Real Analysis Ph.D. Qualifying Exam
Mathematics, Temple University
January 9, 2026
Each problem is worth 20 points

Part I. (Do 3 problems)

1. Prove that on $C[0,1]$ the norms $\|f\|_\infty = \max_{x\in[0,1]}|f(x)|$ and $\|f\|_1 = \int_0^1 |f(x)|\,dx$ are not equivalent.

2. Prove Dini's theorem: Let X be a compact topological space. If $f_n : X \to \mathbb{R}$ is a sequence of continuous functions such that $f_n(x) \to 0$ for each $x \in X$ and $f_n(x) \geq f_{n+1}(x)$ for all x and n, then $f_n \to 0$ uniformly in X.

   HINT: for $\epsilon > 0$ consider $F_n = \{x \in X : f_n(x) < \epsilon\}$.

3. Let $f : [a,b] \to \mathbb{R}$ be Lebesgue integrable and non negative. Prove that

$$\left(\int_a^b f(x)\,\cos x\,dx\right)^2 + \left(\int_a^b f(x)\,\sin x\,dx\right)^2 \leq \left(\int_a^b f(x)\,dx\right)^2.$$

   HINT: write $f(x) = \sqrt{f(x)}\,\sqrt{f(x)}$ and use Cauchy-Schwartz inequality.

4. Let μ be a Borel finite measure in $\mathbb{R}^n$. Suppose $E \subset R^n$ is μ-measurable.

   Prove that the function $F(t) = \mu\left(E \cap \{|x| < t\}\right)$ is continuous from the left; and F is continuous if and only if $\mu(E \cap \{|x| = t\}) = 0$ for all $t \in R$.

Part II. (Do 2 problems)

1. Let $f \in L^1(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} f(x)\,dx = r < 1$. Define $f_k = f * \cdots * f$ where the convolution $*$ is taken k times. Prove that

   (a) $f_k \in L^1(\mathbb{R}^n)$ for all k,

   (b) $f_k \to 0$ in $L^1(\mathbb{R}^n)$ as $k \to \infty$,

   (c) $g(x) := \sum_k |f_k(x)| \in L^1(\mathbb{R}^n)$, and conclude that $f_k(x) \to 0$ a.e.

2. Let f be absolutely continuous on $[a,b]$ and assume that $f' \in L^p([a,b])$ for some $1 < p \leq \infty$. Prove that f is Hölder continuous with exponent $\alpha = 1 - \frac{1}{p}$.

3. Let (E, Σ, μ) be a measure space with $\mu \geq 0$. If $f \in L^p(E,\mu)$, for some $1 \leq p < \infty$, and $E = \cup_{j=1}^\infty E_j$ with $E_j \in \Sigma$, $E_j \subset E_{j+1}$, then prove that $f\,\chi_{E_j} \to f$ in $L^p(E,\mu)$.