{-C^{-1} t 2^{j\alpha}} \|\Delta_j f\|_{L^p(\mathbb{R}^2)} \tag{3.30}$$

holds for all $j \in \mathbb{Z}$. Here Λ^α is the fractional Laplacian of order α defined as a Fourier multiplier with symbol $|\xi|^\alpha$.

4. *Let $R = (R_1, R_2)$ be the Riesz transform, i.e., for $k \in \{1, 2\}$, $R_k = \partial_k \Lambda^{-1}$. For each $p \in [1, \infty]$, there is a positive constant $C > 0$ depending only on p (independent of j) such that*

$$\|\Delta_j R f\|_{L^p(\mathbb{R}^2)} \leq C \|\Delta_j f\|_{L^p(\mathbb{R}^2)} \tag{3.31}$$

holds for all $j \in \mathbb{Z}$. Hence, for $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$, R is bounded from $\dot{B}^s_{p,q}(\mathbb{R}^2)$ to itself.

The following decomposition formula holds.

Proposition 3.2. *Let $f, g \in \mathcal{S}'_h(\mathbb{R}^2)$. Then*

$$\begin{aligned} \Delta_j(fg) &= \sum_{k \geq j-2} \Delta_j(S_{k+1} f \Delta_k g) + \sum_{k \geq j-2} \Delta_j(S_k g \Delta_k f) \\ &= \sum_{k \geq j-2} \Delta_j(S_{k+1} g \Delta_k f) + \sum_{k \geq j-2} \Delta_j(S_k f \Delta_k g) \end{aligned} \tag{3.32}$$

holds for any $j \in \mathbb{Z}$.

Proof: Let $f, g \in \mathcal{S}'_h$. Bony's paraproduct gives the decomposition

$$fg = \sum_{j \in \mathbb{Z}} S_{j-1} f \Delta_j g + \sum_{j \in \mathbb{Z}} S_{j-1} g \Delta_j f + \sum_{|j - j'| \leq 1} \Delta_j f \Delta_{j'} g. \tag{3.33}$$

We note that

$$\sum_{|j-j'|\leq 1}\Delta_j f\Delta_{j'}g = \sum_{j\in\mathbb{Z}}\Delta_j f\Delta_j g + \sum_{j\in\mathbb{Z}}\Delta_j f\Delta_{j-1}g + \sum_{j\in\mathbb{Z}}\Delta_j f\Delta_{j+1}g$$

$$= \sum_{j\in\mathbb{Z}}\Delta_j f\Delta_j g + \sum_{j\in\mathbb{Z}}\Delta_j f\Delta_{j-1}g + \sum_{j\in\mathbb{Z}}\Delta_{j-1} f\Delta_j g$$

$$= \sum_{j\in\mathbb{Z}}(\Delta_{j-1}f + \Delta_j f)\Delta_j g + \sum_{j\in\mathbb{Z}}\Delta_j f\Delta_{j-1}g. \tag{3.34}$$

This implies that

$$fg = \sum_{j\in\mathbb{Z}} S_{j+1}f\Delta_j g + \sum_{j\in\mathbb{Z}} S_j g\Delta_j f. \tag{3.35}$$

Now we apply Δ_j. In view of (3.16), we have

$$k \leq j-2 \Rightarrow \Delta_j(S_k g\Delta_k f) = 0 \tag{3.36}$$

and

$$k \leq j-3 \Rightarrow \Delta_j(S_{k+1} f\Delta_k g) = 0. \tag{3.37}$$

Indeed, for $f, g \in L^1(\mathbb{R}^2)$,

$$\mathcal{F}(\Delta_j(S_k g\Delta_k f)(\xi) = \Psi_j(|\xi|)\mathcal{F}(S_k g\Delta_k f)(\xi)$$

$$= \Psi_j(|\xi|)\left\{\sum_{l\leq k-1}\int_{\mathbb{R}^2}\Psi_l(|\xi-y|)\mathcal{F}g(\xi-y)\Psi_k(|y|)\mathcal{F}f(y)dy\right\}$$

$$= \Psi_j(|\xi|)\left\{\sum_{l\leq k-1}\int_{\frac{2^k}{2}\leq|y|\leq\frac{2^k 5}{4}}\Psi_l(|\xi-y|)\mathcal{F}g(\xi-y)\Psi_k(|y|)\mathcal{F}f(y)dy\right\}$$

$$= \Psi_j(|\xi|)\tilde{\Psi}_k(\xi) \tag{3.38}$$

where

$$\tilde{\Psi}_k(\xi) = \sum_{l\leq k-1}\int_{\frac{2^k}{2}\leq|y|\leq\frac{2^k 5}{4}}\Psi_l(|\xi-y|)\mathcal{F}g(\xi-y)\Psi_k(|y|)\mathcal{F}f(y)dy. \tag{3.39}$$

Fix $l \leq k-1$. Let $y \in \mathbb{R}^2$ be such that $\frac{2^k}{2} \leq |y| \leq \frac{2^k 5}{4}$ and $\Psi_l(|\xi - y|) \neq 0$. This implies that $|\xi - y| \leq \frac{2^l 5}{4}$, thus

$$|\xi| \leq |\xi - y| + |y| \leq \frac{2^l 5}{4} + \frac{2^k 5}{4} \leq \frac{2^{k-1} 5}{4} + \frac{2^k 5}{4} = 2^{k-3} 15. \tag{3.40}$$

Consequently, if $|\xi| > 2^{k-3}15$, then $\Psi_l(|\xi - y|) = 0$ for all $l \leq k-1$ and for all y satisfying $\frac{2^k}{2} \leq |y| \leq \frac{2^k 5}{4}$, and so $\tilde{\Psi}_k(\xi) = 0$. We conclude that the support of $\tilde{\Psi}_k$ is included in the closed ball centered at 0 with radius $2^{k-3}15$. But the support of $\Psi_j(|\cdot|)$ is included in the closed annulus centered at 0 with radii $\frac{2^j}{2}$ and $\frac{2^j 5}{4}$. Therefore, if $k+1 \leq j-1$, then $2^{k-3}15 < 2^{k+1} \leq 2^{j-1}$ and so

$$\mathcal{F}(\Delta_j(S_k g \Delta_k f)) = 0 \tag{3.41}$$

which gives (3.36). The property (3.37) follows from a similar argument. This generalizes to distributions with compact supports, because the support of the convolution of two distributions with compact supports A and B respectively is contained in A + B. Therefore, we obtain the decomposition

$$\Delta_j(fg) = \sum_{k \geq j-2} \Delta_j(S_{k+1} f \Delta_k g) + \sum_{k \geq j-2} \Delta_j(S_k g \Delta_k f). \tag{3.42}$$

This ends the proof of Proposition 3.2.

Throughout this chapter, C (or $C_i, i = 1, 2, \ldots$) denotes a positive constant that may change from line to line in the proofs.

3.3 Well-Posedness in Lebesgue Spaces

We consider the transport and nonlocal diffusion equation

$$\partial_t \rho + u \cdot \nabla \rho + \Lambda \rho = 0 \tag{3.43}$$

in the whole space $\mathbb{R}^2$, where

$$u = -\mathbb{P}(\rho R\rho). \tag{3.44}$$

The initial data are

$$\rho(x,0) = \rho_0(x). \tag{3.45}$$

Here $\mathbb{P}$ is the Leray-Hodge projector, $\Lambda = (-\Delta)^{\frac{1}{2}}$ is the fractional Laplacian, and $R = \nabla\Lambda^{-1}$ is the 2D vector of Riesz transforms.

Definition 3.1. *A solution ρ of the initial value problem* (3.43)–(3.45) *is said to be a weak solution on $[0,T]$ if*

$$\rho \in L^\infty(0,T;L^2(\mathbb{R}^2)) \cap L^2(0,T;\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)) \tag{3.46}$$

and ρ obeys

$$(\rho(t),\Phi)_{L^2} - (\rho_0,\Phi)_{L^2} - \int_0^t (\rho, u\cdot\nabla\Phi)_{L^2} ds + \int_0^t (\Lambda^{\frac{1}{2}}\rho, \Lambda^{\frac{1}{2}}\Phi)_{L^2} ds = 0 \tag{3.47}$$

for all time-independent test functions $\Phi \in H^{\frac{5}{2}}(\mathbb{R}^2)$ and a.e. $t \in [0,T]$.

For $\epsilon \in (0,1]$, let J_ϵ be the standard mollifier operator $J_\epsilon f = J_\epsilon * f$, and let ρ^ϵ be the solution of

$$\partial_t\rho^\epsilon + \widetilde{u}^\epsilon \cdot \nabla\rho^\epsilon + \Lambda\rho^\epsilon - \epsilon\Delta\rho^\epsilon = 0 \tag{3.48}$$

where

$$\widetilde{u}^\epsilon = -J_\epsilon\mathbb{P}(\rho^\epsilon R\rho^\epsilon) \tag{3.49}$$

with smoothed out initial data

$$\rho_0^\epsilon = J_\epsilon\rho_0 \tag{3.50}$$

Remark 3.1. *We note that $\mathbb{P}$ and J_ϵ commutes, hence $\widetilde{u}^\epsilon$ is divergence free.*

Theorem 3.1. *Let $T > 0$ be arbitrary. Let $\rho_0 \in L^2(\mathbb{R}^2)$. Then for each $\epsilon \in (0, 1]$, the mollified initial value problem* (3.48)–(3.50) *has a solution ρ^ϵ on $[0, T]$ satisfying*

$$\frac{1}{2}\|\rho^\epsilon(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{\frac{1}{2}}\rho^\epsilon(s)\|_{L^2}^2 ds \leq \frac{1}{2}\|\rho_0\|_{L^2}^2 \tag{3.51}$$

for all $t \in [0, T]$. Moreover, the sequence $\left\{\rho^{1/n}\right\}_{n=1}^{\infty}$ has a subsequence that converges strongly in $L^2(0, T; L^2(\mathbb{R}^2))$ and weakly in $L^2(0, T; H^{\frac{1}{2}}(\mathbb{R}^2))$ to a function ρ obeying

$$\frac{1}{2}\|\rho(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{\frac{1}{2}}\rho(s)\|_{L^2}^2 ds \leq \frac{1}{2}\|\rho_0\|_{L^2}^2 \tag{3.52}$$

for a.e. $t \in [0, T]$. If $\rho_0 \in L^{2+\delta}(\mathbb{R}^2)$ for some $\delta > 0$, then ρ is a weak solution of (3.43)–(3.45) *on $[0, T]$.*

Proof: We take the L^2 inner product of (3.48) with ρ^ϵ and we obtain

$$\frac{1}{2}\frac{d}{dt}\|\rho^\epsilon\|_{L^2}^2 + \|\Lambda^{\frac{1}{2}}\rho^\epsilon\|_{L^2}^2 + \epsilon\|\nabla\rho^\epsilon\|_{L^2}^2 = 0. \tag{3.53}$$

Here we used the fact that $\widetilde{u}^\epsilon$ is divergence free, which implies that

$$(\widetilde{u}^\epsilon \cdot \nabla\rho^\epsilon, \rho^\epsilon)_{L^2} = 0. \tag{3.54}$$

Integrating (3.53) in time from 0 to t, we obtain (3.51). Therefore, the family $\{\rho^\epsilon : \epsilon \in (0, 1]\}$ is uniformly bounded in $L^2(0, T; H^{\frac{1}{2}})$. Moreover, we have

$$\begin{aligned} |(\Lambda\rho^\epsilon, \Phi)_{L^2}| &= |(\Lambda^{\frac{1}{2}}\rho^\epsilon, \Lambda^{\frac{1}{2}}\Phi)_{L^2}| \\ &\leq \|\Lambda^{\frac{1}{2}}\rho^\epsilon\|_{L^2}\|\Lambda^{\frac{1}{2}}\Phi\|_{L^2} \leq C\|\Lambda^{\frac{1}{2}}\rho^\epsilon\|_{L^2}\|\Phi\|_{H^{\frac{5}{2}}}, \end{aligned} \tag{3.55}$$

$$\epsilon|(-\Delta\rho^\epsilon, \Phi)_{L^2}| = \epsilon|(\rho^\epsilon, -\Delta\Phi)_{L^2}| \leq C\|\rho^\epsilon\|_{L^2}\|\Phi\|_{H^{\frac{5}{2}}}, \tag{3.56}$$

and

$$|(\widetilde{u}^{\epsilon}\cdot\nabla\rho^{\epsilon},\Phi)_{L^2}| = |(\widetilde{u}^{\epsilon}\rho^{\epsilon},\nabla\Phi)_{L^2}|$$

$$\leq \|\widetilde{u}^{\epsilon}\|_{L^2}\|\rho^{\epsilon}\|_{L^2}\|\nabla\Phi\|_{L^\infty} \leq C\|\rho^{\epsilon}\|_{L^4}^2\|\rho^{\epsilon}\|_{L^2}\|\Phi\|_{H^{\frac{5}{2}}} \tag{3.57}$$

for all $\Phi \in H^{\frac{5}{2}}$. Here we used the boundedness of the Riesz operator on L^4, and the continuous Sobolev embedding $H^{\frac{3}{2}} \hookrightarrow L^\infty$. Therefore, we obtain the bound

$$\|\widetilde{u}^{\epsilon}\cdot\nabla\rho^{\epsilon}\|_{H^{-\frac{5}{2}}} + \|\Lambda\rho^{\epsilon}\|_{H^{-\frac{5}{2}}} + \epsilon\|\Delta\rho^{\epsilon}\|_{H^{-\frac{5}{2}}}$$

$$\leq C(\|\rho^{\epsilon}\|_{L^4}^2\|\rho^{\epsilon}\|_{L^2} + \|\rho^{\epsilon}\|_{L^2} + \|\Lambda^{\frac{1}{2}}\rho^{\epsilon}\|_{L^2}). \tag{3.58}$$

In view of the continuous embedding $H^{\frac{1}{2}} \hookrightarrow L^4$, we conclude that the family $\{\partial_t\rho^{\epsilon} : \epsilon \in (0,1]\}$ is uniformly bounded in $L^1(0,T;H^{-\frac{5}{2}})$. Now, we note that the inclusion $H^{\frac{1}{2}} \hookrightarrow L^2$ is compact whereas the inclusion $L^2 \hookrightarrow H^{-\frac{5}{2}}$ is continuous. Let ϵ_n be a decreasing sequence in $(0,1]$ converging to 0. By the Aubin-Lions lemma and (3.51), the sequence $\{\rho^{\epsilon_n}\}_{n=1}^{\infty}$ has a subsequence that converges strongly in $L^2(0,T;L^2)$ and weakly in $L^2(0,T;H^{\frac{1}{2}})$ to some function ρ. By the lower semi-continuity of the norms, we obtain (3.52).

For simplicity of notations, we assume that ρ^{ϵ} converges to ρ strongly in $L^2(0,T;L^2)$ and weakly in $L^2(0,T;H^{\frac{1}{2}})$. We note that

$$(\rho^{\epsilon}(t),\Phi)_{L^2} - (\rho_0,\Phi)_{L^2} + \int_0^t (\widetilde{u}^{\epsilon}\cdot\nabla\rho^{\epsilon},\Phi)_{L^2}ds$$

$$+ \int_0^t (\Lambda^{\frac{1}{2}}\rho^{\epsilon},\Lambda^{\frac{1}{2}}\Phi)_{L^2}ds + \epsilon\int_0^t (\nabla\rho^{\epsilon},\nabla\Phi)_{L^2}ds = 0 \tag{3.59}$$

holds for all $\Phi \in H^{\frac{5}{2}}$ and $t \in [0,T]$. Without loss of generality, we may assume that ρ^{ϵ} converges to ρ in L^2 for a.e. $t \in [0,T]$, and so

$$|(\rho^{\epsilon}(t),\Phi)_{L^2} - (\rho(t),\Phi)_{L^2}| \leq \|\rho^{\epsilon}-\rho\|_{L^2}\|\Phi\|_{L^2} \to 0 \tag{3.60}$$

for all $\Phi \in H^{\frac{5}{2}}$ and a.e. $t \in [0, T]$. By the weak convergence in $L^2(0, T; H^{\frac{1}{2}})$, we obtain

$$\left| \int_0^t (\Lambda^{\frac{1}{2}} \rho^\epsilon, \Lambda^{\frac{1}{2}} \Phi)_{L^2} ds - \int_0^t (\Lambda^{\frac{1}{2}} \rho, \Lambda^{\frac{1}{2}} \Phi)_{L^2} ds \right| \to 0 \tag{3.61}$$

for all $\Phi \in H^{\frac{5}{2}}$ and all $t \in [0, T]$. For the nonlinear term, we let $\Phi \in H^{\frac{5}{2}}$, $t \in [0, T]$ and we write

$$\begin{aligned} &\int_0^t (\widetilde{u}^\epsilon \cdot \nabla \rho^\epsilon, \Phi)_{L^2} ds - \int_0^t (u \cdot \nabla \rho, \Phi)_{L^2} ds \\ &= - \int_0^t ((\rho^\epsilon - \rho) u, \nabla \Phi)_{L^2} ds - \int_0^t ((\widetilde{u}^\epsilon - u) \rho^\epsilon, \nabla \Phi)_{L^2} ds \\ &= I_1 + I_2. \end{aligned} \tag{3.62}$$

We note that

$$|I_1| \le C \|\Phi\|_{H^{\frac{5}{2}}} \int_0^t \|\rho\|_{L^4}^2 \|\rho^\epsilon - \rho\|_{L^2} ds \to 0 \tag{3.63}$$

by the Lebesgue Dominated Convergence theorem. For I_2, we split it as

$$\begin{aligned} I_2 &= \int_0^t ((J_\epsilon \mathbb{P}(\rho(R\rho^\epsilon - R\rho)))\rho^\epsilon, \nabla \Phi)_{L^2} ds + \int_0^t ((J_\epsilon \mathbb{P}((\rho^\epsilon - \rho) R\rho^\epsilon))\rho^\epsilon, \nabla \Phi)_{L^2} ds \\ &= I_{2,1} + I_{2,2}. \end{aligned} \tag{3.64}$$

In view of the boundedness of the Riesz transform on L^2 and the boundedness of the Leray operator on $L^{4/3}$, we have

$$\begin{aligned} |I_{2,1}| &\le C \|\Phi\|_{H^{\frac{5}{2}}} \int_0^t \|\rho^\epsilon\|_{L^4} \|\mathbb{P}(\rho R(\rho^\epsilon - \rho))\|_{L^{4/3}} ds \\ &\le C \|\Phi\|_{H^{\frac{5}{2}}} \int_0^t \|\rho^\epsilon\|_{L^4} \|\rho\|_{L^4} \|\rho^\epsilon - \rho\|_{L^2} ds \\ &\le C \|\Phi\|_{H^{\frac{5}{2}}} \left(\int_0^t \|\rho^\epsilon\|_{L^4}^2 ds \right)^{1/2} \left(\int_0^t \|\rho\|_{L^4}^2 \|\rho^\epsilon - \rho\|_{L^2}^2 ds \right)^{1/2} \to 0 \end{aligned} \tag{3.65}$$

by the Lebesgue Dominated Convergence theorem.

We note that we have not yet used the assumption that $\rho_0 \in L^{2+\delta}$. It will be needed to estimate $|I_{2,2}|$. Indeed, we multiply equation (3.48) by $\rho^\epsilon|\rho^\epsilon|^\delta$ and we integrate in the space variable. We use the Córdoba-Córdoba inequality [22]

$$\int_{\mathbb{R}^2} |\rho^\epsilon|^\delta(\rho^\epsilon \Lambda\rho^\epsilon)dx \geq 0 \tag{3.66}$$

and we obtain the differential inequality

$$\frac{d}{dt}\|\rho^\epsilon(t)\|_{L^{2+\delta}} \leq 0. \tag{3.67}$$

Integrating in time from 0 to t, we end up having the bound

$$\|\rho^\epsilon(t)\|_{L^{2+\delta}} \leq \|\rho_0\|_{L^{2+\delta}} \tag{3.68}$$

for all $t \in [0,T]$. As a consequence,

$$\begin{aligned}
|I_{2,2}| &\leq C\|\Phi\|_{H^{\frac{5}{2}}} \int_0^t \|\rho^\epsilon\|_{L^4}\|\rho^\epsilon\|_{L^{2+\delta}}\|\rho^\epsilon - \rho\|_{L^{\frac{8+4\delta}{2+3\delta}}} ds \\
&\leq C\|\Phi\|_{H^{\frac{5}{2}}}\|\rho_0\|_{L^{2+\delta}} \int_0^t \|\rho^\epsilon\|_{L^4}\|\rho^\epsilon - \rho\|_{L^2}^{\frac{2\delta}{2+\delta}}\|\rho^\epsilon - \rho\|_{L^4}^{\frac{2-\delta}{2+\delta}} ds \\
&\leq C\|\Phi\|_{H^{\frac{5}{2}}}\|\rho_0\|_{L^{2+\delta}} \left(\int_0^t \|\rho^\epsilon\|_{L^4}^2\right)^{\frac{2}{2+\delta}} \left(\int_0^t \|\rho^\epsilon - \rho\|_{L^2}^2 ds\right)^{\frac{\delta}{2+\delta}} \\
&+ C\|\Phi\|_{H^{\frac{5}{2}}}\|\rho_0\|_{L^{2+\delta}} \left(\int_0^t \|\rho^\epsilon\|_{L^4}^2\right)^{1/2} \left(\int_0^t \|\rho\|_{L^4}^2 ds\right)^{\frac{2-\delta}{4+2\delta}} \left(\int_0^t \|\rho^\epsilon - \rho\|_{L^2}^2 ds\right)^{\frac{\delta}{2+\delta}} \\
&\to 0.
\end{aligned}$$

Here we used the interpolation inequality

$$\|f\|_{L^{\frac{8+4\delta}{2+3\delta}}} \leq C\|f\|_{L^2}^{\frac{2\delta}{2+\delta}}\|f\|_{L^4}^{\frac{2-\delta}{2+\delta}} \tag{3.69}$$

that holds for any $f \in L^4$.

Therefore ρ is a weak solution of (3.43). This ends the proof of Theorem 3.1.

As a consequence of the Córdoba-Córdoba inequality [22], the L^p norm of any solution of the equation (3.43)–(3.44) is bounded by the L^p norm of the initial data for any $p \in (2, \infty]$:

Proposition 3.3. *Let* $p > 2$ *and* $\rho_0 \in L^p(\mathbb{R}^2)$. *Suppose* ρ *is a smooth solution of* (3.43)–(3.45) *on* $[0, T]$. *Then*

$$\|\rho(t)\|_{L^p} \leq \|\rho_0\|_{L^p} \tag{3.70}$$

holds for all $t \in [0, T]$.

Proof: We multiply (3.43) by $\rho|\rho|^{p-2}$ and we integrate in the space variable. We obtain the differential inequality

$$\frac{d}{dt}\|\rho\|_{L^p} \leq 0. \tag{3.71}$$

This gives (3.70).

Remark 3.2. *If* $\rho_0 \in L^\infty(\mathbb{R}^2)$ *, then*

$$\|\rho(\cdot, t)\|_{L^\infty} \leq \frac{\|\rho_0\|_{L^\infty}}{1 + Ct\|\rho_0\|_{L^\infty}} \tag{3.72}$$

for $t \in [0, T)$ *(see [22]). This bound is useful to study the long time behavior of solutions.*

Definition 3.2. *A weak solution* ρ *of* (3.43)–(3.45) *is said to be a strong solution on* $[0, T]$ *if it obeys*

$$\rho \in L^\infty(0, T; \dot{H}^2(\mathbb{R}^2)) \cap L^2(0, T; \dot{H}^{\frac{5}{2}}(\mathbb{R}^2)). \tag{3.73}$$

Theorem 3.2. *Let* $\rho_0 \in H^2(\mathbb{R}^2)$. *Then there exists* $T_0 > 0$ *depending only on* $\|\rho_0\|_{H^2}$ *such that a unique strong solution of* (3.43)–(3.45) *exists on* $[0, T_0]$.

Proof: We apply $-\Delta = \Lambda^2$ to (3.48) and we obtain

$$-\partial_t \Delta\rho^\epsilon - \widetilde{u}^\epsilon \cdot \nabla\Delta\rho^\epsilon - 2\nabla\widetilde{u}^\epsilon \nabla\nabla\rho^\epsilon - \Delta\widetilde{u}^\epsilon \cdot \nabla\rho^\epsilon + \Lambda^3\rho^\epsilon + \epsilon\Delta\Delta\rho^\epsilon = 0 \quad (3.74)$$

We multiply (3.74) by $-\Delta\rho^\epsilon$ and we integrate over $\mathbb{R}^2$. In view of the fact that

$$(\widetilde{u}^\epsilon \cdot \nabla\Delta\rho^\epsilon, \Delta\rho^\epsilon)_{L^2} = 0, \quad (3.75)$$

we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta\rho^\epsilon\|_{L^2}^2 + \|\Lambda^{\frac{5}{2}}\rho^\epsilon\|_{L^2}^2 + \epsilon\|\Lambda^3\rho^\epsilon\|_{L^2}^2$$
$$= -2(\nabla\widetilde{u}^\epsilon\nabla\nabla\rho^\epsilon, \Delta\rho^\epsilon)_{L^2} - (\Delta\widetilde{u}^\epsilon \cdot \nabla\rho^\epsilon, \Delta\rho^\epsilon)_{L^2}. \quad (3.76)$$

Using the product rule

$$\|fg\|_{H^s} \le C\|f\|_{H^s}\|g\|_{L^\infty} + C\|g\|_{H^s}\|f\|_{L^\infty} \quad (3.77)$$

that holds for any $f, g \in H^s$, $s > 0$, we estimate

$$\|\nabla\widetilde{u}^\epsilon\|_{L^4} \le C\|\widetilde{u}^\epsilon\|_{H^{\frac{3}{2}}} \le C\|\rho^\epsilon R\rho^\epsilon\|_{H^{\frac{3}{2}}}$$
$$\le C\|\rho^\epsilon\|_{L^\infty}\|R\rho^\epsilon\|_{H^{\frac{3}{2}}} + C\|R\rho^\epsilon\|_{L^\infty}\|\rho^\epsilon\|_{H^{\frac{3}{2}}}$$
$$\le C\|\rho^\epsilon\|_{H^{\frac{3}{2}}}^2. \quad (3.78)$$

Here, we have used the continuous embedding $H^{\frac{1}{2}} \hookrightarrow L^4$, the fact that the Leray projector is bounded on $H^{\frac{3}{2}}$, and the boundedness of the Riesz transforms as operators from $H^{\frac{3}{2}}$ into L^∞. Similarly, we bound

$$\|\Delta\widetilde{u}^\epsilon\|_{L^4} \le C\|\rho^\epsilon R\rho^\epsilon\|_{H^{\frac{5}{2}}}$$
$$\le C\|\rho^\epsilon\|_{L^\infty}\|R\rho^\epsilon\|_{H^{\frac{5}{2}}} + C\|R\rho^\epsilon\|_{L^\infty}\|R\rho^\epsilon\|_{H^{\frac{5}{2}}}$$
$$\le C\|\rho^\epsilon\|_{H^{\frac{3}{2}}}\|\rho^\epsilon\|_{H^{\frac{5}{2}}} \quad (3.79)$$

Consequently,

$$\frac{1}{2}\frac{d}{dt}\|\Delta\rho^\epsilon\|_{L^2}^2 + \|\Lambda^{\frac{5}{2}}\rho^\epsilon\|_{L^2}^2$$
$$\leq 2\|\nabla\widetilde{u}^\epsilon\|_{L^4}\|\nabla\nabla\rho^\epsilon\|_{L^4}\|\Delta\rho^\epsilon\|_{L^2} + \|\Delta\widetilde{u}^\epsilon\|_{L^4}\|\nabla\rho^\epsilon\|_{L^4}\|\Delta\rho^\epsilon\|_{L^2}$$
$$\leq C\|\rho^\epsilon\|_{H^{\frac{3}{2}}}^2\|\rho^\epsilon\|_{H^{\frac{5}{2}}}\|\Delta\rho^\epsilon\|_{L^2} \tag{3.80}$$

and by Young's inequality, we obtain

$$\frac{d}{dt}\|\Delta\rho^\epsilon\|_{L^2}^2 + \|\Lambda^{\frac{5}{2}}\rho^\epsilon\|_{L^2}^2$$
$$\leq C\|\rho^\epsilon\|_{H^{\frac{3}{2}}}^4\|\Delta\rho^\epsilon\|_{L^2}^2 + C\|\rho^\epsilon\|_{H^{\frac{3}{2}}}^2\|\rho^\epsilon\|_{L^2}\|\Delta\rho^\epsilon\|_{L^2}$$
$$\leq C(\|\rho^\epsilon\|_{H^2}^6 + \|\rho^\epsilon\|_{H^2}^4). \tag{3.81}$$

We note that

$$\|\rho^\epsilon\|_{H^2} = \left\|(1+|.|^2)\mathcal{F}(\rho^\epsilon)(\cdot)\right\|_{L^2} \leq C\|\mathcal{F}\rho^\epsilon\|_{L^2} + C\|\Delta\rho^\epsilon\|_{L^2}$$
$$= C\|\rho^\epsilon\|_{L^2} + C\|\Delta\rho^\epsilon\|_{L^2} \leq C\|\rho_0\|_{L^2} + C\|\Delta\rho^\epsilon\|_{L^2} \tag{3.82}$$

in view of Plancherel's theorem and the uniform boundedness of ρ^ϵ in L^2 described by (3.52). Therefore, we obtain the differential inequality

$$\frac{d}{dt}\|\Delta\rho^\epsilon\|_{L^2}^2 + \|\Lambda^{\frac{5}{2}}\rho^\epsilon\|_{L^2}^2 \leq C\|\Delta\rho^\epsilon\|_{L^2}^6 + C_{\rho_0} \tag{3.83}$$

where C_{ρ_0} is a positive constant depending only on ρ_0 and some universal constants. This implies that

$$\frac{d}{dt}\left(\|\Delta\rho^\epsilon\|_{L^2}^2 + 1\right) \leq C_0\left(\|\Delta\rho^\epsilon\|_{L^2}^2 + 1\right)^3 \tag{3.84}$$

for some constant C_0 depending only on the initial data. Diving both sides by $\left(\|\Delta\rho^\epsilon\|_{L^2}^2 + 1\right)^3$ and integrating in time from 0 to t, we get

$$\frac{1}{2\left(\|\Delta\rho^\epsilon(t)\|_{L^2}^2 + 1\right)^2} \geq \frac{1}{2\left(\|\Delta\rho_0\|_{L^2}^2 + 1\right)^2} - C_0T_0 \tag{3.85}$$

for all $t \in [0, T_0]$. We choose a positive time $T_0 > 0$ such that

$$T_0 < \frac{1}{2C_0 \left(\|\Delta\rho_0\|_{L^2}^2 + 1\right)^2} \tag{3.86}$$

and we conclude that

$$\|\Delta\rho^\epsilon(t)\|_{L^2}^2 \leq \frac{\|\Delta\rho_0\|_{L^2}^2 + 1}{\sqrt{1 - 2C_0T_0\left(\|\Delta\rho_0\|_{L^2}^2 + 1\right)^2}} \tag{3.87}$$

for all $t \in [0, T_0]$. In view of the energy inequality (3.83), we also have that

$$\int_0^{T_0} \|\Lambda^{\frac{5}{2}}\rho^\epsilon(t)\|_{L^2}^2 dt \leq \Gamma(\rho_0, T_0) \tag{3.88}$$

where $\Gamma(\rho_0, T_0)$ is a positive constant depending only on the initial data and T_0. This shows that that $\{\rho^\epsilon : \epsilon \in (0,1]\}$ is uniformly bounded in

$$L^\infty(0, T; \dot{H}^2(\mathbb{R}^2)) \cap L^2(0, T; \dot{H}^{\frac{5}{2}}(\mathbb{R}^2)). \tag{3.89}$$

Passing to the limit on a subsequence and using the lower semi-continuity of norms, we conclude that the weak solution ρ, obtained in Theorem 3.1, is strong.

For uniqueness, suppose that ρ_1 and ρ_2 are two strong solutions of (3.43) on $[0, T_0]$ with the same initial condition. Let $\rho = \rho_1 - \rho_2$ and $u = u_1 - u_2$. Then ρ obeys the equation

$$\partial_t\rho + u \cdot \nabla\rho_1 + u_2 \cdot \nabla\rho + \Lambda\rho = 0. \tag{3.90}$$

We take the L^2 inner product with ρ and we obtain

$$\frac{1}{2}\frac{d}{dt}\|\rho\|_{L^2}^2 + \|\Lambda^{\frac{1}{2}}\rho\|_{L^2}^2 = -(u \cdot \nabla\rho_1, \rho)_{L^2}. \tag{3.91}$$

In view of the boundedness of the Riesz transforms on L^4, we have

$$\begin{aligned}\|u\|_{L^4} &\leq \|\mathbb{P}(\rho R\rho_1)\|_{L^4} + \|\mathbb{P}(\rho_2 R\rho)\|_{L^4} \\ &\leq C\|\rho\|_{L^4}\|R\rho_1\|_{L^\infty} + \|\rho_2\|_{L^\infty}\|R\rho\|_{L^4} \\ &\leq C\|\rho\|_{L^4}\left(\|\rho_1\|_{H^{\frac{3}{2}}} + \|\rho_2\|_{H^{\frac{3}{2}}}\right).\end{aligned} \tag{3.92}$$

Hence

$$\begin{aligned}&|(u\cdot\nabla\rho_1,\rho)_{L^2}| \leq \|u\|_{L^4}\|\nabla\rho_1\|_{L^4}\|\rho\|_{L^2} \\ &\leq \frac{1}{2}\|\rho\|^2_{H^{\frac{1}{2}}} + C\left(\|\rho_1\|^2_{H^{\frac{3}{2}}} + \|\rho_2\|^2_{H^{\frac{3}{2}}}\right)\|\rho_1\|^2_{H^{\frac{3}{2}}}\|\rho\|^2_{L^2}.\end{aligned} \tag{3.93}$$

Therefore,

$$\frac{d}{dt}\|\rho\|^2_{L^2} \leq K(t)\|\rho\|^2_{L^2} \tag{3.94}$$

where

$$K(t) = C\left(\|\rho_1\|^2_{H^{\frac{3}{2}}} + \|\rho_2\|^2_{H^{\frac{3}{2}}}\right)\|\rho_1\|^2_{H^{\frac{3}{2}}}. \tag{3.95}$$

We note that $K(t)$ is time-integrable on $[0,T_0]$ since ρ_1 and ρ_2 belong to the space $L^\infty(0,T_0;H^2(\mathbb{R}^2))$. This shows that for each $t \geq 0$, $\rho_1(\cdot,t) = \rho_2(\cdot,t)$ a.e. in $\mathbb{R}^2$, and so we obtain uniqueness. This completes the proof of Theorem 3.2.

3.4 Existence of Global Solutions in Besov Spaces

In this section, we show the existence of a global in time solution in Besov spaces for sufficiently small initial data. The proof uses methods of [4, 11].

Theorem 3.3. *Let* $1 \le p < \infty$. *Let* $\rho_0 \in \dot{B}^{\frac{2}{p}}_{p,1}(\mathbb{R}^2)$ *be sufficiently small. We consider the functional space* E_p *defined by*

$$E_p = \left\{ f(t) \in \mathcal{S}_h'(\mathbb{R}^2) : \|f\|_{E_p} = \|f\|_{\tilde{L}^\infty_t \dot{B}^{\frac{2}{p}}_{p,1}} + \|f\|_{\tilde{L}^1_t \dot{B}^{\frac{2}{p}+1}_{p,1}} < \infty \right\}. \tag{3.96}$$

Then (3.43)–(3.45) *has a unique global in time solution* $\rho \in E_p$.

Proof: Let $\rho^{(0)} = 0$. For each positive integer n, let $\rho^{(n)}$ be the solution of

$$\partial_t \rho^{(n)} + \Lambda \rho^{(n)} = -u^{(n-1)} \cdot \nabla \rho^{(n-1)} \tag{3.97}$$

in $\mathbb{R}^2$, where

$$u^{(n-1)} = -\mathbb{P}(\rho^{(n-1)} R \rho^{(n-1)}), \tag{3.98}$$

with initial data

$$\rho_0^{(n)} = \rho^{(n)}(\cdot, 0) = \rho_0. \tag{3.99}$$

We write $\rho^{(n)}$ in the integral form,

$$\begin{aligned} \rho^{(n)}(t) &= e^{-t\Lambda}\rho_0 - \int_0^t e^{-(t-s)\Lambda} \nabla \cdot (u^{(n-1)} \rho^{(n-1)})(s) ds \\ &= e^{-t\Lambda}\rho_0 - \mathcal{B}(u^{n-1}, \rho^{n-1}) \end{aligned} \tag{3.100}$$

where $\mathcal{B}$ is the bilinear form defined by

$$\mathcal{B}(v, \theta) = \int_0^t e^{-(t-s)\Lambda} \nabla \cdot (v\theta)(s) ds. \tag{3.101}$$

See [11] for a similar approach.

Step 1. Fix a positive integer n. We show that

$$\|\rho^{(n)}\|_{E_p} \le C_1 \|\rho_0\|_{\dot{B}^{\frac{2}{p}}_{p,1}} + C_2 \|\rho^{(n-1)}\|^3_{E_p}. \tag{3.102}$$

We start by estimating $e^{-t\Lambda}\rho_0$ in E_p. We apply Δ_j and we take the L^p norm. In view of the bound (3.30), we have

$$\|e^{-t\Lambda}\Delta_j\rho_0\|_{L^p} \le Ce^{-C^{-1}t2^j}\|\Delta_j\rho_0\|_{L^p}, \tag{3.103}$$

hence

$$\|e^{-t\Lambda}\rho_0\|_{E_p} = \|e^{-t\Lambda}\rho_0\|_{\tilde{L}_t^\infty \dot{B}_{p,1}^{\frac{2}{p}}} + \|e^{-t\Lambda}\rho_0\|_{\tilde{L}_t^1 \dot{B}_{p,1}^{\frac{2}{p}+1}} \le C\|\rho_0\|_{\dot{B}_{p,1}^{\frac{2}{p}}}. \tag{3.104}$$

Now, we estimate the term $\mathcal{B}(u^{(n-1)}, \rho^{(n-1)})$ in E_p. First, we note that

$$\|\mathcal{B}(u^{(n-1)}, \rho^{(n-1)})\|_{E^p} \le C\|u^{(n-1)}\rho^{(n-1)}\|_{\tilde{L}_t^1 \dot{B}_{p,1}^{\frac{2}{p}+1}}. \tag{3.105}$$

Indeed, we apply Δ_j to $\mathcal{B}(u^{(n-1)}, \rho^{(n-1)})$ and we estimate. On one hand,

$$\begin{aligned}\|\Delta_j\mathcal{B}(u^{(n-1)}, \rho^{(n-1)})\|_{L_t^\infty L^p} &\le C2^j\left\|\int_0^t e^{-c^{-1}(t-s)2^j}\|\Delta_j(u^{(n-1)}\rho^{(n-1)})(s)\|_{L^p}ds\right\|_{L_t^\infty} \\ &\le C2^j\|\Delta_j(u^{(n-1)}\rho^{(n-1)})\|_{L_t^1L^p} \end{aligned}\tag{3.106}$$

in view of Bernstein's inequality (3.27) and the bound (3.30). We multiply by $2^{j\frac{2}{p}}$ and we take the ℓ^1 norm. We obtain the bound

$$\|\mathcal{B}(u^{(n-1)}, \rho^{(n-1)})\|_{\tilde{L}_t^\infty \dot{B}_{p,1}^{\frac{2}{p}}} \le C\|u^{(n-1)}\rho^{(n-1)}\|_{\tilde{L}_t^1 \dot{B}_{p,1}^{\frac{2}{p}+1}}. \tag{3.107}$$

On the other hand,

$$\begin{aligned}&\|\Delta_j\mathcal{B}(u^{(n-1)}, \rho^{(n-1)})\|_{L_t^1L^p} \le C\left\|\int_0^t 2^je^{-c^{-1}(t-s)2^j}\|\Delta_j(u^{(n-1)}\rho^{(n-1)})(s)\|_{L^p}ds\right\|_{L_t^1} \\ &\le C\int_0^\infty\left(\int_0^\infty 2^je^{-c^{-1}(t-s)2^j}\chi_{[0,t]}(s)dt\right)\|\Delta_j(u^{(n-1)}\rho^{(n-1)})(s)\|_{L^p}ds \\ &\le C\|\Delta_j(u^{(n-1)}\rho^{(n-1)})\|_{L_t^1L^p}\end{aligned}\tag{3.108}$$

where χ_E denotes the characteristic function of the set E. Multiplying by $2^{j\left(\frac{2}{p}+1\right)}$ and taking the ℓ^1 norm yields the bound

$$\|\mathcal{B}(u^{(n-1)}, \rho^{(n-1)})\|_{\tilde{L}_t^1 \dot{B}_{p,1}^{\frac{2}{p}+1}} \le C\|u^{(n-1)}\rho^{(n-1)}\|_{\tilde{L}_t^1 \dot{B}_{p,1}^{\frac{2}{p}+1}}. \tag{3.109}$$

Combining (3.107) and (3.109), we obtain (3.105). Accordingly, our next goal is to show that

$$\|u^{(n-1)}\rho^{(n-1)}\|_{\tilde{L}^1_t \dot{B}^{\frac{2}{p}+1}_{p,1}} \le C\|\rho^{(n-1)}\|^3_{E^p} \tag{3.110}$$

which gives (3.102). In order to establish the bound (3.110), we use the decomposition (3.32)

$$\begin{aligned}\Delta_j(u^{(n-1)}\rho^{(n-1)}) = &\sum_{k\ge j-2} \Delta_j(S_k u^{(n-1)} \Delta_k \rho^{(n-1)})\\ &+ \sum_{k\ge j-2} \Delta_j(S_{k+1}\rho^{(n-1)}\Delta_k u^{(n-1)}).\end{aligned} \tag{3.111}$$

We apply the $L^1_t L^p$ norm, we use the bound

$$\|\Delta_j f\|_{L^p} \le C\|f\|_{L^p} \tag{3.112}$$

that holds for any $f \in \mathcal{S}'_h$ where C is a positive universal constant independent of j, and we obtain

$$\begin{aligned}\|\Delta_j(u^{(n-1)}\rho^{(n-1)})\|_{L^1_t L^p} \le C &\sum_{k\ge j-2} \|S_k u^{(n-1)}\|_{L^\infty_t L^\infty} \|\Delta_k \rho^{(n-1)}\|_{L^1_t L^p}\\ &+ C\sum_{k\ge j-2} \|S_{k+1}\rho^{(n-1)}\|_{L^\infty_t L^\infty}\|\Delta_k u^{(n-1)}\|_{L^1_t L^p}.\end{aligned} \tag{3.113}$$

In view of Bernstein's inequality (3.28), we have

$$\begin{aligned}\|S_{k+1}\rho^{(n-1)}\|_{L^\infty_t L^\infty} &\le \sum_{l\le k} \|\Delta_l \rho^{(n-1)}\|_{L^\infty_t L^\infty}\\ &\le C\sum_{l\le k} 2^{l\frac{2}{p}} \|\Delta_l \rho^{(n-1)}\|_{L^\infty_t L^p} \le C\|\rho^{(n-1)}\|_{\tilde{L}^\infty_t \dot{B}^{\frac{2}{p}}_{p,1}}.\end{aligned} \tag{3.114}$$

We show below that

$$\|S_k u^{(n-1)}\|_{L^\infty_t L^\infty} \le C\|\rho^{(n-1)}\|^2_{\tilde{L}^\infty_t \dot{B}^{\frac{2}{p}}_{p,1}} \tag{3.115}$$

and

$$\|\Delta_k u^{(n-1)}\|_{L^1_t L^p} \le C\|\rho^{(n-1)}\|_{\tilde{L}^\infty_t \dot{B}^{\frac{2}{p}}_{p,1}} \left(\sum_{m\ge k-2} \|\Delta_m \rho^{(n-1)}\|_{L^1_t L^p} \right). \tag{3.116}$$

Using the bounds (3.115) and (3.116), we obtain

$$\begin{aligned} \|\Delta_j(u^{(n-1)}\rho^{(n-1)})\|_{L^1_t L^p} &\le C\|\rho^{(n-1)}\|^2_{\tilde{L}^\infty_t \dot{B}^{\frac{2}{p}}_{p,1}} \sum_{k\ge j-2} \|\Delta_k \rho^{(n-1)}\|_{L^1_t L^p} \\ &+ C\|\rho^{(n-1)}\|^2_{\tilde{L}^\infty_t \dot{B}^{\frac{2}{p}}_{p,1}} \sum_{k\ge j-2} \sum_{m\ge k-2} \|\Delta_m \rho^{(n-1)}\|_{L^1_t L^p}. \end{aligned} \tag{3.117}$$

We multiply (3.117) by $2^{j\left(\frac{2}{p}+1\right)}$ and we take the ℓ^1 norm. In view of Young's convolution inequality, we have in the first term

$$\begin{aligned} &\sum_{j\in\mathbb{Z}} \sum_{k\ge j-2} 2^{j\left(\frac{2}{p}+1\right)} \|\Delta_k \rho^{(n-1)}\|_{L^1_t L^p} \\ &= \sum_{j\in\mathbb{Z}} \sum_{k\ge j-2} 2^{-(k-j)\left(\frac{2}{p}+1\right)} 2^{k\left(\frac{2}{p}+1\right)} \|\Delta_k \rho^{(n-1)}\|_{L^1_t L^p} \\ &\le \left(\sum_{j\ge -2} 2^{-j\left(\frac{2}{p}+1\right)} \right) \left(\sum_{j\in\mathbb{Z}} 2^{j\left(\frac{2}{p}+1\right)} \|\Delta_j \rho^{(n-1)}\|_{L^1_t L^p} \right) \\ &\le C\|\rho^{(n-1)}\|_{\tilde{L}^1_t \dot{B}^{\frac{2}{p}+1}_{p,1}}. \end{aligned} \tag{3.118}$$

For the second summation on the right hand side of (3.117), we apply Fubini's theorem and write it as

$$\begin{aligned} &\sum_{j\in\mathbb{Z}} \sum_{k\ge j-2} \sum_{m\ge k-2} 2^{j\left(\frac{2}{p}+1\right)} \|\Delta_m \rho^{(n-1)}\|_{L^1_t L^p} \\ &= \sum_{j\in\mathbb{Z}} \sum_{m\ge j-4} \sum_{j-2\le k\le m+2} 2^{-(m-j)\left(\frac{2}{p}+1\right)} 2^{m\left(\frac{2}{p}+1\right)} \|\Delta_m \rho^{(n-1)}\|_{L^1_t L^p} \\ &= \sum_{j\in\mathbb{Z}} \sum_{m\ge j-4} (m-j+5) 2^{-(m-j)\left(\frac{2}{p}+1\right)} 2^{m\left(\frac{2}{p}+1\right)} \|\Delta_m \rho^{(n-1)}\|_{L^1_t L^p}. \end{aligned}$$

Now estimate as in (3.118) and obtain

$$\sum_{j\in\mathbb{Z}}\sum_{k\geq j-2}\sum_{m\geq k-2}2^{j\left(\frac{2}{p}+1\right)}\|\Delta_m\rho^{(n-1)}\|_{L^1_tL^p}$$

$$\leq C\sum_{j\in\mathbb{Z}}\sum_{m\geq j-4}2^{-(m-j)\left(\frac{1}{p}+\frac{1}{2}\right)}2^{m\left(\frac{2}{p}+1\right)}\|\Delta_m\rho^{(n-1)}\|_{L^1_tL^p}$$

$$+5\sum_{j\in\mathbb{Z}}\sum_{m\geq j-4}2^{-(m-j)\left(\frac{2}{p}+1\right)}2^{m\left(\frac{2}{p}+1\right)}\|\Delta_m\rho^{(n-1)}\|_{L^1_tL^p}$$

$$\leq C\|\rho^{(n-1)}\|_{\tilde{L}^1_t\dot{B}^{\frac{2}{p}+1}_{p,1}}. \tag{3.119}$$

Here, we have used the fact that $x2^{-x}\leq C2^{-\frac{x}{2}}$ for all $x\in\mathbb{R}$. Putting (3.118) and (3.119) together, we obtain (3.110).

We end the proof of Step 1 by showing the estimates (3.115) and (3.116). For each $l\in\mathbb{Z}$, we use again paraproducts to decompose $\Delta_l(\rho^{(n-1)}R\rho^{(n-1)})$ as

$$\Delta_l(\rho^{(n-1)}R\rho^{(n-1)})=\sum_{m\geq l-2}\Delta_l(S_{m+1}\rho^{(n-1)}\Delta_mR\rho^{(n-1)})$$

$$+\sum_{m\geq l-2}\Delta_l(S_mR\rho^{(n-1)}\Delta_m\rho^{(n-1)}). \tag{3.120}$$

In view of the boundedness of the Riesz transform (3.31) and the definition of the Leray projector as

$$\mathbb{P}=I+R\otimes R, \tag{3.121}$$

we bound

$$\|S_ku^{(n-1)}\|_{L^\infty_tL^\infty}\leq\sum_{l\leq k-1}\|\Delta_lu^{(n-1)}\|_{L^\infty_tL^\infty}\leq C\sum_{l\leq k-1}2^{l\frac{2}{p}}\|\Delta_lu^{(n-1)}\|_{L^\infty_tL^p}$$

$$\leq C\sum_{l\leq k-1}2^{l\frac{2}{p}}\|\Delta_l(\rho^{(n-1)}R\rho^{(n-1)})\|_{L^\infty_tL^p} \tag{3.122}$$

for any $p \in [1,\infty]$ and using the paraproduct decomposition (3.120), we obtain

$$\|S_k u^{(n-1)}\|_{L_t^\infty L^\infty} \le C \sum_{l\le k-1} 2^{l\frac{2}{p}} \sum_{m\ge l-2} \|S_{m+1}\rho^{(n-1)}\|_{L_t^\infty L^\infty} \|\Delta_m R\rho^{(n-1)}\|_{L_t^\infty L^p}$$

$$+ C \sum_{l\le k-1} 2^{l\frac{2}{p}} \sum_{m\ge l-2} \|S_m R\rho^{(n-1)}\|_{L_t^\infty L^\infty} \|\Delta_m \rho^{(n-1)}\|_{L_t^\infty L^p}. \tag{3.123}$$

We note that

$$\|S_{m+1}\rho^{(n-1)}\|_{L_t^\infty L^\infty} \le C\|\rho^{(n-1)}\|_{\tilde{L}_t^\infty \dot{B}_{p,1}^{\frac{2}{p}}} \tag{3.124}$$

as shown in (3.114). Moreover, in view of (3.31), we have

$$\|S_m R\rho^{(n-1)}\|_{L_t^\infty L^\infty} \le \sum_{z\le m-1} \|\Delta_z R\rho^{(n-1)}\|_{L_t^\infty L^\infty}$$

$$\le C \sum_{z\le m-1} 2^{z\frac{2}{p}} \|\Delta_z R\rho^{(n-1)}\|_{L_t^\infty L^p} \le C \sum_{z\le m-1} 2^{z\frac{2}{p}} \|\Delta_z \rho^{(n-1)}\|_{L_t^\infty L^p}$$

$$\le C\|\rho^{(n-1)}\|_{\tilde{L}_t^\infty \dot{B}_{p,1}^{\frac{2}{p}}}. \tag{3.125}$$

Now we use the assumption that $p < \infty$ which implies that $\frac{2}{p} > 0$ and so we can apply Young's convolution inequality to obtain

$$\|S_k u^{(n-1)}\|_{L_t^\infty L^\infty} \le C\|\rho^{(n-1)}\|_{\tilde{L}_t^\infty \dot{B}_{p,1}^{\frac{2}{p}}} \left\{ \sum_{l\le k-1} 2^{l\frac{2}{p}} \sum_{m\ge l-2} \|\Delta_m \rho^{(n-1)}\|_{L_t^\infty L^p} \right\}$$

$$= C\|\rho^{(n-1)}\|_{\tilde{L}_t^\infty \dot{B}_{p,1}^{\frac{2}{p}}} \left\{ \sum_{l\le k-1} \sum_{m\ge l-2} 2^{-(m-l)\frac{2}{p}} 2^{m\frac{2}{p}} \|\Delta_m \rho^{(n-1)}\|_{L_t^\infty L^p} \right\}$$

$$\le C\|\rho^{(n-1)}\|^2_{\tilde{L}_t^\infty \dot{B}_{p,1}^{\frac{2}{p}}} \tag{3.126}$$

which proves (3.115). We proceed to show (3.116). Using the paraproduct decomposition (3.120) and the bound (3.31), we have

$$\begin{aligned}\|\Delta_k u^{(n-1)}\|_{L^1_t L^p} &\le C\|\Delta_k(\rho^{(n-1)} R\rho^{(n-1)})\|_{L^1_t L^p} \\ &\le C \sum_{m\ge k-2} \|S_{m+1}\rho^{(n-1)}\|_{L^\infty_t L^\infty} \|\Delta_m R\rho^{(n-1)}\|_{L^1_t L^p} \\ &+ C \sum_{m\ge k-2} \|S_m R\rho^{(n-1)}\|_{L^\infty_t L^\infty} \|\Delta_m \rho^{(n-1)}\|_{L^1_t L^p} \\ &\le C\|\rho^{(n-1)}\|_{\tilde{L}^\infty_t \dot{B}^{\frac{2}{p}}_{p,1}} \left(\sum_{m\ge k-2} \|\Delta_m \rho^{(n-1)}\|_{L^1_t L^p} \right) \end{aligned} \tag{3.127}$$

yielding (3.116). This ends the proof of Step 1.

Step 2. We show that there exists an $\epsilon > 0$ sufficiently small such that if $C_1\|\rho_0\|_{\dot{B}^{\frac{2}{p}}_{p,1}} < \epsilon$, then the sequence $\left\{\rho^{(n)}\right\}_{n=1}^{\infty}$ converges to a unique solution ρ of (3.43)–(3.45) obeying $\|\rho\|_{E_p} < 2\epsilon$.

First, choose an $\epsilon > 0$ such that $C_2(2\epsilon)^3 < \epsilon$, where C_2 is the constant in (3.102), and suppose that $C_1\|\rho_0\|_{\dot{B}^{\frac{2}{p}}_{p,1}} < \epsilon$. Then an inductive argument yields

$$\|\rho^{(n)}\|_{E_p} < 2\epsilon \tag{3.128}$$

for all $n \ge 1$. Indeed,

$$\|\rho^{(1)}\|_{E_p} \le C_1\|\rho_0\|_{\dot{B}^{\frac{2}{p}}_{p,1}} < \epsilon < 2\epsilon \tag{3.129}$$

in view of (3.102). Suppose that

$$\|\rho^{(n-1)}\|_{E_p} < 2\epsilon. \tag{3.130}$$

Then

$$\|\rho^{(n)}\|_{E_p} < \epsilon + C_2(2\epsilon)^3 < \epsilon + \epsilon = 2\epsilon. \tag{3.131}$$

Therefore, we obtain (3.128)

Now, we show that the sequence $\left\{\rho^{(n)}\right\}_{n=1}^{\infty}$ is Cauchy. Indeed, the difference $\rho^{(n)} - \rho^{(n-1)}$ obeys

$$
\begin{aligned}
&(\rho^{(n)} - \rho^{(n-1)})(t) \\
&= \int_0^t e^{-(t-s)\Lambda}\nabla \cdot \left[u^{(n)}(\rho^{(n)} - \rho^{(n-1)}) - (u^{(n-1)} - u^{(n)})\rho^{(n-1)}\right](s)ds \\
&= \mathcal{B}(u^{(n)}, \rho^{(n)} - \rho^{(n-1)}) - \mathcal{B}(u^{(n-1)} - u^{(n)}, \rho^{(n-1)}). \qquad (3.132)
\end{aligned}
$$

As in Step 1 and using (3.128), it can be shown that

$$
\begin{aligned}
&\|\rho^{(n)} - \rho^{(n-1)}\|_{E_p} \\
&\leq \|\mathcal{B}(u^{(n)}, \rho^{(n)} - \rho^{(n-1)})\|_{E_p} + \|\mathcal{B}(u^{(n-1)} - u^{(n)}, \rho^{(n-1)})\|_{E_p} \\
&\leq C(\epsilon)\|\rho^{(n-1)} - \rho^{(n-2)}\|_{E_p} \qquad (3.133)
\end{aligned}
$$

where $C(\epsilon)$ is a constant depending on ϵ obeying $C(\epsilon) < 1$ for a sufficiently small ϵ. Therefore, the sequence $\left\{\rho^{(n)}\right\}_{n=1}^{\infty}$ is Cauchy in E_p and converges to a solution ρ of (3.43)–(3.45). Uniqueness follows from a similar estimate to (3.133). This finishes the proof of Step 2. Therefore the proof of Theorem 3.3 is complete.

3.5 Regularity of Solutions for Arbitrary Initial Data

In this section, we prove that any solution of (3.43)–(3.45) is smooth for arbitrary initial data, provided that it satisfies a certain regularity condition.

Theorem 3.4. *Let ρ be a weak solution of* (3.43)–(3.45) *on* $[0,\infty)$. *Let* $0 < t_0 < t < \infty$. *If*

$$\rho \in L^{\infty}([t_0, t]; C^{\delta}(\mathbb{R}^2)), \qquad (3.134)$$

for some $\delta \in (0,1)$*, then*

$$\rho \in C^\infty((t_0,t] \times \mathbb{R}^2). \tag{3.135}$$

Proof: We sketch the main ideas. Let us note first that

$$u \in L^\infty([t_0,t]; C^\delta(\mathbb{R}^2)). \tag{3.136}$$

where

$$u = -\mathbb{P}(\rho R\rho). \tag{3.137}$$

Indeed, for any $s \in [t_0,t]$, we have

$$\begin{aligned} &\|u(s)\|_{C^\delta} \le C\|\rho(s)R\rho(s)\|_{C^\delta} \\ &\le C\|\rho(s)\|_{L^\infty}\|R\rho(s)\|_{L^\infty} + C\|\rho(s)\|_{L^\infty}\|R\rho(s)\|_{C^\delta} + C\|R\rho(s)\|_{L^\infty}\|\rho(s)\|_{C^\delta} \\ &\le C\|\rho(s)\|_{C^\delta}^2 \end{aligned} \tag{3.138}$$

in view of the boundedness of the Leray projector and Riesz transforms on the Hölder space C^δ. Consequently, the Hölder regularity of ρ imposed in (3.134) gives (3.136).

Next, we show that

$$\rho \in L^\infty([t_0,t]; \dot{B}^{\delta_1}_{p,\infty}(\mathbb{R}^2) \cap C^{\delta_1}(\mathbb{R}^2)) \tag{3.139}$$

and

$$u \in L^\infty([t_0,t]; \dot{B}^{\delta_1}_{p,\infty}(\mathbb{R}^2) \cap C^{\delta_1}(\mathbb{R}^2)) \tag{3.140}$$

for any $p \geq 2$ and $\delta_1 = \delta\left(1 - \frac{2}{p}\right)$. Indeed, for any $s \in [t_0, t]$, we have

$$\begin{aligned}
\|\rho(s)\|_{\dot{B}^{\delta_1}_{p,\infty}} &= \sup_{j\in\mathbb{Z}} \left(2^{\delta_1 j}\|\Delta_j \rho(s)\|_{L^p}\right) \\
&\leq \sup_{j\in\mathbb{Z}} \left(2^{\delta_1 j}\|\Delta_j \rho(s)\|_{L^\infty}^{1-\frac{2}{p}} \|\Delta_j \rho(s)\|_{L^2}^{\frac{2}{p}}\right) \\
&\leq C\left(\|\rho(s)\|_{\dot{B}^{\delta}_{\infty,\infty}}\right)^{1-\frac{2}{p}} \|\rho(s)\|_{L^2}^{\frac{2}{p}} \leq C\left(\|\rho(s)\|_{C^\delta}\right)^{1-\frac{2}{p}} \|\rho(s)\|_{L^2}^{\frac{2}{p}} \qquad (3.141)
\end{aligned}$$

and similarly

$$\begin{aligned}
\|u(s)\|_{\dot{B}^{\delta_1}_{p,\infty}} &\leq C\left(\|u(s)\|_{\dot{B}^{\delta}_{\infty,\infty}}\right)^{1-\frac{2}{p}} \|u(s)\|_{L^2}^{\frac{2}{p}} \\
&\leq C\left(\|u(s)\|_{C^\delta}\right)^{1-\frac{2}{p}} \|\rho(s)\|_{L^4}^{\frac{4}{p}}. \qquad (3.142)
\end{aligned}$$

The last inequality holds in view of the boundedness of the Leray projector on L^2 followed by an application of Hölder's inequality with exponents $4, 4$. The interpolation inequality

$$\|\rho(s)\|_{L^4} \leq \|\rho(s)\|_{L^\infty}^{1/2} \|\rho(s)\|_{L^2}^{1/2} \qquad (3.143)$$

together with (3.136) and (3.134) gives (3.139) and (3.140).

Now, we proceed as in [20]. We apply Δ_j to (3.43), we multiply the resulting equation by $p|\Delta_j \rho|^{p-2}\Delta_j \rho$, we integrate first in the space variable $x \in \mathbb{R}^2$ and then in time from t_0 to t. We obtain the bound

$$\begin{aligned}
\|\Delta_j \rho(t)\|_{L^p} &\leq C e^{-c2^j(t-t_0)}\|\Delta_j \rho(t_0)\|_{L^p} \\
&\quad + C\int_{t_0}^{t} e^{-c2^j(t-s)} 2^{(1-2\delta_1)j} \|\rho(s)\|_{C^{\delta_1}} \|u(s)\|_{\dot{B}^{\delta_1}_{p,\infty}} \\
&\quad + C\int_{t_0}^{t} e^{-c2^j(t-s)} 2^{(1-2\delta_1)j} \|u(s)\|_{C^{\delta_1}} \|\rho(s)\|_{\dot{B}^{\delta_1}_{p,\infty}} ds \qquad (3.144)
\end{aligned}$$

(see [20] for details). We multiply by $2^{2\delta_1 j}$ and we take the ℓ^∞ norm in j. This yields the bound

$$\begin{aligned}\|\rho(t)\|_{\dot{B}^{2\delta_1}_{p,\infty}} \leq C \sup_{j\in\mathbb{Z}} \left\{2^{\delta_1 j} e^{-c2^j(t-t_0)}\right\} \|\rho(t_0)\|_{\dot{B}^{\delta_1}_{p,\infty}} \\ + C \sup_{j\in\mathbb{Z}} \left\{1 - e^{-c2^j(t-t_0)}\right\} \sup_{s\in[t_0,t]} \|\rho(s)\|_{C^{\delta_1}} \|u(s)\|_{\dot{B}^{\delta_1}_{p,\infty}} \\ + C \sup_{j\in\mathbb{Z}} \left\{1 - e^{-c2^j(t-t_0)}\right\} \sup_{s\in[t_0,t]} \|u(s)\|_{C^{\delta_1}} \|\rho(s)\|_{\dot{B}^{\delta_1}_{p,\infty}} \qquad (3.145)\end{aligned}$$

Therefore

$$\rho(\cdot,t) \in \dot{B}^{2\delta_1}_{p,\infty}(\mathbb{R}^2). \qquad (3.146)$$

for any $p \geq 2$. In view of the continuous Besov embedding (3.29), we have the continuous inclusion

$$\dot{B}^{2\delta_1}_{p,\infty}(\mathbb{R}^2) \hookrightarrow \dot{B}^{2\delta_1 - \frac{2}{p}}_{\infty,\infty}(\mathbb{R}^2) \qquad (3.147)$$

for any $p \geq 2$. We choose $p > \frac{2+2\delta}{\delta}$ so that $2\delta_1 - \frac{2}{p} > \delta_1$, hence

$$\rho(\cdot,t) \in \dot{B}^{\delta_2}_{p,\infty}(\mathbb{R}^2) \cap C^{\delta_2}(\mathbb{R}^2) \qquad (3.148)$$

where $\delta_2 > \delta_1$. In fact, the spacial regularity (3.148) holds at any s in $[t_0, t]$ because the pointwise-in-time estimate (3.144) holds at those times as well. Now we iterate the above process infinitely many times to upgrade the spacial regularity of the solution and we simultaneously use the PDE (3.43) to upgrade their time regularity. This yields the desired smoothness (3.135), completing the proof of Theorem 3.4.

3.6 Periodic Case

In this section, we consider the initial value problem (3.43)–(3.45) posed on the torus $\mathbb{T}^2$ with periodic boundary conditions. We assume the initial data ρ_0 have zero mean. We prove existence and regularity of solutions.

Theorem 3.5. *Let* $1 \le p < \infty$. *Let* $\rho_0 \in \dot{B}^{\frac{2}{p}}_{p,1}(\mathbb{T}^2)$ *be sufficiently small. We consider the functional space* E_p *defined by*

$$E_p(\mathbb{T}^2) = \left\{ f(t) \in \mathcal{D}_0'(\mathbb{T}^2) : \|f\|_{E_p(\mathbb{T}^2)} < \infty \right\} \tag{3.149}$$

where

$$\|f\|_{E_p(\mathbb{T}^2)} = \|f\|_{\tilde{L}^\infty_t \dot{B}^{\frac{2}{p}}_{p,1}(\mathbb{T}^2)} + \|f\|_{\tilde{L}^1_t \dot{B}^{\frac{2}{p}+1}_{p,1}(\mathbb{T}^2)}, \tag{3.150}$$

and $\mathcal{D}_0'(\mathbb{T}^2)$ *is the dual space of*

$$\mathcal{D}_0(\mathbb{T}^2) = \left\{ f \in C^\infty(\mathbb{T}^2) : \int_{\mathbb{T}^2} f(x)dx = 0 \right\}.$$

Then (3.43)–(3.45) *has a unique global in time solution* $\rho \in E_p(\mathbb{T}^2)$.

The proof of Theorem 3.5 follows from the proof of Theorem 3.3.

In view of the Besov embedding and Theorem 3.5, we conclude that if $\rho_0 \in \dot{B}^1_{2,1}(\mathbb{T}^2)$ is sufficiently small, then there is a constant $C > 0$ depending only on the initial data such that the unique solution ρ of (3.43)–(3.45) obeys

$$\sup_{t>0} \|\nabla\rho(t)\|_{L^2(\mathbb{T}^2)} + \int_0^\infty \|\Delta\rho(t)\|_{L^2(\mathbb{T}^2)} dt \le C. \tag{3.151}$$

Using this latter estimate, we end this section by showing that the $L^2(\mathbb{T}^2)$ norm of $\Lambda^{\frac{1}{2}}\rho$ converges exponentially in time to zero.

Corollary 3.1. *Let $\rho_0 \in \dot{B}^1_{2,1}(\mathbb{T}^2)$ be sufficiently small. Then there is a constant $C > 0$ depending only on the initial data such that the unique solution ρ of* (3.43)–(3.45) *obeys*

$$\|\Lambda^{\frac{1}{2}}\rho(t)\|^2_{L^2(\mathbb{T}^2)} \le Ce^{-t} \tag{3.152}$$

for all $t \ge 0$.

Proof: We take the inner product in $L^2(\mathbb{T}^2)$ of (3.43) with $\Lambda\rho$ to obtain

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^{\frac{1}{2}}\rho(t)\|^2_{L^2(\mathbb{T}^2)} + \|\Lambda\rho(t)\|^2_{L^2(\mathbb{T}^2)} = -\int_{\mathbb{T}^2}(u\cdot\nabla\rho)\Lambda\rho dx. \tag{3.153}$$

We estimate the nonlinear term

$$\begin{aligned}\left|\int_{\mathbb{T}^2}(u\cdot\nabla\rho)\Lambda\rho dx\right| &\le C\|\rho\|_{L^\infty(\mathbb{T}^2)}\|\rho\|_{L^4(\mathbb{T}^2)}\|\nabla\rho\|_{L^4(\mathbb{T}^2)}\|\nabla\rho\|_{L^2(\mathbb{T}^2)}\\ &\le C\|\rho\|_{L^4(\mathbb{T}^2)}\|\nabla\rho\|^2_{L^4(\mathbb{T}^2)}\|\nabla\rho\|_{L^2(\mathbb{T}^2)}\\ &\le C\|\rho\|^{\frac{1}{2}}_{L^2(\mathbb{T}^2)}\|\nabla\rho\|^{\frac{5}{2}}_{L^2(\mathbb{T}^2)}\|\Delta\rho\|_{L^2(\mathbb{T}^2)}\end{aligned} \tag{3.154}$$

in view of the boundedness of the Leray projector and Riesz transforms on $L^4(\mathbb{T}^2)$, the continuous embedding $W^{1,4}(\mathbb{T}^2) \hookrightarrow L^\infty(\mathbb{T}^2)$, and the Ladyzhenskaya interpolation inequality.

Since $H^1(\mathbb{T}^2)$ is continuously embedded in $H^{\frac{1}{2}}(\mathbb{T}^2)$, we have

$$\|\Lambda^{\frac{1}{2}}\rho\|_{L^2(\mathbb{T}^2)} \le C\|\Lambda\rho\|_{L^2(\mathbb{T}^2)}, \tag{3.155}$$

yielding the differential inequality

$$\frac{d}{dt}\|\Lambda^{\frac{1}{2}}\rho\|_{L^2(\mathbb{T}^2)} + C_1\|\Lambda^{\frac{1}{2}}\rho\|_{L^2(\mathbb{T}^2)} \le C_2\|\rho\|^{\frac{1}{2}}_{L^2(\mathbb{T}^2)}\|\nabla\rho\|^{\frac{5}{2}}_{L^2(\mathbb{T}^2)}\|\Delta\rho\|_{L^2(\mathbb{T}^2)}. \tag{3.156}$$

We note that

$$\|\rho(t)\|_{L^2(\mathbb{T}^2)} \le C\|\rho_0\|_{L^2(\mathbb{T}^2)}e^{-ct} \tag{3.157}$$

for all $t \geq 0$. Indeed, we multiply (3.43) by ρ and we integrate in the space variable. Then we use the cancellation of the nonlinear term and the continuous embedding of $H^{\frac{1}{2}}(\mathbb{T}^2)$ in $L^2(\mathbb{T}^2)$ to obtain

$$\frac{d}{dt}\|\rho(t)\|_{L^2(\mathbb{T}^2)} + C\|\rho(t)\|_{L^2(\mathbb{T}^2)} \leq 0 \tag{3.158}$$

which gives (3.157).

Now we go back to the differential inequality (3.156). Using the bounds (3.151) and (3.157) together with Lemma 1.1, we obtain (3.152).

3.7 Subcritical Periodic Case

In this section, we consider the subcritical case where the dissipation is given by Λ^α for $\alpha \in (1, 2]$, that is, we consider the equation

$$\partial_t \rho + u \cdot \nabla \rho + \Lambda^\alpha \rho = 0 \tag{3.159}$$

posed on $\mathbb{T}^2$, where

$$u = -\mathbb{P}(\rho R \rho). \tag{3.160}$$

The initial data are given by

$$\rho(x, 0) = \rho_0(x) \tag{3.161}$$

and have zero mean.

Global weak solutions exist:

Theorem 3.6. *Let* $\alpha \in (1, 2]$. *Let* $T > 0$ *be arbitrary. Let* $\rho_0 \in L^2(\mathbb{T}^2)$. *Then* (3.159)–(3.161) *has a weak solution* ρ *on* $[0, T]$ *obeying*

$$\frac{1}{2}\|\rho(t)\|_{L^2(\mathbb{T}^2)}^2 + \int_0^t \|\Lambda^{\frac{\alpha}{2}}\rho(s)\|_{L^2(\mathbb{T}^2)}^2 ds \le \frac{1}{2}\|\rho_0\|_{L^2(\mathbb{T}^2)}^2 \tag{3.162}$$

for $t \in [0, T]$.

The proof is similar to that of Theorem 3.1, and so we omit the details.

We note that the regularity of the initial data imposed in the critical case ($\alpha = 1$), namely $\rho_0 \in L^{2+\delta}$ for some $\delta > 0$, is not required in the subcritical case in view of the fact that ρ obeys

$$\rho \in L^2(0, T; H^{\frac{\alpha}{2}}(\mathbb{T}^2)). \tag{3.163}$$

The following proposition is the analogue of Proposition 3.3:

Proposition 3.4. *Let* $\alpha \in (1, 2]$. *Let* $p > 2$ *and* $\rho_0 \in L^p(\mathbb{T}^2)$. *Suppose* ρ *is a smooth solution of* (3.159)–(3.161) *on* $[0, T]$. *Then*

$$\|\rho(t)\|_{L^p(\mathbb{T}^2)} \le \|\rho_0\|_{L^p(\mathbb{T}^2)} \tag{3.164}$$

holds for all $t \in [0, T]$. *Moreover, if* $\rho_0 \in L^\infty(\mathbb{T}^2)$, *then*

$$\|\rho(t)\|_{L^\infty(\mathbb{T}^2)} \le \|\rho_0\|_{L^\infty(\mathbb{T}^2)} \tag{3.165}$$

holds for all $t \in [0, T]$.

The solution of the initial value problem (3.159)–(3.161) with large smooth data are globally regular.

Theorem 3.7. *Let* $\alpha \in (1, 2]$, $s > 0$. *Let* $T > 0$ *be arbitrary. Let* $\rho_0 \in H^s(\mathbb{T}^2) \cap L^\infty(\mathbb{T}^2)$. *Then there are positive constants* C_1, C_2 *and* C_3 *depending only on*

$\|\rho_0\|_{L^\infty(\mathbb{T}^2)}$ *such that the solution of* (3.159)–(3.160) *with initial data* ρ_0 *exists and satisfies*

$$\|\Lambda^s \rho(t)\|_{L^2(\mathbb{T}^2)} \leq \|\Lambda^s \rho_0\|_{L^2(\mathbb{T}^2)} e^{C_1 t} \tag{3.166}$$

and

$$\int_0^t \|\Lambda^{s+\frac{\alpha}{2}} \rho(\tau)\|_{L^2(\mathbb{T}^2)}^2 d\tau \leq \|\Lambda^s \rho_0\|_{L^2(\mathbb{T}^2)}^2 + C_2 \|\Lambda^s \rho_0\|_{L^2(\mathbb{T}^2)}^2 (e^{C_3 t} - 1) \tag{3.167}$$

for $t \in [0, T]$.

Proof: Fix a small $\epsilon \in (0,1)$ such that $\alpha \geq \epsilon + 1$. We multiply (3.159) by $\Lambda^{2s}\rho$ and we integrate in the space variable over $\mathbb{T}^2$. We obtain the equation

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^s \rho\|_{L^2(\mathbb{T}^2)}^2 + \|\Lambda^{s+\frac{\alpha}{2}} \rho\|_{L^2(\mathbb{T}^2)}^2 = -\int_{\mathbb{T}^2} (u \cdot \nabla \rho) \Lambda^{2s} \rho dx. \tag{3.168}$$

We estimate the nonlinear term. Integrating by parts and using Hölder's inequality, we have

$$\begin{aligned} \left| \int_{\mathbb{T}^2} (u \cdot \nabla \rho) \Lambda^{2s} \rho dx \right| &= \left| \int_{\mathbb{T}^2} \Lambda^{s-\frac{\alpha}{2}} \nabla \cdot (u\rho) \Lambda^{s+\frac{\alpha}{2}} \rho dx \right| \\ &\leq \|\Lambda^{s-\frac{\alpha}{2}+1}(u\rho)\|_{L^2(\mathbb{T}^2)} \|\Lambda^{s+\frac{\alpha}{2}} \rho\|_{L^2(\mathbb{T}^2)}. \end{aligned} \tag{3.169}$$

In view of the fractional estimate

$$\|\Lambda^s(fg)\|_{L^p(\mathbb{T}^2)} \leq C\|g\|_{L^{p_1}(\mathbb{T}^2)}\|\Lambda^s f\|_{L^{p_2}(\mathbb{T}^2)} + C\|\Lambda^s g\|_{L^{p_3}(\mathbb{T}^2)}\|f\|_{L^{p_4}(\mathbb{T}^2)} \tag{3.170}$$

that holds for any mean zero functions $f, g \in C^\infty(\mathbb{T}^2), s > 0, p \in (1, \infty)$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}, p_2, p_3 \in (1, \infty)$ (see [19]), we estimate

$$\begin{aligned} &\|\Lambda^{s-\frac{\alpha}{2}+1}(u\rho)\|_{L^2(\mathbb{T}^2)} \\ &\leq C\|u\|_{L^{\frac{2}{\epsilon}}(\mathbb{T}^2)} \|\Lambda^{s-\frac{\alpha}{2}+1}\rho\|_{L^{\frac{2}{1-\epsilon}}(\mathbb{T}^2)} + C\|\rho\|_{L^\infty(\mathbb{T}^2)}\|\Lambda^{s-\frac{\alpha}{2}+1} u\|_{L^2(\mathbb{T}^2)}. \end{aligned} \tag{3.171}$$

In view of the boundedness of the Riesz transforms (and hence the Leray projector) on $L^p(\mathbb{T}^2)$ for $p \in (1, \infty)$ and Proposition 3.4, we bound

$$\begin{aligned} \|u\|_{L^{\frac{2}{\epsilon}}(\mathbb{T}^2)} &\le C\|\rho R\rho\|_{L^{\frac{2}{\epsilon}}(\mathbb{T}^2)} \le C\|\rho\|_{L^\infty(\mathbb{T}^2)}\|\rho\|_{L^{\frac{2}{\epsilon}}(\mathbb{T}^2)} \\ &\le C\|\rho\|_{L^\infty(\mathbb{T}^2)}^2 \le C\|\rho_0\|_{L^\infty(\mathbb{T}^2)}^2. \end{aligned} \tag{3.172}$$

By the fractional estimate (3.170), we have

$$\begin{aligned} &\|\Lambda^{s-\frac{\alpha}{2}+1}u\|_{L^2(\mathbb{T}^2)} \le C\|\Lambda^{s-\frac{\alpha}{2}+1}(\rho R\rho)\|_{L^2(\mathbb{T}^2)} \\ &\le C\|\rho\|_{L^\infty(\mathbb{T}^2)}\|\Lambda^{s-\frac{\alpha}{2}+1}R\rho\|_{L^2(\mathbb{T}^2)} + C\|R\rho\|_{L^{\frac{2}{\epsilon}}(\mathbb{T}^2)}\|\Lambda^{s-\frac{\alpha}{2}+1}\rho\|_{L^{\frac{2}{1-\epsilon}}(\mathbb{T}^2)} \\ &\le C\|\rho_0\|_{L^\infty(\mathbb{T}^2)}\|\Lambda^{s-\frac{\alpha}{2}+1}\rho\|_{L^2(\mathbb{T}^2)} + C\|\rho_0\|_{L^\infty(\mathbb{T}^2)}\|\Lambda^{s-\frac{\alpha}{2}+1}\rho\|_{L^{\frac{2}{1-\epsilon}}(\mathbb{T}^2)} \end{aligned} \tag{3.173}$$

Hence

$$\begin{aligned} \|\Lambda^{s-\frac{\alpha}{2}+1}(u\rho)\|_{L^2(\mathbb{T}^2)} &\le C\|\rho_0\|_{L^\infty(\mathbb{T}^2)}^2\|\Lambda^{s-\frac{\alpha}{2}+1}\rho\|_{L^{\frac{2}{1-\epsilon}}(\mathbb{T}^2)} \\ &\quad + C\|\rho_0\|_{L^\infty(\mathbb{T}^2)}^2\|\Lambda^{s-\frac{\alpha}{2}+1}\rho\|_{L^2(\mathbb{T}^2)}. \end{aligned} \tag{3.174}$$

In view of the continuous Sobolev embedding

$$H^\epsilon(\mathbb{T}^2) \hookrightarrow L^{\frac{2}{1-\epsilon}}(\mathbb{T}^2), \tag{3.175}$$

we obtain the bound

$$\begin{aligned} \|\Lambda^{s-\frac{\alpha}{2}+1}(u\rho)\|_{L^2(\mathbb{T}^2)} &\le C\|\rho_0\|_{L^\infty(\mathbb{T}^2)}^2\|\Lambda^{s-\frac{\alpha}{2}+1+\epsilon}\rho\|_{L^2(\mathbb{T}^2)} \\ &\quad + C\|\rho_0\|_{L^\infty(\mathbb{T}^2)}^2\|\Lambda^{s-\frac{\alpha}{2}+1}\rho\|_{L^2(\mathbb{T}^2)}. \end{aligned} \tag{3.176}$$

Using the Sobolev interpolation inequality

$$\|\Lambda^{s_1}f\|_{L^2(\mathbb{T}^2)} \le C\|\Lambda^{s_0}f\|_{L^2(\mathbb{T}^2)}^{1-\sigma}\|\Lambda^{s_2}f\|_{L^2(\mathbb{T}^2)}^{\sigma} \tag{3.177}$$

that holds for any mean zero function $f \in H^{s_2}(\mathbb{T}^2)$ and $s_1 = (1-\sigma)s_0 + \sigma s_2$, $\sigma \in [0, 1]$, we estimate

$$\|\Lambda^{s-\frac{\alpha}{2}+1}\rho\|_{L^2(\mathbb{T}^2)} \le C\left(\|\Lambda^s\rho\|_{L^2(\mathbb{T}^2)}\right)^{\frac{2(\alpha-1)}{\alpha}}\left(\|\Lambda^{s+\frac{\alpha}{2}}\rho\|_{L^2(\mathbb{T}^2)}\right)^{\frac{2}{\alpha}-1} \tag{3.178}$$

and

$$\|\Lambda^{s-\frac{\alpha}{2}+1+\epsilon}\rho\|_{L^2(\mathbb{T}^2)} \leq C\left(\|\Lambda^s\rho\|_{L^2(\mathbb{T}^2)}\right)^{\frac{2(\alpha-\epsilon-1)}{\alpha}}\left(\|\Lambda^{s+\frac{\alpha}{2}}\rho\|_{L^2(\mathbb{T}^2)}\right)^{\frac{2(\epsilon+1)}{\alpha}-1}. \quad (3.179)$$

Consequently,

$$\begin{aligned}
&\|\Lambda^{s-\frac{\alpha}{2}+1}(u\rho)\|_{L^2(\mathbb{T}^2)}\|\Lambda^{s+\frac{\alpha}{2}}\rho\|_{L^2(\mathbb{T}^2)} \\
&\leq C\|\rho_0\|^2_{L^\infty(\mathbb{T}^2)}\left(\|\Lambda^s\rho\|_{L^2(\mathbb{T}^2)}\right)^{\frac{2(\alpha-\epsilon-1)}{\alpha}}\left(\|\Lambda^{s+\frac{\alpha}{2}}\rho\|_{L^2(\mathbb{T}^2)}\right)^{\frac{2(\epsilon+1)}{\alpha}} \\
&+ C\|\rho_0\|^2_{L^\infty(\mathbb{T}^2)}\left(\|\Lambda^s\rho\|_{L^2(\mathbb{T}^2)}\right)^{\frac{2(\alpha-1)}{\alpha}}\left(\|\Lambda^{s+\frac{\alpha}{2}}\rho\|_{L^2(\mathbb{T}^2)}\right)^{\frac{2}{\alpha}}. \quad (3.180)
\end{aligned}$$

By Young's inequality, we end up with

$$\left|\int_{\mathbb{T}^2}(u\cdot\nabla\rho)\Lambda^{2s}\rho dx\right| \leq C_{\rho_0}\|\Lambda^s\rho\|^2_{L^2(\mathbb{T}^2)} + \frac{1}{2}\|\Lambda^{s+\frac{\alpha}{2}}\rho\|^2_{L^2(\mathbb{T}^2)} \quad (3.181)$$

where C_{ρ_0} is a constant depending on the L^∞ norm of the initial data ρ_0.

Therefore, we obtain the differential inequality,

$$\frac{d}{dt}\|\Lambda^s\rho\|^2_{L^2(\mathbb{T}^2)} + \|\Lambda^{s+\frac{\alpha}{2}}\rho\|^2_{L^2(\mathbb{T}^2)} \leq 2C_{\rho_0}\|\Lambda^s\rho\|^2_{L^2(\mathbb{T}^2)} \quad (3.182)$$

which gives (3.166) and (3.167).

We have shown existence of global smooth solutions in the subcritical case, provided that the initial data is smooth enough. No smallness condition is imposed on the size of the initial data.

Remark 3.3. *The solutions in the subcritical case are unique. This is obtained by following the same argument as for the uniqueness of local strong solutions in the critical case.*

Remark 3.4. *The results obtained in Theorem 3.7 holds as well in the whole space* $\mathbb{R}^2$ *when the initial data is smooth. The proof of Theorem 3.7 is mainly based on fractional estimates* (3.170) *which hold in the whole space (see [30]), the uni-*

form boundedness of the L^p norms of solutions to the subcritical equation which is obtained in $\mathbb{R}^2$ (see Proposition 3.3 and Remark 3.2), and periodic Sobolev interpolation inequalities given by (3.177) *which, in the whole space setting, becomes*

$$\|f\|_{H^{s_1}(\mathbb{R}^2)} \leq C\|f\|_{H^{s_0}(\mathbb{R}^2)}^{1-\sigma}\|f\|_{H^{s_1}(\mathbb{R}^2)}^{\sigma} \tag{3.183}$$

for $f \in H^{s_2}(\mathbb{R}^2)$ *and* $s_1 = (1-\sigma)s_0 + \sigma s_2$, $\sigma \in [0,1]$. *Therefore, the differential inequality* (3.182) *becomes*

$$\frac{d}{dt}\|\Lambda^s \rho\|_{L^2(\mathbb{R}^2)}^2 + \|\Lambda^{s+\frac{\alpha}{2}}\rho\|_{L^2(\mathbb{R}^2)}^2 \leq C_1^0\|\Lambda^s\rho\|_{L^2(\mathbb{T}^2)}^2 + C_2^0 \tag{3.184}$$

where C_1^0 *and* C_2^0 *are constants depending only on the initial data, yielding the desired bounds.*

CHAPTER 4

Nernst-Planck-Navier-Stokes Equations

We consider long time dynamics of solutions of 2D periodic Nernst-Planck-Navier-Stokes systems forced by body charges and body forces. We show that, in the absence of body charges, but in the presence of fluid body forces, the charge density of the ions converges exponentially in time to zero, and the ion concentrations converge exponentially in time to equal time independent constants. This happens while the fluid continues to be dynamically active for all time. In the general case of body charges and body forces, the solutions converge in time to an invariant finite dimensional compact set in phase space.

4.1 Introduction

Electrodiffusion of ions in fluids, described by the Nernst-Planck-Navier-Stokes (NPNS) equations [36], is a broad subject, extensively studied in the chemical-physics, bio-physics and engineering literature. From mathematical point of view, the Nernst-Planck system without added charges and without fluid possesses global smooth solutions which converge to unique stationary states in bounded domains in two dimensions [7, 12, 26]. These results are obtained in situations in which boundaries are impermeable to the ions, where the relevant blocking boundary conditions require the vanishing of the normal fluxes of ions through the boundary. The NPNS system with blocking boundary conditions and with no applied voltage at the boundary is globally well posed in 2D [39]. Furthermore, the NPNS system was proved to have globally smooth and stable solutions in 2D with blocking boundary conditions and nonzero applied voltage [16]. In [40], weak solutions in three dimensions were shown to exist for homogeneous Neumann boundary conditions for the potential. Recently, in [33], the authors established the existence of weak solutions in the whole space, $\Omega = \mathbb{R}^3$. All these results concern situations without forcing in which there is a unique stable stationary solution.

Numerical simulations [37, 44] and experiments [38] show that instabilities occur in regimes when the system is forced. The lack of stability was suggested to lead to chaotic, and even turbulent behavior [24], analogous to fluid turbulence.

In this chapter, we consider the issue of long time dynamics of solutions of the NPNS system with forcing of two kinds: added charges and fluid body forces. Two ionic species, with concentrations c_1 and c_2, with valences $z_1 = 1$ and $z_2 = -1$ respectively, and with equal diffusivities $D > 0$, evolve according the Nernst-Planck equations

$$(\partial_t + u \cdot \nabla)c_i = D\mathrm{div}(\nabla c_i + z_i c_i \nabla \Phi), \tag{4.1}$$

$i = 1, 2$. The ionic species concentrations $c_i(x, t)$ are nonnegative functions of the two variables, position x and time t. The potential Φ obeys the Poisson equation

$$-\epsilon \Delta \Phi = \rho + N \tag{4.2}$$

driven by the charge density

$$\rho = c_1 - c_2 \tag{4.3}$$

and by the added charge density N, which we take smooth and time independent. The constant $\epsilon > 0$ is proportional to the square of the Debye length. The velocity u of the fluid obeys the Navier-Stokes equations

$$\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla p = -(\rho + N)\nabla\Phi + f \tag{4.4}$$

with the divergence free condition

$$\nabla \cdot u = 0. \tag{4.5}$$

The variable p represents the pressure. The positive constant ν is the kinematic viscosity. The body forces f are time independent, smooth, and divergence free. We consider the NPNS system in the two dimensional periodic domain

$$\mathbb{T}^2 = [-\pi, \pi] \times [-\pi, \pi] \tag{4.6}$$

with periodic boundary conditions.

Our main results are as follows. In the absence of forcing of any kind ($f = N = 0$), we prove that solutions are global and regular. The velocity converges exponentially in time to zero, the concentrations converge exponentially in time to equal constant values and the charge density converges exponentially in time to zero. In the case of body forces, but in the absence of added charge densities ($f \neq 0$, $N = 0$), we prove that the solutions are global, regular and the ionic concentrations still converge exponentially in time to equal constant values, while the charge density converges exponentially in time to zero. This is interesting in view of the fact that the Navier-Stokes evolution is forced and the velocity does not cease to be dynamically active. In all cases of forced equations, including $f \neq 0$ and $N \neq 0$, we prove that all solutions converge in time to a global attractor, which is an invariant compact set in phase space with finite Hausdorff and fractal dimension.

The chapter is organized as follows. Section 4.2 is devoted to preliminaries. We describe the asymptotic behavior of eigenvalues of the dissipative operator $\mathcal{A} = (\nu A, -D\Delta, -D\Delta)$, where A is the Stokes operator and Δ is the Laplacian. In section 4.3, we prove, as in [18], that

$$\int_0^T \int_{\mathbb{T}^2} \left(|c_1(x)|^2 + |c_2(x)|^2\right) dxdt < \infty, \tag{4.7}$$

for all $T > 0$ is a necessary and sufficient condition for the persistence of global regular solutions of the NPNS system (4.1)–(4.5). Under condition (4.7), the non-negativity of the initial ionic concentrations is preserved for all positive times. In section 4.4, we discuss the case where no body forces f are present in the fluid and no added charge densities N take part in generating the electric field. We prove that the concentrations decay exponentially in all L^p spaces ($p \in [2;\infty)$) independently of the velocity u, implying, together with the exponential decay of the L^p norms of u, the existence of a single point attractor. We prove further that the solutions decay exponentially in H^2. In section 4.5, we consider added body forces, and we establish that the concentrations converge exponentially to equal constant steady states, and the charge density vanishes in the limit of large times. We address the evolution of the system in a phase space corresponding to strong solutions (H^1).

We show that there exists a compact set (a ball in a the stronger norm H^2) which is an absorbing ball. This means that starting from any initial data w_0 in phase space, there exists a time t_0, depending locally uniformly on the norm of the initial data in the phase space, such that solution $\mathcal{S}(t)w_0$ belongs to the absorbing ball for times larger than t_0. We study further the properties of the nonlinear solution map $\mathcal{S}(t)$ corresponding to the NPNS system. We establish Lipschitz continuity of $\mathcal{S}(t)$ in various norms, including a smoothing property for positive times (see Theorem 4.5). We prove the injectivity of the solution map $\mathcal{S}(t)$ in Appendix A. Exponential decay of volume elements is proved in Appendix B. The existence of a finite dimensional global attractor is thus established for the case $N = 0$, $f \neq 0$. The global attractor is a set which is invariant under the solution map, and such that all solutions converge to it as time tends to infinity. In section 4.6 we treat the general case with an added charge density N. In this case the concentrations and the charge density are no longer convergent in time, but we still obtain the properties of existence of a compact absorbing ball, Lipschitz continuity and smoothing properties of the solution map. The injectivity and decay of volume elements are valid as well, and we obtain the existence of a global attractor with finite Hausdorff and fractal dimension.

4.2 Preliminaries

We consider the Hilbert space

$$\mathcal{H} = H \oplus L^2 \oplus L^2 \tag{4.8}$$

where H is the space of L^2 periodic vector fields which are divergence free and have mean zero. We define

$$\mathcal{A}w = (\nu A u, -D\Delta c_1, -D\Delta c_2) \tag{4.9}$$

where Δ is the Laplacian operator with periodic boundary conditions on $\mathbb{T}^2$, and $A = \mathbb{P}(-\Delta)$ is the Stokes operator. Here, $\mathbb{P}$ denotes the Leray-Hopf projector. We recall that $\mathbb{P}$ and $-\Delta$ commute on $\mathbb{T}^2$. The domain of definition of $\mathcal{A}$ is

$$\mathcal{D}(\mathcal{A}) = (H^2 \cap H) \oplus H^2 \oplus H^2. \tag{4.10}$$

By the spectral theorem for Hilbert spaces, and since 0 is not an eigenvalue, there is an orthonormal basis of $\mathcal{H}$ formed by a sequence of eigenvectors ω_k of $\mathcal{A}$ with corresponding eigenvalues μ_k counted with multiplicity such that $0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_k \to \infty$.

Proposition 4.1. *There exists a constant $C > 0$ such that $\mu_k \geq Ck$ for all $k \geq 1$.*

Proof: We denote by $\{\lambda_j\}$ the eigenvalues of $-\Delta$ with periodic boundary conditions on $\mathbb{T}^2$ counted with multiplicity, $0 < \lambda_1 \leq \lambda_2 \leq \ldots$. There exists a constant $c > 0$ such that $j \leq c\lambda_j$ for all $j \in \mathbb{N}$, and $\{\nu\lambda_j\}$ and $\{D\lambda_j\}$ are the eigenvalues of νA and $D(-\Delta)$ respectively counted with multiplicity. We write

$$\{\mu_i : i = 1, \ldots, N\} = \{\nu\lambda_i : i = 1, \ldots, j\} \cup \{D\lambda_i : i = 1, \ldots, k\} \tag{4.11}$$

and we note that if $\mu_N = \nu\lambda_j$, then $j \leq \frac{c}{\nu}\mu_N$, whereas if $\mu_N = D\lambda_k$, then $k \leq \frac{c}{D}\mu_N$. Consequently, $N = j + k \leq c\left(\frac{1}{\nu} + \frac{1}{D}\right)\mu_N$, which completes the proof of the lemma.

4.3 Existence and Uniqueness of Solutions

We consider the system

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u - (\rho + N)\nabla \Phi + f \\ \nabla \cdot u = 0 \\ \rho = c_1 - c_2 \\ -\epsilon \Delta \Phi = \rho + N \\ \partial_t c_1 + u \cdot \nabla c_1 = D\Delta c_1 + D\nabla \cdot (c_1 \nabla \Phi) \\ \partial_t c_2 + u \cdot \nabla c_2 = D\Delta c_2 - D\nabla \cdot (c_2 \nabla \Phi) \end{cases} \tag{4.12}$$

in $\mathbb{T}^2 \times [0, \infty)$, where ν, D, ϵ are positive constants. The body forces f are smooth, divergence free, time independent, and have mean zero. The added charge density N is smooth and time independent. We assume that the initial fluid velocity u_0 has mean zero. We also assume that the initial concentrations $c_1(x, 0)$ and $c_2(x, 0)$ have space averages $\bar{c}_1$ and $\bar{c}_2$ satisfying

$$\bar{c}_2 - \bar{c}_1 = \bar{N} \tag{4.13}$$

where $\bar{N}$ is the space average of the charge density N.

Remark 4.1. *We note that ρ maintains a space average equal to $-\bar{N}$ whereas u maintains a space average equal to zero for all $t \geq 0$. This follows by integrating* *the equations satisfied by ρ and u and by using*

$$\int(\rho+N)\nabla\Phi = -\frac{1}{\epsilon}\int(\rho+N)\nabla\Lambda^{-2}(\rho+N)$$
$$= -\frac{1}{\epsilon}\int\Lambda^{-1/2}(\rho+N)R\Lambda^{-1/2}(\rho+N) = 0 \tag{4.14}$$

where the last equality holds because the Riesz operator $R = \nabla\Lambda^{-1}$ is antisymmetric.

We use the following convention regarding constants: we denote by C a positive constant that might depend on the parameters of the problem or universal constants, C_N a positive constant depending, in addition, on the charge density N. Following the same pattern, we denote by $C_{N,f}$ a constant depending on N and f. These constants may change from line to line along the proofs.

Theorem 4.1. *(Local Solution) Suppose $u_0 \in H^1$ and $c_i(0) \in L^2$. Then, there exists T_0 depending on $\|u_0\|_{H^1}, \|c_i(0)\|_{L^2}$ and the parameters of the problem such that system* (4.12) *has a unique solution such that $u \in L^\infty(0,T;H^1)\cap L^2(0,T;H^2)$ and $c_i \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1)$ on* $[0,T_0]$.

Proof: We start by taking the L^2 inner product of the equation satisfied by u with $-\Delta u$. We use the identity

$$Tr(G^TG^2) = 0 \tag{4.15}$$

for the two-by-two traceless matrix G with entries $G_{ij} = \frac{\partial u_i}{\partial x_j}$, and we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \nu\|\Delta u\|_{L^2}^2 = \int(\rho+N)\nabla\Phi\cdot\Delta u - \int f\Delta u. \tag{4.16}$$

In view of Hölder's, Ladyzhenskaya's, Poincaré's, and Young's inequalities, we have

$$\begin{aligned}\int (\rho+N)\nabla\Phi\cdot\Delta u &\le \|\rho+N\|_{L^2}\|\nabla\Phi\|_{L^\infty}\|\Delta u\|_{L^2}\\ &\le C\|\Delta u\|_{L^2}[\|\rho\|_{L^2}+\|N\|_{L^2}][\|\rho\|_{L^4}+\|N\|_{L^4}]\\ &\le \frac{\nu}{4}\|\Delta u\|_{L^2}^2+\frac{D}{8}\|\nabla\rho\|_{L^2}^2+C\|\rho\|_{L^2}^6+C_N. \end{aligned}\tag{4.17}$$

and consequently, we obtain the differential inequality

$$\frac{d}{dt}\|\nabla u\|_{L^2}^2+\nu\|\Delta u\|_{L^2}^2\le \frac{D}{4}\|\nabla\rho\|_{L^2}^2+C\|\rho\|_{L^2}^6+C_{N,f} \tag{4.18}$$

Let $\sigma = c_1 + c_2$. Then, σ and ρ obey the system

$$\begin{cases}\partial_t\sigma+u\cdot\nabla\sigma=D\Delta\sigma+D\nabla\cdot(\rho\nabla\Phi)\\ \partial_t\rho+u\cdot\nabla\rho=D\Delta\rho+D\nabla\cdot(\sigma\nabla\Phi).\end{cases}\tag{4.19}$$

Taking the L^2 inner product of the first equation with σ and of the second equation with ρ, adding them, and noting that

$$\begin{aligned}\left|\int\rho\Delta\Phi\sigma\right| &\le C\|\rho\|_{L^4}\|\sigma\|_{L^4}\|\rho+N\|_{L^2}\\ &\le \frac{D}{2}[\|\nabla\rho\|_{L^2}^2+\|\nabla\sigma\|_{L^2}^2]+C\|\sigma\|_{L^2}^4+C\|\rho\|_{L^2}^4+C_N\end{aligned}\tag{4.20}$$

by Ladyzhenskaya's and Young's inequalities, we obtain the differential inequality

$$\frac{d}{dt}(\|\sigma\|_{L^2}^2+\|\rho\|_{L^2}^2)+D(\|\nabla\sigma\|_{L^2}^2+\|\nabla\rho\|_{L^2}^2)\le C[\|\sigma\|_{L^2}^4+\|\rho\|_{L^2}^4]+C_N. \tag{4.21}$$

Let

$$M(t)=\|\nabla u\|_{L^2}^2+\|\rho\|_{L^2}^2+\|\sigma\|_{L^2}^2. \tag{4.22}$$

Adding (4.21) to (4.18), we obtain

$$M'(t)+\frac{D}{2}(\|\nabla\sigma(t)\|_{L^2}^2+\|\nabla\rho(t)\|_{L^2}^2)+\nu\|\Delta u(t)\|_{L^2}^2\le CM(t)^3+C_{N,f}. \tag{4.23}$$

This latter differential inequality gives short time control of the desired norms. For uniqueness, suppose (u_1, c_1^1, c_2^1) and (u_2, c_1^2, c_2^2) are two solutions of (4.12). Let $\rho_1 = c_1^1 - c_2^1$, $\rho_2 = c_1^2 - c_2^2$, $\sigma_1 = c_1^1 + c_2^1$, $\sigma_2 = c_1^2 + c_2^2$. We write $u = u_1 - u_2$, $\rho = \rho_1 - \rho_2$ and $\sigma = \sigma_1 - \sigma_2$. Then u, ρ and σ obey the system

$$\begin{cases} \partial_t u + u_1 \cdot \nabla u_1 - u_2 \cdot \nabla u_2 + \nabla(p_1 - p_2) \\ \qquad = \nu \Delta u - [\rho_1 \nabla \Phi_1 - \rho_2 \nabla \Phi_2] \\ \partial_t \rho + u_1 \cdot \nabla \rho_1 - u_2 \cdot \nabla \rho_2 = D\Delta \rho + D\nabla \cdot (\sigma_1 \nabla \Phi_1 - \sigma_2 \nabla \Phi_2) \\ \partial_t \sigma + u_1 \cdot \nabla \sigma_1 - u_2 \cdot \nabla \sigma_2 = D\Delta \sigma + D\nabla \cdot (\rho_1 \nabla \Phi_1 - \rho_2 \nabla \Phi_2) \end{cases} \tag{4.24}$$

We take the L^2 inner product of the first equation of (4.24) with u to obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \nu\|\nabla u\|_{L^2}^2 = &- \int (u_1 \cdot \nabla u_1 - u_2 \cdot \nabla u_2) \cdot u \, dx \\ &- \int (\rho_1 \nabla \Phi_1 - \rho_2 \nabla \Phi_2) \cdot u \, dx. \end{aligned} \tag{4.25}$$

We estimate the term

$$\begin{aligned} &\left| \int (u_1 \cdot \nabla u_1 - u_2 \cdot \nabla u_2) \cdot u \, dx \right| = \left| \int [u \cdot \nabla u_1 + u_2 \cdot \nabla u] \cdot u \, dx \right| \\ &\le C\|u\|_{L^2}^{3/2} \|\nabla u\|_{L^2}^{1/2} \|\nabla u_1\|_{L^2}^{1/2} \|\Delta u_1\|_{L^2}^{1/2} \end{aligned} \tag{4.26}$$

using Ladyzhenskaya's inequality. In view of elliptic regularity

$$\|\nabla \Phi\|_{L^\infty} \le C\|\rho\|_{L^4}, \tag{4.27}$$

we have

$$\begin{aligned} &\left| \int (\rho_1 \nabla \Phi_1 - \rho_2 \nabla \Phi_2) \cdot u \, dx \right| = \left| \int [\rho \nabla \Phi_1 + \rho_2 \nabla \Phi] \cdot u \, dx \right| \\ &\le C[\|\nabla \Phi_1\|_{L^\infty} \|\rho\|_{L^2} \|u\|_{L^2} + \|\rho_2\|_{L^2} \|\rho\|_{L^2}^{1/2} \|\nabla \rho\|_{L^2}^{1/2} \|u\|_{L^2}]. \end{aligned} \tag{4.28}$$

Now, we take the L^2 inner product of the second equation of (4.24) with ρ, and we get

$$\frac{1}{2}\frac{d}{dt}\|\rho\|_{L^2}^2 + D\|\nabla\rho\|_{L^2}^2 = -\int (u_1\cdot\nabla\rho_1 - u_2\cdot\nabla\rho_2)\,\rho + D\int \nabla\cdot(\sigma_1\nabla\Phi_1 - \sigma_2\nabla\Phi_2)\rho. \tag{4.29}$$

We have

$$\left|\int (u_1\cdot\nabla\rho_1 - u_2\cdot\nabla\rho_2)\,\rho\right| = \left|\int [u\cdot\nabla\rho_1 + u_2\cdot\nabla\rho]\rho\right| \le C\|\nabla\rho_1\|_{L^2}\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2}\|\rho\|_{L^2}^{1/2}\|\nabla\rho\|_{L^2}^{1/2} \tag{4.30}$$

and

$$\left|\int \nabla\cdot(\sigma_1\nabla\Phi_1 - \sigma_2\nabla\Phi_2)\rho\right| \le C\left[\|\nabla\Phi_1\|_{L^\infty}\|\sigma\|_{L^2}\|\nabla\rho\|_{L^2} + \|\sigma_2\|_{L^2}\|\rho\|_{L^2}^{1/2}\|\nabla\rho\|_{L^2}^{3/2}\right]. \tag{4.31}$$

Finally, we take the L^2 inner product of the third equation of (4.24) with σ to obtain

$$\frac{1}{2}\frac{d}{dt}\|\sigma\|_{L^2}^2 + D\|\nabla\sigma\|_{L^2}^2 = -\int (u_1\cdot\nabla\sigma_1 - u_2\cdot\nabla\sigma_2)\,\sigma + D\int \nabla\cdot(\rho_1\nabla\Phi_1 - \rho_2\nabla\Phi_2)\sigma. \tag{4.32}$$

We estimate the first term on the right hand side of (4.32) as in (4.30). For the second term, as in (4.31), we have

$$\left|\int \nabla\cdot(\rho_1\nabla\Phi_1 - \rho_2\nabla\Phi_2)\sigma\,dx\right| \le C\left[\|\nabla\Phi_1\|_{L^\infty}\|\rho\|_{L^2}\|\nabla\sigma\|_{L^2} + \|\rho_2\|_{L^2}\|\rho\|_{L^2}^{1/2}\|\nabla\rho\|_{L^2}^{1/2}\|\nabla\sigma\|_{L^2}\right]. \tag{4.33}$$

Putting (4.25)–(4.33) together, and applying Young's inequality, we obtain a differential inequality of the form

$$\frac{d}{dt}\left[\|u\|_{L^2}^2 + \|\rho\|_{L^2}^2 + \|\sigma\|_{L^2}^2\right] \le CC(t)\left[\|u\|_{L^2}^2 + \|\rho\|_{L^2}^2 + \|\sigma\|_{L^2}^2\right] \tag{4.34}$$

where

$$C(t) = \|\nabla u_1\|_{L^2}^{2/3} \|\Delta u_1\|_{L^2}^{2/3} + \|\nabla \rho_1\|_{L^2}^2 + \|\nabla \sigma_1\|_{L^2}^2$$

$$+ \|\rho_1 + N\|_{L^3}^2 + \|\sigma_2\|_{L^2}^4 + \|\rho_2\|_{L^2}^4 + 1. \tag{4.35}$$

Since

$$\int_0^t C(s)ds < \infty. \tag{4.36}$$

for any $t \in [0, T_0]$, we obtain uniqueness.

Theorem 4.1 shows existence of local solutions. The calculations can be done rigorously using Galerkin approximations. Namely, we consider an orthonormal basis of L^2 consisting of the eigenfunctions $\{\Phi_k\}_{k=1}^{\infty}$ of the Stokes operator

$$-\Delta \Phi_k + \nabla \xi_k = \mu_k \Phi_k \tag{4.37}$$

with periodic boundary condition on $\mathbb{T}^2$, and such that

$$\nabla \cdot \Phi_k = 0 \qquad \forall k \in \mathbb{N}. \tag{4.38}$$

The functions Φ_k are C^∞, divergence free, and have mean zero. We also consider an orthonormal basis of L^2 consisting of the eigenfunctions $\{w_k\}_{k=1}^{\infty}$ of the Laplacian operator

$$-\Delta w_k = \lambda_k w_k \tag{4.39}$$

with periodic boundary condition on $\mathbb{T}^2$. The functions w_k are C^∞ and have mean zero. Let

$$\mathbb{P}_n u = \sum_{k=1}^{n} (u, \Phi_k)_H \Phi_k \tag{4.40}$$

and

$$\mathbb{P}_n c_i = \sum_{k=1}^{n} (c_i, w_k)_{L^2} w_k + \bar{c}_i = \sum_{k=0}^{n} (c_i, w_k)_{L^2} w_k \tag{4.41}$$

be the Galerkin approximations of u and c_i for $i \in \{1, 2\}$. Here, $\bar{c}_i$ is the constant average of c_i over $\mathbb{T}^2$, and $w_0 = 1/2\pi$. We fix m and n and write the system of nonlinear ODE's obeyed by the coefficients of the Galerkin approximations. A solution of this latter system exists if it is bounded in some norm. To show that, we multiply the equations of this latter system by Φ_i and w_i correspondingly and we sum. We obtain the approximate system

$$\begin{cases} \partial_t u_n + \mathbb{P}_n(u_n \cdot \nabla u_n) - \nu \Delta u_n = -\mathbb{P}_n((\rho_n + \mathbb{P}_n N)\nabla \Phi_n) + \mathbb{P}_n f \\ \partial_t c_n^1 + \mathbb{P}_n(u_n \cdot \nabla c_n^1) - D\Delta c_n^1 = D\mathbb{P}_n(\nabla \cdot (c_n^1 \nabla \Phi_n)) \\ \partial_t c_n^2 + \mathbb{P}_n(u_n \cdot \nabla c_n^2) - D\Delta c_n^2 = D\mathbb{P}_n(\nabla \cdot (c_n^2 \nabla \Phi_n)) \\ -\epsilon \Delta \Phi_n = \rho_n + \mathbb{P}_n N \\ \rho_n = c_n^1 - c_n^2 \end{cases} \tag{4.42}$$

with $u_n(0) = \mathbb{P}_n u_0, c_n^i(0) = \mathbb{P}_n c_i(0), i = 1, 2$. We establish a priori estimates by taking suitable scalar products in L^2 and integrating in time. Then, we pass to the limit via the Aubin-Lions lemma.

Theorem 4.2. *Let $u_0 \in H^1$ and $c_i(0) \in H^1$. Let $T > 0$. Suppose (u, c_1, c_2) solves* (4.12) *on the interval $[0, T]$ with*

$$\int_0^T (\|c_1(t)\|_{L^2}^2 + \|c_2(t)\|_{L^2}^2) dt < \infty. \tag{4.43}$$

Then, $u \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2)$ and $c_i \in L^\infty(0, T; H^1) \cap L^2(0, T, H^2)$.

Proof: The following calculations can be done rigorously using Galerkin approximations.

The differential inequality (4.21) gives

$$\frac{d}{dt}(\|\sigma\|_{L^2}^2 + \|\rho\|_{L^2}^2) \le C(\|\sigma\|_{L^2}^2 + \|\rho\|_{L^2}^2)^2 + C_N. \tag{4.44}$$

Thus, under the assumption (4.43), we obtain that $c_i \in L^\infty(0,T;L^2)\cap L^2(0,T;H^1)$. Moreover, the differential inequality (4.18) allows us to conclude that $u \in L^\infty(0,T;H^1)\cap L^2(0,T;H^2)$.

Now, we taking the L^2 inner product of the equation satisfied by ρ in (4.19) with $-\Delta\rho$, and we obtain the equation

$$\frac{1}{2}\frac{d}{dt}\|\nabla\rho\|_{L^2}^2 + D\|\Delta\rho\|_{L^2}^2 = \int (u\cdot\nabla\rho)\Delta\rho - D\int \nabla\cdot(\sigma\nabla\Phi)\Delta\rho. \tag{4.45}$$

We estimate

$$\left|\int \sigma\Delta\Phi\Delta\rho\right| \le \frac{1}{4}\|\Delta\rho\|_{L^2}^2 + C\|\sigma\|_{L^2}^2\|\nabla\sigma\|_{L^2}^2 + C\|\sigma\|_{L^2}^4 + C\|\nabla\rho\|_{L^2}^4 + C_N, \tag{4.46}$$

$$\left|\int (\nabla\sigma\cdot\nabla\Phi)\Delta\rho\right| \le \frac{1}{4}\|\Delta\rho\|_{L^2}^2 + C\|\nabla\rho\|_{L^2}^4 + C\|\nabla\sigma\|_{L^2}^4 + C_N \tag{4.47}$$

and

$$\left|\int (u\cdot\nabla\rho)\Delta\rho\right| = \left|\int \nabla u\nabla\rho\nabla\rho\right| \le \frac{D}{4}\|\Delta\rho\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2\|\nabla\rho\|_{L^2}^2 \tag{4.48}$$

where we used elliptic regularity together with Ladyzhenskaya's inequality and Poincaré's inequality applied to the mean zero function $\rho + N$.

Finally, we take the L^2 inner product of the equation obeyed by σ in (4.19) with $-\Delta\sigma$ to get

$$\frac{1}{2}\frac{d}{dt}\|\nabla\sigma\|_{L^2}^2 + D\|\Delta\sigma\|_{L^2}^2 = \int (u\cdot\nabla\sigma)\Delta\sigma - D\int \nabla\cdot(\rho\nabla\Phi)\Delta\sigma \tag{4.49}$$

and proceeding in the same fashion as above, we obtain

$$\left|\int \rho\Delta\Phi\Delta\sigma\right| \le \frac{1}{4}\|\Delta\sigma\|_{L^2}^2 + C\|\rho\|_{L^2}^4 + C\|\nabla\rho\|_{L^2}^4 + C_N, \tag{4.50}$$

$$\left|\int (\nabla\rho\cdot\nabla\Phi)\Delta\sigma\right| \le \frac{1}{4}\|\Delta\sigma\|_{L^2}^2 + C\|\nabla\rho\|_{L^2}^4 + C_N \tag{4.51}$$

and

$$\left|\int (u\cdot\nabla\sigma)\Delta\sigma\right| \le \frac{D}{4}\|\Delta\sigma\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2\|\nabla\sigma\|_{L^2}^2. \tag{4.52}$$

Putting (4.45)–(4.52) together, we conclude that c_i lies in $L^\infty(0,T;H^1)$ and $L^2(0,T;H^2)$ with bounds depending on the initial data and T.

Remark 4.2. *Note that if we assume that* $u_0 \in H^2$ *and* $c_i(0) \in H^2$*, then the regularity of the solutions is upgraded to* $u \in L^\infty(0,T_0;H^2) \cap L^2(0,T_0;H^3)$ *and* $c_i \in L^\infty(0,T_0;H^2) \cap L^2(0,T_0,H^3)$.

Remark 4.3. *Under the conditions of Theorem 4.2, if* $c_i(0) \ge 0$*, then* $c_i(t) \ge 0$ *for* $0 \le t \le T$ *(see [16]).*

4.4 NPNS System without Body Forces nor Charge Densities

In this section, we treat the case where $f = N = 0$. We consider the system

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u - \rho \nabla \Phi \\ \nabla \cdot u = 0 \\ \rho = c_1 - c_2 \\ -\epsilon \Delta \Phi = \rho \\ \partial_t c_1 + u \cdot \nabla c_1 = D\Delta c_1 + D\nabla \cdot (c_1 \nabla \Phi) \\ \partial_t c_2 + u \cdot \nabla c_2 = D\Delta c_2 - D\nabla \cdot (c_2 \nabla \Phi) \end{cases} \tag{4.53}$$

in $\mathbb{T}^2 \times [0, \infty)$. We prove global regularity and asymptotic behavior of solutions.

We start with a priori L^2 bounds.

Proposition 4.2. *Let $u_0 \in H, c_i(0) \in L^2$. We assume that $c_i(t) \geq 0$ holds for all $t \geq 0$. Then, there exists an absolute constant $C > 0$ such that*

$$\|\sigma(t) - \bar{\sigma}\|_{L^2}^2 + \|\rho(t)\|_{L^2}^2 \leq (2\|\sigma_0\|_{L^2}^2 + 2\|\bar{\sigma}\|_{L^2}^2 + \|\rho_0\|_{L^2}^2)e^{-2CDt} \tag{4.54}$$

holds for all $t \geq 0$. Moreover,

$$\int_t^{t+T} \left(\|\nabla \rho(s)\|_{L^2}^2 + \|\nabla \sigma(s)\|_{L^2}^2 + \frac{1}{\epsilon}\|\rho(s)\|_{L^3}^3 \right) ds$$
$$\leq \frac{1}{2D}(2\|\sigma_0\|_{L^2}^2 + 2\|\bar{\sigma}\|_{L^2}^2 + \|\rho_0\|_{L^2}^2)Te^{-2CDt} \tag{4.55}$$

holds for any $t \geq 0, T > 0$.

Proof: We recall that σ and ρ obey

$$\begin{cases} \partial_t \sigma + u \cdot \nabla \sigma = D\Delta\sigma + D\nabla \cdot (\rho \nabla \Phi) \\ \partial_t \rho + u \cdot \nabla \rho = D\Delta \rho + D\nabla \cdot (\sigma \nabla \Phi). \end{cases} \tag{4.56}$$

We take the L^2 inner product of the first equation of system (4.56) with σ and of the second equation with ρ, we add them and we use the fact that

$$\int \rho\Delta\Phi\sigma = -\frac{1}{\epsilon}\int \sigma(\rho)^2 \tag{4.57}$$

and that $c_i > 0$ for $i = 1, 2$, to obtain the differential inequality

$$\frac{d}{dt}(\|\sigma - \bar{\sigma}\|_{L^2}^2 + \|\rho\|_{L^2}^2) + 2D(\|\nabla\sigma\|_{L^2}^2 + \|\nabla\rho\|_{L^2}^2) + \frac{2D}{\epsilon}\|\rho\|_{L^3}^3 \leq 0. \tag{4.58}$$

In view of Poincaré's inequality, we get (4.54). Going back to (4.58) and integrating, we obtain (4.55).

Theorem 4.3. *Let $u_0 \in H^1$ be divergence free, and let $c_i(0) \in H^1$ be nonnegative $c_i(0) \geq 0$. Let $T > 0$. Then there exists a unique solution (u, c_1, c_2) satisfying $u \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2)$ and $c_i \in L^\infty(0, T; H^1) \cap L^2(0, T, H^2)$. Moreover $c_i(t) \geq 0$ holds on $[0, T]$.*

Proof: By the local existence theorem (Theorem 4.1) there exists $T_0 > 0$ depending only on the norms of initial data in H^1 such that the solution exists and belongs to H^1. The condition (4.43) holds, and therefore, by Remark 4.3 $c_i(t) \geq 0$. The inequality (4.58) is valid on $[0, T_0]$. By Theorem 4.2 the solution is bounded in H^1. We apply the local existence theorem again, starting from T_0, and deduce that the solution can be extended for $T_1 > T_0$. The inequality (4.58) holds on $[0, T_1]$. Because the inequality (4.58) holds as long as $c_i \geq 0$, reasoning by contradiction we see that the solution extends to the whole interval $[0, T]$.

Corollary 4.1. *Under the assumptions of Proposition 4.2 there exists a positive constant $a = a(D, \nu)$ depending on D and ν, and a positive constant A depending* *on* $\|\rho_0\|_{L^2}, \|\sigma_0\|_{L^2}, \|u_0\|_{L^2}$*, the parameters of the problem and universal constants, such that*

$$\|u(t)\|_{L^2} \le Ae^{-at} \tag{4.59}$$

holds for all $t \ge 0$*.*

Proof: We take the L^2 inner product of the first equation in (4.53) with u, and we get

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \nu\|\nabla u\|_{L^2}^2 = -\int \rho\nabla\Phi \cdot u. \tag{4.60}$$

We estimate

$$\left|\int \rho\nabla\Phi \cdot u \, dx\right| \le \|\rho\|_{L^2}\|\nabla\Phi\|_{L^\infty}\|u\|_{L^2} \le C\|\rho\|_{L^2}\|\rho\|_{L^3}\|u\|_{L^2} \tag{4.61}$$

and thus, we obtain the differential inequality

$$\frac{d}{dt}\|u\|_{L^2} + \nu\|u\|_{L^2} \le C\|\rho\|_{L^2}\|\rho\|_{L^3}. \tag{4.62}$$

By Proposition 4.2 and Lemma 1.1, using (4.55), we obtain (4.59).

Remark 4.4. *In the case* $f = N = 0$*, the global attractor exists and is the singleton* $(0, \bar{\sigma}/2, \bar{\sigma}/2)$*. That is, for all initial data, the solution* (u, c_1, c_2) *converges to* $(0, \bar{\sigma}/2, \bar{\sigma}/2)$*.*

Proposition 4.3. *Let* $u_0 \in H^1$ *and* $c_i(0) \in H^1$*. Let* $p > 2$*. Then, there exist positive constants* a_1, a_2 *depending on* D*,* ϵ*,* $\bar{\sigma}$*, and* λ *(the constant in Proposition 2.1), and positive constants* $C_1^p(\|\rho_0\|_{L^p}, \|\sigma_0\|_{L^2})$ *and* $C_2^p(\|\sigma_0\|_{L^p}, \|\rho_0\|_{L^2})$ *depending on the corresponding initial data,* $\bar{\sigma}$*,* p *and universal constants, such that*

$$\|\rho(t)\|_{L^p} \le C_1^p e^{-a_1 t} \tag{4.63}$$

and

$$\|\sigma(t) - \bar{\sigma}\|_{L^p} \le C_2^p e^{-a_2 t} \tag{4.64}$$

hold for all $t \ge 0$.

Proof: The equation (4.56) for ρ is equivalent to

$$\partial_t \rho + u \cdot \nabla \rho + \frac{D\bar{\sigma}}{\epsilon}\rho - D\Delta\rho = D\nabla \cdot ((\sigma - \bar{\sigma})\nabla\Phi). \tag{4.65}$$

Taking the L^2 inner product of equation (4.65) with $\rho|\rho|^{p-2}$ gives

$$\frac{1}{p}\frac{d}{dt}\|\rho\|_{L^p}^p + \frac{D\bar{\sigma}}{\epsilon}\|\rho\|_{L^p}^p + D(p-1)\int |\nabla\rho|^2|\rho|^{p-2}dx$$
$$= -D(p-1)\int (\sigma - \bar{\sigma})\nabla\Phi \cdot |\rho|^{p-2}\nabla\rho. \tag{4.66}$$

By Hölder's inequality with exponents 2, p, 2p/(p-2), followed by Young's inequality, we get

$$\left|\int (\sigma - \bar{\sigma})\nabla\Phi \cdot |\rho|^{p-2}\nabla\rho\right| \le \|\nabla\Phi\|_{L^\infty}\||\rho|^{\frac{p-2}{2}}\nabla\rho\|_{L^2}\|\sigma - \bar{\sigma}\|_{L^p}\||\rho|^{\frac{p-2}{2}}\|_{L^{\frac{2p}{p-2}}}$$
$$\le \frac{1}{2}\||\rho|^{\frac{p-2}{2}}\nabla\rho\|_{L^2}^2 + \frac{1}{2}\|\nabla\Phi\|_{L^\infty}^2\|\sigma - \bar{\sigma}\|_{L^p}^2\|\rho\|_{L^p}^{p-2}. \tag{4.67}$$

In view of the Gagliardo-Nirenberg inequality, we have

$$\|\sigma - \bar{\sigma}\|_{L^p} \le C_p\|\sigma - \bar{\sigma}\|_{H^1}^{\frac{p-2}{p}}\|\sigma - \bar{\sigma}\|_{L^2}^{\frac{2}{p}} \tag{4.68}$$

where C_p is a constant that depends on p. Therefore, we get the differential inequality

$$\frac{d}{dt}\|\rho\|_{L^p}^2 + \frac{2D\bar{\sigma}}{\epsilon}\|\rho\|_{L^p}^2 \le C_p^2 D(p-1)\|\nabla\Phi\|_{L^\infty}^2\|\sigma - \bar{\sigma}\|_{H^1}^{\frac{2(p-2)}{p}}\|\sigma - \bar{\sigma}\|_{L^2}^{\frac{4}{p}}. \tag{4.69}$$

If $p = 3$, then elliptic regularity, an application of Young's inequality with exponents 3, 3/2 and Poincaré inequality imply that

$$\begin{aligned}&\|\nabla\Phi\|_{L^\infty}^2\|\sigma-\bar\sigma\|_{H^1}^{\frac{2}{3}}\|\sigma-\bar\sigma\|_{L^2}^{\frac{4}{3}}\\ &\le C\left(\|\rho\|_{L^3}^3\|\sigma-\bar\sigma\|_{L^2}+\|\nabla\sigma\|_{L^2}^2\|\sigma-\bar\sigma\|_{L^2}^2\right). \qquad (4.70)\end{aligned}$$

In view of (4.55), (4.69) and Lemma 1.1, we obtain (4.63) for $p = 3$.

Now, we go back to the differential inequality (4.69). We estimate

$$\|\nabla\Phi\|_{L^\infty}^2\|\sigma-\bar\sigma\|_{H^1}^{\frac{2(p-2)}{p}}\|\sigma-\bar\sigma\|_{L^2}^{\frac{4}{p}}\le C\|\rho\|_{L^3}^2\|\nabla\sigma\|_{L^2}^2 \qquad (4.71)$$

where we have used elliptic regularity and Poincaré's inequality. Therefore, (4.69) and an application of Lemma 1.1 give (4.63) for any $p > 2$.

Next, we note that the equation satisfied by $\sigma - \bar\sigma$ is given by

$$\partial_t(\sigma-\bar\sigma)+u\cdot\nabla(\sigma-\bar\sigma)=D\Delta(\sigma-\bar\sigma)+D\nabla\cdot(\rho\nabla\Phi). \qquad (4.72)$$

We take the L^2 inner product of equation (4.72) with $(\sigma-\bar\sigma)|\sigma-\bar\sigma|^{p-2}$ and we get the equation

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\|\sigma-\bar\sigma\|_{L^p}^p-D\int|\sigma-\bar\sigma|^{p-2}(\sigma-\bar\sigma)\Delta(\sigma-\bar\sigma)dx\\ &=-D\int\rho\nabla\Phi\cdot\nabla((\sigma-\bar\sigma)|\sigma-\bar\sigma|^{p-2})dx \qquad (4.73)\end{aligned}$$

By Hölder's inequality with exponents 2, p, 2p/(p-2), followed by Young's inequality, we obtain

$$\begin{aligned}&\left|\int\rho\nabla\Phi\cdot\nabla((\sigma-\bar\sigma)|\sigma-\bar\sigma|^{p-2})dx\right|\\ &\le(p-1)\|\nabla\Phi\|_{L^\infty}\|\rho\|_{L^p}\||\sigma-\bar\sigma|^{\frac{p-2}{2}}\|_{L^{\frac{2p}{p-2}}}\||\sigma-\bar\sigma|^{\frac{p-2}{2}}\nabla(\sigma-\bar\sigma)\|_{L^2}\\ &\le(p-1)\left[\frac{1}{2}\||\sigma-\bar\sigma|^{\frac{p-2}{2}}\nabla(\sigma-\bar\sigma)\|_{L^2}^2+\frac{1}{2}\|\nabla\Phi\|_{L^\infty}^2\|\rho\|_{L^p}^2\|\sigma-\bar\sigma\|_{L^p}^{p-2}\right]. \qquad (4.74)\end{aligned}$$

Thus, we have the differential inequality

$$\frac{1}{p}\frac{d}{dt}\|\sigma-\bar{\sigma}\|_{L^p}^p+\frac{D(p-1)}{2}\int|\nabla(\sigma-\bar{\sigma})|^2|\sigma-\bar{\sigma}|^{p-2}dx$$
$$\leq\frac{D(p-1)}{2}\|\nabla\Phi\|_{L^\infty}^2\|\rho\|_{L^p}^2\|\sigma-\bar{\sigma}\|_{L^p}^{p-2}. \tag{4.75}$$

We note that

$$D(p-1)\int|\nabla(\sigma-\bar{\sigma})|^2|\sigma-\bar{\sigma}|^{p-2}dx$$
$$=-D\int|\sigma-\bar{\sigma}|^{p-2}(\sigma-\bar{\sigma})\Delta(\sigma-\bar{\sigma})\geq D\lambda\|\sigma-\bar{\sigma}\|_{L^p}^p \tag{4.76}$$

if p is an even number greater than 2. This follows from Proposition 2.1. Thus, for any even number $p>2$,

$$\frac{d}{dt}\|\sigma-\bar{\sigma}\|_{L^p}^2+D\lambda\|\sigma-\bar{\sigma}\|_{L^p}^2\leq D(p-1)\|\rho\|_{L^3}^2\|\rho\|_{L^p}^2. \tag{4.77}$$

In view of Lemma 1.1, we obtain (4.64) for any even number $p>2$. An L^p estimate when p is not even can be obtained by an application of Hölder's inequality.

Proposition 4.4. *Let $u_0\in H^2, c_i(0)\in H^2$. Then, there exist positive constants c_3,c_4,c_5,c_6 depending on D, ϵ and ν, and positive constants C_3,C_4,C_5,C_6 depending on the initial data $\|u_0\|_{H^2}$, $\|c_1(0)\|_{H^2}$, $\|c_2(0)\|_{H^2}$, $\bar{\sigma}$ and universal constants, such that*

$$\|\nabla u(t)\|_{L^2}^2\leq C_3e^{-c_3t}, \tag{4.78}$$

$$\|\nabla\rho(t)\|_{L^2}^2+\|\nabla\sigma(t)\|_{L^2}^2\leq C_4e^{-c_4t}, \tag{4.79}$$

$$\|\Delta u(t)\|_{L^2}^2\leq C_5e^{-c_5t}, \tag{4.80}$$

and

$$\|\Delta\rho(t)\|_{L^2}^2 + \|\Delta\sigma(t)\|_{L^2}^2 \le C_6 e^{-c_6 t} \tag{4.81}$$

hold for all $t \ge 0$.

Proof: We take the L^2 inner product of the equation satisfied by u in (4.53) with $-\Delta u$, and we apply Hölder's and Young's inequalities to get

$$\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \nu\|\Delta u\|_{L^2}^2 \le C\|\rho\|_{L^3}^2\|\rho\|_{L^2}^2 \tag{4.82}$$

and so we obtain (4.78) by an application of Lemma 1.1.

Now, we take the L^2 inner product of equation (4.65) obeyed by ρ with $-\Delta\rho$ and we estimate

$$\left|\int(\sigma-\bar{\sigma})\Delta\Phi\Delta\rho\right| \le C\|\Delta\rho\|_{L^2}\|\rho\|_{L^2}^{1/2}\|\nabla\rho\|_{L^2}^{1/2}\|\sigma-\bar{\sigma}\|_{L^2}^{1/2}\|\nabla\sigma\|_{L^2}^{1/2}, \tag{4.83}$$

$$\left|\int(\nabla\sigma\cdot\nabla\Phi)\Delta\rho\right| \le C\|\Delta\rho\|_{L^2}\|\nabla\sigma\|_{L^2}\|\rho\|_{L^3} \tag{4.84}$$

and

$$\left|\int(u\cdot\nabla\rho)\Delta\rho\right| \le C\|\nabla\rho\|_{L^2}^{1/2}\|\Delta\rho\|_{L^2}^{3/2}\|\nabla u\|_{L^2} \tag{4.85}$$

in view of Ladyzhenskaya's inequality. This gives

$$\begin{aligned}&\frac{d}{dt}\|\nabla\rho\|_{L^2}^2 + D\|\Delta\rho\|_{L^2}^2\\&\le C\left[(\|\rho\|_{L^2}^2+\|\nabla u\|_{L^2}^4)\|\nabla\rho\|_{L^2}^2 + (\|\rho\|_{L^3}^2+\|\sigma-\bar{\sigma}\|_{L^2}^2)\|\nabla\sigma\|_{L^2}^2\right].\end{aligned} \tag{4.86}$$

Next, we take the L^2 inner product of the equation satisfied by σ, and proceeding as above, we obtain

$$\begin{aligned}
&\frac{d}{dt}\|\nabla\sigma\|_{L^2}^2 + D\|\Delta\sigma\|_{L^2}^2 \\
&\leq C\left[(\|\rho\|_{L^2}^2 + \|\rho\|_{L^3}^2)\|\nabla\rho\|_{L^2}^2 + \|\nabla u\|_{L^2}^4\|\nabla\sigma\|_{L^2}^2\right]. \qquad (4.87)
\end{aligned}$$

Adding (4.87) to (4.86) and using (4.55), we obtain (4.79).

Then, we apply $-\Delta$ to the equation obeyed by u in (4.53) and we take the L^2 inner product of the resulting equation with $-\Delta u$. We obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta u\|_{L^2}^2 + \nu\|\nabla\Delta u\|_{L^2}^2 = -\int \Delta(u\cdot\nabla u)\cdot\Delta u - \int \Delta(\rho\nabla\Phi)\cdot\Delta u. \qquad (4.88)$$

In view of Ladyzehsnkaya's inequality, we have

$$\left|\int \Delta(u\cdot\nabla u)\cdot\Delta u\right| \leq C\|\nabla\Delta u\|_{L^2}\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}. \qquad (4.89)$$

Moreover,

$$\left|\int \Delta(\rho\nabla\Phi)\cdot\Delta u\right| \leq C\|\nabla\Delta u\|_{L^2}(\|\rho\|_{L^2}\|\nabla\rho\|_{L^2} + \|\rho\|_{L^3}\|\nabla\rho\|_{L^2}). \qquad (4.90)$$

Here we have used the fact that the Riesz transforms are bounded in L^4, so

$$\|\nabla\nabla\Phi\|_{L^4} = \frac{1}{\epsilon}\|\nabla\nabla\Lambda^{-2}\rho\|_{L^4} \leq C\|\rho\|_{L^4}. \qquad (4.91)$$

Consequently, we obtain

$$\begin{aligned}
&\frac{d}{dt}\|\Delta u\|_{L^2}^2 + \nu\|\nabla\Delta u\|_{L^2}^2 \\
&\leq C\left[\|\nabla u\|_{L^2}^2\|\Delta u\|_{L^2}^2 + \|\rho\|_{L^2}^2\|\nabla\rho\|_{L^2}^2 + \|\rho\|_{L^3}^2\|\nabla\rho\|_{L^2}^2\right]. \qquad (4.92)
\end{aligned}$$

In view of (4.82) and Lemma 1.1, we deduce (4.80).

Finally, we apply $-\Delta$ to the equations satisfied by ρ and σ in (4.56) and we take the L^2 inner product of the resulting equations with $-\Delta\rho$ and $-\Delta\sigma$ respectively.

We obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta\rho\|_{L^2}^2 + \frac{D\bar{\sigma}}{\epsilon}\|\Delta\rho\|_{L^2}^2 + D\|\nabla\Delta\rho\|_{L^2}^2$$
$$= D\int \Delta\nabla\cdot((\sigma-\bar{\sigma})\nabla\Phi)\Delta\rho - \int \Delta(u\cdot\nabla\rho)\Delta\rho \quad (4.93)$$

and

$$\frac{1}{2}\frac{d}{dt}\|\Delta\sigma\|_{L^2}^2 + D\|\nabla\Delta\sigma\|_{L^2}^2 = D\int \Delta\nabla\cdot(\rho\nabla\Phi)\Delta\sigma - \int \Delta(u\cdot\nabla\sigma)\Delta\sigma. \quad (4.94)$$

We estimate

$$\left|\int \Delta(u\cdot\nabla\rho)\Delta\rho\right| \le \|\nabla\Delta\rho\|_{L^2}\|\nabla u\|_{L^4}\|\nabla\rho\|_{L^4}$$
$$\le C\|\nabla\Delta\rho\|_{L^2}\|\nabla u\|_{L^2}^{1/2}\|\Delta u\|_{L^2}^{1/2}\|\nabla\rho\|_{L^2}^{1/2}\|\Delta\rho\|_{L^2}^{1/2} \quad (4.95)$$

and similarly

$$\left|\int \Delta(u\cdot\nabla\sigma)\Delta\sigma\right| \le C\|\nabla\Delta\sigma\|_{L^2}\|\nabla u\|_{L^2}^{1/2}\|\Delta u\|_{L^2}^{1/2}\|\nabla\sigma\|_{L^2}^{1/2}\|\Delta\sigma\|_{L^2}^{1/2}. \quad (4.96)$$

Now, we have

$$\left|\int \Delta((\sigma-\bar{\sigma})\Delta\Phi)\Delta\rho\right|$$
$$\le \|\nabla\Delta\rho\|_{L^2}\left[\|\nabla\sigma\|_{L^4}\|\Delta\Phi\|_{L^4} + \|\sigma-\bar{\sigma}\|_{L^4}\|\nabla\Delta\Phi\|_{L^4}\right]$$
$$\le C\|\nabla\Delta\rho\|_{L^2}\left[\|\nabla\sigma\|_{L^2}^{1/2}\|\Delta\sigma\|_{L^2}^{1/2}\|\nabla\rho\|_{L^2} + \|\nabla\sigma\|_{L^2}\|\nabla\rho\|_{L^2}^{1/2}\|\Delta\rho\|_{L^2}^{1/2}\right] \quad (4.97)$$

whereas

$$\left|\int \Delta(\nabla(\sigma-\bar{\sigma})\cdot\nabla\Phi)\Delta\rho\right|$$
$$\le \|\nabla\Delta\rho\|_{L^2}\left[\|\nabla\nabla\sigma\|_{L^2}\|\nabla\Phi\|_{L^\infty} + \|\nabla\sigma\|_{L^4}\|\nabla\nabla\Phi\|_{L^4}\right]$$
$$\le C\|\nabla\Delta\rho\|_{L^2}\left[\|\Delta\sigma\|_{L^2}\|\rho\|_{L^3} + \|\nabla\sigma\|_{L^2}^{1/2}\|\Delta\sigma\|_{L^2}^{1/2}\|\nabla\rho\|_{L^2}\right]. \quad (4.98)$$

Here, we have used the fact that the Riesz transforms are bounded in L^2, and so

$$\|\nabla\nabla\sigma\|_{L^2} = \|\nabla\Lambda^{-1}\nabla\Lambda^{-1}\Delta\sigma\|_{L^2} \le C\|\Delta\sigma\|_{L^2} \quad (4.99)$$

Similarly, we have the bounds

$$\left|\int \Delta(\rho\Delta\Phi)\Delta\sigma\right|$$
$$\leq C\|\nabla\Delta\sigma\|_{L^2}\left[\|\nabla\rho\|_{L^2}^{1/2}\|\Delta\rho\|_{L^2}^{1/2}\|\nabla\rho\|_{L^2} + \|\nabla\rho\|_{L^2}\|\nabla\rho\|_{L^2}^{1/2}\|\Delta\rho\|_{L^2}^{1/2}\right] \quad (4.100)$$

and

$$\left|\int \Delta(\nabla\rho\cdot\nabla\Phi)\Delta\sigma\right|$$
$$\leq C\|\nabla\Delta\sigma\|_{L^2}\left[\|\Delta\rho\|_{L^2}\|\rho\|_{L^3} + \|\nabla\rho\|_{L^2}^{1/2}\|\Delta\rho\|_{L^2}^{1/2}\|\nabla\rho\|_{L^2}\right]. \quad (4.101)$$

Putting (4.93)–(4.101) together, and applying Young's and Poincaré's inequalities, we have the differential inequality

$$\frac{d}{dt}(\|\Delta\rho\|_{L^2}^2 + \|\Delta\sigma\|_{L^2}^2) + D(\|\nabla\Delta\rho\|_{L^2}^2 + \|\nabla\Delta\sigma\|_{L^2}^2)$$
$$\leq C(\|\Delta u\|_{L^2}^2 + \|\nabla\rho\|_{L^2}^2 + \|\nabla\sigma\|_{L^2}^2)\|\Delta\rho\|_{L^2}^2$$
$$+ C(\|\Delta u\|_{L^2}^2 + \|\nabla\rho\|_{L^2}^2)\|\Delta\sigma\|_{L^2}^2 \quad (4.102)$$

Consequently, (4.81) follows from (4.86), (4.87), and Lemma 1.1.

We denote by $C^{0,\gamma}$ the space of γ-Hölder continuous functions on $\mathbb{T}^2$ with the norm

$$\|v\|_{C^{0,\gamma}} = \|v\|_{L^\infty} + \sup_{x,y\in\mathbb{T}^2, x\neq y}\frac{|v(x)-v(y)|}{|x-y|^\gamma}. \quad (4.103)$$

Corollary 4.2. *Let $u_0 \in H^2, c_i(0) \in H^2$. Then, there exists a positive constant c_8 depending on D, ϵ, ν, and a positive constant C_8 depending on $\|u_0\|_{H^2}, \|c_i(0)\|_{H^2}$, $\|c_i(0)\|_{H^2}$, $\bar{\sigma}$ and universal constants, such that*

$$\|u(t)\|_{C^{0,1/2}} + \|\rho(t)\|_{C^{0,1/2}} + \|\sigma(t)-\bar{\sigma}\|_{C^{0,1/2}} \leq C_8 e^{-c_8 t} \quad (4.104)$$

holds for all $t \geq 0$.

Proof: The estimate (4.104) follows from the bound

$$\|v\|_{C^{0,1/2}} \le C\|v\|_{W^{1,4}} \le C[\|v\|_{L^4} + \|\nabla v\|_{L^4}]$$
$$\le C[\|v\|_{L^2}^{1/2}\|\nabla v\|_{L^2}^{1/2} + \|\nabla v\|_{L^2}^{1/2}\|\Delta v\|_{L^2}^{1/2}], \tag{4.105}$$

which holds for all $v \in W^{1,4}(\mathbb{T}^2)$ with mean zero, and from Proposition 4.4.

Remark 4.5. *In Proposition 4.3, we assumed that $u_0 \in H^1, c_i(0) \in H^1$ which guarantee by Theorem 4.3 the global existence of solutions and the nonnegativity of the concentrations c_i, and obtained the exponential decay of the L^p norm of ρ and $\sigma - \bar{\sigma}$. In Corollary 4.2, we have assumed higher regularity of the initial data to get the exponential decay of the L^∞ norm of u, ρ and $\sigma - \bar{\sigma}$. However, if we assume in this latter corollary that the initial data are only in H^1, then from* (4.86)*,* (4.87)*, and* (4.82) *we deduce the existence of t_0 such that*

$$\|\Delta u(t_0)\|_{L^2}^2 + \|\Delta \rho(t_0)\|_{L^2}^2 + \|\Delta \sigma(t_0)\|_{L^2}^2 < \infty \tag{4.106}$$

and so we obtain (4.81) *and* (4.80) *for all $t \ge t_0$. We also note that the constants C_1^p and C_2^p in Proposition 4.3 are independent of u, depending only on the L^p norm of the $c_1(0)$ and $c_2(0)$, whereas the constants C_4 and C_6 in Proposition 4.4 depend on the H^2 norm of all initial data.*

4.5 Added Body Forces

In this section, we consider the Navier-Stokes equations driven the electrical force and a smooth, mean zero, divergence-free body force,

$$
\begin{cases}
\partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u - \rho \nabla \Phi + f \\
\nabla \cdot u = 0 \\
\rho = c_1 - c_2 \\
-\epsilon \Delta \Phi = \rho \\
\partial_t c_1 + u \cdot \nabla c_1 = D \Delta c_1 + D \nabla \cdot (c_1 \nabla \Phi) \\
\partial_t c_2 + u \cdot \nabla c_2 = D \Delta c_2 - D \nabla \cdot (c_2 \nabla \Phi)
\end{cases}
\tag{4.107}
$$

in $\mathbb{T}^2 \times [0, \infty)$, with $u_0, c_1(0), c_2(0) \in H^1$. We assume that u_0 has mean zero, and $c_1(0)$ and $c_2(0)$ have equal mean. We take $c_i(0) \geq 0$, and by Theorem 4.3 which is valid in this case as well, the concentrations c_i are nonnegative for all time $t > 0$.

Proposition 4.5. *Let $p \geq 2$. $u_0, c_1(0), c_2(0) \in H^1$ There exist positive constants a_1, a_2 depending on D, ϵ, $\bar{\sigma}$, and λ (the constant in Proposition 2.1), and positive constants $C_1^p(\|\rho_0\|_{L^p}, \|\sigma_0\|_{L^2})$ and $C_2^p(\|\sigma_0\|_{L^p}, \|\rho_0\|_{L^2})$ depending on the corresponding initial data, $\bar{\sigma}$, p and universal constants, such that*

$$
\|\rho(t)\|_{L^p} \leq C_1^p e^{-a_1 t} \tag{4.108}
$$

and

$$
\|\sigma(t) - \bar{\sigma}\|_{L^p} \leq C_2^p e^{-a_2 t} \tag{4.109}
$$

hold for all $t \geq 0$. Furthermore,

$$
\int_t^{t+T} \left(\|\nabla \rho(s)\|_{L^2}^2 + \|\nabla \sigma(s)\|_{L^2}^2 + \frac{1}{\epsilon} \|\rho(s)\|_{L^3}^3 \right) ds
$$

$$
\leq \frac{1}{2D} (2\|\sigma_0\|_{L^2}^2 + 2\|\bar{\sigma}\|_{L^2}^2 + \|\rho_0\|_{L^2}^2) T e^{-2CDt} \tag{4.110}
$$

holds for any $t \geq 0, T > 0$.

The proof follows along the lines of the proofs of Propositions 4.2 and 4.3. Indeed, multiplying the ρ and $\sigma - \bar{\sigma}$ equations by $\rho|\rho|^{p-2}$ and $(\sigma - \bar{\sigma})|\sigma - \bar{\sigma}|^{p-2}$ respectively, the terms involving u cancel out and we conclude that the estimates for the L^p norms of ρ and σ (4.108) and (4.109) hold for any $p \geq 2$. In particular, (4.43) is satisfied.

The following proposition shows that adding a body force to the Navier-Stokes equation does not change the exponential decay of the H^2 norms of ρ and $\sigma - \bar{\sigma}$ but results in the velocity u being bounded in H^2.

Proposition 4.6. *Let* $u_0 \in H^2, c_i(0) \in H^2$. *Then, there exist positive constants* c'_3, c'_4, c'_5, c'_6 *depending on* D, ϵ *and* ν, *and positive constants* C'_3 *and* C'_5 *depending on the initial data* $\|u_0\|_{H^2}$, $\|c_1(0)\|_{H^2}$, $\|c_2(0)\|_{H^2}$ *and* $\bar{\sigma}$, *and positive constants* C'_4 *and* C'_6 *depending in addition on the forces* f, *and positive constants* R_3 *and* R_5 *depending on* f *such that*

$$\|\nabla u(t)\|_{L^2}^2 \leq C'_3 e^{-c'_3 t} + R_3, \tag{4.111}$$

$$\|\nabla \rho(t)\|_{L^2}^2 + \|\nabla \sigma(t)\|_{L^2}^2 \leq C'_4 e^{-c'_4 t}, \tag{4.112}$$

$$\|\Delta u(t)\|_{L^2}^2 \leq C'_5 e^{-c'_5 t} + R_5, \tag{4.113}$$

and

$$\|\Delta \rho(t)\|_{L^2}^2 + \|\Delta \sigma(t)\|_{L^2}^2 \leq C'_6 e^{-c'_6 t} \tag{4.114}$$

hold for all $t \geq 0$.

Moreover, there exists a positive constant $L > 0$ *depending on* $\|u_0\|_{H^1}$, $\|c_1(0)\|_{H^1}$, $\|c_2(0)\|_{H^1}$, f *and universal constants such that*

$$\int_0^t (\|\Delta u(s)\|_{L^2}^2 + \|\Delta\rho(s)\|_{L^2}^2 + \|\Delta\sigma(s)\|_{L^2}^2) ds \leq L \tag{4.115}$$

for all $t \geq 0$.

We note that the estimate (4.115) requires only that $u_0, c_1(0), c_2(0) \in H^1$. No additional regularity of the initial data is required.

The proof is similar to the proof of Proposition 4.4. We omit the details.

Corollary 4.3. *Let* $u_0 \in H^2, c_i(0) \in H^2$. *Then, there exist positive constants* c_8' *and* c_9' *depending on* D, ϵ, ν, *and a positive constant* C_8' *depending on* $\|u_0\|_{H^2}$, $\|c_i(0)\|_{H^2}$, $\|c_i(0)\|_{H^2}$, *and* $\bar{\sigma}$, *a positive constant* C_9' *depending in addition on the body forces* f, *and a positive constant* R_9 *depending on* f *such that*

$$\|u\|_{C^{0,1/2}} \leq C_8' e^{-c_8' t} + R_9 \tag{4.116}$$

and

$$\|\rho(t)\|_{C^{0,1/2}} + \|\sigma(t) - \bar{\sigma}\|_{C^{0,1/2}} \leq C_9' e^{-c_9' t} \tag{4.117}$$

holds for all $t \geq 0$.

This follows from Proposition 4.6, see the proof of Corollary 4.2.

Theorem 4.4. *(Absorbing Ball) Let* $u_0, c_1(0), c_2(0) \in H^1$ *such that* u_0 *and* $(c_1 - c_2)(0)$ *have mean zero. Suppose that* (u, c_1, c_2) *solves* (4.107). *Then, there exists an* $R > 0$ *depending on* f, *and* $t_0 > 0$ *depending on* $\|u_0\|_{H^1}$, $\|c_1(0)\|_{H^1}$, $\|c_2(0)\|_{H^1}$

and the parameters of the problem, such that

$$\|\Delta u(t)\|_{L^2}^2 + \|\Delta c_1(t)\|_{L^2}^2 + \|\Delta c_2(t)\|_{L^2}^2 \leq R \tag{4.118}$$

holds for all $t \geq t_0$.

Proof: In view of equation (4.115), there exists $t_0 \in [0, 1]$ such that

$$\|\Delta u(t_0)\|_{L^2}^2 + \|\Delta \rho(t_0)\|_{L^2}^2 + \|\Delta \sigma(t_0)\|_{L^2} \leq L. \tag{4.119}$$

Thus, the result follows from equations (4.113), (4.114), and from the parallelogram law

$$\|\Delta \rho\|_{L^2}^2 + \|\Delta \sigma\|_{L^2}^2 = 2\|\Delta c_1\|_{L^2}^2 + 2\|\Delta c_2\|_{L^2}^2. \tag{4.120}$$

Let $\mathcal{V} = H^1 \cap H \oplus H^1 \oplus H^1 \subset \mathcal{H}$. Let $\mathcal{V}'$ be the convex subset of $\mathcal{V}$ consisting of vectors (u, c_1, c_2) such that u is divergence free with mean zero and $c_1 \geq 0$, $c_2 \geq 0$ a.e. with $\int c_1 = \int c_2$. Let

$$\mathcal{S}(t) : \mathcal{V}' \mapsto \mathcal{V}' \tag{4.121}$$

be the solution map

$$\mathcal{S}(t)(u_0, c_1(0), c_2(0)) = (u(t), c_1(t), c_2(t)) \tag{4.122}$$

corresponding to system (4.107). As a consequence of Theorem 4.2, $\mathcal{S}(t)$ is well-defined on $\mathcal{V}'$ for every $t \geq 0$. Moreover, the uniqueness of solutions implies that

$$\mathcal{S}(t + s)w_0 = \mathcal{S}(t)(\mathcal{S}(s)w_0) \tag{4.123}$$

for all $t, s \geq 0$, i.e., $\mathcal{S}(t)$ is a semigroup. We proceed to investigate other properties of the map $\mathcal{S}(t)$.

We consider the natural topology on $\mathcal{H}$

$$\|w\|_{\mathcal{H}}^2 = \|u\|_{L^2}^2 + \|c_1\|_{L^2}^2 + \|c_2\|_{L^2}^2 \tag{4.124}$$

and the natural topology on $\mathcal{V}'$

$$\|w\|_{\mathcal{V}'}^2 = \|u\|_{H^1}^2 + \|c_1\|_{H^1}^2 + \|c_2\|_{H^1}^2. \tag{4.125}$$

We address the continuity of the map $\mathcal{S}(t)$.

Theorem 4.5. *(Continuity) Let* $w_1^0 = (u_1(0), c_1^1(0), c_2^1(0))$, $w_2^0 = (u_2(0), c_1^2(0), c_2^2(0))$ $\in \mathcal{V}'$. *Let* $t > 0$. *There exist constants* $K_1(t)$, $K_2(t)$ *and* $K_3(t)$, *locally uniformly bounded as functions of* $t \geq 0$, *and locally bounded as initial data* w_1^0, w_2^0 *are varied in* $\mathcal{V}'$, *such that* $\mathcal{S}(t)$ *is Lipschitz continuous in* $\mathcal{H}$ *obeying*

$$\|\mathcal{S}(t)w_1^0 - \mathcal{S}(t)w_2^0\|_{\mathcal{H}}^2 \leq K_1(t)\|w_1^0 - w_2^0\|_{\mathcal{H}}^2, \tag{4.126}$$

$\mathcal{S}(t)$ *is Lipschitz continuous in* $\mathcal{V}'$ *obeying*

$$\|\mathcal{S}(t)w_1^0 - \mathcal{S}(t)w_2^0\|_{\mathcal{V}'}^2 \leq K_2(t)\|w_1^0 - w_2^0\|_{\mathcal{V}'}^2, \tag{4.127}$$

and $\mathcal{S}(t)$ *is Lipschitz continuous for* $t > 0$ *from* $\mathcal{H}$ *to* $\mathcal{V}'$ *obeying*

$$t\|\mathcal{S}(t)w_1^0 - \mathcal{S}(t)w_2^0\|_{\mathcal{V}'}^2 \leq K_3(t)\|w_1^0 - w_2^0\|_{\mathcal{H}}^2. \tag{4.128}$$

Proof: We write $\mathcal{S}(t)w_1^0 = (u_1(t), c_1^1(t), c_2^1(t))$ and $\mathcal{S}(t)w_2^0 = (u_2(t), c_1^2(t), c_2^2(t))$. Let $\rho_1 = c_1^1 - c_2^1$, $\rho_2 = c_1^2 - c_2^2$, $\sigma_1 = c_1^1 + c_2^1$, $\sigma_2 = c_1^2 + c_2^2$. We write $u = u_1 - u_2$, $\rho = \rho_1 - \rho_2$ and $\sigma = \sigma_1 - \sigma_2$.

We note that u, ρ and σ obey system (4.24). Following the proof of uniqueness in Theorem 4.1, we obtain a differential inequality of the form

$$\begin{aligned}&\frac{d}{dt}\left[\|u\|_{L^2}^2 + \|\rho\|_{L^2}^2 + \|\sigma\|_{L^2}^2\right] + \nu\|\nabla u\|_{L^2}^2 + D\|\nabla\rho\|_{L^2}^2 + D\|\nabla\sigma\|_{L^2}^2\\ &\leq k_1(t)\left[\|u\|_{L^2}^2 + \|\rho\|_{L^2}^2 + \|\sigma\|_{L^2}^2\right]\end{aligned} \tag{4.129}$$

where

$k_1(t) = C[\|\nabla u_1\|_{L^2}^{2/3}\|\Delta u_1\|_{L^2}^{2/3} + \|\nabla \rho_1\|_{L^2}^2 + \|\nabla \sigma_1\|_{L^2}^2 + \|\rho_1\|_{L^3}^2 + \|\sigma_2\|_{L^2}^4 + \|\rho_2\|_{L^2}^4 + 1].$

Letting

$$K_1(t) = 4\exp\left\{\int_0^t k_1(s)ds\right\}, \tag{4.130}$$

we obtain (4.126).

Now, we take the L^2 inner product of the three equations of system (4.24) with $-\Delta u$, $-\Delta\rho$ and $-\Delta\sigma$ respectively, and we add them. We obtain the differential inequality

$$\begin{aligned}
&\frac{d}{dt}\left[\|\nabla u\|_{L^2}^2 + \|\nabla\rho\|_{L^2}^2 + \|\nabla\sigma\|_{L^2}^2\right] \\
&+ \nu\|\Delta u\|_{L^2}^2 + D\|\Delta\rho\|_{L^2}^2 + D\|\Delta\sigma\|_{L^2}^2 \\
&\le C\left[\|u_1\cdot\nabla u_1 - u_2\cdot\nabla u_2\|_{L^2}^2 + \|u_1\cdot\nabla\rho_1 - u_2\cdot\nabla\rho_2\|_{L^2}^2\right] \\
&+ C\left[\|u_1\cdot\nabla\sigma_1 - u_2\cdot\nabla\sigma_2\|_{L^2}^2 + \|\rho_1\nabla\Phi_1 - \rho_2\nabla\Phi_2\|_{L^2}^2\right] \\
&+ C\left[\|\nabla\cdot(\sigma_1\nabla\Phi_1 - \sigma_2\nabla\Phi_2)\|_{L^2}^2 + \|\nabla\cdot(\rho_1\nabla\Phi_1 - \rho_2\nabla\Phi_2)\|_{L^2}^2\right].
\end{aligned} \tag{4.131}$$

We estimate

$$\begin{aligned}
&\|u_1\cdot\nabla u_1 - u_2\cdot\nabla u_2\|_{L^2}^2 = \|u\cdot\nabla u_1 + u_2\cdot\nabla u\|_{L^2}^2 \\
&\le C[\|\nabla u_1\|_{L^4}^2\|\nabla u\|_{L^2}^2 + \|u_2\|_{L^\infty}^2\|\nabla u\|_{L^2}^2],
\end{aligned} \tag{4.132}$$

$$\begin{aligned}
&\|u_1\cdot\nabla\rho_1 - u_2\cdot\nabla\rho_2\|_{L^2}^2 = \|u\cdot\nabla\rho_1 + u_2\cdot\nabla\rho\|_{L^2}^2 \\
&\le C[\|\nabla\rho_1\|_{L^4}^2\|\nabla u\|_{L^2}^2 + \|u_2\|_{L^\infty}^2\|\nabla\rho\|_{L^2}^2]
\end{aligned} \tag{4.133}$$

and

$$\|u_1 \cdot \nabla\sigma_1 - u_2 \cdot \nabla\sigma_2\|_{L^2}^2 = \|u \cdot \nabla\sigma_1 + u_2 \cdot \nabla\sigma\|_{L^2}^2$$

$$\leq C[\|\nabla\sigma_1\|_{L^4}^2 \|\nabla u\|_{L^2}^2 + \|u_2\|_{L^\infty}^2 \|\nabla\sigma\|_{L^2}^2] \tag{4.134}$$

using Poincaré and Ladyzhenskaya's interpolation inequalities. Using in addition elliptic regularity, we have

$$\|\rho_1 \nabla\Phi_1 - \rho_2 \nabla\Phi_2\|_{L^2}^2 = \|\rho\nabla\Phi_1 + \rho_2 \nabla\Phi\|_{L^2}^2$$

$$\leq C[\|\nabla\Phi_1\|_{L^\infty}^2 + \|\rho_2\|_{L^2}^2] \|\nabla\rho\|_{L^2}^2. \tag{4.135}$$

We also estimate

$$\|\nabla \cdot (\sigma_1 \nabla\Phi_1 - \sigma_2 \nabla\Phi_2)\|_{L^2}^2$$

$$= \|\sigma\Delta\Phi_1 + \sigma_2 \Delta\Phi + \nabla\sigma \cdot \nabla\Phi_1 + \nabla\sigma_2 \cdot \nabla\Phi\|_{L^2}^2 \tag{4.136}$$

$$\leq C[\|\rho_1\|_{L^\infty}^2 \|\sigma\|_{L^2}^2 + \|\nabla\Phi_1\|_{L^\infty}^2 \|\nabla\sigma\|_{L^2}^2]$$

$$+ C(\|\sigma_2\|_{L^\infty}^2 + \|\nabla\sigma_2\|_{L^2}^2)\|\nabla\rho\|_{L^2}^2 \tag{4.137}$$

and

$$\|\nabla \cdot (\rho_1 \nabla\Phi_1 - \rho_2 \nabla\Phi_2)\|_{L^2}^2$$

$$= \|\rho\Delta\Phi_1 + \rho_2 \Delta\Phi + \nabla\rho \cdot \nabla\Phi_1 + \nabla\rho_2 \cdot \nabla\Phi\|_{L^2}^2$$

$$\leq C[\|\rho_1\|_{L^\infty}^2 + \|\rho_2\|_{L^\infty}^2 + \|\nabla\Phi_1\|_{L^\infty}^2 + \|\nabla\rho_2\|_{L^2}^2] \|\nabla\rho\|_{L^2}^2. \tag{4.138}$$

In view of (4.129), we obtain a differential inequality of the form

$$\frac{d}{dt}[\|u\|_{H^1}^2 + \|\rho\|_{H^1}^2 + \|\sigma\|_{H^1}^2] \leq k_2(t)[\|u\|_{H^1}^2 + \|\rho\|_{H^1}^2 + \|\sigma\|_{H^1}^2] \tag{4.139}$$

where

$$k_2(t) = k_1(t) + C[\|\nabla u_1\|_{L^4}^2 + \|\nabla \rho_1\|_{L^4}^2 + \|\nabla \sigma_1\|_{L^4}^2 + \|\nabla \rho_2\|_{L^2}^2]$$
$$+ C[\|\nabla \sigma_2\|_{L^2}^2 + \|u_2\|_{L^\infty}^2 + \|\sigma_2\|_{L^\infty}^2 + \|\rho_2\|_{L^\infty}^2]. \tag{4.140}$$

Letting

$$K_2(t) = 4 \exp\left\{ \int_0^t k_2(s) ds \right\}, \tag{4.141}$$

we obtain (4.127).

The derivation of (4.128) is a little different. The sum of the equations resulting from taking L^2 inner product of the u, ρ and σ equations with $-\Delta u$, $-\Delta \rho$ and $-\Delta \sigma$ respectively gives

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}[\|\nabla u\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2 + \|\nabla \sigma\|_{L^2}^2] \\
&+ \nu \|\Delta u\|_{L^2}^2 + D\|\Delta \rho\|_{L^2}^2 + D\|\Delta \sigma\|_{L^2}^2 \\
&= \int (u \cdot \nabla u_1 + u_2 \cdot \nabla u) \cdot \Delta u + \int (u \cdot \nabla \rho_1 + u_2 \cdot \nabla \rho) \Delta \rho \\
&+ \int (u \cdot \nabla \sigma_1 + u_2 \cdot \nabla \sigma) \Delta \sigma + \int (\rho \nabla \Phi_1 + \rho_2 \nabla \Phi) \cdot \Delta u \\
&- D \int (\nabla \cdot (\sigma \nabla \Phi_1 + \sigma_2 \nabla \Phi)) \Delta \rho - D \int (\nabla \cdot (\rho \nabla \Phi_1 + \rho_2 \nabla \Phi)) \Delta \sigma. \qquad (4.142)
\end{aligned}$$

In order to get (4.128), we let $w(t) = (u(t), \rho(t), \sigma(t))$, and we show that w obeys a differential inequality of the type

$$\frac{d}{dt}\|w\|_{H^1}^2 \leq Z_1(t)\|w\|_{H^1}^2 + Z_2(t)\|w\|_{L^2}^2 \tag{4.143}$$

such that

$$\|w(t)\|_{L^2}^2 \leq Z_3(t)\|w_0\|_{L^2}^2 \tag{4.144}$$

and

$$\int_0^t \|w(s)\|_{H^1}^2 ds \le C(Z_4(t)+1)\|w_0\|_{L^2}^2 \tag{4.145}$$

where $Z_1(t), Z_3(t)$ and $Z_4(t)$ are locally bounded functions in time, $Z_2(t)$ is a locally integrable function in time, and C is a positive constant. Then, multiplying (4.143) by t and integrating by parts in time from 0 to t, we obtain

$$t\|w(t)\|_{H^1}^2 \le C'(Z_5(t)+1)\|w_0\|_{L^2}^2 \tag{4.146}$$

where $Z_5(t)$ is a locally bounded function in time, and $C' > 0$ is a positive constant.

We start by integrating (4.129). Using (4.126), we obtain

$$\int_0^t (\|\nabla u(s)\|_{L^2}^2 + \|\nabla\rho(s)\|_{L^2}^2 + \|\nabla\sigma(s)\|_{L^2}^2)ds$$
$$\le C\left[1+\int_0^t k_1(s)K_1(s)ds\right]\|w_1^0 - w_2^0\|_{L^2}^2. \tag{4.147}$$

This is the analogue of (4.145). Then, we estimate

$$\left|\int (u\cdot\nabla u_1 + u_2\cdot\nabla u)\cdot\Delta u\right| \le C\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2}\|\nabla u_1\|_{L^2}^{1/2}\|\Delta u_1\|_{L^2}^{1/2}\|\Delta u\|_{L^2}$$
$$+ C\|u_2\|_{L^2}^{1/2}\|\nabla u_2\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2}\|\Delta u\|_{L^2}^{3/2}, \tag{4.148}$$

$$\left|\int (u\cdot\nabla\rho_1 + u_2\cdot\nabla\rho)\Delta\rho\right| \le C\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2}\|\nabla \rho_1\|_{L^2}^{1/2}\|\Delta\rho_1\|_{L^2}^{1/2}\|\Delta\rho\|_{L^2}$$
$$+ C\|u_2\|_{L^2}^{1/2}\|\nabla u_2\|_{L^2}^{1/2}\|\nabla\rho\|_{L^2}^{1/2}\|\Delta\rho\|_{L^2}^{3/2} \tag{4.149}$$

and

$$\left|\int (u\cdot\nabla\sigma_1 + u_2\cdot\nabla\sigma)\Delta\sigma\right| \le C\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2}\|\nabla \sigma_1\|_{L^2}^{1/2}\|\Delta\sigma_1\|_{L^2}^{1/2}\|\Delta\sigma\|_{L^2}$$
$$+ C\|u_2\|_{L^2}^{1/2}\|\nabla u_2\|_{L^2}^{1/2}\|\nabla\sigma\|_{L^2}^{1/2}\|\Delta\sigma\|_{L^2}^{3/2}. \tag{4.150}$$

In view of the fact that

$$\|\nabla\Phi\|_{L^4} \le C\|\rho\|_{L^2}, \tag{4.151}$$

we have

$$\left|\int (\rho\nabla\Phi_1 + \rho_2\nabla\Phi)\cdot\Delta u\right|$$

$$\leq C[\|\rho_1\|_{L^3}\|\rho\|_{L^2}\|\Delta u\|_{L^2} + \|\rho_2\|_{L^4}\|\rho\|_{L^2}\|\Delta u\|_{L^2}] \tag{4.152}$$

Moreover,

$$\left|\int (\nabla\cdot(\sigma\nabla\Phi_1 + \sigma_2\nabla\Phi))\Delta\rho\right|$$

$$\leq C[(\|\sigma\|_{L^2}^{1/2}\|\nabla\sigma\|_{L^2}^{1/2} + |\sigma\|_{L^2})\|\nabla\rho_1\|_{L^2} + \|\sigma_2\|_{L^\infty}\|\rho\|_{L^2}]\|\Delta\rho\|_{L^2}$$

$$+ C[\|\rho_1\|_{L^3}\|\nabla\sigma\|_{L^2} + \|\nabla\sigma_2\|_{L^2}\|\nabla\rho\|_{L^2}]\|\Delta\rho\|_{L^2} \tag{4.153}$$

and

$$\left|\int (\nabla\cdot(\rho\nabla\Phi_1 + \rho_2\nabla\Phi))\Delta\sigma\right|$$

$$\leq C[\|\nabla\rho\|_{L^2}\|\nabla\rho_1\|_{L^2} + \|\rho_2\|_{L^\infty}\|\rho\|_{L^2}]\|\Delta\sigma\|_{L^2}$$

$$+ C[\|\rho_1\|_{L^3}\|\nabla\rho\|_{L^2} + \|\nabla\rho_2\|_{L^2}\|\nabla\rho\|_{L^2}]\|\Delta\sigma\|_{L^2}. \tag{4.154}$$

We apply Young's inequality and we use (4.129) to obtain

$$\frac{d}{dt}[\|u\|_{H^1}^2 + \|\rho\|_{H^1}^2 + \|\sigma\|_{H^1}^2]$$

$$\leq CM_1(t)\|u\|_{L^2}^2 + CM_2(t)(\|\rho\|_{L^2}^2 + \|\sigma\|_{L^2}^2)$$

$$+ CM_3(t)(\|\nabla u\|_{L^2}^2 + \|\nabla\rho\|_{L^2}^2 + \|\nabla\sigma\|_{L^2}^2), \tag{4.155}$$

where

$$M_1(t) = k_1 + \|\nabla u_1\|_{L^2}^2\|\Delta u_1\|_{L^2}^2 + \|\nabla\rho_1\|_{L^2}^2\|\Delta\rho_1\|_{L^2}^2$$

$$+ \|\nabla\sigma_1\|_{L^2}^2\|\Delta\sigma_1\|_{L^2}^2, \tag{4.156}$$

$$M_2(t) = k_1 + \|\nabla\rho_1\|_{L^2}^2 + \|\nabla\rho_2\|_{L^2}^2 + \|\sigma_2\|_{L^\infty}^2 + \|\rho_2\|_{L^\infty}^2, \tag{4.157}$$

and

$$M_3(t) = 1 + \|\nabla u_2\|_{L^2}^4 + \|\nabla \rho_1\|_{L^2}^2 + \|\nabla \rho_2\|_{L^2}^2 + \|\nabla \sigma_2\|_{L^2}^2. \tag{4.158}$$

This is a differential inequality of type (4.143), with $w(t) = (u(t), \rho(t), \sigma(t))$ satisfying (4.144) and (4.145). Therefore, we obtain (4.128).

We proceed to show that the solution map $\mathcal{S}(t)$ is injective on $\mathcal{V}'$.

Theorem 4.6. *(Backward Uniqueness) Let $w_1^0, w_2^0 \in \mathcal{V}'$. If there exists $T > 0$ such that $\mathcal{S}(T)w_1^0 = \mathcal{S}(T)w_2^0$, then $w_1^0 = w_2^0$.*

Proof: Let $w(t) = \mathcal{S}(t)w_1^0 - \mathcal{S}(t)w_2^0 = (u(t), c_1(t), c_2(t))$ and $\widetilde{w}(t) = \frac{1}{2}(\mathcal{S}(t)w_1^0 + \mathcal{S}(t)w_2^0) = (\widetilde{u}(t), \widetilde{c}_1(t), \widetilde{c}_2(t))$. Let $\rho = c_1 - c_2$, $\widetilde{\rho} = \widetilde{c}_1 - \widetilde{c}_2$, $\Phi = \frac{1}{\epsilon}\Lambda^{-2}\rho$ and $\widetilde{\Phi} = \frac{1}{\epsilon}\Lambda^{-2}\widetilde{\rho}$.

We note that $w(t)$ obeys the equation

$$\partial_t w + \mathcal{A}w + L(\widetilde{w})w = 0 \tag{4.159}$$

where

$$\mathcal{A}w = (\nu A u, -D\Delta c_1, -D\Delta c_2) \tag{4.160}$$

and

$$L(\widetilde{w})w = (L_1(\widetilde{w})w, L_2(\widetilde{w})w, L_3(\widetilde{w})w) \tag{4.161}$$

with

$$L_1(\widetilde{w})w = B(\widetilde{u}, u) + B(u, \widetilde{u}) + \mathbb{P}(\rho\nabla\widetilde{\Phi} + \widetilde{\rho}\nabla\Phi)), \tag{4.162}$$

$$L_2(\widetilde{w})w = u \cdot \nabla\widetilde{c}_1 + \widetilde{u} \cdot \nabla c_1 - D\nabla \cdot (c_1\nabla\widetilde{\Phi} + \widetilde{c}_1\nabla\Phi), \tag{4.163}$$

$$L_3(\widetilde{w})w = u \cdot \nabla \widetilde{c}_2 + \widetilde{u} \cdot \nabla c_2 + D\nabla \cdot (c_2 \nabla \widetilde{\Phi} + \widetilde{c}_2 \nabla \Phi). \tag{4.164}$$

We consider the evolution of the norm

$$E_0 = \|u\|_{L^2}^2 + \|c_1\|_{L^2}^2 + \|c_2\|_{L^2}^2 = \|w\|_H^2 \tag{4.165}$$

obtained by taking the inner product in $\mathcal{H}$ of equation (4.159) with $(u, c_1, c_2) = w$, and we note that E_0 obeys the equation

$$\frac{1}{2}\frac{d}{dt}E_0 + E_1 + (L(\widetilde{w})w, w)_{\mathcal{H}} = 0 \tag{4.166}$$

where

$$E_1 = \nu \|A^{\frac{1}{2}}u\|_H^2 + D\|\nabla c_1\|_{L^2}^2 + D\|\nabla c_2\|_{L^2}^2 = (w, \mathcal{A}w)_{\mathcal{H}}. \tag{4.167}$$

We observe that

$$\frac{1}{2}\frac{d}{dt}\log\left(\frac{1}{E_0}\right) = \frac{E_1}{E_0} + \frac{(L(\widetilde{w})w, w)_{\mathcal{H}}}{E_0} \tag{4.168}$$

Let

$$Y(t) = \log\left(\frac{1}{E_0}\right) \tag{4.169}$$

and so

$$\frac{1}{2}\frac{d}{dt}Y(t) = \frac{E_1}{E_0} + \frac{(L(\widetilde{w})w, w)_{\mathcal{H}}}{E_0}. \tag{4.170}$$

We proceed to show that $Y(t)$ cannot reach the value $+\infty$ in finite time. We start by noting that the derivative of E_1/E_0 obeys

$$\frac{d}{dt}\frac{E_1}{E_0} = E_0^{-1}\frac{d}{dt}E_1 - \frac{E_1}{E_0}\frac{d}{dt}\log E_0 = E_0^{-1}\frac{d}{dt}E_1 + \frac{E_1}{E_0}\frac{d}{dt}Y. \tag{4.171}$$

Taking the inner product of equation (4.159) in $\mathcal{H}$ with $\mathcal{A}w$ leads to

$$\frac{1}{2}\frac{d}{dt}E_1 + \|\mathcal{A}w\|_{\mathcal{H}}^2 + (L(\widetilde{w})w, \mathcal{A}w)_{\mathcal{H}} = 0 \tag{4.172}$$

which implies that

$$\frac{1}{2}\frac{d}{dt}\frac{E_1}{E_0} = -\frac{\|\mathcal{A}w\|_{\mathcal{H}}^2 + (L(\widetilde{w})w, \mathcal{A}w)_{\mathcal{H}}}{E_0} + \frac{E_1}{E_0}\left(\frac{E_1 + (L(\widetilde{w})w, w)_{\mathcal{H}}}{E_0}\right) \quad (4.173)$$

Since

$$\frac{E_1^2}{E_0^2} - \frac{\|\mathcal{A}w\|_{\mathcal{H}}^2}{E_0} = -\left\|\left(\mathcal{A} - \frac{E_1}{E_0}\right)\frac{w}{E_0^{1/2}}\right\|_{\mathcal{H}}^2, \quad (4.174)$$

we obtain

$$\frac{1}{2}\frac{d}{dt}\frac{E_1}{E_0} = -E_0^{-1}\|(\mathcal{A}-E_1E_0^{-1})w\|_{\mathcal{H}}^2 - E_0^{-1}\left(L(\widetilde{w})w, \left(A - E_1E_0^{-1}\right)w\right)_{\mathcal{H}}. \quad (4.175)$$

Now, we claim that

$$|(L(\widetilde{w})w, w)_{\mathcal{H}}| \le A_1(t)E_1 + A_0(t)E_0 \quad (4.176)$$

with

$$\int_0^T (A_0(t) + A_1(t))dt < \infty. \quad (4.177)$$

To prove this claim, we note first that

$$(B(\widetilde{u}, u), u)_{L^2} = (\widetilde{u}\cdot\nabla c_1, c_1)_{L^2} = (\widetilde{u}\cdot\nabla c_2, c_2)_{L^2} = 0. \quad (4.178)$$

Since u has mean zero, an application of Ladyzhenskaya's inequality followed by Poincaré's inequality gives

$$|(B(u, \widetilde{u}), u)_{L^2}| \le \|\nabla\widetilde{u}\|_{L^2}\|u\|_{L^4}^2 \le C\|\nabla\widetilde{u}\|_{L^2}\|\nabla u\|_{L^2}^2 \le C\|\nabla\widetilde{u}\|_{L^2}E_1. \quad (4.179)$$

Using in addition elliptic regularity and the fact that ρ has mean zero, we obtain

$$\begin{aligned} |(\mathbb{P}(\rho\nabla\widetilde{\Phi} + \widetilde{\rho}\nabla\Phi), u)_{L^2}| &\le \|u\|_{L^4}\|\rho\|_{L^4}\|\nabla\widetilde{\Phi}\|_{L^2} + \|u\|_{L^4}\|\widetilde{\rho}\|_{L^2}\|\nabla\Phi\|_{L^4} \\ &\le C\|\nabla u\|_{L^2}(\|\nabla c_1\|_{L^2} + \|\nabla c_2\|_{L^2})(\|\nabla\widetilde{\Phi}\|_{L^2} + \|\widetilde{\rho}\|_{L^2}) \\ &\le C\left(1 + \|\nabla\widetilde{\Phi}\|_{L^2}^2 + \|\widetilde{\rho}\|_{L^2}^2\right)E_1. \end{aligned} \quad (4.180)$$

Now, we estimate

$$|(u\cdot\nabla\widetilde{c}_1, c_1)_{L^2}| = |(u\cdot\nabla c_1, \widetilde{c}_1)_{L^2}| \le C\|\nabla u\|_{L^2}\|\nabla c_1\|_{L^2}\|\widetilde{c}_1\|_{L^4}$$

$$\le C(1+\|\widetilde{c}_1\|_{L^4}^2)E_1, \tag{4.181}$$

$$|(u\cdot\nabla\widetilde{c}_2, c_2)_{L^2}| \le C(1+\|\widetilde{c}_2\|_{L^4}^2)E_1, \tag{4.182}$$

$$|(\nabla\cdot(c_1\nabla\widetilde{\Phi}+\widetilde{c}_1\nabla\Phi), c_1)_{L^2}|$$

$$\le C(\|c_1\|_{L^2}\|\nabla\widetilde{\Phi}\|_{L^\infty}\|\nabla c_1\|_{L^2} + \|\widetilde{c}_1\|_{L^2}\|\nabla c_1\|_{L^2}(\|\nabla c_1\|_{L^2}+\|\nabla c_2\|_{L^2}))$$

$$\le C(\|\nabla\widetilde{\Phi}\|_{L^\infty}^2 + \|\widetilde{c}_1\|_{L^2} + \|\widetilde{c}_1\|_{L^2}^2 + 1)E_1 + E_0 \tag{4.183}$$

and

$$|(\nabla\cdot(c_2\nabla\widetilde{\Phi}+\widetilde{c}_2\nabla\Phi), c_2)_{L^2}| \le C(\|\nabla\widetilde{\Phi}\|_{L^\infty}^2+\|\widetilde{c}_2\|_{L^2}+\|\widetilde{c}_2\|_{L^2}^2+1)E_1+E_0 \tag{4.184}$$

This ends the proof of the first claim.

Next, we claim that

$$\|L(\widetilde{w})w\|_{\mathcal{H}}^2 \le B_1(t)E_1 + B_0(t)E_0 \tag{4.185}$$

with

$$\int_0^T (B_0(t)+B_1(t))dt < \infty. \tag{4.186}$$

Since u and ρ have mean zero, then elliptic regularity together with an application of Hölder, Ladyzhenskaya, Poincaré and Young inequalities gives

$$\|B(\widetilde{u},u)+B(u,\widetilde{u})\|_{L^2}^2 \le C(\|\widetilde{u}\|_{L^\infty}^2\|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\|\nabla\widetilde{u}\|_{L^4}^2)$$

$$\le C(\|\widetilde{u}\|_{L^\infty}^2 + \|\nabla\widetilde{u}\|_{L^4}^2)E_1, \tag{4.187}$$

$$\|\mathbb{P}(\rho\nabla\widetilde{\Phi}+\widetilde{\rho}\nabla\Phi)\|_{L^2}^2 \le C(\|\nabla\rho\|_{L^2}^2\|\nabla\widetilde{\Phi}\|_{L^\infty}^2 + \|\widetilde{\rho}\|_{L^2}^2\|\nabla\rho\|_{L^2}^2)$$

$$\le C(\|\nabla\widetilde{\Phi}\|_{L^\infty}^2 + \|\widetilde{\rho}\|_{L^2}^2)E_1, \tag{4.188}$$

$$\|u \cdot \nabla \widetilde{c}_1 + \widetilde{u} \cdot \nabla c_1\|_{L^2}^2 \leq C(\|\nabla u\|_{L^2}^2 \|\nabla \widetilde{c}_1\|_{L^4}^2 + \|\widetilde{u}\|_{L^\infty}^2 \|\nabla c_1\|_{L^2}^2)$$

$$\leq C(\|\nabla \widetilde{c}_1\|_{L^4}^2 + \|\widetilde{u}\|_{L^\infty}^2) E_1, \tag{4.189}$$

$$\|u \cdot \nabla \widetilde{c}_2 + \widetilde{u} \cdot \nabla c_2\|_{L^2}^2 \leq C(\|\nabla \widetilde{c}_2\|_{L^4}^2 + \|\widetilde{u}\|_{L^\infty}^2) E_1, \tag{4.190}$$

$$\|\nabla \cdot (c_1 \nabla \widetilde{\Phi} + \widetilde{c}_1 \nabla \Phi)\|_{L^2}^2$$

$$= \|c_1 \Delta \widetilde{\Phi} + \nabla c_1 \nabla \widetilde{\Phi} + \widetilde{c}_1 \Delta \Phi + \nabla \widetilde{c}_1 \nabla \Phi\|_{L^2}^2$$

$$\leq C((\|c_1\|_{L^2} \|\nabla c_1\|_{L^2} + \|c_1\|_{L^2}^2) \|\widetilde{\rho}\|_{L^2} \|\nabla \widetilde{\rho}\|_{L^2} + \|\nabla c_1\|_{L^2}^2 \|\nabla \widetilde{\Phi}\|_{L^\infty}^2)$$

$$+ C(\|\widetilde{c}_1\|_{L^4}^2 \|\rho\|_{L^2} \|\nabla \rho\|_{L^2} + \|\nabla \widetilde{c}_1\|_{L^2}^2 \|\nabla \rho\|_{L^2}^2)$$

$$\leq C(\|\widetilde{\rho}\|_{L^2}^2 \|\nabla \widetilde{\rho}\|_{L^2}^2 + \|\nabla \widetilde{\Phi}\|_{L^\infty}^2 + \|\widetilde{c}_1\|_{L^4}^2 + \|\nabla \widetilde{c}_1\|_{L^2}^2) E_1$$

$$+ \|\widetilde{\rho}\|_{L^2} \|\nabla \widetilde{\rho}\|_{L^2} E_0, \tag{4.191}$$

and

$$\|\nabla \cdot (c_2 \nabla \widetilde{\Phi} + \widetilde{c}_2 \nabla \Phi)\|_{L^2}^2$$

$$\leq C(\|\widetilde{\rho}\|_{L^2}^2 \|\nabla \widetilde{\rho}\|_{L^2}^2 + \|\nabla \widetilde{\Phi}\|_{L^\infty}^2 + \|\widetilde{c}_2\|_{L^4}^2 + \|\nabla \widetilde{c}_2\|_{L^2}^2) E_1$$

$$+ \|\widetilde{\rho}\|_{L^2} \|\nabla \widetilde{\rho}\|_{L^2} E_0. \tag{4.192}$$

Thus, the second claim is proved.

As a consequence of the above claims and Schwarz inequality, we deduce the differential inequalities

$$\frac{d}{dt} \frac{E_1}{E_0} \leq 2B_1(t) \frac{E_1}{E_0} + 2B_0(t) \tag{4.193}$$

and

$$\frac{d}{dt} Y(t) \leq (2A_1(t) + 1) \frac{E_1}{E_0} + 2A_0(t) \tag{4.194}$$

which imply that $Y(t) \in L^\infty(0,T)$. This ends the proof.

Now, we fix $M > 0$, and we let $\mathcal{V}_M$ to be the subset of $\mathcal{V}'$ consisting of vectors (u, c_1, c_2) such that u is divergence free with mean zero and c_1 and c_2 are non-negative functions a.e. with equal space averages less than or equal to M. As a consequence of Theorem 4.4, there exists $R_1 > 0$ depending only on f such that for any initial data $w_0 = (u_0, c_1(0), c_2(0)) \in \mathcal{V}_M$, there exists $t_0 > 0$ depending on $\|u_0\|_{H^1}, \|c_1(0)\|_{H^1}, \|c_2(0)\|_{H^1}$ and the parameters of the problem such that for all $t \geq t_0$, we have $\mathcal{S}(t)w_0 \in \mathcal{B}_{R_1}^M$, where

$$\mathcal{B}_{R_1}^M = \{w = (u, c_1, c_2) \in \mathcal{V}_M : \|u\|_{H^2} + \|c_1 - \bar{c}_1\|_{H^2} + \|c_2 - \bar{c}_2\|_{H^2} \leq R_1\}.$$

Remark 4.6. *We note that there exists $T > 0$ depending only on R_1 and M and the parameters of the problem such that*

$$\mathcal{S}(t)B_{R_1}^M \subset B_{R_1}^M \tag{4.195}$$

for all $t \geq T$.

Remark 4.7. *$\mathcal{B}_{R_1}^M$ is compact in $\mathcal{H}$ because the space averages of all the concentrations c_1 and c_2 such that $(u, c_1, c_2) \in \mathcal{V}_M$ are uniformly bounded by M.*

Remark 4.8. *The set $\mathcal{V}_M$ is convex. Consequently, $\mathcal{B}_{R_1}^M$ is a convex set, and so it is connected.*

The properties of the map $\mathcal{S}(t)$ listed and proved above, together with the connectedness and compactness properties of $\mathcal{B}_{R_1}^M$, imply the existence of a global attractor.

Theorem 4.7. *(Global Attractor) Let*

$$X_M = \bigcap_{t>0} \mathcal{S}(t)\mathcal{B}_{R_1}^M \tag{4.196}$$

Then:

(a) X_M *is compact in* $\mathcal{H}$.

(b) $\mathcal{S}(t)X_M = X_M$ *for all* $t \geq 0$.

(c) If Z *is bounded in* $\mathcal{V}_M$ *in the norm of of* $\mathcal{V}$*, and* $\mathcal{S}(t)Z = Z$ *for all* $t \geq 0$*, then* $Z \subset X_M$.

(d) For every $w_0 \in \mathcal{V}_M$, $\lim_{t\to\infty} dist_{\mathcal{H}}(\mathcal{S}(t)w_0, X_M) = 0$.

(e) X_M *is connected.*

The proof is omitted and follows the proof of Theorem 1.7

We end this section by showing that X_M has finite fractal dimension. The abstract formulation of the system is

$$\begin{cases} \partial_t u + \nu A u + B(u,u) + \mathbb{P}(\rho\nabla\Phi) = f, \\ \partial_t c_1 + u \cdot \nabla c_1 - D\Delta c_1 - D\nabla \cdot (c_1 \nabla\Phi) = 0, \\ \partial_t c_2 + u \cdot \nabla c_2 - D\Delta c_2 + D\nabla \cdot (c_2 \nabla\Phi) = 0, \\ -\epsilon\Delta\Phi = \rho, \\ \rho = c_1 - c_2 \end{cases} \tag{4.197}$$

where $\mathbb{P}$ is the Leray-Hopf projector, $A = \mathbb{P}(-\Delta)$ is the Stokes operator, and $B(u,v) = \mathbb{P}(u.\nabla v)$.

We consider a solution $\widetilde{w} = S(t)\widetilde{w}_0 = (\widetilde{u}(t), \widetilde{c}_1(t), \widetilde{c}_2(t))$ of (4.197) with initial data $\widetilde{w}_0$ in $\mathcal{B}_{R_1}^M$. We consider the linearization of $S(t)$ along $\widetilde{w}(t)$

$$w_0 \mapsto w(t) = S'(t, \widetilde{w})w_0 \tag{4.198}$$

viewed as an operator on $\mathcal{H}$. The function $w(t) = (u(t), c_1(t), c_2(t))$ solves

$$\partial_t w + \mathcal{A}w + L(\widetilde{w})w = 0 \tag{4.199}$$

where

$$\mathcal{A}w = (\nu Au, -D\Delta c_1, -D\Delta c_2) \tag{4.200}$$

and

$$L(\widetilde{w})w = (L_1(\widetilde{w})w, L_2(\widetilde{w})w, L_3(\widetilde{w})w) \tag{4.201}$$

with

$$L_1(\widetilde{w})w = B(\widetilde{u}, u) + B(u, \widetilde{u}) + \mathbb{P}(\rho\nabla\widetilde{\Phi} + \widetilde{\rho}\nabla\Phi)), \tag{4.202}$$

$$L_2(\widetilde{w})w = u \cdot \nabla\widetilde{c}_1 + \widetilde{u} \cdot \nabla c_1 - D\nabla \cdot (c_1\nabla\widetilde{\Phi} + \widetilde{c}_1\nabla\Phi), \tag{4.203}$$

$$L_3(\widetilde{w})w = u \cdot \nabla\widetilde{c}_2 + \widetilde{u} \cdot \nabla c_2 + D\nabla \cdot (c_2\nabla\widetilde{\Phi} + \widetilde{c}_2\nabla\Phi). \tag{4.204}$$

We consider the scalar product in $\wedge^n\mathcal{H}$ given by

$$(w_1 \wedge \cdots \wedge w_n, y_1 \wedge \cdots \wedge y_n)_{\wedge^n\mathcal{H}} = det(w_i, y_j)_{\mathcal{H}} \tag{4.205}$$

and the volume elements given by

$$V_n(t) = \|w_1(t) \wedge \cdots \wedge w_n(t)\|_{\wedge^n\mathcal{H}}. \tag{4.206}$$

We note that the monomial $w_1(t) \wedge \cdots \wedge w_n(t)$ evolves according to the equation

$$\partial_t(w_1(t) \wedge \cdots \wedge w_n(t)) + (\mathcal{A} + L(\widetilde{w}))_n(w_1(t) \wedge \cdots \wedge w_n(t)) = 0 \tag{4.207}$$

where

$$(A+L(\widetilde{w}))_n(w_1(t)\wedge\cdots\wedge w_n(t))$$
$$=(A+L(\widetilde{w}))w_1\wedge\cdots\wedge w_n+\cdots+w_1\wedge\cdots\wedge(\mathcal{A}+L(\widetilde{w}))w_n. \qquad (4.208)$$

Thus, the volume element evolves according to the ODE

$$\frac{d}{dt}V_n+\mathrm{Trace}((\mathcal{A}+L(\widetilde{w}))Q_n)V_n=0 \qquad (4.209)$$

where Q_n is the orthogonal projection in $\mathcal{H}$ onto the linear space spanned by the vectors $w_1,\dots,w_n$.

Theorem 4.8. *(Decay of Volume Elements) There exists a positive integer N_0 depending on R_1 and M such that for any $\widetilde{w}_0\in\mathcal{B}_{R_1}$, and for any $n\geq N_0$, and for any $w_1(0),\dots,w_n(0)\in\mathcal{H}$*

$$\|S'(t,\widetilde{w})w_1(0)\wedge\cdots\wedge S'(t,\widetilde{w})w_n(0)\|_{\Lambda^n\mathcal{H}}\leq V_n(0)e^{-cnt} \qquad (4.210)$$

holds for any $t\geq t_0$ with t_0 depending on R_1.

Proof: For each t, choose an orthonormal basis $b_i=(v_i,r_i^1,r_i^2)$ of the linear span of $w_1,\dots,w_n$. Then

$$\mathrm{Trace}((\mathcal{A}+L(\widetilde{w}))Q_n)=\sum_{i=1}^n(\mathcal{A}b_i,b_i)_{\mathcal{H}}+\sum_{i=1}^n(L(\widetilde{w})b_i,b_i)_{\mathcal{H}}. \qquad (4.211)$$

We note that

$$\mathrm{Trace}(\mathcal{A}Q_n)=\sum_{i=1}^n(\mathcal{A}b_i,b_i)_{\mathcal{H}}$$
$$=\sum_{i=1}^n\left[(\nu Av_i,v_i)_{\mathcal{H}}+(-D\Delta r_i^1,r_i^1)_{L^2}+(-D\Delta r_i^2,r_i^2)_{L^2}\right]$$
$$\geq\mu_1+\cdots+\mu_n \qquad (4.212)$$

where μ_i are eigenvalues of $\mathcal{A}$ in $\mathcal{H}$. By Proposition 4.1, there exists a constant C such that $\mu_k \geq Ck$ for all $k \geq 1$. It follows that $\mathrm{Trace}(\mathcal{A}Q_n) \geq C_0 n^2$ for some positive constant C_0.

Let $\rho_i = r_i^1 - r_i^2$ and $\Phi_i = \frac{1}{\epsilon}\Lambda^{-2}\rho_i$. In view of Hölder's inequality, Ladyzhenskaya's inequality, elliptic regularity and the fact that $\|b_i\|_{\mathcal{H}} = 1$ for all i, we have the bounds

$$\begin{aligned} |\sum_{i=1}^{n}(B(v_i,\widetilde{u}),v_i)_{L^2}| &\leq \sum_{i=1}^{n}\|v_i\|_{L^4}^2\|\nabla\widetilde{u}\|_{L^2} \\ &\leq C\|\nabla\widetilde{u}\|_{L^2}n^{1/2}\left(\sum_{i=1}^{n}\|\nabla v_i\|_{L^2}^2\right)^{1/2} \end{aligned} \tag{4.213}$$

and

$$\begin{aligned} &|\sum_{i=1}^{n}(\mathbb{P}(\rho_i\nabla\widetilde{\Phi}+\widetilde{\rho}\nabla\Phi_i),b_i)_{L^2}| \\ &\leq \sum_{i=1}^{n}\left(\|\nabla\widetilde{\Phi}\|_{L^\infty}\|\rho_i\|_{L^2}\|b_i\|_{L^2}+\|\nabla\Phi_i\|_{L^\infty}\|\widetilde{\rho}\|_{L^2}\|b_i\|_{L^2}\right) \\ &\leq \sum_{i=1}^{n}\left(2\|\nabla\widetilde{\Phi}\|_{L^\infty}+C\|\widetilde{\rho}\|_{L^2}\|\nabla r_i^1\|_{L^2}^{1/2}+C\|\widetilde{\rho}\|_{L^2}\|\nabla r_i^2\|_{L^2}^{1/2}\right) \\ &\leq 2\|\nabla\widetilde{\Phi}\|_{L^\infty}n \\ &\quad + C\|\widetilde{\rho}\|_{L^2}n^{3/4}\left[\left(\sum_{i=1}^{n}\|\nabla r_i^1\|_{L^2}^2\right)^{1/4}+\left(\sum_{i=1}^{n}\|\nabla r_i^2\|_{L^2}^2\right)^{1/4}\right] \end{aligned} \tag{4.214}$$

and

$$\begin{aligned} |\sum_{i=1}^{n}(v_i\cdot\nabla\widetilde{c}_1,r_i^1)_{L^2}| &\leq \sum_{i=1}^{n}C\|\nabla v_i\|_{L^2}^{1/2}(\|\nabla r_i^1\|_{L^2}^{1/2}+1)\|\nabla\widetilde{c}_1\|_{L^2} \\ &\leq C\|\nabla\widetilde{c}_1\|_{L^2}n^{1/2}\left(\sum_{i=1}^{n}\|\nabla v_i\|_{L^2}^2\right)^{1/4}\left(\sum_{i=1}^{n}\|\nabla r_i^1\|_{L^2}^2\right)^{1/4} \\ &\quad + C\|\nabla\widetilde{c}_1\|_{L^2}n^{3/4}\left(\sum_{i=1}^{n}\|\nabla v_i\|_{L^2}^2\right)^{1/4} \end{aligned} \tag{4.215}$$

and

$$
\begin{aligned}
&|\sum_{i=1}^{n}(v_i\cdot\nabla\widetilde{c}_2, r_i^2)_{L^2}| \\
&\le C\|\nabla\widetilde{c}_2\|_{L^2} n^{1/2}\left(\sum_{i=1}^{n}\|\nabla v_i\|_{L^2}^2\right)^{1/4}\left(\sum_{i=1}^{n}\|\nabla r_i^2\|_{L^2}^2\right)^{1/4} \\
&\quad + C\|\nabla\widetilde{c}_2\|_{L^2} n^{3/4}\left(\sum_{i=1}^{n}\|\nabla v_i\|_{L^2}^2\right)^{1/4}.
\end{aligned}
\tag{4.216}
$$

Now, using the triangle inequality, we have

$$
\begin{aligned}
&\left|\sum_{i=1}^{n}\left[-(\nabla\cdot(r_i^1\nabla\widetilde{\Phi}+\widetilde{c}_1\nabla\Phi_i), r_i^1)_{L^2} + (\nabla\cdot(r_i^2\nabla\widetilde{\Phi}+\widetilde{c}_2\nabla\Phi_i), r_i^2)_{L^2}\right]\right| \\
&\le \left|\sum_{i=1}^{n}\left[(r_i^1\nabla\widetilde{\Phi}, \nabla r_i^1)_{L^2} - (r_i^2\nabla\widetilde{\Phi}, \nabla r_i^2)_{L^2}\right]\right| \\
&+ \left|\sum_{i=1}^{n}\left[((\widetilde{c}_1-\overline{\widetilde{c}_1})\nabla\Phi_i, \nabla r_i^1)_{L^2} - ((\widetilde{c}_2-\overline{\widetilde{c}_2})\nabla\Phi_i, \nabla r_i^2)_{L^2}\right]\right| \\
&+ \left|\sum_{i=1}^{n}(\bar{c}\nabla\Phi_i, \nabla(r_i^1-r_i^2))_{L^2}\right|
\end{aligned}
\tag{4.217}
$$

where $\bar{c}=\overline{\widetilde{c}_1}=\overline{\widetilde{c}_2}$, and using the same inequalities as above, we obtain

$$
\begin{aligned}
&|\sum_{i=1}^{n}[(r_i^1\nabla\widetilde{\Phi}, \nabla r_i^1)_{L^2} - (r_i^2\nabla\widetilde{\Phi}, \nabla r_i^2)_{L^2}]| \\
&\le \sum_{i=1}^{n}\left[\|\nabla\widetilde{\Phi}\|_{L^\infty}\|\nabla r_i^1\|_{L^2} + \|\nabla\widetilde{\Phi}\|_{L^\infty}\|\nabla r_i^2\|_{L^2}\right] \\
&\le \|\nabla\widetilde{\Phi}\|_{L^\infty} n^{1/2}\left[\left(\sum_{i=1}^{n}\|\nabla r_i^1\|_{L^2}^2\right)^{1/2} + \left(\sum_{i=1}^{n}\|\nabla r_i^2\|_{L^2}^2\right)^{1/2}\right]
\end{aligned}
\tag{4.218}
$$

and

$$
\begin{aligned}
&\left|\sum_{i=1}^{n}((\widetilde{c}_1-\overline{\widetilde{c}_1})\nabla\Phi_i, \nabla r_i^1)_{L^2}\right| \le \sum_{i=1}^{n} C\|\nabla r_i^1\|_{L^2}\|\nabla\rho_i\|_{L^2}^{1/2}\|\widetilde{c}_1-\overline{\widetilde{c}_1}\|_{L^2} \\
&\le Cn^{1/4}\|\widetilde{c}_1-\overline{\widetilde{c}_1}\|_{L^2}\left(\sum_{i=1}^{n}\|\nabla r_i^1\|_{L^2}^2\right)^{3/4} \\
&\quad + Cn^{1/4}\|\widetilde{c}_1-\overline{\widetilde{c}_1}\|_{L^2}\left(\sum_{i=1}^{n}\|\nabla r_i^2\|^2\right)^{1/4}\left(\sum_{i=1}^{n}\|\nabla r_i^1\|_{L^2}^2\right)^{1/2}
\end{aligned}
\tag{4.219}
$$

and

$$\left| \sum_{i=1}^{n} ((\widetilde{c}_2 - \overline{\widetilde{c}_2}) \nabla \Phi_i, \nabla r_i^2)_{L^2} \right|$$
$$\leq C n^{1/4} \|\widetilde{c}_2 - \overline{\widetilde{c}_2}\|_{L^2} \left(\sum_{i=1}^{n} \|\nabla r_i^2\|_{L^2}^2 \right)^{3/4}$$
$$+ C n^{1/4} \|\widetilde{c}_2 - \overline{\widetilde{c}_2}\|_{L^2} \left(\sum_{i=1}^{n} \|\nabla r_i^1\|^2 \right)^{1/4} \left(\sum_{i=1}^{n} \|\nabla r_i^2\|_{L^2}^2 \right)^{1/2} \tag{4.220}$$

and

$$\left| \sum_{i=1}^{n} (\bar{c} \nabla \Phi_i, \nabla (r_i^1 - r_i^2))_{L^2} \right| = \sum_{i=1}^{n} \frac{\bar{c}}{\epsilon} \|\nabla \Lambda^{-1} (r_i^1 - r_i^2)\|_{L^2}^2$$
$$\leq \sum_{i=1}^{n} C \bar{c} \|r_i^1 - r_i^2\|_{L^2}^2 \leq 4 C \bar{c} n. \tag{4.221}$$

Since $\widetilde{w}_0 \in \mathcal{B}_{R_1}^M$, there exists t_0 depending on R_1 such that $\widetilde{w}(t) = \mathcal{S}(t)\widetilde{w}_0 \in \mathcal{B}_{R_1}^M$ for all $t \geq t_0$.

Combining the bounds (4.213)–(4.221) and applying Young's inequality give

$$\frac{1}{t} \int_0^t \mathrm{Trace}((\mathcal{A} + L(\widetilde{w}))Q_n) ds \geq \frac{1}{4} \mathrm{Trace}(\mathcal{A} Q_n) - C_1 \bar{c} n - C_2 C(R_1) n$$
$$\geq n \left(\frac{1}{4} C_0 n - C_1 \bar{c} - C_2 C(R_1) \right) \tag{4.222}$$

for all $t \geq t_0$. Here, C_1, C_2 are universal positive constants, $C(R_1)$ is a constant depending on R_1, and $0 \leq \bar{c} \leq M$. Thus, choosing

$$n \geq \frac{4}{C_0} (1 + C_1 M + C_2 C(R_1)) \tag{4.223}$$

ends the proof.

As a consequence, and following the proof of the similar result in [35], we conclude that

Theorem 4.9. *The global attractor X_M has a finite fractal dimension in $\mathcal{H}$.*

We end this section with the following result:

Theorem 4.10. *The global attractor X_M has a finite fractal dimension in $\mathcal{V}$.*

Proof: Since $\mathcal{B}_{R_1}^M$ is bounded in H^2, we conclude by Rellich compactness theorem that $\mathcal{S}(t)B_{R_1}^M$ is compact in $\mathcal{V}$ for all $t \geq T$, see Remark 4.6. Hence, the property (4.128), together with the fact that X_M has a finite fractal dimension in $\mathcal{H}$, allows us to conclude that X_M has a finite fractal dimension in $\mathcal{V}$.

4.6 Added Body Forces and Added Charge Density

In this section, we consider the general case

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u - (\rho + N) \nabla \Phi + f \\ \nabla \cdot u = 0 \\ \rho = c_1 - c_2 \\ -\epsilon \Delta \Phi = \rho + N \\ \partial_t c_1 + u \cdot \nabla c_1 = D \Delta c_1 + D \nabla \cdot (c_1 \nabla \Phi) \\ \partial_t c_2 + u \cdot \nabla c_2 = D \Delta c_2 - D \nabla \cdot (c_2 \nabla \Phi) \end{cases} \tag{4.224}$$

where the body forces f are smooth, divergence-free, time independent, and have mean zero, and the added charge density N is smooth and time independent. We assume that u_0 has mean zero, and that the initial concentrations $c_1(x,0)$ and $c_2(x,0)$ have space averages $\bar{c}_1$ and $\bar{c}_2$ satisfying $\bar{c}_2 - \bar{c}_1 = \bar{N}$. We consider initial data

$(u_0, c_1(0), c_2(0)) \in H^1$. We also assume that the initial concentrations are nonnegative functions and we recall that this property is preserved for all positive times t by Theorem 4.3, which holds in this case as well.

Proposition 4.7. *Let $u_0 \in H$ and $c_i(0) \in L^2$. Then, there exists $C > 0$ such that*

$$\|\sigma(t) - \bar{\sigma}\|_{L^2}^2 + \|\rho(t) - \bar{\rho}\|_{L^2}^2$$
$$\leq (\|\sigma_0 - \bar{\sigma}\|_{L^2}^2 + \|\rho_0 - \bar{\rho}\|_{L^2}^2)e^{-Dt} + \|\bar{\sigma}\|_{L^2}^2 + C\|N\|_{L^6}^6 \tag{4.225}$$

holds for all $t \geq 0$. Moreover,

$$\int_t^{t+T} \left(\|\nabla\rho(s)\|_{L^2}^2 + \|\nabla\sigma(s)\|_{L^2}^2 + \frac{1}{\epsilon}\|\rho(s)\|_{L^3}^3 \right) ds$$
$$\leq \frac{1}{D}((\|\sigma_0 - \bar{\sigma}\|_{L^2}^2 + \|\rho_0 - \bar{\rho}\|_{L^2}^2)e^{-Dt}$$
$$+ C(T+1)(\|\bar{\sigma}\|_{L^2}^2 + \|N\|_{L^6}^6) \tag{4.226}$$

holds for any $t \geq 0, T > 0$.

Proof. We recall that σ and ρ obey

$$\begin{cases} \partial_t\sigma + u \cdot \nabla\sigma = D\Delta\sigma + D\nabla \cdot (\rho\nabla\Phi) \\ \partial_t\rho + u \cdot \nabla\rho = D\Delta\rho + D\nabla \cdot (\sigma\nabla\Phi). \end{cases} \tag{4.227}$$

We take the L^2 inner product of the equations obeyed by σ and ρ with σ and ρ respectively, we add, and use the fact that

$$\int \rho\Delta\Phi\sigma = -\frac{1}{\epsilon}\int \sigma(\rho)^2 - \frac{1}{\epsilon}\int N\rho\sigma \tag{4.228}$$

to get the equation

$$\frac{1}{2}\frac{d}{dt}(\|\sigma\|_{L^2}^2 + \|\rho\|_{L^2}^2) + D(\|\nabla\sigma\|_{L^2}^2 + \|\nabla\rho\|_{L^2}^2)$$
$$+ \frac{D}{\epsilon}\int \sigma(\rho)^2 = -\frac{D}{\epsilon}\int N\rho\sigma. \tag{4.229}$$

We estimate

$$\left|\frac{D}{\epsilon}\int N\rho\sigma\right| \le \frac{D}{\epsilon}\|N\|_{L^6}\|\rho\|_{L^3}\|\sigma\|_{L^2} \le \frac{D}{2\epsilon}\|\rho\|_{L^3}^3 + \frac{D}{4}\|\sigma\|_{L^2}^2 + C\|N\|_{L^6}^6$$

$$\le \frac{D}{2\epsilon}\|\rho\|_{L^3}^3 + \frac{D}{2}\|\sigma - \bar{\sigma}\|_{L^2}^2 + \frac{D}{2}\|\bar{\sigma}\|_{L^2}^2 + C\|N\|_{L^6}^6 \qquad (4.230)$$

in view of Hölder's and Young's inequalities. We obtain the differential inequality

$$\frac{1}{2}\frac{d}{dt}(\|\sigma - \bar{\sigma}\|_{L^2}^2 + \|\rho - \bar{\rho}\|_{L^2}^2) + \frac{D}{2}(\|\nabla\sigma\|_{L^2}^2 + \|\nabla\rho\|_{L^2}^2)$$

$$+ \frac{D}{2\epsilon}\|\rho\|_{L^3}^3 \le \frac{D}{2}\|\bar{\sigma}\|_{L^2}^2 + C\|N\|_{L^6}^6. \qquad (4.231)$$

In view of Poincaré inequality, we get

$$\frac{d}{dt}(\|\sigma - \bar{\sigma}\|_{L^2}^2 + \|\rho - \bar{\rho}\|_{L^2}^2) + D(\|\sigma - \bar{\sigma}\|_{L^2}^2 + \|\rho - \bar{\rho}\|_{L^2}^2)$$

$$\le D\|\bar{\sigma}\|_{L^2}^2 + C\|N\|_{L^6}^6. \qquad (4.232)$$

This gives (4.225). Integrating (4.231), we obtain (4.226).

Proposition 4.8. *Let $u_0 \in H^1, c_i(0) \in H^1$. Then, there exist positive constants M_1, M_2, M_3, M_4 and M_5 depending on the initial data and the parameters of the problem, and positive constants ξ_1, ξ_2, and ξ_3 depending on f, N and $\bar{\sigma}$ such that*

$$\|\nabla u\|_{L^2}^2 \le M_1(\|\nabla u_0\|_{L^2}, \|\sigma_0\|_{L^2}, \|\rho_0\|_{L^2})e^{-Dt} + \xi_1(f, N, \bar{\sigma}), \qquad (4.233)$$

$$\|\rho\|_{L^3}^2 \le M_2(\|\rho_0\|_{L^3}, \|\sigma_0\|_{L^2})e^{-Dt} + \xi_2(f, N, \bar{\sigma}), \qquad (4.234)$$

and

$$\|\nabla\rho\|_{L^2}^2 + \|\nabla\sigma\|_{L^2}^2$$

$$\le M_3(\|\nabla\rho_0\|_{L^2}, \|\nabla\sigma_0\|_{L^2}, \|\rho_0\|_{L^3}, \|\nabla u_0\|_{L^2})e^{-Dt} + \xi_3(f, N, \bar{\sigma}) \qquad (4.235)$$

hold for any $t \geq 0$. *Moreover,*

$$\int_t^{t+T} (\|\Delta\rho\|_{L^2}^2 + \|\Delta\sigma\|_{L^2}^2)ds \leq M_4(\|\nabla\rho_0\|_{L^2}, \|\nabla\sigma_0\|_{L^2}, \|\rho_0\|_{L^3}, \|\nabla u_0\|_{L^2})e^{-Dt} + \xi_3\left(f, N, \bar{\sigma}\right)\left(T+1\right) \tag{4.236}$$

and

$$\int_t^{t+T} \|\Delta u\|_{L^2}^2 ds \leq M_5(\|\nabla u_0\|_{L^2}, \|\sigma_0\|_{L^2}, \|\rho_0\|_{L^2})e^{-Dt} + \xi_1(f, N, \bar{\sigma})(T+1) \tag{4.237}$$

hold for any $t \geq 0$, $T > 0$.

Proof. The proof is similar to that of Proposition 4.4. We briefly sketch the main ideas. Taking the L^2 inner product of the u-equation with $-\Delta u$ leads to the differential inequality

$$\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \nu\|\Delta u\|_{L^2}^2 \leq C\|\rho\|_{L^2}^6 + C\|\rho\|_{L^3}^3 + C_{f,N}. \tag{4.238}$$

An application of Lemma 1.1 gives (4.237). Integrating (4.238) gives (4.237).

Taking the L^2 inner product of the ρ-equation (4.65) with $\rho|\rho|$ and estimating the resulting terms gives

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\rho\|_{L^3}^2 + \frac{D\bar{\sigma}}{\epsilon}\|\rho\|_{L^3}^2 &\leq C\|\sigma - \bar{\sigma}\|_{H^1}^{2/3}\|\sigma - \bar{\sigma}\|_{L^2}^{4/3}\|\rho\|_{L^3}^2 + C_N \\ &\leq C\|\rho\|_{L^3}^3 + C\|\nabla\sigma\|_{L^2}^2\|\sigma - \bar{\sigma}\|_{L^2}^4 + C_N. \end{aligned} \tag{4.239}$$

Thus, Lemma 1.1 gives (4.234).

Finally, taking the L^2 inner product of the ρ-equation (4.65) and of the σ-equation with $-\Delta\rho$ and $-\Delta\sigma$ respectively, adding the resulting equations, and estimating the obtained terms give the differential inequality

$$\frac{d}{dt}(\|\nabla\rho\|_{L^2}^2 + \|\nabla\sigma\|_{L^2}^2) + D(\|\Delta\rho\|_{L^2}^2 + \|\Delta\sigma\|_{L^2}^2)$$
$$\leq C(\|\nabla u\|_{L^2}^4 + \|\rho\|_{L^3}^2 + +\|N\|_{L^3}^2)(\|\nabla\rho\|_{L^2}^2 + \|\nabla\sigma\|_{L^2}^2)$$
$$+ C\|\sigma - \bar{\sigma}\|_{L^2}^2\|\nabla\sigma\|_{L^2}^2 + C\|\rho\|_{L^2}^2\|\nabla\rho\|_{L^2}^2 + +\|\rho\|_{L^2}^4 + C_N. \quad (4.240)$$

Lemma 1.1 gives (4.235). Integrating (4.240) gives (4.236).

Proposition 4.9. *Let $u_0 \in H^2, c_i(0) \in H^2$. Then, there exist positive constants M_6 and M_7 depending on the initial data and the parameters of the problem, and positive constants ξ_4 and ξ_5 depending on f, N and $\bar{\sigma}$ such that*

$$\|\Delta u\|_{L^2}^2 \leq M_6(\|\Delta u_0\|_{L^2}, \|\nabla\sigma_0\|_{L^2}, \|\nabla\rho_0\|_{L^2})e^{-Dt} + \xi_4(f, N, \bar{\sigma}) \quad (4.241)$$

and

$$\|\Delta\rho\|_{L^2}^2 + \|\Delta\rho\|_{L^2}^2 \leq M_7(\|\Delta\rho_0\|_{L^2}, \|\Delta\sigma_0\|_{L^2}, \|\nabla u_0\|_{L^2})e^{-Dt}$$
$$+ \xi_5(f, N, \bar{\sigma}) \quad (4.242)$$

hold for all $t \geq 0$.

Proof. The proof follows the derivation of (4.80) and (4.81) in Proposition 4.4. We omit the details.

Let $\mathcal{V}''$ be the convex subset of $\mathcal{V} = H^1 \oplus H \oplus H^1 \oplus H^1$ consisting of vectors (u, c_1, c_2) such that u is divergence free with mean zero and c_1 and c_2 are non-negative functions a.e. whose difference has a space average equal to $-\bar{N}$. We define the solution map

$$\mathcal{O}(t) : \mathcal{V}'' \mapsto \mathcal{V}'' \quad (4.243)$$

corresponding to system (4.224) by

$$\mathcal{O}(t)(u_0, c_1(0), c_2(0)) = (u(t), c_1(t), c_2(t)). \tag{4.244}$$

For each $M > 0$, we consider the convex subset $\mathcal{V}'_M$ of $\mathcal{V}''$ consisting of vectors (u, c_1, c_2) such that u is divergence free with mean zero and c_1 and c_2 are non-negative functions a.e. whose space averages are less than or equal to M and whose difference has a space average equal to $-\bar{N}$. By Proposition 4.9, there exists $R_2 > 0$ depending on the body forces f, the added charge density N, and the positive constant M, such that for any $w_0 = (u_0, c_1(0), c_2(0)) \in \mathcal{V}'_M$, there exists $t'_0 > 0$ depending on $\|u_0\|_{H^1}, \|c_1(0)\|_{H^1}, \|c_2(0)\|_{H^1}$ such that for all $t \geq t'_0$, we have $\mathcal{O}(t)w_0 \in \mathcal{B}^M_{R_2}$, where

$$\mathcal{B}^M_{R_2} = \{(u, c_1, c_2) \in \mathcal{V}'_M : \|u\|_{H^2} + \|c_1 - \bar{c}_1\|_{H^2} + \|c_2 - \bar{c}_2\|_{H^2} \leq R_2\}. \tag{4.245}$$

We note that the map $\mathcal{O}(t)$ has the same properties as the map $\mathcal{S}(t)$, namely the existence of a compact absorbing ball, continuity properties (cf. Theorem 4.5) and injectivity (cf. Theorem 4.6). The existence of a global attractor is proved as in Theorem 4.7 and its finite dimensionality follows from decay of volume elements (Theorem 4.8) like in Theorems 4.9 and 4.10. The proofs of these theorems are similar to the proofs of the respective results for $N = 0$, and are omitted.

Theorem 4.11. *There exists a global attractor X which is compact in $\mathcal{V}''$ and has finite fractal dimension, such that*

$$\lim_{t\to\infty} dist_{\mathcal{V}}(\mathcal{O}(t)w_0, X) = 0 \tag{4.246}$$

holds uniformly for w_0 in bounded sets in $\mathcal{V}''$.

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