Numerical Analysis Qualifying Written Exam (January 2026)

Part I: do 3 of 4

1.
(a) Give definitions and formulas for the Chebyshev polynomials and prove that they are orthogonal in an appropriate inner product (you need to show such inner product).
(b) Using the roots of these polynomials (appropriately scaled for the interval of interest) is known to be better than using equidistant points for interpolation given $n+1$ points, and absisas for them.
In what sense is this better? Discuss two different improvements in the results of the interpolation that are obatined by using the Chebyschev points.
(c) Mention an interpolation method with which you can use these non-uniformly distributed points. Provide formulas.

2.
(a) The QR factorization of a matrix A is known to be backwards stable. What does backwards stable mean for this QR factorization?
(b) Show that the floating point addition of two real numbers is backward stable. Is the same true for the subtraction of two real numbers? Why or why not?
(c) Show that in floating point arithmetic, the sum is not associative. What about the product?

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3. Consider a linear r-step (multistep) method $\sum_{i=0}^{r} \alpha_i U^{n+i} = k \sum_{i=0}^{r} \beta_i f(U^{n+i})$.
(a) Define consistency and zero-stability for above method.
(b) Prove whether $U^{n+2} - \frac{4}{3}U^{n+1} + \frac{1}{3}U^n = \frac{2}{3}kf(U^{n+2})$ is consistent and zero-stable.
(c) Explain whether consistency is sufficient for a method to be convergent. You can use an example to illustrate your arguments.

4.

(a) Explain why, in practice, we need the notion of absolute stability in addition to zero-stability.

(b) What is the definition of the region of absolute stability for Runge-Kutta or linear multistep schemes? Explain the reasoning behind this definition.

(c) Based on the region of absolute stability for the method $U^{n+2} - U^n = 2kf(U^{n+1})$ explain whether this method is a good choice for solving the dynamical system

$$\begin{bmatrix} u \\ v \end{bmatrix}' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

Part II: do 2 of 3

1.
Consider the Newton Method for the solution of $F(x) = 0$, when $F : \mathbb{R}^n \to \mathbb{R}^n$, i.e., the n-dimensional case.
(a) First consider the case $n = 1$. Describe the method. Present the appropriate assumptions to show that the method converges at least linearly. Give detailed conditions on F and on the initial vector x_0 for these assumptions.
(b) Present the appropriate assumptions to show that the method converges quadratically. Give detailed conditions on F and on the initial vector x_0 for these assumptions.
(c) For $n > 1$, describe the method, and give the appropriate assumptions for its (linear or quadratic) convergence.

2.
Let $F_i : \mathbb{R}^2 \to \mathbb{R}^2, i = 1, 2$ given by

$$F_1(x) = \begin{bmatrix} x_1 \cos x_2 \\ x_2 \cos x_1 \end{bmatrix}, \quad F_2(x) = \begin{bmatrix} x_1^2 - x_2^2 + 1 \\ x_1^2 + x_2^2 - 1 \end{bmatrix}.$$

(a) For these particular F_i and for an initial vector x_0 give conditions so that Newton's method converges for these functions if the conditions are satisfied, or show that there may not be convergence.
(b) Describe the Bisection Method for the scalar case, that is, $f : \mathbb{R} \to \mathbb{R}$. Discuss its convergence properties.
(c) How would you generalize the Bisection Method to $\mathbb{R}^2$, so you could apply it to the functions F_1 and F_2 defined at the beginning of this question.

3.
(a) Give an example of a stiff and an example of a non-stiff dynamical system, explain the difference between them, and conclude with a definition of stiffness.
(b) Define A-stability and L-stability.
(c) Use parts (a) and (b) to propose suitable time-stepping schemes for your examples. Explain mathematically why these are good choices.