Comprehensive Examination in Algebra
Department of Mathematics, Temple University

August 2025

Part I. Do three of these problems.

I.1 Let X be a G-set. Recall that a function $f : X \to X$ is a map of G-sets if $f(g(x)) = g(f(x))$ for every $g \in G$. Let H be a subgroup of G and X be the set G/H of left cosets of H in G. We consider X with the standard left action of G: $g(aH) := (ga)H$. Prove that the group $\mathrm{Aut}(X)$ of automorphisms of the G-set $X := G/H$ is isomorphic to the quotient group $N_G(H)/H$, where $N_G(H)$ is the normalizer of H in G.

I.2 Let F be a field and V be a vector space over F.

(a) Prove that, for every pair of finite dimensional subspaces $U_1, U_2 \subset V$, we have

$$\dim(U_1 + U_2) = \dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2).$$

(b) Let U_1, U_2, U_3 be finite dimensional subspaces of V. Is this true that

$$\begin{aligned}\dim(U_1 + U_2 + U_3) &= \dim(U_1) + \dim(U_2) + \dim(U_3) \\ &\quad - \dim(U_1 \cap U_2) - \dim(U_2 \cap U_3) - \dim(U_1 \cap U_3) + \dim(U_1 \cap U_2 \cap U_3)\,?\end{aligned}$$

Please prove this identity or disprove it by giving a counterexample.

I.3 Let R be a commutative ring and M be an R-module. Recall that M is called *torsion free* if $rw = 0$ (with $r \in R$ and $w \in M$) implies that $r = 0$ or $w = 0$. Consider an integral domain R and assume that M is a free R-module.

(a) Is this true that M is torsion free? Prove or give a counterexample.

(b) Is this true that every submodule $N \subset M$ is free? Prove or give a counterexample.

I.4 This problem has two parts:

(a) Let $K \supset F$ be fields and $\alpha_1, \alpha_2, \ldots, \alpha_n \in K$ be algebraic over F. Prove that, if $K = F(\alpha_1, \alpha_2, \ldots, \alpha_n)$, then K is a finite extension of F.

(b) Let $p_1, p_2, \ldots, p_n$ be n distinct prime integers and $K := \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_n})$. Prove that

$$[K : \mathbb{Q}] = 2^n.$$

Part II. Do two of these problems.

II.1 Let w, $\tilde{w}$ be elements of the free group F_k on k generators $x_1, x_2, \ldots, x_k$ with $k \geq 2$.

(a) Prove that, if $w^n = \tilde{w}^n$ for some $n \in \mathbb{Z}_{\geq 1}$, then $w = \tilde{w}$.

(b) Deduce that the identity element is the only element of finite order in F_k.

(c) Let n, m be non-zero integers. Prove that, if $w^n \tilde{w}^m = \tilde{w}^m w^n$, then $w\tilde{w} = \tilde{w}w$.

II.2 Let $K \supset F$ be a finite extension of fields, $\alpha \in K$ and $T_\alpha : K \to K$ be the following F-linear map:

$$T_\alpha(\beta) := \alpha\beta, \qquad \beta \in K. \tag{1}$$

Let A be the matrix of T_α with respect to a basis of K over F.

(a) Show that the minimal polynomial of A coincides with the minimal polynomial $m_{\alpha,F} \in F[x]$ of α over F.

(b) Let $f \in F[x]$ be the characteristic polynomial of A. Prove that

$$f = (m_{\alpha,F})^n,$$

where $m_{\alpha,F}$ is the minimal polynomial of α over F and $n = [K : F(\alpha)]$.

II.3 Let $F \supset K$ be fields and R be the following subring of $F[x]$:

$$R := \{f \in F[x] \mid f(0) \in K\}.$$

In other words, R consists of polynomials in $F[x]$ whose constant term belongs to K. Thus we have a tower of rings $K[x] \subset R \subset F[x]$. Prove that the following statements are equivalent:

(a) $[F : K]$ is finite.

(b) R is a Noetherian ring.

(c) R is a Noetherian $K[x]$-module.

(d) $F[x]$ is a Noetherian $K[x]$-module.