Real Analysis Ph.D. Qualifying Exam
Mathematics, Temple University
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Part I. (Do 3 problems)

1. Fix an arbitrary $n \in \mathbb{N}$. For each part below, give an example of a function $f : \mathbb{R}^n \to \mathbb{R}$ satisfying the stated conditions.

   (a) $f \in L^1(\mathbb{R}^n)$ but $f \notin L^2(\mathbb{R}^n)$.

   (b) $f \in L^2(\mathbb{R}^n)$ but $f \notin L^1(\mathbb{R}^n)$.

2. Define a sequence of functions $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \frac{ne^{-x}}{1 + n^2x^2}.$$

   Prove that $\lim_{n\to\infty} f_n(x) = 0$ for $x \in (0,1]$, $\lim_{n\to\infty} f_n(0) = +\infty$, and evaluate

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx.$$

3. Let $f(x) = x^2 \sin(1/x^3)$ for $x \in [-1,1]$, $x \neq 0$, and $f(0) = 0$. Show that f is differentiable on $[-1,1]$ but f' is unbounded on $[-1,1]$.

4. Let $E \subset \mathbb{R}^n$ be a bounded Lebesgue measurable set. Let $v \in \mathbb{R}^n$ be fixed and define H to be the hyperplane with equation $H = \{x : x \cdot v = 0\}$.

   Given $t \in \mathbb{R}$, let $H_t = \{x : x \cdot v \leq t\}$. Prove that there exists $t_0 \in R$ such that $|E \cap H_{t_0}| = |E \cap (H_{t_0})^c|$. That is, for each v there exists a hyperplane with normal v dividing E into two sets of equal measure.

   HINT: Prove and use that the function $G(t) = |E \cap H_t|$ is continuous.

Part II. (Do 2 problems)

1. Consider the space $X = [0,1]$ and the $\sigma-$algebra of Borel sets of X. Let m denote the Lebesgue measure and μ the counting measure (i.e. for any Borel set E of X, $\mu(E) =$ the number of elements in E if E is finite, and $\mu(E) = \infty$ if E is infinite).

   Prove that $m \ll \mu$, but there is no f such that $m(E) = \int_E f d\mu$. Explain why this does not contradict the Radon-Nikodym Theorem.

2. Suppose μ is σ-finite measure in X and $f_n \to f$ a.e. with f and f_n measurable for all n. Given $\epsilon > 0$ prove that there exists a sequence of sets $E_k \subset X$ such that $\mu\left(X \setminus \cup_{k=1}^{\infty} E_k\right) < \epsilon$ with $f_n \to f$ uniformly on each E_k.

3. Let $E \subset \mathbb{R}^n$ be a Lebesgue measurable set. Let $1 \leq p < \infty$, $f_k, f \in L^p(E)$, and let $a_k = \int_E |f_k(x) - f(x)|^p\, dx$. Here dx denotes, as usual, integration with respect to the Lebesgue measure in $\mathbb{R}^n$. Suppose that $\sum_{k=1}^{\infty} a_k < \infty$. Prove that $f_k \to f$ a.e. in E as $k \to \infty$.