Comprehensive Examination in Geometry & Topology
Department of Mathematics, Temple University

January 2025

Part I. Solve three of the following problems.

I.1 Prove or give a counterexample to each of the following:

(a) Every connected cell complex has a connected double cover.

(b) Every continuous map $f\colon T^2 \to T^2 \setminus \{p\}$ is null homotopic, where T^2 is the 2-torus.

(c) For any smooth embedded $f\colon S^1 \to S^4$, the manifold $S^4 \setminus f(S^1)$ is simply connected.

I.2 Fix $a, b \in (0, \infty)$ and consider the subset of $\mathbb{R}^4$ defined by

$$N = \left\{(x, y, z, w) \in \mathbb{R}^4 \ : \ x^2 + y^6 = a,\ z^4 - w^2 = b\right\}.$$

(a) Prove that N is a 2-dimensional manifold.

(b) Compute the tangent space $T_{(x,y,z,w)}N$ at $(x, y, z, w) \in N$ by giving an explicit basis.

I.3 Let $F = F(a, b, c)$ be the free group of rank 3.

(a) Find a normal subgroup H such that $F/H \cong \mathbb{Z}/3$.

(b) Give, with proof, a minimal set of generators for H.

I.4 Consider the submanifold

$$X = \left\{(x, y, z, w) \in \mathbb{R}^4 \ : \ x^2 + y^2 = z^2 + w^2 = 1\right\}$$

of $\mathbb{R}^4$ and the 2-form

$$\omega = (-y\ dx + x\ dy) \wedge (-w\ dz + z\ dw)$$

on X.

(a) Compute $\displaystyle\int_X \omega$.

(b) Prove that ω is an orientation form on X.

Part II. Solve two of the following problems.

II.1 Let M be a smooth manifold.

(a) Suppose that M is connected and that $p, q \in M$. Show that there is a diffeomorphism $F\colon M \to M$ such that $F(p) = q$.

(b) Is the same true if M is disconnected? Prove or give a counterexample.

II.2 Let α be a nonseparating simple closed curve on a torus T. Let X be obtained by gluing a Möbius band M to T via a map $f\colon \partial M \to \alpha$ of degree 3.

(a) Compute $\pi_1(X)$.

(b) Show $\pi_1(X)$ is infinite and not abelian.

(c) Compute the singular homology groups of X.

II.3 Answer the following:

(a) Show that a closed (i.e. compact without boundary) orientable manifold cannot be homotopy equivalent to a closed nonorientable manifold.

(b) Is the same true for noncompact manifolds?

(c) Suppose that M is a compact orientable n-manifold with boundary $\partial M \neq \emptyset$. Prove that if M is contractable, then $H_i(\partial M) = H_i(S^{n-1})$ for all $i \geq 0$. (Hint: Use Poincaré–Lefschetz duality.)