# Comprehensive Examination in Geometry & Topology
# Department of Mathematics, Temple University

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## Part I. Solve three of the following problems.

**I.1** Let $X$ be the cell complex obtained by starting with a rose whose four petals are labeled $a, b, c, d$, considered as generators for $\pi_1(X)$, and attaching three 2-cells along the words $aba^{-1}b^{-1}$, $cdc^{-1}d^{-1}$, and $aba^{-1}c^{-1}$.

**(a)** Find a presentation for $\pi_1(X)$.

**(b)** Prove that $\pi_1(X)$ is not abelian.

**(c)** Compute the homology groups $H_n(X)$ for each $n$.

**I.2**

**(a)** State Sard's theorem for smooth maps between smooth manifolds.

**(b)** Prove the following simpler statement, without appealing to Sard's theorem, using only standard facts about Lebesgue measure on $\mathbb{R}^n$:

If $M$ and $N$ are compact smooth manifolds of dimensions $m$ and $n$, respectively, with $m < n$, then for any smooth map $f : M \to N$, the image $f(M)$ has measure zero in $N$.

**I.3** Let $S_g$ be the closed, orientable surface of genus $g$. Show that $f_*(\pi_1(S_2))$ has infinite index in $\pi_1(S_5)$ for any continuous map $f : S_2 \to S_5$.

**I.4** Let $\mathbb{S}^2$ be the standard unit sphere in $\mathbb{R}^3$. Compute

$$\int_{\mathbb{S}^2} \omega$$

for $\omega = (e^{x^3+y^2} + z)dy \wedge dx + \cos(y^3 + z)dy \wedge dz + (x + z^4)dz \wedge dx$.

**Part II. Solve two of the following problems.**

**II.1** Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function with $f(x) > 0$ for all $x \in \mathbb{R}$.

**(a)** Prove that the surface of revolution

$$y^2 + z^2 = f(x)^2$$

is a smooth hypersurface in $H \subseteq \mathbb{R}^3$.

**(b)** If $x \in \mathbb{R}$ is a critical point of $f(x)$ and $p = (x, y, z) \in H$, prove that the tangent space to $H$ at $p$ is spanned by $\frac{\partial}{\partial x}$ and $z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}$, where $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\}$ are the standard coordinate directions generating $T_p\mathbb{R}^3$.

**(c)** Prove that the map

$$(x, y, z) \mapsto \left(\frac{y}{f(x)}, \frac{z}{f(x)}\right)$$

defines a smooth submersion from $H$ onto the standard unit circle in the $yz$-plane.

**II.2** Let $N$ be a closed, connected 5-manifold with $\pi_1(N) \cong \mathbb{Z}/7$ and $H_2(N) \cong \mathbb{Z}^3 \oplus \mathbb{Z}/4$.

**(a)** Prove that $N$ is orientable.

**(b)** Compute the singular homology groups of $N$.

**II.3** Let $X$ be a connected cell complex, $x \in X$ be a base point, and $p\colon \widetilde{X} \to X$ be a (possibly disconnected) covering. For any $\gamma \in \pi_1(X, x)$ and any $y \in p^{-1}(x)$ define

$$y \cdot \gamma = \widetilde{\gamma}_y(1),$$

where $\widetilde{\gamma}_y$ is the path lift of $\gamma$ starting at $y$.

**(a)** Show that this defines a right action of $\pi_1(X, x)$ on a fiber $p^{-1}(x)$, called the *monodromy action.* (A right action means, $y \cdot 1 = y$ and $(y \cdot \gamma) \cdot \delta = y \cdot (\gamma\delta)$ for all $\gamma, \delta \in \pi_1(X, x)$.)

**(b)** Show that the monodromy action is transitive on $p^{-1}(x)$ if and only if $\widetilde{X}$ is connected. (An action is transitive if for any two points there is a group element mapping one to the other.)

**(c)** Show that if $\widetilde{X}$ is connected and $y \in p^{-1}(x)$, then $(\widetilde{X}, y) \to (X, x)$ is the based covering space corresponding to the subgroup $H$ stabilizing $y$ with respect to the monodromy action.