Comprehensive Examination in Geometry & Topology
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Part I. Solve three of the following problems.

I.1 Prove or give a counterexample to each of the following:

1. Every smooth manifold M with $H_1(M) = \mathbb{Z}/5$ is orientable.
2. If $Y \subset X$ is a retract, then $H_i(X) \cong H_i(Y)$ for all i.
3. Every subgroup of F_2 is free.
4. Every finite normal cover of a two-petal rose whose deck group is abelian embeds in a 2-torus.

I.2 Let S be a closed (compact, without boundary) orientable surface and $f\colon S \to \mathbb{R}$ be a smooth function. Prove that

$$\int_S f d\alpha = -\int_S df \wedge \alpha$$

for every smooth 1-form α on S.

I.3 Let $x^1, \dots, x^N$ be standard coordinates on $\mathbb{R}^N$ and M be a smooth k-dimensional submanifold. Prove that every point $x \in M$ has a neighborhood on which the restriction of some k coordinate functions $x^{i_1}, \dots, x^{i_k}$ form local coordinates on M.

I.4 Classify all closed (compact, without boundary) connected surfaces that finitely cover the closed orientable surface of genus 5.

Part II. Solve two of the following problems.

II.1 Suppose $F(x_0, \ldots, x_n)$ is a nonzero homogeneous polynomial of degree $k \geq 1$, meaning that

$$F(\lambda x_0, \ldots, \lambda x_n) = \lambda^k F(x_0, \ldots, x_n)$$

for all $\lambda \in \mathbb{R}$ and $(x_0, \ldots, x_n) \in \mathbb{R}^{n+1}$.

1. Suppose that the gradient vector $\nabla F(x_0, \ldots, x_n)$ is nonzero for all $(x_0, \ldots, x_n)$ for which $F(x_0, \ldots, x_n) = 0$. Prove that

$$Z_F = \Big\{ [x_0 : \cdots : x_n] \in \mathbb{RP}^n \;:\; F(x_0, \ldots, x_n) = 0 \Big\} \subseteq \mathbb{RP}^n$$

   is a well-defined smooth submanifold and compute its dimension, where $[x_0 : \cdots : x_n]$ denotes standard homogeneous coordinates in $\mathbb{RP}^n$. (Note that F itself is a function defined on $\mathbb{R}^{n+1}$, not $\mathbb{RP}^n$.)

2. Give an example where $\mathbb{RP}^n$ is *not* orientable but Z_F is orientable (as an abstract smooth manifold).

II.2

1. Construct a Δ-complex Y whose nonzero integer homology groups are:

$$H_0(Y) = \mathbb{Z}^2$$
$$H_1(Y) = \mathbb{Z}^2 \oplus \mathbb{Z}/7$$
$$H_3(Y) = \mathbb{Z}$$

2. Can such a Y be homotopy equivalent to a closed, orientable manifold?

II.3 Let T^3 be the 3–torus (i.e. $T^3 = S^1 \times S^1 \times S^1$) and let X be the space obtained by taking two copies of T^3 and gluing them together by identifying the 2–torus $S^1 \times S^1 \times \{\theta\}$ in each (for some fixed $\theta \in S^1$).

1. Compute $\pi_1(X)$.
2. Show that $\pi_1(X)$ is nonabelian.
3. Describe the universal cover of X.