Numerical Analysis Qualifying Written Exam (August 2025)

Part I: do 3 of 4

1.
We want to interpolate a continuous function with continuous derivative in the interval $I = [0, 1]$. We are only given the values of the function at the nodes $x_i \in I$, i.e., we know $f(x_i)$, but not the function f itself. We are also given the values of the derivative at the end points, i.e., $f'(0)$ and $f'(1)$. We also know that the original function is such that

$$\max_{x \in I} |f^{(k)}(x)| > 10, \;\; k \geq 3.$$

Our goal is to produce an interpolating function $g(x) \in C^1$ and we want to guarantee that $|f(x) - g(x)|$ is small across the interval.
(a) Describe a method you would use to solve this interpolation problem.
(b) Give a formula for $g(x), x \in [0, 1]$, and explain in detail how you obtain it.
(c) Provide an estimate of the error $|f(x) - g(x)|, x \in [0, 1]$.

2.
Consider the function $f(x) = \cos^2(x) - 1/2$, $x \in [0, \pi]$, and a root x_* such that $f(x_*) = 0$. In this question, you are asked to provide a method of the form $x_{k+1} = g(x_k)$, so that $x_k \to x_*$ quadratically.
(a) Explain what quadratic convergence means for the sequence of iterates given by x_k.
(b) Describe your proposed method in detail, and how would you use it for this particular function f.
(c) Prove the quadratic convergence of your proposed method, and give condition on the initial value x_0 that guaratntees the convergence for this particular function f.

3.
Consider the dynamical system

$$\begin{bmatrix} u \\ v \end{bmatrix}' = \begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

(a) Explain *absolute stability* of a numerical time-stepping scheme.
(b) Derive the regions of absolute stability for the Forward Euler (FE) and the leapfrog

(LF) method ($U^{n+1} = U^{n-1} + 2kf(U^n)$).
(c) Which of the two methods, FE or LF, would you use for the above system? Explain.

4.
Consider the initial value problem $u' = -u$; $u(0) = 1$.
(a) Calculate and sketch the graph of the growth factor $R(z)$ (note, z is a real number in this problem) for both the Backward Euler (BE) and Crank-Nicholson (CN) scheme.
(b) What can you say about the behavior of BE and CN for larger time steps k? How does this behavior affect the numerical solution of u?
(c) Define L-stability and, based on (a) and (b), argue the importance of L-stability.

Part II: do 2 of 3

1.
Consider an interpolation problem with the $n+1$ points $x_0, x_1, \ldots, x_n$, with abcisas $f_0, \ldots, f_n$. Let the interpolation polynomial of degree n be $p(x)$. Suppose that once you have $p(x)$ you have a new piece of data, namely f'_j for some $0 \le j \le n$. One is seeking an new interpolating polynomial $q(x)$ so that $q(x_i) = f_i$, $i = 0, \ldots, n$ and $q'(x_j) = f'_j$.
(a) Can you use the polynomial already computed $p(x)$ and just compute a new polynomial $r(x)$ so that $q(x) = p(x) + r(x)$? If yes, explain how to do it. If not, why not?
(b) What is the degree of $r(x)$?
(c) Is this polynomial unique? Why?

2.
We wish to approximate the numerical value of the integral $\int_a^b f(x)dx$ using the evaluation of the function at n points, x_i, $i = 1, \ldots, n$..
(a) Give the expression for the general Newton-Cotes rule using n points.
(b) Give a formula for the error obtained with the formula provided in part (a).
(c) Let $f(x) = p(x)$ be a polynomial. Given the formula from part (a), for what degree of $p(x)$ is the formula exact?.
(d) Let α_i be the weights for the Newton-Cotes rule. Assume that the points x_i are equidistant. Prove that

$$\sum_{i=1}^{n} \alpha_i = n.$$

(e) Describe a numerical method using n points to evaluate integrals of the form $\int_a^b \omega(x) f(x)dx$ for some weight function $\omega(x)$ so that the method is exact for $f(x)$ being a polynomial of degree $2n-1$.

3.
Consider the system

$$\begin{bmatrix} u \\ v \end{bmatrix}' = \begin{bmatrix} -500 & 499 \\ 499 & -500 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} u \cdot v \\ -u \cdot v \end{bmatrix}.$$

(a) Using the above system, explain the properties of linearity/nonlinearity and stiffness. Identify which part of the right hand side can be considered stiff.
(b) Suggest a time-stepping scheme for stiff problems and explain why or why not it is a good choice. In this context also explain the effect of order reduction when dealing

with stiff problems.
(c) Explain the idea of semi-implicit methods.
(d) Give an example of a semi-implicit method and apply it to the above system.