Graduate School
TEMPLE UNIVERSITY

DOCTORAL DISSERTATION APPROVAL
ON BEHALF OF THE GRADUATE BOARD

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Title of Dissertation: The Limiting Distribution of Decoherent Quantum Random Walks

Date of Defense: 4/18/2007

By signing below, I indicate my approval of this dissertation and of the candidate's oral defense.

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Date Dissertation Submitted to the Graduate School:	4/30/07

Accepted by the Graduate Board of Temple University in partial fulfillment of the requirements for the degree of: Doctor of Philosophy

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THE LIMITING DISTRIBUTION OF DECOHERENT QUANTUM RANDOM WALKS

A Dissertation
Submitted to
the Temple University Graduate Board

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in Partial Fulfillment
of the Requirements for the Degree of
DOCTOR OF PHILOSOPHY

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by
Kai Zhang
May, 2007

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ABSTRACT

THE LIMITING DISTRIBUTION OF DECOHERENT QUANTUM RANDOM WALKS

Kai Zhang
DOCTOR OF PHILOSOPHY

Temple University, May, 2007

Professor Wei-Shih Yang, Chair

Although the position distributions of one-dimensional quantum random walks are strikingly different from those of classical random walks, when decoherence is involved, simulations suggest that the resulting position distributions take on many classical features over time. Our research aims to investigate this phenomenon analytically. We establish the connection between pure quantum random walks and decoherent ones through a decoherence equation. From this equation, we obtain exact analytical formulae of the generating functions of decoherent quantum walks, for two different initial states. Using these formulae, we show that when time $t \to \infty$, the limiting position distributions of both walks are Gaussian. These results explicitly describe the relationship between the system and the level of decoherence.

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ACKNOWLEDGEMENTS

My sincere gratitude goes to my parents, all my teachers, and all my friends for their love, support and patience over the last few years.

I wish to thank my advisor, Professor Wei-Shih Yang for his expert guidance. His precise comments are extremely perceptive, helpful, and inspirational. His strict training will benefit my entire career.

I am grateful to my thesis committee members: Professor Shiferaw Berhanu, Professors Janos Galambos, and Professor Yuan Shi for their insightful advice. Professor Shiferaw Berhanu was the graduate chair and offered me great help when I came to the department. Professor Janos Galambos taught three graduate courses on advanced probability theory which influenced me greatly; he also provided me immense help during the final step of my graduate work. I sincerely appreciate their support.

I am thankful to Professor Seymour Lipschutz. He taught me when I was an undergraduate student and he led me to this department. He is always willing to help me not only with academic ideas but also with personal advice. I am indebted to him for his continuous guidance.

Many thank to Professor Gary Bowman, Professor Boris Datskovsky, Professor Cristian Gutierrez, Professor Martin Lorenz, Professor Sanat Sarkar and Professor Daniel Szyld for their dedicated teaching and guiding over the years. They provided me both deep understanding of the theories and fundamental tools in research.

I would like to thank Professor Shiferaw Berhanu, Professor Eric Grinberg, Professor Cristian Gutierrez, Professor Omar Hijab and Professor Jack Schiller for their great efforts in creating this excellent environment for me to study here.

I also thank the staff members at the department of mathematics for their cooperation.

My years of graduate study here would not have been this colorful without my friends in this department. They gave me so many important suggestions.

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I thank particularly for Xiuhong Du, Zhongzhi Liu, Shuo Qiu, Zhan Shen and Sheng Xiong for their help in writing up the thesis.

Special thanks to my girlfriend, Caiyan Li for her love and encouragement during the last year. I thank her from the bottom of my heart.

My final, and most heartfelt, acknowledgment must go to my parents. This dissertation would not have been possible without their unconditional love and support. This dissertation is dedicated to them.

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To my parents.

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CHAPTER 1

Introduction

Quantum random walks recently gained great interest from probabilists, quantum physicists, and computer scientists. The interest was sparked by their important roles in developing the highly efficient quantum algorithms. For instance, Grover's search algorithm [11] has time cost $O(\sqrt{N})$, in contrast to the ordinary search algorithm which has a cost of $O(N)$. This quantum searching algorithm was proved to be closely related to the behaviors of quantum random walks in [2] and [26]. As another example, Shor's algorithm also improved the speed of factorization dramatically [27]. The high efficiency of quantum algorithms is discussed in [12] and [13]. Experimental implementations of the algorithms are discussed in [7] and [8].

Besides their important applications, quantum random walks are very attractive themselves due to their dramatic non-classical behaviors. After quantum random walks are defined in [3], many articles ([4], [10], [19], [21], [22], etc.) to study the distribution of quantum random walks have been published and it is known that the non-classical behaviors are due to the quantum coherence evolution [23]. One of the most shocking differences [4] is that the variances of quantum random walks are $O(t^2)$ as time t grows, while the ones of classical random walks are $O(t)$. Various limit theorems of quantum random walks are established ([9], [10], [18], [19]). An excellent reference could be found in [15].

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One of the most concerns about the quantum walk system is that it is very sensitive to the inevitable decoherence effect. This effect could be caused by many reasons, such as interaction with environment and system imperfections ([5], [6], [25]). Decoherence could make the quantum behaviors disappear. Physicists tried to study this phenomenon as well as its influence. For the one-dimensional case, in the model in [5], decoherence is caused by measurements on the particle's chirality. Long-time first and second moments of the walk were obtained and numerical results showed that the distributions look like classical normal distributions. Similar scenes are found in other models ([14], [16], [17], [20], [23], [24]). If we denote the position random variable of the decoherent quantum random walk by X_t, all of above papers mentioned the fact that the variance of the simulated X_t grows linearly in time t. In particular, in [23] it is shown analytically that the variance is indeed linear in t.

These results stimulated us to prove that the one-dimensional decoherent quantum random walk $\frac{X_t}{\sqrt{t}}$ converges to a normal distribution. Our work focuses on the one-dimensional discrete-time Hadamard walk with measurements taken on both position and chirality at each time step. This kind of decoherence is studied numerically in [7], [14], [16] and [17] but we will study it fully analytically.

We shall see that when the particle is not measured, then the system is pure quantum and $\frac{X_t}{\sqrt{t}}$ does not converge. However, even when the particle is measured subject to a very small probability, $\frac{X_t}{\sqrt{t}}$ will converge to a normal distribution. Eventually, when the particle is measured for sure at each step, then the system becomes purely classical and $\frac{X_t}{\sqrt{t}}$ converges to the standard normal distribution.

In the next section, we introduce the basic notations and definitions. We then introduce our methodology of generating functions and the decoherence equation. We list our results and proofs after them. Finally, we summarize our work and give some discussions.

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CHAPTER 2

Statement of Results

2.1 Notations and Definitions

2.1.1 Quantum Random Walks

We start with a brief description of the one-dimensional pure quantum random walk system. Recall that in the classical random walks, the particle moves to the right or left depending on the result of a coin toss. However, in the quantum random walks, the particle has its chirality $\{right, left\}$ as another degree of freedom. At each time step, a unitary transformation is applied to the chirality state of the particle and the particle moves according to its new chirality state.

Formal definitions are as follows.

Definition 2.1 *(The Space of the Quantum Random Walk)*
The ***position space*** H_p *is defined as the complex Hilbert space spanned by the orthonormal basis* $\{|y>, y \in \mathbb{Z}\}$. *The* ***coin space*** H_c *is defined as the complex Hilbert space spanned by the orthonormal basis* $\{|l>, l = 1, 2\}$. *The* ***state space*** H *is defined as*

$$H = H_p \otimes H_c.$$

A vector $\psi \in H$ *with* $||\psi||_{l_2} = 1$ *is called a* ***state***.

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The state tells us the particle's position and chirality. A **basis** in H is denoted by $\{\phi_{y_l} = |y > \otimes |l >: y \in \mathbb{Z}, l = 1, 2\}$ where y is the particle's position and l is its chirality. $l = 1$ means "left" and $l = 2$ means "right".

Now we introduce the evolution operator which drives the particle.

Definition 2.2 *(The Evolution of the Quantum Random Walk)*
*The **shift operator** $S : H \to H$ is defined by*

$$S(\phi_{y_l}) = \begin{cases} \phi_{(y+1)_1} & l = 1 \\ \phi_{(y-1)_2} & l = 2. \end{cases}$$

*The **coin operator** $A : H_c \to H_c$ is a unitary operator. The **evolution operator** $U : H \to H$ is defined by*

$$U = S(I_p \otimes A),$$

where I_p is the identity in the position space.

The coin operator here is an analogue to the flipping coin in the classical walk. We now define the quantum random walk as follows.

Definition 2.3 *(The Quantum Random Walk)*
*Let $\psi_0 \in H$ be the initial state and let $\psi_t = U^t \psi_0$. The sequence $\{\psi_t\}_0^\infty$ is called a **one-dimensional quantum random walk**.*

The most famous and best-studied example of quantum random walks is the Hadamard walk.

Example 2.1 *(The One-Dimensional Hadamard Walk)*
Let A be the 2×2 Hadamard matrix

$$H_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

*The quantum random walk is associated with A is called a **one-dimensional Hadamard walk**.*

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We will focus on Hadamard walk in this article and we may assume that the particle starts at the origin.

We now define the quantum probability as follows.

Definition 2.4 *(The Quantum Probability)*
The ***probability*** *of a particle at state* ψ *to be found at state* η *is defined by* $| < \eta, \psi > |^2$.

In particular, the probability of the quantum random walk, starting from 0, with type i, to be found at x with type j is

$$W_j^i(x,t) = | < \phi_{x_j}, U^t \phi_{0_i} > |^2. \tag{2.1}$$

2.1.2 Decoherence

We focus on decoherence caused by measurements. The definition of measurements is

Definition 2.5 *(Measurements)*
$\{A_i, i \in \mathbb{A}\}$ *is called a* ***measurement*** *if*

$$\sum_{i \in \mathbb{A}} A_i^* A_i = I, \tag{2.2}$$

where $\mathbb{A}$ *is some index set and* A^* *is the adjoint operator of* A*, i.e., the complex conjugate of transposed matrix of* A.

In this work, we consider the measurements similar as in [5]. Let p be a real number in $[0,1]$ to denote the probability of the random walk being measured at each step. We let $A_c : H \to H$ be s.t. $A_c = \sqrt{1-p}I$ to be the coherence projection. We also let $A_{x_i} : H \to H$ be s.t. $A_{x_i} = \sqrt{p}E_{x_i,x_i}$ to be the decoherence projection to the subspace $span\{\phi_{x_i}\}$. Under this setup, the index set $\mathbb{A}$ is $\mathbb{A} = \{c\} \bigcup \{x_i : x \in \mathbb{Z}, i = 1,2\}$.

Now we define the decoherent quantum random walk as follows.

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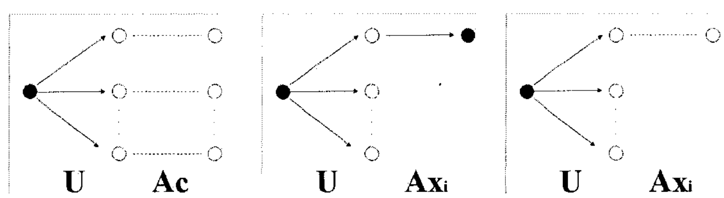
Definition 2.6 *(Decoherent Quantum Random Walk)*
Let ψ be a state in H. The random variable X_t^ψ over $\mathbb{Z}$ is called the ***decoherent quantum random walk*** *starting at ψ at time t if its probability mass function at x is given by*

$$P(X_t^\psi = x) = \sum_i \sum_{j_t \in \mathbb{A}} \sum_{j_{t-1} \in \mathbb{A}} \cdots \sum_{j_1 \in \mathbb{A}} | < \phi_{x_i}, (A_{j_t}U)(A_{j_{t-1}}U) \ldots (A_{j_1}U)\psi > |^2. \tag{2.3}$$

In other words, the walk starts at ψ, then we apply the evolution operator U, then we try to measure it, then we let the evolution go again and try to measure again. Figure 2.1.2 illustrates the possibilities that could happen in a single step. The process repeats until the t th step is finished. We then consider the position distribution of the particle. We call each $(j_1, j_2, \ldots, j_t, x_i)$ a path. We also call $< \phi_{x_i}, (A_{j_t}U)(A_{j_{t-1}}U) \ldots (A_{j_1}U)\psi >$ an amplitude function of the particle associated with the path. Many paths yield 0 amplitude due to the decoherence projections, A_{x_j}'s. However, the summation in (2.3) over all paths $(j_1, j_2, \ldots j_t, x_i)$ gives the probability of the particle to be found at position state $|x>$ at time t.



(a) No measurement is taken. (b) A measurement is taken and the particle is found. (c) A measurement is taken but the particle is not found.

Figure 2.1: Three possibilities in a step of the decoherent quantum walk.

At each step of a path, the walk is either not measured with probability $q = 1 - p$ or is measured at ϕ_{x_i} with probability p. So when $p = 0$, then

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the walk is not measured and the system is the same as the pure quantum random walk defined before. When $p = 1$, the particle is interfered at each step, hence the quantum behavior essentially disappears and the system is exactly classical.

Throughout this paper, we shall work on the decoherent Hadamard walk starting from position 0. We use

$$P_t^{\psi}(\phi) = \sum_{j_t \in \mathbb{A}} \sum_{j_{t-1} \in \mathbb{A}} \cdots \sum_{j_1 \in \mathbb{A}} | < \phi, (A_{j_t}U)(A_{j_{t-1}}U) \ldots (A_{j_1}U)\psi > |^2 \qquad (2.4)$$

to denote the probability of at time t, a particle in the decoherent quantum random walk starting from ψ to be found at state ϕ. In particular, we denote the probability that at time t, the particle starting at ϕ_{0_i} to be found at ϕ_{x_j} by

$$P_j^i(x,t) = \sum_{j_1, j_2, \ldots, j_t \in \mathbb{A}} | < \phi_{x_j}, (A_{j_t}U)(A_{j_{t-1}}U) \ldots (A_{j_1}U)\phi_{0_i} > |^2. \qquad (2.5)$$

Since we are interested in the limiting distribution of the walk, we focus on the Fourier transform of the above probabilities,

$$\widehat{P}_j^i(k,t) = \sum_x P_j^i(x,t)e^{ikx}, \qquad (2.6)$$

where i in e^{ikx} is the standard notation for the complex number such that $i^2 = -1$, while i in $P_j^i(x,t)$ means index.

We shall consider two types of walks. We first consider the walk starting at the state $\phi_0 = \frac{1}{\sqrt{2}}\phi_{0_1} + i\frac{1}{\sqrt{2}}\phi_{0_2}$. We call this walk "centered" and denote it by X_t. Note that

$$\begin{aligned}
P_t^{\phi_0}(\phi_{x_j}) = & \sum_{j_1, j_2, \ldots, j_t \in \mathbb{A}} | < \phi_{x_j}, (A_{j_t}U)(A_{j_{t-1}}U) \ldots (A_{j_1}U)(\frac{1}{\sqrt{2}}\phi_{0_1} + i\frac{1}{\sqrt{2}}\phi_{0_2}) > |^2 \\
= & \sum_{j_1, j_2, \ldots, j_t \in \mathbb{A}} | < \phi_{x_j}, (A_{j_t}U)(A_{j_{t-1}}U) \ldots (A_{j_1}U)\frac{1}{\sqrt{2}}\phi_{0_1} > |^2 + \\
& + \sum_{j_1, j_2, \ldots, j_t \in \mathbb{A}} | < \phi_{x_j}, (A_{j_t}U)(A_{j_{t-1}}U) \ldots (A_{j_1}U)i\frac{1}{\sqrt{2}}\phi_{0_2} > |^2 \\
= & \frac{1}{2}(P_j^1(x,t) + P_j^2(x,t)).
\end{aligned} \qquad (2.7)$$

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Hence,

$$\begin{aligned} P(X_t = x) =& \sum_j P_t^{\phi_0}(\phi_{x_j}) \\ =& \frac{1}{2} \sum_i \sum_j P_j^i(x,t) \end{aligned} \tag{2.8}$$

and the characteristic function of X_t would be

$$\widehat{P}(k,t) = \frac{1}{2} \sum_i \sum_j \widehat{P}_j^i(k,t). \tag{2.9}$$

From the above equation, we can see that its characteristic function is obtained by taking average of those of the initial chirality state i's. Further more, in [4], it is shown that the pure quantum random walk starting with ϕ_0 has a symmetric position distribution. These are the reasons why we call it "centered".

We also consider the walk that starts at ϕ_{0_1}, $\tilde{X}_t = X_t^{\phi_{0_1}}$. It is the walk starting at 0 with chirality "right". Therefore, the characteristic function of $\tilde{X}_t$ is

$$\widehat{\tilde{P}}(k,t) = \sum_j \widehat{P}_j^1(k,t). \tag{2.10}$$

Our goals are to show that as $t \to \infty$,

$$\frac{1}{2} \sum_i \sum_j \widehat{P}_j^i(\frac{k}{\sqrt{t}},t) \to e^{-\frac{1}{2}vk^2},$$

for some positive number v in the centered walk case, as well as to show that as $t \to \infty$,

$$\sum_j \widehat{P}_j^1(\frac{k}{\sqrt{t}},t) \to e^{-\frac{1}{2}vk^2}$$

in this specific initial state case.

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2.2 Generating Functions and the Decoherence Equation

2.2.1 Generating Functions

The direct calculation is with very complicated combinatorics and is formidable. Therefore, we introduce the idea of the generating functions.

Definition 2.7 *(Generating Functions)*
*The **generating function** of the decoherent quantum random walk is*

$$P_j^i(x,z) = \sum_{t=0}^{\infty} P_j^i(x,t)z^t. \tag{2.11}$$

The Fourier transform of the generating function is

$$\widehat{P}_j^i(k,z) = \sum_{x} P_j^i(x,z)e^{ikx}. \tag{2.12}$$

Note that for z in the unit disk $\{z : |z| < 1\}$, since $|\widehat{P}_j^i(k,t)| \le 1$ and $|P_j^i(x,t)| \le 1$ for every t, $\sum_{t=0}^{\infty} \widehat{P}_j^i(k,t)z^t$'s and $P_j^i(x,z)$'s are analytic. Furthermore,

$$\sum_{x}\sum_{t=0}^{\infty} |P_j^i(x,t)e^{ikx}z^t|^2 < \infty.$$

Hence, by Fubini's theorem, we have

$$\begin{aligned}
\widehat{P}_j^i(k,z) &= \sum_{x}\sum_{t=0}^{\infty} P_j^i(x,t)e^{ikx}z^t \\
&= \sum_{t=0}^{\infty}\sum_{x} P_j^i(x,t)e^{ikx}z^t \\
&= \sum_{t=0}^{\infty} \widehat{P}_j^i(k,t)z^t,
\end{aligned}$$

i.e., $\widehat{P}_j^i(k,z)$'s are analytic and $\widehat{P}_j^i(k,t)$'s are the coefficients of z^t in the expansions of $\widehat{P}_j^i(k,z)$'s. Therefore, instead of finding $\widehat{P}_j^i(k,t)$'s directly, we first

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find the explicit formulae of $\widehat{P}^i_j(k,z)$'s, then we apply the Cauchy's Theorem

$$\widehat{P}^i_j(k,t) = \frac{1}{2\pi i}\oint_{|z|=r} \frac{\widehat{P}^i_j(k,z)}{z^{t+1}} dz, \tag{2.13}$$

for some $0 < r < 1$, to get $\widehat{P}^i_j(k,t)$.

2.2.2 The Decoherence Equation

We now introduce the functions $\widehat{Q}^i_j(k,z)$ and $Q^i_j(x,z)$. These two functions are very important in the proofs. We let $\widehat{W}^i_j(k,t) = \sum_x W^i_j(x,t)e^{ikx}$ be the Fourier transform of the pure Hadamard walk. We also let $\widehat{Q}^i_j(k,z) = \frac{p}{q}\sum_{t=1}^{\infty}\widehat{W}^i_j(k,t)(qz)^t$ for $0 < p \leq 1$ and $q = 1-p$. Note that $|\widehat{W}^i_j(k,t)| \leq 1$. Hence, for $z \in \{z : |z| < \frac{1}{q}\}$, $|\widehat{Q}^i_j(k,z)| < \infty$. Therefore, $\widehat{Q}^i_j(k,z)$ is analytic in $\{z : |z| < \frac{1}{q}\}$. Furthermore, let $Q^i_j(x,z) = \frac{p}{q}\sum_{t=1}^{\infty} W^i_j(x,t)(qz)^t$, by Fubini's theorem again we have

$$\widehat{Q}^i_j(k,z) = \frac{p}{q}\sum_{t=1}^{\infty}\widehat{W}^i_j(k,t)(qz)^t = \sum_x Q^i_j(x,z)e^{ikx}. \tag{2.14}$$

With above notations, we derive the following theorem.

Theorem 2.1 *(The Decoherence Equation)*
The function $\widehat{P}^i_j(k,z)$'s are analytic in $\{z : |z| < 1\}$ and are meromorphic in $\{z : |z| < \frac{1}{q}\}$. Furthermore, if we denote the matrices of $(\widehat{P}^i_j(k,z))$ and $(\widehat{Q}^i_j(k,z))$ by P and Q respectively, then

$$P = -\frac{q}{p}I + \frac{1}{p}(I-Q)^{-1}. \tag{2.15}$$

This equation established the relationship between the decoherent quantum random walk (left hand side) and the pure quantum random walk (right hand side). By working on the Fourier transform of the pure quantum random walk, we get the formulae of $\widehat{P}^i_j(k,z)$'s from this equation. The following diagram is an illustration of our methodology.

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$$\begin{array}{ccccc}
P^i_j(x,t) & \xrightarrow{\text{Generating Functions}} & P^i_j(x,z) & & W^i_j(x,t) \\
\Big\downarrow \text{Fourier Transform} & & \Big\downarrow \text{Fourier Transform} & & \Big\downarrow \text{Fourier Transform} \\
\widehat{P}^i_j(k,t) & \xleftarrow[\text{Cauchy Theorem}]{} & \widehat{P}^i_j(k,z) & \xleftarrow[\text{Decoherence Equation}]{} & \widehat{W}^i_j(k,t)
\end{array}$$

2.3 Main Results

2.3.1 Classical Walks Revisited

We first consider the classical walk case, i.e., the $p = 1$ case, as an illustration of our approach. Note that for a classical random walk C_t, we have the result that $\frac{C_t}{\sqrt{t}} \to N(0,1)$ as $t \to \infty$.

When $p = 1$ and $q = 0$, $\widehat{Q}^i_j(k,z) = \widehat{W}^i_j(k,1)z$. Let $\mathbf{W}$ denote the matrix of $(\widehat{W}^i_j(k,1))$ and let $\mathbf{Q}$ denote the matrix of $(\widehat{Q}^i_j(k,1))$, then $\mathbf{Q} = z\mathbf{W}$. The decoherence equation becomes

$$\begin{aligned}
\widehat{P}^i_j(k,z) &= ((I - \mathbf{Q})^{-1})^i_j(k,z) \\
&= \sum_{t=0}^{\infty} (\mathbf{Q}^t)^i_j(k,z) \\
&= \sum_{t=0}^{\infty} ((\mathbf{W})^t)^i_j z^t.
\end{aligned}$$

By comparing the coefficients of z^t we see that $\widehat{P}^i_j(k,t) = (\mathbf{W}^t)^i_j$. Since only the first step of the pure quantum walk is involved in $\mathbf{W}$, we can find $\widehat{P}^i_j(k,t)$'s directly. We do this by first calculating $W^i_j(x,1)$. Note that

$$W^i_j(x,1) = |<\phi_{x_j}, U\phi_{0_i}>|^2,$$

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we have

$$\begin{aligned} W_1^1(x,1) &= \frac{1}{2}\delta_x^1, \\ W_2^1(x,1) &= \frac{1}{2}\delta_x^{-1}, \\ W_1^2(x,1) &= \frac{1}{2}\delta_x^1, \\ W_2^2(x,1) &= \frac{1}{2}\delta_x^{-1}. \end{aligned}$$

Hence, after the Fourier transform,

$$\begin{aligned} \widehat{W}_1^1(k,1) &= \frac{1}{2}e^{ik}, \\ \widehat{W}_2^1(k,1) &= \frac{1}{2}e^{-ik}, \\ \widehat{W}_1^2(k,1) &= \frac{1}{2}e^{ik}, \\ \widehat{W}_2^2(k,1) &= \frac{1}{2}e^{-ik}. \end{aligned}$$

Therefore,

$$\mathbf{W} = \frac{1}{2}\begin{pmatrix} e^{ik} & e^{-ik} \\ e^{ik} & e^{-ik} \end{pmatrix}$$

and

$$W^t = \frac{(\cos k)^t}{1+e^{2ik}}\begin{pmatrix} e^{2ik} & 1 \\ e^{2ik} & 1 \end{pmatrix}.$$

Hence,

$$\begin{aligned} &\widehat{P}(\frac{k}{\sqrt{t}},t) \\ =& \frac{1}{2}\sum_{i,j}\widehat{P}_j^i(\frac{k}{\sqrt{t}},t) \\ =& (\cos\frac{k}{\sqrt{t}})^t \\ =& (1-\frac{1}{2}\frac{k^2}{t}+o(\frac{1}{t}))^t \\ \to& e^{-\frac{1}{2}k^2}, \end{aligned}$$

which is the characteristic function of the standard normal distribution, the limiting distribution of the well known classical walk. It is also clear that the walk starting at ϕ_{0_i} is exactly classical.

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2.3.2 Results for the Centered Decoherent Quantum Random Walk

Now we look into the case of the centered walk with general p. We first show that the position distribution of the walk is symmetric with respect to the origin.

Theorem 2.2 *Let X_t be the centered decoherent Hadamard walk on the line with $0 < p \leq 1$ and $q = 1 - p$. Then $E(X_t) = 0, \forall t$.*

We then consider the limiting distribution. We shall apply the same method as in the previous section to find the characteristic function of the centered walk, $\widehat{P}(k,t) = \frac{1}{2}\sum_{i,j}\widehat{P}^i_j(k,t)$. We derive the following theorem for the limiting distribution of the centered decoherent Hadamard walk.

Theorem 2.3 *Let X_t be the centered decoherent Hadamard walk on the line with $0 < p \leq 1$ and $q = 1 - p$. Then the characteristic function $\widehat{P}(k,t)$ of X_t satisfies*

$$\widehat{P}(\frac{k}{\sqrt{t}}, t) = \exp\left(-\frac{p + 2\sqrt{1+q^2} - 2}{2p}k^2\right) + O(t^{-1}) \tag{2.16}$$

as $t \longrightarrow \infty$, i.e.,

$$\frac{X_t}{\sqrt{t}} \longrightarrow N(0, \frac{p + 2\sqrt{1+q^2} - 2}{p}) \tag{2.17}$$

in distribution as $t \longrightarrow \infty$.

This theorem states that after long time t, the position distribution of the particle is Gaussian. We see from the variance of the distribution that it is a mixture of the quantum and classical distribution. When p is nearly 1, the variance approaches 1 and this implies that the behaviors of the walk are as of a classical one. When p is very small, we see that the variance goes to infinity, meaning that $\frac{X_t}{\sqrt{t}}$ does not converge. A remark on the speed of convergence is discussed in the following chapter.

We also find the long time variance of X_t as follows.

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Theorem 2.4 *Let X_t be the centered decoherent Hadamard walk on the line with $0 < p \leq 1$ and $q = 1 - p$. Then*

$$Var(X_t) = \frac{p + 2\sqrt{1+q^2} - 2}{p} t - \frac{2q^2}{p\sqrt{1+q^2}} - \frac{2}{p^2}(1 + q^2 - \sqrt{1+q^2}) + O(e^{-ct}), \tag{2.18}$$

for some $c > 0$, as $t \to \infty$.

This theorem shows that for fixed p, the magnitude of the walk is growing linearly in $\sqrt{t}$. We compare our analytical results with simulated results in [16]. For $p = 0.1$, the standard deviation at $t = 500$ is about 62.0. For $p = 0.08$, the standard deviation at $t = 500$ is about 69.4. For $p = 0.06$, the standard deviation at $t = 500$ is about 80.0. For $p = 0.04$, the standard deviation at $t = 500$ is about 97.2. For $p = 0.02$, the standard deviation at $t = 500$ is about 132.8. Those above results match well with the figure 1 in [16].

2.3.3 Results for a Specific Initial State

Now we consider the decoherent walk starting at ϕ_{0_1}. As before we first find its expectation.

Theorem 2.5 *Let $\tilde{X}_t$ be the decoherent Hadamard walk starting at ϕ_{0_1} with $0 < p \leq 1$ and $q = 1-p$. Let $\mu_t = E(\tilde{X}_t)$, then we have $\mu_t = \frac{\sqrt{1+q^2}-1}{p} + O(e^{-dt})$ for some $d > 0$, as $t \to \infty$.*

This theorem shows that the limiting position expectation of the decoherent Hadamard walk is to the right of the origin, if the initial coin state is "right". We see that when $p \to 0$, $\mu_t \to \infty$. It is consistent with the result in [4] that the pure quantum random walk starting with chirality "right" is drifted to the right.

Then we show that the limiting distribution of the decoherent Hadamard walk starting at ϕ_{0_1} is also Gaussian.

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Theorem 2.6 *Let $\tilde{X}_t$ be the decoherent Hadamard walk starting at ϕ_{0_1} with $0 < p \leq 1$ and $q = 1 - p$. Then the characteristic function $\widehat{\tilde{P}}(k,t)$ of $\tilde{X}_t$ satisfies*

$$\widehat{\tilde{P}}(\frac{k}{\sqrt{t}}, t) = \exp\left(-\frac{p + 2\sqrt{1+q^2} - 2}{2p} k^2\right) + O(t^{-\frac{1}{2}}) \tag{2.19}$$

as $t \to \infty$, i.e.,

$$\frac{\tilde{X}_t - \mu_t}{\sqrt{t}} \to N(0, \frac{p + 2\sqrt{1+q^2} - 2}{p}) \tag{2.20}$$

in distribution as $t \to \infty$.

Remark 2.1 *Note that here the converging speed is $O(t^{-\frac{1}{2}})$ while we have $O(t^{-1})$ for the centered walk. This is because when one takes the average of the $\widehat{P}^i_j(\frac{k}{\sqrt{t}}, t)$'s, the error terms in $t^{-\frac{1}{2}}$ cancel. This result shows that the centered walk converges faster.*

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CHAPTER 3

Proofs of the Theorems

3.1 Proof of the Decoherence Equation

We start with an observation of the decoherent quantum random walk and get a recursive formula. Then we apply that formula to $\widehat{P}_j^i(k,z)$'s to establish the decoherence equation.

For any state $\phi \in H$, ϕ can be written as $\phi = \sum_{y,l} < \phi_{y_l}, \phi > \phi_{y_l}$. By definition, for $t \geq 1$,

$$\begin{aligned}
& P_{t+1}^{\phi_{0_i}}(\phi) \\
= & \sum_{j_1,j_2,\ldots,j_t \in \mathbb{A}} (| < \phi, A_c U(A_{j_t}U)(A_{j_{t-1}}U)\ldots(A_{j_1}U)\phi_{0_i} > |^2 + \\
& + \sum_{y,l} | < \phi, A_{y_l} U(A_{j_t}U)(A_{j_{t-1}}U)\ldots(A_{j_1}U)\phi_{0_i} > |^2) \\
= & \sum_{j_1,j_2,\ldots,j_t \in \mathbb{A}} (q| < U^*\phi, (A_{j_t}U)(A_{j_{t-1}}U)\ldots(A_{j_1}U)\phi_{0_i} > |^2 + \\
& + \sum_{y,l} | < U^* A_{y_l}^* \phi, (A_{j_t}U)(A_{j_{t-1}}U)\ldots(A_{j_1}U)\phi_{0_i} > |^2) \\
= & \; q \sum_{j_1,j_2,\ldots,j_t \in \mathbb{A}} | < U^*\phi, (A_{j_t}U)(A_{j_{t-1}}U)\ldots(A_{j_1}U)\phi_{0_i} > |^2 + \\
& p \sum_{y,l} | < \phi_{y_l}, \phi > |^2 \sum_{j_1,j_2,\ldots,j_t \in \mathbb{A}} | < U^*\phi_{y_l}, (A_{j_t}U)(A_{j_{t-1}}U)\ldots(A_{j_1}U)\phi_{0_i} > |^2.
\end{aligned}$$

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That is

$$P_{t+1}^{\phi_{0_i}}(\phi) = qP_t^{\phi_{0_i}}(U^*\phi) + p\sum_{y,l} |<\phi_{y_l},\phi>|^2 P_t^{\phi_{0_i}}(U^*\phi_{y_l}). \tag{3.1}$$

In particular, for $\phi = \phi_{x_j}$, we have

$$P_{t+1}^{\phi_{0_i}}(\phi_{x_j}) = P_t^{\phi_{0_i}}(U^*\phi_{x_j}), \tag{3.2}$$

which in turn gives

$$P_{t+1}^{\phi_{0_i}}(\phi) = qP_t^{\phi_{0_i}}(U^*\phi) + p\sum_{y,l} |<\phi_{y_l},\phi>|^2 P_{t+1}^{\phi_{0_i}}(\phi_{y_l}). \tag{3.3}$$

This is our recursive formula. Also, for $t = 1$, we have

$$P_1^{\phi_{0_i}}(\phi) = q|<\phi, U\phi_{0_i}>|^2 + p\sum_{y,l} |<\phi_{y_l},\phi>|^2 P_1^{\phi_{0_i}}(U^*\phi_{y_l}), \tag{3.4}$$

and

$$P_1^{\phi_{0_i}}(\phi_{x_j}) = |<\phi_{x_j}, U\phi_{0_i}>|^2. \tag{3.5}$$

Apply the recursive formula (3.3) and (3.4) repeatedly, we have the following equation

$$\begin{aligned} &P_j^i(x,t) = P_t^{\phi_{0_i}}(\phi_{x_j}) \\ =&\sum_{s=1}^{t-1} pq^{s-1}\sum_{y,l} |<\phi_{y_l},(U^*)^s\phi_{x_j}>|^2 P_{t-s}^{\phi_{0_i}}(\phi_{y_l}) + q^{t-1}|<\phi_{x_j},U^t\phi_{0_i}>|^2. \end{aligned} \tag{3.6}$$

Note that by the definition of $W_j^i(x,t)$, we have

$$\begin{aligned} & |<\phi_{y_l},(U^*)^s\phi_{x_j}>|^2 \\ = & |<\phi_{x_j},U^s\phi_{y_l}>|^2 \\ = & W_j^l(x-y,s) \end{aligned}$$

and that

$$|<\phi_{x_j},U^t\phi_{0_i}>|^2 = W_j^i(x,t).$$

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Therefore, (3.6) becomes

$$P_j^i(x,t) = \sum_{s=1}^{t-1} pq^{s-1} \sum_{y,l} W_j^l(x-y,s) P_l^i(y,t-s) + q^{t-1} W_j^i(x,t). \tag{3.7}$$

Now, by (3.7), for $z \in \{z : |z| < \frac{1}{q}\}$,

$$\begin{aligned}
P_j^i(x,z) = &\sum_{t=0}^{\infty} P_j^i(x,t) z^t \\
=&\delta_{x_j}^{0_i} + \sum_{t=1}^{\infty} q^{t-1} W_j^i(x,t) z^t + \\
&+ \sum_{t=1}^{\infty} \sum_{s=1}^{t-1} \frac{p}{q} (qz)^s \sum_{y,l} W_j^l(x-y,s) P_l^i(y,t-s) z^{t-s} \\
=&\delta_{x_j}^{0_i} + \frac{1}{p} Q_j^i(x,z) + \\
&+ \sum_{s=1}^{\infty} \frac{p}{q} (qz)^s \sum_{y,l} W_j^l(x-y,s) \sum_{t=s-1}^{\infty} P_l^i(y,t-s) z^{t-s} \\
=&\delta_{x_j}^{0_i} + \frac{1}{p} Q_j^i(x,z) + \\
&+ \sum_{s=1}^{\infty} \frac{p}{q} (qz)^s \sum_{y,l} W_j^l(x-y,s) (P_l^i(y,z) - \delta_{y_l}^{0_i}) \\
=&\delta_{x_j}^{0_i} + \frac{1}{p} Q_j^i(x,z) - \sum_{s=1}^{\infty} \frac{p}{q} (qz)^s W_j^i(x,s) + \\
&+ \sum_{s=1}^{\infty} \frac{p}{q} (qz)^s \sum_{y,l} W_j^l(x-y,s) P_l^i(y,z) \\
=&\delta_{x_j}^{0_i} + \frac{1}{p} Q_j^i(x,z) - Q_j^i(x,z) + \\
&+ \sum_{y,l} Q_j^l(x-y,z) P_l^i(y,z),
\end{aligned} \tag{3.8}$$

where

$$\delta_{\beta_n}^{\alpha_m} = \begin{cases} 1, & \alpha = \beta, m = n \\ 0, & otherwise \end{cases} .$$

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Finally, we take the Fourier transform on (3.8),

$$\begin{aligned}
\widehat{P}^i_j(k,z) =& \sum_x P^i_j(x,z)e^{ikx} \\
=& \sum_x \delta^{0_i}_{x_j} e^{ikx} + \frac{q}{p}\sum_x Q^i_j(x,z)e^{ikx} + \\
&+ \sum_x \sum_{y,l} Q^l_j(x-y,z)P^i_l(y,z)e^{ikx} \\
=& \delta^i_j + \frac{q}{p}\widehat{Q}^i_j(k,z) + \\
&+ \sum_l \sum_y P^i_l(y,z)e^{iky} \sum_x Q^l_j(x-y,z)e^{ik(x-y)} \\
=& \delta^i_j + \frac{q}{p}\widehat{Q}^i_j(k,z) + \\
&+ \sum_l \sum_y P^i_l(y,z)e^{iky}\widehat{Q}^l_j(k,z) \\
=& \delta^i_j + \frac{q}{p}\widehat{Q}^i_j(k,z) + \\
&+ \sum_l \widehat{P}^i_l(k,z)\widehat{Q}^l_j(k,z).
\end{aligned} \tag{3.9}$$

The interchanges of summations are justified since the series absolutely converges. Now, denoting the matrices $(\widehat{P}^i_j(k,z))$ and $(\widehat{Q}^i_j(k,z))$ by P and Q, we have the following equation.

$$P = I + \frac{q}{p}Q + PQ, \tag{3.10}$$

which is

$$P(I-Q) = -\frac{q}{p}(I-Q) + \frac{1}{p}I. \tag{3.11}$$

We complete the proof by the following lemma.

Lemma 3.1 *For $z \in \{z : |z| < 1\}$, the matrix $I - Q$ is invertible.*

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Proof. For $z \in \{z : |z| < 1\}$, if we let $Q^i_j = \widehat{Q}^i_j(k,z)$, we have

$$\begin{aligned}
|Q^i_1| + |Q^i_2| =& \frac{p}{q}\sum_{t=1}^{\infty} |\widehat{W}^i_1(k,t)(qz)^t| + \frac{p}{q}\sum_{t=1}^{\infty} |\widehat{W}^i_2(k,t)(qz)^t| \\
<& \frac{p}{q}\sum_{t=1}^{\infty} q^t(|\widehat{W}^i_1(k,t)| + |\widehat{W}^i_2(k,t)|) \\
\leq& \frac{p}{q}\sum_{t=1}^{\infty} q^t(|\widehat{W}^i_1(0,t)| + |\widehat{W}^i_2(0,t)|) \\
\leq& \frac{p}{q}\frac{q}{p} = 1, \forall i.
\end{aligned} \tag{3.12}$$

(3.12) implies that $\|Q\|_\infty = \max_i \sum_j |Q^i_j| < 1$. Therefore,

$$\begin{aligned}
\|\sum_{n=0}^{\infty} Q^n\|_\infty \leq& \sum_{n=0}^{\infty} \|Q^n\|_\infty \\
<& \infty,
\end{aligned} \tag{3.13}$$

i.e., the series $\sum_{n=0}^{\infty} Q^n$ converges. This implies that $(I-Q)^{-1}$ exists and

$$(I-Q)^{-1} = \sum_{n=0}^{\infty} Q^n.$$

By Lemma 3.1, $I - Q$ is invertible and together with (3.11) we have

$$P = -\frac{q}{p}I + \frac{1}{p}(I-Q)^{-1}, \tag{3.14}$$

which is exactly (2.15).

For $z \in \{z : |z| < \frac{1}{q}\}$, $|\det(I-Q)| < \infty$. Hence, $\det(I-Q)$ is analytic. Note also that

$$\begin{aligned}
\widehat{P}^1_1(k,z) &= -\frac{q}{p} + \frac{1-Q^2_2}{p\det(I-Q)}, \\
\widehat{P}^1_2(k,z) &= \frac{Q^1_2}{p\det(I-Q)}, \\
\widehat{P}^2_1(k,z) &= \frac{Q^2_1}{p\det(I-Q)}, \\
\widehat{P}^2_2(k,z) &= -\frac{q}{p} + \frac{1-Q^1_1}{p\det(I-Q)}.
\end{aligned}$$

Therefore, $\widehat{P}^i_j(k,z)$'s are meromorphic functions for $z \in \{z : |z| < \frac{1}{q}\}$.

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3.2 Proofs in the Centered Walk Case

3.2.1 Preliminary

To make use of the decoherence equation (2.15), first we need to know the formulae of $\widehat{W}_j^i(k,t)$, i.e., we look at the pure quantum walk in the Fourier transform.

Similar as the setup in [4], we let the initial state be ϕ_{0_i}, and we let $\Psi_j^i(x,t) =< \phi_{x_j}, U^t\phi_{0_i} >$ be the coefficient of the walk at time t at coordinate ϕ_{x_j}, then $W_j^i(x,t) = |\Psi_j^i(x,t)|^2$. We also introduce $\widehat{\Psi}_j^i(k,t) = \sum_x \Psi_j^i(x,t)e^{ikx}$ and $\widehat{\Psi}^i(k,t) = (\widehat{\Psi}_1^i(k,t), \widehat{\Psi}_2^i(k,t))^T$ in the Fourier transform as in [4]. The evolution operator in k space, $U(k)$, is defined s.t. $\widehat{\Psi}^i(k,t+1) = U(k)\widehat{\Psi}^i(k,t)$. It is obtained in [4] that

$$U(k) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{ik} & e^{ik} \\ e^{-ik} & -e^{-ik} \end{pmatrix}.$$

Therefore,

$$U^t(k) = \begin{pmatrix} U_{11}^t & U_{12}^t \\ U_{21}^t & U_{22}^t \end{pmatrix},$$

with

$$\begin{aligned} U_{11}^t &= \frac{1}{2}(1+\frac{\cos k}{\sqrt{1+\cos^2 k}})e^{i\omega_k t} + \frac{(-1)^t}{2}(1-\frac{\cos k}{\sqrt{1+\cos^2 k}})e^{-i\omega_k t}, \\ U_{21}^t &= \frac{e^{-ik}}{2\sqrt{1+\cos^2 k}}(e^{i\omega_k t} - (-1)^t e^{-i\omega_k t}), \\ U_{12}^t &= \frac{e^{ik}}{2\sqrt{1+\cos^2 k}}(e^{i\omega_k t} - (-1)^t e^{-i\omega_k t}), \\ U_{22}^t &= \frac{1}{2}(1-\frac{\cos k}{\sqrt{1+\cos^2 k}})e^{i\omega_k t} + \frac{(-1)^t}{2}(1+\frac{\cos k}{\sqrt{1+\cos^2 k}})e^{-i\omega_k t}, \end{aligned}$$

where $\omega_k \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ is s.t. $\sin\omega_k = \frac{\sin k}{\sqrt{2}}$ and $\cos\omega_k = \sqrt{\frac{1+\cos^2 k}{2}}$.

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Let $A_k = \frac{1}{2} + \frac{\cos k}{2\sqrt{1+\cos^2 k}}$ and $C_k = \frac{e^{-ik}}{2\sqrt{1+\cos^2 k}}$. Then for $t = 2n-1$, we have

$$\begin{aligned} U^t(k) &= \begin{pmatrix} (2A_k - 1)\cos\omega_k t + i\sin\omega_k t & 2\bar{C}_k\cos\omega_k t \\ 2C_k\cos\omega_k t & (1-2A_k)\cos\omega_k t + i\sin\omega_k t \end{pmatrix} \\ &= \begin{pmatrix} -e^{-i\omega_k t} + 2A_k\cos\omega_k t & 2\bar{C}_k\cos\omega_k t \\ 2C_k\cos\omega_k t & e^{i\omega_k t} - 2A_k\cos\omega_k t \end{pmatrix}, \end{aligned}$$

and, for $t = 2n$, we have

$$\begin{aligned} U^t(k) &= \begin{pmatrix} i(2A_k - 1)\sin\omega_k t + \cos\omega_k t & 2i\bar{C}_k\sin\omega_k t \\ 2iC_k\sin\omega_k t & i(1-2A_k)\sin\omega_k t + \cos\omega_k t \end{pmatrix} \\ &= \begin{pmatrix} e^{-i\omega_k t} + 2A_k i\sin\omega_k t & 2i\bar{C}_k\sin\omega_k t \\ 2iC_k\sin\omega_k t & e^{i\omega_k t} - 2A_k i\sin\omega_k t \end{pmatrix}. \end{aligned}$$

Now, note that

$$\widehat{\Psi}^i_j(k,0) = \sum_x < \phi_{x_j}, \phi_{0_i} > e^{ikx} = \delta^i_j, \tag{3.15}$$

and that $\widehat{\Psi}^i(k,t) = (U(k))^t\widehat{\Psi}^i(k,0)$, we conclude that $\widehat{\Psi}^i_j(k,t) = ((U(k))^t)^j_i$.

Hence, for $t = 2n-1$,

$$\begin{aligned} \widehat{\Psi}^1_1(k,t) &= -e^{-i\omega_k t} + 2A_k\cos\omega_k t, \\ \widehat{\Psi}^1_2(k,t) &= 2C_k\cos\omega_k t, \\ \widehat{\Psi}^2_1(k,t) &= 2\bar{C}_k\cos\omega_k t, \\ \widehat{\Psi}^2_2(k,t) &= e^{i\omega_k t} - 2A_k\cos\omega_k t. \end{aligned}$$

For $t = 2n$,

$$\begin{aligned} \widehat{\Psi}^1_1(k,t) &= e^{-i\omega_k t} + 2A_k i\sin\omega_k t, \\ \widehat{\Psi}^1_2(k,t) &= 2iC_k\sin\omega_k t, \\ \widehat{\Psi}^2_1(k,t) &= 2i\bar{C}_k\sin\omega_k t, \\ \widehat{\Psi}^2_2(k,t) &= e^{i\omega_k t} - 2A_k i\sin\omega_k t. \end{aligned}$$

Since $W^i_j(x,t) = |\Psi^i_j(x,t)|^2$, in the Fourier transform,

$$\widehat{W}^i_j(k,t) = \frac{1}{2\pi}\int_0^{2\pi} \widehat{\Psi}^i_j(s,t)\widehat{\Psi}^i_j(k-s,t)ds. \tag{3.16}$$

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We use the above equation to find $\widehat{W_j^i}(k,t)$.

For $t = 2n-1$,

$$\widehat{W_1^1}(k,t) = \frac{1}{2\pi}\int_0^{2\pi}(e^{-i(\omega_s+\omega_{k-s})t} - 2A_s\cos\omega_s t e^{-i\omega_{k-s}t} -2A_{k-s}\cos\omega_{k-s}te^{-i\omega_s t} + 4A_sA_{k-s}\cos\omega_s t\cos\omega_{k-s}t)ds,$$

$$\widehat{W_2^2}(k,t) = \frac{1}{2\pi}\int_0^{2\pi}(e^{i(\omega_s+\omega_{k-s})t} - 2A_s\cos\omega_s t e^{i\omega_{k-s}t} -2A_{k-s}\cos\omega_{k-s}te^{i\omega_s t} + 4A_sA_{k-s}\cos\omega_s t\cos\omega_{k-s}t)ds,$$

$$\widehat{W_2^1}(k,t) = \frac{1}{2\pi}\int_0^{2\pi}4C_sC_{k-s}\cos\omega_k t\cos\omega_{k-s}tds,$$

$$\widehat{W_1^2}(k,t) = \frac{1}{2\pi}\int_0^{2\pi}4\bar{C}_s\bar{C}_{k-s}\cos\omega_k t\cos\omega_{k-s}tds.$$

Similarly, for $t = 2n$,

$$\widehat{W_1^1}(k,t) = \frac{1}{2\pi}\int_0^{2\pi}(e^{-i(\omega_s+\omega_{k-s})t} + 2A_si\sin\omega_s t e^{-i\omega_{k-s}t} +2A_{k-s}i\sin\omega_{k-s}te^{-i\omega_s t} - 4A_sA_{k-s}\sin\omega_s t\sin\omega_{k-s}t)ds,$$

$$\widehat{W_2^2}(k,t) = \frac{1}{2\pi}\int_0^{2\pi}(e^{i(\omega_s+\omega_{k-s})t} - 2A_si\sin\omega_s t e^{i\omega_{k-s}t} -2A_{k-s}i\sin\omega_{k-s}te^{i\omega_s t} - 4A_sA_{k-s}\sin\omega_s t\sin\omega_{k-s}t)ds,$$

$$\widehat{W_2^1}(k,t) = -\frac{1}{2\pi}\int_0^{2\pi}4C_sC_{k-s}\sin\omega_k t\sin\omega_{k-s}tds,$$

$$\widehat{W_1^2}(k,t) = -\frac{1}{2\pi}\int_0^{2\pi}4\bar{C}_s\bar{C}_{k-s}\sin\omega_k t\sin\omega_{k-s}tds.$$

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We separate the real and imaginary parts of $\widehat{W}_j^i(k,t)$. For $t = 2n-1$,

$$Re(\widehat{W}_1^1(k,t)) = Re(\widehat{W}_2^2(k,t))$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}\frac{\cos\omega_s t\cos\omega_{k-s}t}{\cos\omega_s\cos\omega_{k-s}}\cos s\cos(k-s)ds - \frac{1}{2\pi}\int_0^{2\pi}\sin\omega_s t\sin\omega_{k-s}tds,$$

$$Re(\widehat{W}_2^1(k,t)) = Re(\widehat{W}_1^2(k,t))$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2}\frac{\cos k}{\cos\omega_s\cos\omega_{k-s}}\cos\omega_s t\cos\omega_{k-s}tds,$$

$$Im(\widehat{W}_1^1(k,t)) = -Im(\widehat{W}_2^2(k,t))$$
$$= \frac{1}{2\pi}\int_0^{2\pi}(\frac{1}{\sqrt{2}}\frac{\cos s}{\cos\omega_s}\cos\omega_s t\sin\omega_{k-s}t + \frac{1}{\sqrt{2}}\frac{\cos(k-s)}{\cos\omega_{k-s}}\cos\omega_{k-s}t\sin\omega_s t)ds,$$

$$Im(\widehat{W}_2^1(k,t)) = -Im(\widehat{W}_1^2(k,t))$$
$$= -\frac{1}{2\pi}\int_0^{2\pi}\frac{\sin k}{2}\frac{\cos\omega_s t\cos\omega_{k-s}t}{\cos\omega_s\cos\omega_{k-s}}ds.$$

For $t = 2n$,

$$Re(\widehat{W}_1^1(k,t)) = Re(\widehat{W}_2^2(k,t))$$
$$= -\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2}\frac{\sin\omega_s t\sin\omega_{k-s}t}{\cos\omega_s\cos\omega_{k-s}}\cos s\cos(k-s)ds + \frac{1}{2\pi}\int_0^{2\pi}\cos\omega_s t\cos\omega_{k-s}tds,$$

$$Re(\widehat{W}_2^1(k,t)) = Re(\widehat{W}_1^2(k,t))$$
$$= -\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2}\frac{\cos k}{\cos\omega_s\cos\omega_{k-s}}\sin\omega_s t\sin\omega_{k-s}tds,$$

$$Im(\widehat{W}_1^1(k,t)) = -Im(\widehat{W}_2^2(k,t))$$
$$= \frac{1}{2\pi}\int_0^{2\pi}(\frac{1}{\sqrt{2}}\frac{\cos s}{\cos\omega_s}\sin\omega_s t\cos\omega_{k-s}t + \frac{1}{\sqrt{2}}\frac{\cos(k-s)}{\cos\omega_{k-s}}\sin\omega_{k-s}t\cos\omega_s t)ds,$$

$$Im(\widehat{W}_2^1(k,t)) = -Im(\widehat{W}_1^2(k,t))$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\frac{\sin k}{2}\frac{\sin\omega_s t\sin\omega_{k-s}t}{\cos\omega_s\cos\omega_{k-s}}ds.$$

Now we are ready to find $\widehat{P}_j^i(k,z)$'s formulae.

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3.2.2 The Introduction of Σ_i's

To find the formulae of $\widehat{P}^i_j(k,z)$'s, we first introduce several short notations. We introduce the Σ_i's for $z \in \{z : |z| < \frac{1}{q}\}$. Let

$$\begin{aligned}\Sigma_1 &= Re(Q^1_1) \\ &= \frac{p}{q}\sum_{t=1}^{\infty}[Re(\widehat{W}^1_1(k,t))](qz)^t,\end{aligned}$$

$$\begin{aligned}\Sigma_2 &= Re(Q^1_2) \\ &= \frac{p}{q}\sum_{t=1}^{\infty}[Re(\widehat{W}^1_2(k,t))](qz)^t,\end{aligned}$$

$$\begin{aligned}\Sigma_3 &= Im(Q^1_1) \\ &= \frac{p}{q}\sum_{t=1}^{\infty}[Im(\widehat{W}^1_1(k,t))](qz)^t,\end{aligned}$$

$$\begin{aligned}\Sigma_4 &= Im(Q^1_2) \\ &= \frac{p}{q}\sum_{t=1}^{\infty}[Im(\widehat{W}^1_2(k,t))](qz)^t.\end{aligned}$$

Since $|\widehat{W}^i_j(k,t)| \leq 1$, for $z \in \{z : |z| < \frac{1}{q}\}$, the above series all converge. Therefore, Σ_i's are all analytic in $\{z : |z| < \frac{1}{q}\}$.

Now $\det(I-Q)$ can be written as

$$\begin{aligned}&\det(I-Q) \\ =\ & 1 - Q^1_1 - Q^2_2 + Q^1_1 Q^2_2 - Q^1_2 Q^2_1 \\ =\ & (1-\Sigma_1)^2 - {\Sigma_2}^2 + {\Sigma_3}^2 - {\Sigma_4}^2.\end{aligned}$$

Note that $\widehat{P}(k,z) = \frac{1}{2}\sum_{i,j}\widehat{P}^i_j(k,z)$. By the decoherence equation (2.15), this

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function can be written as

$$
\begin{aligned}
&\widehat{P}(k,z)\\
&= -\frac{q}{p}+\frac{1}{2p}\frac{2-Q_1^1+Q_2^1+Q_1^2-Q_2^2}{\det(I-Q)}\\
&= -\frac{q}{p}+\frac{1}{p\det(I-Q)}(1-\frac{p}{q}\sum_{t=1}^{\infty}[Re(\widehat{W}_1^1(k,t))](qz)^t+\frac{p}{q}\sum_{t=1}^{\infty}[Re(\widehat{W}_2^1(k,t))](qz)^t)\\
&= -\frac{q}{p}+\frac{1-\Sigma_1+\Sigma_2}{p((1-\Sigma_1)^2-{\Sigma_2}^2+{\Sigma_3}^2-{\Sigma_4}^2)}.
\end{aligned}
\tag{3.17}
$$

Therefore, once we have the formulae of Σ_i's, we have the formulae of $\widehat{P}(k,z)$. To find Σ_i's formulae, we first look for the formulae for $z \in (-\frac{1}{q}, \frac{1}{q})$ as a real number. Then we show that they are the desired formulae for all $z \in \{z : |z| < \frac{1}{q}\}$. Let

$$
\begin{aligned}
I_1 &= \sum_{n=1}^{\infty} \cos[(2n-1)\omega_s]\cos[(2n-1)\omega_{k-s}](qz)^{2n-1},\\
I_2 &= \sum_{n=1}^{\infty} \sin[(2n-1)\omega_s]\sin[(2n-1)\omega_{k-s}](qz)^{2n-1},\\
I_3 &= \sum_{n=1}^{\infty} \cos[(2n)\omega_s]\cos[(2n)\omega_{k-s}](qz)^{2n},\\
I_4 &= \sum_{n=1}^{\infty} \sin[(2n)\omega_s]\sin[(2n)\omega_{k-s}](qz)^{2n},\\
I_5 &= \sum_{n=1}^{\infty} \cos[(2n-1)\omega_s]\sin[(2n-1)\omega_{k-s}](qz)^{2n-1},\\
I_6 &= \sum_{n=1}^{\infty} \sin[(2n-1)\omega_s]\cos[(2n-1)\omega_{k-s}](qz)^{2n-1},\\
I_7 &= \sum_{n=1}^{\infty} \sin[(2n)\omega_s]\cos[(2n)\omega_{k-s}](qz)^{2n},\\
I_8 &= \sum_{n=1}^{\infty} \cos[(2n)\omega_s]\sin[(2n)\omega_{k-s}](qz)^{2n}.
\end{aligned}
$$

Since Σ_i's are bounded, we can interchange the integral and the summation

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to write Σ_i's as

$$\begin{aligned}
\Sigma_1 &= \frac{p}{q}\frac{1}{2\pi}\int_0^{2\pi}(\frac{1}{2}\frac{\cos s\cos(k-s)}{\cos\omega_s\cos\omega_{k-s}}(I_1-I_4)-I_2+I_3)ds, \\
\Sigma_2 &= \frac{p}{q}\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2}\frac{\cos k}{\cos\omega_s\cos\omega_{k-s}}(I_1-I_4)ds, \\
\Sigma_3 &= \frac{p}{q}\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{\sqrt{2}}[\frac{\cos s}{\cos\omega_s}(I_5+I_7)+\frac{\cos(k-s)}{\cos\omega_{k-s}}(I_6+I_8)]ds, \\
\Sigma_4 &= \frac{p}{q}\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2}\frac{\sin k}{\cos\omega_s\cos\omega_{k-s}}(-I_1+I_4)ds.
\end{aligned}$$

Then we have

$$\begin{aligned}
& I_1-I_4 \\
= & \frac{1}{D}\cos\omega_s\cos\omega_{k-s}qz(1-q^2z^2),
\end{aligned}$$

$$\begin{aligned}
& -I_2+I_3 \\
= & \frac{1}{D}[-\frac{1}{2}\sin s\sin(k-s)qz+q^2z^2[\cos^2 s+\cos^2(k-s)-1] \\
& -\frac{3}{2}\sin s\sin(k-s)q^3z^3-q^4z^4],
\end{aligned}$$

$$\begin{aligned}
& I_5+I_7 \\
= & \frac{1}{D}\frac{1}{\sqrt{2}}qz\cos\omega_s[\sin(k-s)+2qz\sin s+q^2z^2\sin(k-s)],
\end{aligned}$$

$$\begin{aligned}
& I_6+I_8 \\
= & \frac{1}{D}\frac{1}{\sqrt{2}}qz\cos\omega_{k-s}[\sin s+2qz\sin(k-s)+q^2z^2\sin s],
\end{aligned}$$

where

$$\begin{aligned}
& D \\
= & (1-2\cos(\omega_s+\omega_{k-s})qz+q^2z^2)(1+2\cos(\omega_s+\omega_{k-s})qz+q^2z^2) \\
= & \cos(k-2s)(q^3z^3-2\cos kq^2z^2+qz)+q^4z^4-\cos kq^3z^3-\cos kqz+1.
\end{aligned}$$

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Therefore,

$$\begin{aligned}
\Sigma_1 &= pz\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{D}[\cos(k-2s)qz(\cos k - qz) + \frac{1}{2}\cos k \\
&\quad + \frac{1}{2}\cos kq^2z^2 - q^3z^3]ds, \\
\Sigma_2 &= \frac{1}{2}pz\cos k(1-q^2z^2)\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{D}ds, \\
\Sigma_3 &= pz\sin k\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{D}[\cos(k-2s)qz + \frac{1}{2} + \frac{1}{2}q^2z^2]ds, \\
\Sigma_4 &= \frac{1}{2}pz\sin k(1-q^2z^2)\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{D}ds.
\end{aligned}$$

By the integral formula

$$\int\frac{dx}{b+c\cos ax} = \frac{2}{a\sqrt{b^2-c^2}}\arctan\left(\sqrt{\frac{b-c}{b+c}}\tan\left(\frac{1}{2}ax\right)\right)$$

for $b > c$ and the fact that

$$q^4z^4 - \cos kq^3z^3 - \cos kqz + 1 > q^3z^3 - 2\cos kq^2z^2 + qz$$

for $z \in (-\frac{1}{q}, \frac{1}{q})$, we have

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{D}ds \\
= &\frac{1}{2\pi}\int_0^{2\pi}\frac{ds}{\cos(k-2s)(q^3z^3 - 2\cos kq^2z^2 + qz) + q^4z^4 - \cos kq^3z^3 - \cos kqz + 1} \\
= &\frac{1}{\sqrt{(q^4z^4 - \cos kq^3z^3 - \cos kqz + 1)^2 - (q^3z^3 - 2\cos kq^2z^2 + qz)^2}} \\
= &\frac{1}{(1+qz)(1-qz)\sqrt{(q^2z^2 - (1+\cos k)qz + 1)(q^2z^2 + (1+\cos k)qz + 1)}}.
\end{aligned}$$

Letting

$$E = \sqrt{(q^2z^2 - (1+\cos k)qz + 1)(q^2z^2 + (1+\cos k)qz + 1)},$$

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we have

$$\begin{aligned}\Sigma_1 =& \frac{pz}{q^2z^2 - 2\cos kqz + 1}[\cos k - qz \\ & - \frac{\cos k - 2qz + \cos kq^2z^2}{2E}], \\ \Sigma_2 =& pz\cos k\frac{1}{2E}, \\ \Sigma_3 =& \frac{pz\sin k}{q^2z^2 - 2\cos kqz + 1}[1 - \frac{1-q^2z^2}{2E}], \\ \Sigma_4 =& -pz\sin k\frac{1}{2E}.\end{aligned} \tag{3.18}$$

Now that we have obtained the formulae of Σ_i's for $z \in (\frac{-1}{q}, \frac{1}{q})$, we can check easily by taking the principal branch of log, the formulae are analytic in $\{z : |z| < \frac{1}{q}\}$. Hence, by the Analytic Continuation Theorem, they are the desired formulae for $z \in \{z : |z| < \frac{1}{q}\}$.

3.2.3 Proofs in Subsection 2.3.2

We begin with a lemma showing the formula of $\widehat{P}(k, z)$.

Lemma 3.2 *The generating function of the centered decoherent quantum random walk, $\widehat{P}(k, z)$, is given by*

$$\begin{aligned}&\widehat{P}(k,z) \\ =& \frac{q(q-\cos^2 k)z^2 + p\cos kz + (1 - z\cos k)E}{pq\cos kz^3 - (pq+p)z^2 + p\cos kz + (z^2 - 2\cos kz + 1)E}.\end{aligned}$$

Proof. The formula is obtained by applying (3.18) to (3.17).

Proof of theorem 2.2. Note that from the formula above, for some $r < 1$, we have

$$\begin{aligned}& E(X_t) \\ =& \frac{1}{i}\partial_k \widehat{P}(0,t) \\ =& \frac{1}{2\pi i}\oint_{|z|=r} \frac{\partial_k \widehat{P}(0,z)}{iz^{t+1}}dz \\ =& 0.\end{aligned}$$

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The change of the order of integration and differentiation is justified since $\partial_k \widehat{P}(k,z)$ is continuous on the contour.

Proof of theorem 2.3. The denominator of $\widehat{P}(k,z)$ has less than eight isolated roots. We shall now look for the root with the smallest absolute value. This root has no closed form. However, since we concentrate on the asymptotic behaviors, we need only to know its behaviors around $k = 0$. The properties of this root are summarized in the following lemma.

Lemma 3.3 *Let $D(k,z)$ denote the denominator of $\widehat{P}(k,z)$. Then the root of $D(k,z) = 0$ in z, with $z = 1$ when $k = 0$ is of the smallest absolute value in a neighborhood of $k = 0$. If we denote it by $z(k)$, then $z(k)$ has multiplicity one and can be written as follows.*

$$z(k) = 1 + \partial_k z(0)k + o(k).$$

Proof. For $k = 0$, $D(0,z) = (1-z)(1-qz)(pz + (1-z)\sqrt{1+q^2z^2})$. By solving this equation we can see that $z = 1$ has the smallest absolute value. The root of the second smallest absolute value has a closed form expression, which we put in the appendix. We denote this root by $\tilde{z}(p)$. An expansion of the root around $p = 0$ is

$$\tilde{z}(p) = 1 + \frac{\sqrt{2}}{2}p + \frac{1}{2}(\frac{1}{2} + \frac{1}{\sqrt{2}})p^2 + o(p^2). \tag{3.19}$$

Now, by continuity of k, $z(k)$ has the smallest absolute value in a neighborhood of $k = 0$.

Since $\partial_z D(k,z)|_{k=0,z=1} \neq 0$, $z(k)$ has multiplicity one. We then apply the Implicit Function Theorem to find its derivatives.

Remark 3.1 *For $p \to 1$, $D(0,z) \to 1 - z$, for all $z \in \{z : |z| < \frac{1}{q}\}$, which implies that other roots go to infinity and eventually disappear.*

Now we utilize the Implicit Function Theorem to find $\partial_k z(0)$ and ${\partial_k}^2 z(0)$.

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Let $F(k)$ be the denominator of $\widehat{P}(k,z)$ and the function $z(k)$ is defined implicitly by $F(k) \equiv 0$.

Take the first derivative and we get

$$\begin{aligned} 0 &= \partial_k F(k) \\ &= -pq \sin k z^3 + pq \cos k 3z^2 \partial_k z - (pq+p) 2z \partial_k z \\ &\quad -p \sin k z + p \cos k \partial_k z \\ &\quad +E(2z\partial_k z + 2\sin kz - 2\cos k \partial_k z) + \partial_k E(z^2 - 2\cos kz + 1). \end{aligned}$$

Put in $k=0$ and $z=1$, the equation becomes

$$(pq-p)\partial_k z = 0,$$

which implies that

$$\partial_k z(0) = 0. \tag{3.20}$$

Now, for ${\partial_k}^2 z(0)$, we can take the second derivative on $F(k)$ to get

$$\begin{aligned} 0 &= {\partial_k}^2 F \\ &= -pq \cos k z^3 - 2pq \sin k 3z^2 \partial_k z + pq \cos k (3z^2 {\partial_k}^2 z + 6z(\partial_k z)^2) \\ &\quad -(pq+p)(2z{\partial_k}^2 z + 2(\partial_k z)^2) - p\cos kz - 2p \sin k \partial_k z + p \cos k {\partial_k}^2 z \\ &\quad +E(2z{\partial_k}^2 z + 2\cos kz - 2\cos k {\partial_k}^2 z) \\ &\quad +2\partial_k E(2z\partial_k z + 2\sin k \partial_k z - 2\cos k \partial_k z) \\ &\quad {\partial_k}^2 E(z^2 - 2\cos kz + 1), \end{aligned}$$

which in turn gives

$${\partial_k}^2 z(0) = \frac{p + 2\sqrt{1+q^2} - 2}{p}. \tag{3.21}$$

Similarly, taking the third derivative of $F(k) = 0$ gives

$${\partial_k}^3 z(0) = 0.$$

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Also, by taking the fourth derivative we get

$$\begin{aligned}\partial_k{}^4 z(0) =& \frac{1}{p^3(1+q^2)^{\frac{1}{2}}}(76q^4 - 83q^3(1+q^2)^{\frac{1}{2}} + \\ &+ 16q^3 + 68q^2 - q^2(1+q^2)^{\frac{1}{2}} - 37q(1+q^2)^{\frac{1}{2}} + \\ &+ 16q - 23(1+q^2)^{\frac{1}{2}} + 28).\end{aligned}$$

Hence we have the expansion of $z(k)$ at $k = 0$

$$z(k) = 1 + \frac{p + 2\sqrt{1+q^2} - 2}{2p}k^2 + O(k^4). \tag{3.22}$$

The residue of $\frac{\widehat{P}(k,z)}{z^{t+1}}$ is

$$Res(\frac{\widehat{P}(k,z)}{z^{t+1}}, z(k)) = (\frac{1}{z(k)})^{t+1} \lim_{z \to z(k)} (z - z(k))\widehat{P}(k,z). \tag{3.23}$$

We then show another lemma.

Lemma 3.4

$$\lim_{z \to z(k)} (z(k) - z)\widehat{P}(k,z) = 1 + O(k^2) \tag{3.24}$$

as $k \to 0$.

Proof. Note that $\forall z \neq 1$,

$$\lim_{k \to 0} (z(k) - z)\widehat{P}(k,z) = 1,$$

i.e., $\forall \epsilon > 0$, $\exists \delta$, s.t.,

$$|(z(k) - z)\widehat{P}(k,z) - 1| < \epsilon \tag{3.25}$$

for $|k| < \delta$. (3.25) implies that

$$\lim_{z \to z(k)} |(z(k) - z)\widehat{P}(k,z) - 1| \leq \epsilon$$

for $|k| < \delta$. Hence,

$$|\lim_{z \to z(k)} (z(k) - z)\widehat{P}(k,z) - 1| \leq \epsilon$$

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for $|k| < \delta$, i.e.,

$$\lim_{k\to 0}\lim_{z\to z(k)}(z(k)-z)\widehat{P}(k,z)=1.$$

Now, for a small $r_1 > 0$ s.t. $z(k)$ is the only pole in the circle $|z-1| = r_1$, we have

$$\begin{aligned}
&\lim_{k\to 0}\lim_{z\to z(k)}\frac{1}{k}((z(k)-z)\widehat{P}(k,z)-1)\\
=&\lim_{k\to 0}\frac{1}{k}(\frac{1}{2\pi i}\oint_{|z-1|=r_1}\widehat{P}(k,z)dz-1)\\
=&\frac{1}{2\pi i}\oint_{|z-1|=r_1}\partial_k\widehat{P}(0,z)dz\\
=&0.
\end{aligned}\tag{3.26}$$

Similarly, we also have

$$\begin{aligned}
&\lim_{k\to 0}\lim_{z\to z(k)}\frac{1}{k^2}((z(k)-z)\widehat{P}(k,z)-1)\\
=&\frac{1}{2\pi i}\oint_{|z-1|=r_1}\partial_k{}^2\widehat{P}(0,z)dz\\
=&Res(\frac{z}{(1-z)^2}+\frac{2z^2(-1+\sqrt{1+q^2z^2})}{(1-z)^2(pz+(1-z)\sqrt{1+q^2z^2})},1)\\
=&\frac{p+2\sqrt{1+q^2}-2}{p}-\frac{2q^2}{p\sqrt{1+q^2}}-\frac{2}{p^2}(1+q^2-\sqrt{1+q^2}).
\end{aligned}\tag{3.27}$$

Therefore, $\lim_{z\to z(k)}(z(k)-z)\widehat{P}(k,z) = 1 + O(k^2)$.

Now for any fixed $k \in [0, 2\pi]$, the characteristic function of $\frac{X_t}{\sqrt{t}}$ is $\widehat{P}(\frac{k}{\sqrt{t}}, t)$. Since the roots of $D(k,z)$ are isolated, we can set $r(p) = 1 + \frac{\sqrt{2}}{2}p$ s.t. $|z(\frac{k}{\sqrt{t}})| < r(p)$ and other roots are outside the circle $\{|z| = r(p)\}$. Furthermore, when t is large, $\frac{k}{\sqrt{t}}$ is small, hence the lemmas are applicable. We define the contour C as $C = \{z : |z| = r_0\}\bigcup\{z : |z| = r(p)\}$, where $r_0 < 1$.

By definition,

$$\widehat{P}(\frac{k}{\sqrt{t}},t)=\frac{1}{2\pi i}\oint_{|z|=r_0}\frac{\widehat{P}(\frac{k}{\sqrt{t}},z)}{z^{t+1}}dz.\tag{3.28}$$

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Since $z(\frac{k}{\sqrt{t}})$ is the only pole in the contour, we have

$$-Res(\frac{\widehat{P}(\frac{k}{\sqrt{t}},z)}{z^{t+1}},z(\frac{k}{\sqrt{t}})) = \frac{1}{2\pi i}(\oint_{|z|=r_0}\frac{\widehat{P}(\frac{k}{\sqrt{t}},z)}{z^{t+1}}dz - \oint_{|z|=r(p)}\frac{\widehat{P}(\frac{k}{\sqrt{t}},z)}{z^{t+1}}dz).$$

For fixed $0 < p \leq 1$, $\sup_{k,|z|=r(p)} |\widehat{P}(\frac{k}{\sqrt{t}},z)| < \infty$. Hence,

$$\oint_{|z|=r(p)}\frac{\widehat{P}(\frac{k}{\sqrt{t}},z)}{z^{t+1}}dz = O(r(p)^{-t}).$$

We have

$$\widehat{P}(\frac{k}{\sqrt{t}},t) = -Res(\frac{\widehat{P}(\frac{k}{\sqrt{t}},z)}{z^{t+1}},z(\frac{k}{\sqrt{t}})) + O(r(p)^{-t}). \tag{3.29}$$

Note that by (3.24), we have

$$\lim_{t\to\infty}\lim_{z\to z(\frac{k}{\sqrt{t}})}(z(\frac{k}{\sqrt{t}})-z)\widehat{P}(\frac{k}{\sqrt{t}},z) = 1 + O(t^{-1}). \tag{3.30}$$

Note also that by (3.22),

$$z(\frac{k}{\sqrt{t}}) = 1 + \frac{p+2\sqrt{1+q^2}-2}{2p}\frac{k^2}{t} + O(t^{-2}), \tag{3.31}$$

which implies that

$$\begin{aligned}
&(\frac{1}{z(\frac{k}{\sqrt{t}})})^{t+1} \\
=\ &(1+\frac{p+2\sqrt{1+q^2}-2}{2p}\frac{k^2}{t}+O(t^{-2}))^{-(t+1)} \\
=\ &(1-\frac{p+2\sqrt{1+q^2}-2}{2p}\frac{k^2}{t}+O(t^{-2}))^{t+1} \\
=\ &\exp\{-\frac{p+2\sqrt{1+q^2}-2}{2p}k^2\} + O(t^{-1}), \forall k.
\end{aligned}$$

Therefore, by (3.23),

$$\widehat{P}(z(\frac{k}{\sqrt{t}}),t) = \exp\{-\frac{p+2\sqrt{1+q^2}-2}{2p}k^2\} + O(t^{-1}), \forall k \in [0,2\pi] \tag{3.32}$$

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as $t \to \infty$. Hence, the limiting distribution is Gaussian with variance $v = \frac{p+2\sqrt{1+q^2}-2}{p}$.

Remark 3.2 *We put some comment on the speed of convergence in the theorem, i.e., we compare the speed of $O(r(p)^{-t})$ and $O(t^{-1})$. Note that by (3.19) that we can pick $r(p)$ as large as $r(p) = 1 + \frac{p}{\sqrt{2}}$. Note also that the equation*

$$r(p)^t = t \tag{3.33}$$

has two solutions if $1 < r(p) < e^{\frac{1}{e}} \approx 1.44$. Therefore, when $p > \sqrt{2}(e^{\frac{1}{e}} - 1) \approx 0.62$, $r(p)^{-t} < t^{-1}$ for all t, which implies that the error term is of order $O(\frac{1}{t})$. However, for p small, there is an interesting phenomenon. Although for t very large, $r(p)^{-t} < t^{-1}$, there is an interval of t such that $r(p)^{-t} > t^{-1}$. A rough approximation of the bigger root, T, in equation (3.33) is

$$T > \frac{1}{\log(r(p))}(-\log(\log(r(p))). \tag{3.34}$$

For instance, if $r(p) = 1.01$ then $r(p)^{-t} > t^{-1}$ for $2 \leq t \leq 651$. This implies that when the system is nearly not interfered, the error term can be very large and the classical behaviors are not significant for a long time.
Numerical studies on this phenomenon can be found in [16] and [17]. They showed that for $pT \ll 1$ and $T \gg 1$, the distribution is highly uniform in the time interval $-\frac{T}{\sqrt{2}} < x < \frac{T}{\sqrt{2}}$ and the variance of the walk is about T^2, which are very similar to the one of pure quantum walk.

Proof of Theorem 2.4.: For X_t, we can also find its long time variance. Let C be the same contour as before, when t is large, $z = 1$ is the closest root to 0 among all that of the denominator of $\widehat{P}(k, z)$.

Note that

$$\begin{aligned} &\partial_k^2 \widehat{P}(0, z) \\ = \;& \frac{z}{(1-z)^2} + \frac{2z^2(-1+\sqrt{1+q^2z^2})}{(1-z)^2(pz+(1-z)\sqrt{1+q^2z^2})}, \end{aligned}$$

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and that

$$Res(\frac{1}{z^{t+1}}(\frac{z}{(1-z)^2}+\frac{2z^2(-1+\sqrt{1+q^2z^2})}{(1-z)^2(pz+(1-z)\sqrt{1+q^2z^2})}),1)$$
$$= (-1-\frac{2(\sqrt{1+q^2}-1)}{p})t+\frac{2q^2}{p\sqrt{1+q^2}}+\frac{2}{p^2}(1+q^2-\sqrt{1+q^2}).$$

Hence,

$$\begin{aligned}
& Var(X_t) \\
= & -\partial_k{}^2\widehat{P}(0,t) \\
= & -\frac{1}{2\pi i}\oint_C \frac{\partial_k{}^2\widehat{P}(0,z)}{z^{t+1}}dz \\
= & -Res(\frac{1}{z^{t+1}}(\frac{z}{(1-z)^2}+\frac{2z^2(-1+\sqrt{1+q^2z^2})}{(1-z)^2(pz+(1-z)\sqrt{1+q^2z^2})}),1)+O(r(p)^{-t}) \\
= & \frac{p+2\sqrt{1+q^2}-2}{p}t-\frac{2q^2}{p\sqrt{1+q^2}}-\frac{2}{p^2}(1+q^2-\sqrt{1+q^2})+O(r(p)^{-t}).
\end{aligned}$$

The change of the order of integration and differentiation is justified since $\partial_k{}^2\widehat{P}(k,z)$ is continuous on the contour. Hence, the long time variance of the walk is $\frac{p+2\sqrt{1+q^2}-2}{p}t-\frac{2q^2}{p\sqrt{1+q^2}}-\frac{2}{p^2}(1+q^2-\sqrt{1+q^2})+O(r(p)^{-t})$.

3.3 Proofs in Subsection 2.3.3

From the decoherence equation we have

$$\begin{aligned}
\widehat{P}_1^1(k,z) &= -\frac{q}{p}+\frac{1}{p}\frac{1-Q_2^2}{\det(I-Q)}, \\
\widehat{P}_2^1(k,z) &= \frac{1}{p}\frac{Q_2^1}{\det(I-Q)}.
\end{aligned}$$

Let $\tilde{X}_t$ be the walk starting with type 1 at time t and $\widehat{\widetilde{P}}(k,z)=\widehat{P}_1^1(k,z)+\widehat{P}_2^1(k,z)$ be the generating function of it. Then

$$\begin{aligned}
& \widehat{\widetilde{P}}(k,z) \\
= & -\frac{q}{p}+\frac{1-\Sigma_1+i\Sigma_3+\Sigma_2+i\Sigma_4}{p\det(I-Q)} \\
= & \widehat{P}(k,z)+i\frac{\Sigma_3+\Sigma_4}{p\det(I-Q)}.
\end{aligned}$$

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Note that Σ_3 and Σ_4 both have a factor of $\sin k$, we denote $\frac{\Sigma_3}{\sin k}$ and $\frac{\Sigma_4}{\sin k}$ by $\tilde{\Sigma_3}$ and $\tilde{\Sigma_4}$ respectively.

Proof of Theorem 2.5. Note that

$$\begin{aligned}
& \partial_k \widehat{\tilde{P}}(0,z) \\
= \ & \partial_k \widehat{P}(0,z) + i\partial_k(\sin k \frac{\tilde{\Sigma_3}+\tilde{\Sigma_4}}{p\det(I-Q)})|_{k=0} \\
= \ & i(\frac{\tilde{\Sigma_3}+\tilde{\Sigma_4}}{p})|_{k=0} \\
= \ & i\frac{z(\sqrt{1+q^2z^2}-1)}{(1-z)(pz+(1-z)\sqrt{1+q^2z^2})}.
\end{aligned}$$

Let C be the same contour as before, when t is large, $z=1$ is closest to 0 among the roots of the above denominator. Hence, for fixed p,

$$\begin{aligned}
& E(\tilde{X}_t) \\
= \ & \frac{1}{i}\partial_k \widehat{\tilde{P}}(0,t) \\
= \ & \frac{1}{2\pi i}\oint_C \frac{\partial_k \widehat{\tilde{P}}(0,z)}{iz^{t+1}}dz \\
= \ & \frac{1}{2\pi i}\oint_C \frac{(\sqrt{1+q^2z^2}-1)}{z^t(1-z)(pz+(1-z)\sqrt{1+q^2z^2})}dz \\
= \ & Res(\frac{(\sqrt{1+q^2z^2}-1)}{z^t(1-z)(pz+(1-z)\sqrt{1+q^2z^2})},1) + O(r(p)^{-t}) \\
= \ & \frac{\sqrt{1+q^2}-1}{p} + O(r(p)^{-t}).
\end{aligned}$$

Proof of Theorem 2.6. Let $\mu_t = E(\tilde{X}_t)$. We want to show that $\frac{\tilde{X}_t-\mu_t}{\sqrt{t}} \to N(0,v)$. The long time characteristic function is

$$\begin{aligned}
& \widehat{\tilde{P}}(\frac{k}{\sqrt{t}},t)e^{-i\frac{\mu_t k}{\sqrt{t}}} \\
= \ & \widehat{P}(\frac{k}{\sqrt{t}},t)e^{-i\frac{\mu_t k}{\sqrt{t}}} + e^{-i\frac{\mu_t k}{\sqrt{t}}}\sin\frac{k}{\sqrt{t}}\frac{1}{2\pi i}\oint_C \frac{1}{z^{t+1}}(\frac{\tilde{\Sigma_3}+\tilde{\Sigma_4}}{p\det(I-Q)})(\frac{k}{\sqrt{t}},z)dz \\
= \ & \exp\{-\frac{p+2\sqrt{1+q^2}-2}{2p}k^2\} + O(t^{-\frac{1}{2}}).
\end{aligned}$$

Hence, the limiting distribution of the decoherent quantum random walk starting with type 1 is Gaussian with variance $v = \frac{p+2\sqrt{1+q^2}-2}{p}$.

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Remark 3.3 *Note again that there is an interval of* t *such that* $r(p)^{-t} > t^{-\frac{1}{2}}$. *The length of this interval* T *has the rough estimation of*

$$T > -\frac{\log(2p)}{2p}. \tag{3.35}$$

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CHAPTER 4

Conclusions

We have investigated the quantum walk with decoherence on both position and chirality states. Long time limits are obtained for both centered walk and the walk starting at 0 with chirality "right". Analytical explanations of the dynamics of the decoherent quantum walk system are given and we see that the system is indeed a mixture of quantum and classical ones. The limiting distributions of quantum random walks are Gaussian if decoherence occurs.

We want to emphasize that our work is done through a purely analytical approach. The method of establishing an equation between the decoherent system and the pure quantum system is new in this area of study. We believe this method is widely applicable in future studies.

From Remark 3.2 and Remark 3.3, we also see that when p is small, the system remains non-classical for a very long time. If a quantum algorithm can be finished before the classical features appear, then we call it a "pseudoquantum" algorithm. However, we do not know how fast the "pseudoquantum" algorithms are as compared to the classical ones. It would be a very interesting problem for future studies.

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APPENDIX A

Appendix

The other three roots of $pz + (1-z)\sqrt{1+q^2z^2} = 0$ are

```
zp1=1/2+1/6*3^(1/2)/q*((3*q^2*((27-50*q+27*q^2+3*3^(1/2)*(1
46*q^2+27-100*q-100*q^3+27*q^4)^(1/2))*q^2)^(1/3)-4*q*((27-
50*q+27*q^2+3*3^(1/2)*(146*q^2+27-100*q-100*q^3+27*q^4)^(1/
2))*q^2)^(1/3)+2^(1/3)*((27-50*q+27*q^2+3*3^(1/2)*(146*q^2+
27-100*q-100*q^3+27*q^4)^(1/2))*q^2)^(2/3)+2*2^(2/3)*q^2)/(
(27-50*q+27*q^2+3*3^(1/2)*(146*q^2+27-100*q-100*q^3+27*q^4)
^(1/2))*q^2)^(1/3))^(1/2)+1/6*3^(1/2)/q*((6*q^2*((27-50*q+2
7*q^2+3*3^(1/2)*(146*q^2+27-100*q-100*q^3+27*q^4)^(1/2))*q^
2)^(1/3)*((3*q^2*((27-50*q+27*q^2+3*3^(1/2)*(146*q^2+27-100
*q-100*q^3+27*q^4)^(1/2))*q^2)^(1/3)-4*q*((27-50*q+27*q^2+3
*3^(1/2)*(146*q^2+27-100*q-100*q^3+27*q^4)^(1/2))*q^2)^(1/3
)+2^(1/3)*((27-50*q+27*q^2+3*3^(1/2)*(146*q^2+27-100*q-100*
q^3+27*q^4)^(1/2))*q^2)^(2/3)+2*2^(2/3)*q^2)/((27-50*q+27*q
^2+3*3^(1/2)*(146*q^2+27-100*q-100*q^3+27*q^4)^(1/2))*q^2)^
(1/3))^(1/2)-8*q*((27-50*q+27*q^2+3*3^(1/2)*(146*q^2+27-100
*q-100*q^3+27*q^4)^(1/2))*q^2)^(1/3)*((3*q^2*((27-50*q+27*q
^2+3*3^(1/2)*(146*q^2+27-100*q-100*q^3+27*q^4)^(1/2))*q^2)^
(1/3)-4*q*((27-50*q+27*q^2+3*3^(1/2)*(146*q^2+27-100*q-100*
q^3+27*q^4)^(1/2))*q^2)^(1/3)+2^(1/3)*((27-50*q+27*q^2+3*3^
(1/2)*(146*q^2+27-100*q-100*q^3+27*q^4)^(1/2))*q^2)^(2/3)+2
*2^(2/3)*q^2)/((27-50*q+27*q^2+3*3^(1/2)*(146*q^2+27-100*q-
100*q^3+27*q^4)^(1/2))*q^2)^(1/3))^(1/2)-((3*q^2*((27-50*q+
27*q^2+3*3^(1/2)*(146*q^2+27-100*q-100*q^3+27*q^4)^(1/2))*q
^2)^(1/3)-4*q*((27-50*q+27*q^2+3*3^(1/2)*(146*q^2+27-100*q-
100*q^3+27*q^4)^(1/2))*q^2)^(1/3)+2^(1/3)*((27-50*q+27*q^2+
3*3^(1/2)*(146*q^2+27-100*q-100*q^3+27*q^4)^(1/2))*q^2)^(2/
```

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```
3)+2*2^(2/3)*q^2)/((27-50*q+27*q^2+3*3^(1/2)*(146*q^2+27-10
0*q-100*q^3+27*q^4)^(1/2))*q^2)^(1/3))^(1/2)*2^(1/3)*((27-5
0*q+27*q^2+3*3^(1/2)*(146*q^2+27-100*q-100*q^3+27*q^4)^(1/2
))*q^2)^(2/3)-2*((3*q^2*((27-50*q+27*q^2+3*3^(1/2)*(146*q^2
+27-100*q-100*q^3+27*q^4)^(1/2))*q^2)^(1/3)-4*q*((27-50*q+2
7*q^2+3*3^(1/2)*(146*q^2+27-100*q-100*q^3+27*q^4)^(1/2))*q^
2)^(1/3)+2^(1/3)*((27-50*q+27*q^2+3*3^(1/2)*(146*q^2+27-100
*q-100*q^3+27*q^4)^(1/2))*q^2)^(2/3)+2*2^(2/3)*q^2)/((27-50
*q+27*q^2+3*3^(1/2)*(146*q^2+27-100*q-100*q^3+27*q^4)^(1/2)
)*q^2)^(1/3))^(1/2)*2^(2/3)*q^2+12*3^(1/2)*q*((27-50*q+27*q
^2+3*3^(1/2)*(146*q^2+27-100*q-100*q^3+27*q^4)^(1/2))*q^2)^
(1/3)+6*3^(1/2)*q^3*((27-50*q+27*q^2+3*3^(1/2)*(146*q^2+27-
100*q-100*q^3+27*q^4)^(1/2))*q^2)^(1/3)-12*3^(1/2)*q^2*((27
-50*q+27*q^2+3*3^(1/2)*(146*q^2+27-100*q-100*q^3+27*q^4)^(1
/2))*q^2)^(1/3))/((27-50*q+27*q^2+3*3^(1/2)*(146*q^2+27-100
*q-100*q^3+27*q^4)^(1/2))*q^2)^(1/3)/((3*q^2*((27-50*q+27*q
^2+3*3^(1/2)*(146*q^2+27-100*q-100*q^3+27*q^4)^(1/2))*q^2)^
(1/3)-4*q*((27-50*q+27*q^2+3*3^(1/2)*(146*q^2+27-100*q-100*
q^3+27*q^4)^(1/2))*q^2)^(1/3)+2^(1/3)*((27-50*q+27*q^2+3*3^
(1/2)*(146*q^2+27-100*q-100*q^3+27*q^4)^(1/2))*q^2)^(2/3)+2
*2^(2/3)*q^2)/((27-50*q+27*q^2+3*3^(1/2)*(146*q^2+27-100*q-
100*q^3+27*q^4)^(1/2))*q^2)^(1/3))^(1/2))^(1/2)
```

```
zp2=1/2-1/6*3^(1/2)/q*((3*q^2*((27-50*q+27*q^2+3*3^(1/2)*(1
46*q^2+27-100*q-100*q^3+27*q^4)^(1/2))*q^2)^(1/3)-4*q*((27-
50*q+27*q^2+3*3^(1/2)*(146*q^2+27-100*q-100*q^3+27*q^4)^(1/
2))*q^2)^(1/3)+2^(1/3)*((27-50*q+27*q^2+3*3^(1/2)*(146*q^2+
27-100*q-100*q^3+27*q^4)^(1/2))*q^2)^(2/3)+2*2^(2/3)*q^2)/(
(27-50*q+27*q^2+3*3^(1/2)*(146*q^2+27-100*q-100*q^3+27*q^4)
^(1/2))*q^2)^(1/3))^(1/2)+1/6/q*(-(-18*q^2*((27-50*q+27*q^2
+3*3^(1/2)*(146*q^2+27-100*q-100*q^3+27*q^4)^(1/2))*q^2)^(1
/3)*((3*q^2*((27-50*q+27*q^2+3*3^(1/2)*(146*q^2+27-100*q-10
0*q^3+27*q^4)^(1/2))*q^2)^(1/3)-4*q*((27-50*q+27*q^2+3*3^(1
/2)*(146*q^2+27-100*q-100*q^3+27*q^4)^(1/2))*q^2)^(1/3)+2^(
1/3)*((27-50*q+27*q^2+3*3^(1/2)*(146*q^2+27-100*q-100*q^3+2
7*q^4)^(1/2))*q^2)^(2/3)+2*2^(2/3)*q^2)/((27-50*q+27*q^2+3*
3^(1/2)*(146*q^2+27-100*q-100*q^3+27*q^4)^(1/2))*q^2)^(1/3)
)^(1/2)+24*q*((27-50*q+27*q^2+3*3^(1/2)*(146*q^2+27-100*q-1
00*q^3+27*q^4)^(1/2))*q^2)^(1/3)*((3*q^2*((27-50*q+27*q^2+3
*3^(1/2)*(146*q^2+27-100*q-100*q^3+27*q^4)^(1/2))*q^2)^(1/3
)-4*q*((27-50*q+27*q^2+3*3^(1/2)*(146*q^2+27-100*q-100*q^3+
```

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```
27*q^4)^(1/2))*q^2)^(1/3)+2^(1/3)*((27-50*q+27*q^2+3*3^(1/2
)*(146*q^2+27-100*q-100*q^3+27*q^4)^(1/2))*q^2)^(2/3)+2*2^(
2/3)*q^2)/((27-50*q+27*q^2+3*3^(1/2)*(146*q^2+27-100*q-100*
q^3+27*q^4)^(1/2))*q^2)^(1/3))^(1/2)+3*((3*q^2*((27-50*q+27
*q^2+3*3^(1/2)*(146*q^2+27-100*q-100*q^3+27*q^4)^(1/2))*q^2
)^(1/3)-4*q*((27-50*q+27*q^2+3*3^(1/2)*(146*q^2+27-100*q-10
0*q^3+27*q^4)^(1/2))*q^2)^(1/3)+2^(1/3)*((27-50*q+27*q^2+3*
3^(1/2)*(146*q^2+27-100*q-100*q^3+27*q^4)^(1/2))*q^2)^(2/3)
+2*2^(2/3)*q^2)/((27-50*q+27*q^2+3*3^(1/2)*(146*q^2+27-100*
q-100*q^3+27*q^4)^(1/2))*q^2)^(1/3))^(1/2)*2^(1/3)*((27-50*
q+27*q^2+3*3^(1/2)*(146*q^2+27-100*q-100*q^3+27*q^4)^(1/2))
*q^2)^(2/3)+6*((3*q^2*((27-50*q+27*q^2+3*3^(1/2)*(146*q^2+2
7-100*q-100*q^3+27*q^4)^(1/2))*q^2)^(1/3)-4*q*((27-50*q+27*
q^2+3*3^(1/2)*(146*q^2+27-100*q-100*q^3+27*q^4)^(1/2))*q^2)
^(1/3)+2^(1/3)*((27-50*q+27*q^2+3*3^(1/2)*(146*q^2+27-100*q
-100*q^3+27*q^4)^(1/2))*q^2)^(2/3)+2*2^(2/3)*q^2)/((27-50*q
+27*q^2+3*3^(1/2)*(146*q^2+27-100*q-100*q^3+27*q^4)^(1/2))*
q^2)^(1/3))^(1/2)*2^(2/3)*q^2+36*3^(1/2)*q*((27-50*q+27*q^2
+3*3^(1/2)*(146*q^2+27-100*q-100*q^3+27*q^4)^(1/2))*q^2)^(1
/3)+18*3^(1/2)*q^3*((27-50*q+27*q^2+3*3^(1/2)*(146*q^2+27-1
00*q-100*q^3+27*q^4)^(1/2))*q^2)^(1/3)-36*3^(1/2)*q^2*((27-
50*q+27*q^2+3*3^(1/2)*(146*q^2+27-100*q-100*q^3+27*q^4)^(1/
2))*q^2)^(1/3))/((27-50*q+27*q^2+3*3^(1/2)*(146*q^2+27-100*
q-100*q^3+27*q^4)^(1/2))*q^2)^(1/3)/((3*q^2*((27-50*q+27*q^
2+3*3^(1/2)*(146*q^2+27-100*q-100*q^3+27*q^4)^(1/2))*q^2)^(
1/3)-4*q*((27-50*q+27*q^2+3*3^(1/2)*(146*q^2+27-100*q-100*q
^3+27*q^4)^(1/2))*q^2)^(1/3)+2^(1/3)*((27-50*q+27*q^2+3*3^(
1/2)*(146*q^2+27-100*q-100*q^3+27*q^4)^(1/2))*q^2)^(2/3)+2*
2^(2/3)*q^2)/((27-50*q+27*q^2+3*3^(1/2)*(146*q^2+27-100*q-1
00*q^3+27*q^4)^(1/2))*q^2)^(1/3))^(1/2))^(1/2)
```

```
zp3=1/2-1/6*3^(1/2)/q*((3*q^2*((27-50*q+27*q^2+3*3^(1/2)*(1
46*q^2+27-100*q-100*q^3+27*q^4)^(1/2))*q^2)^(1/3)-4*q*((27-
50*q+27*q^2+3*3^(1/2)*(146*q^2+27-100*q-100*q^3+27*q^4)^(1/
2))*q^2)^(1/3)+2^(1/3)*((27-50*q+27*q^2+3*3^(1/2)*(146*q^2+
27-100*q-100*q^3+27*q^4)^(1/2))*q^2)^(2/3)+2*2^(2/3)*q^2)/(
(27-50*q+27*q^2+3*3^(1/2)*(146*q^2+27-100*q-100*q^3+27*q^4)
^(1/2))*q^2)^(1/3))^(1/2)-1/6/q*(-(-18*q^2*((27-50*q+27*q^2
+3*3^(1/2)*(146*q^2+27-100*q-100*q^3+27*q^4)^(1/2))*q^2)^(1
/3)*((3*q^2*((27-50*q+27*q^2+3*3^(1/2)*(146*q^2+27-100*q-10
0*q^3+27*q^4)^(1/2))*q^2)^(1/3)-4*q*((27-50*q+27*q^2+3*3^(1
```

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```
/2)*(146*q^2+27-100*q-100*q^3+27*q^4)^(1/2))*q^2)^(1/3)+2^(
1/3)*((27-50*q+27*q^2+3*3^(1/2)*(146*q^2+27-100*q-100*q^3+2
7*q^4)^(1/2))*q^2)^(2/3)+2*2^(2/3)*q^2)/((27-50*q+27*q^2+3*
3^(1/2)*(146*q^2+27-100*q-100*q^3+27*q^4)^(1/2))*q^2)^(1/3)
)^(1/2)+24*q*((27-50*q+27*q^2+3*3^(1/2)*(146*q^2+27-100*q-1
00*q^3+27*q^4)^(1/2))*q^2)^(1/3)*((3*q^2*((27-50*q+27*q^2+3
*3^(1/2)*(146*q^2+27-100*q-100*q^3+27*q^4)^(1/2))*q^2)^(1/3
)-4*q*((27-50*q+27*q^2+3*3^(1/2)*(146*q^2+27-100*q-100*q^3+
27*q^4)^(1/2))*q^2)^(1/3)+2^(1/3)*((27-50*q+27*q^2+3*3^(1/2
)*(146*q^2+27-100*q-100*q^3+27*q^4)^(1/2))*q^2)^(2/3)+2*2^(
2/3)*q^2)/((27-50*q+27*q^2+3*3^(1/2)*(146*q^2+27-100*q-100*
q^3+27*q^4)^(1/2))*q^2)^(1/3))^(1/2)+3*((3*q^2*((27-50*q+27
*q^2+3*3^(1/2)*(146*q^2+27-100*q-100*q^3+27*q^4)^(1/2))*q^2
)^(1/3)-4*q*((27-50*q+27*q^2+3*3^(1/2)*(146*q^2+27-100*q-10
0*q^3+27*q^4)^(1/2))*q^2)^(1/3)+2^(1/3)*((27-50*q+27*q^2+3*
3^(1/2)*(146*q^2+27-100*q-100*q^3+27*q^4)^(1/2))*q^2)^(2/3)
+2*2^(2/3)*q^2)/((27-50*q+27*q^2+3*3^(1/2)*(146*q^2+27-100*
q-100*q^3+27*q^4)^(1/2))*q^2)^(1/3))^(1/2)*2^(1/3)*((27-50*
q+27*q^2+3*3^(1/2)*(146*q^2+27-100*q-100*q^3+27*q^4)^(1/2))
*q^2)^(2/3)+6*((3*q^2*((27-50*q+27*q^2+3*3^(1/2)*(146*q^2+2
7-100*q-100*q^3+27*q^4)^(1/2))*q^2)^(1/3)-4*q*((27-50*q+27*
q^2+3*3^(1/2)*(146*q^2+27-100*q-100*q^3+27*q^4)^(1/2))*q^2)
^(1/3)+2^(1/3)*((27-50*q+27*q^2+3*3^(1/2)*(146*q^2+27-100*q
-100*q^3+27*q^4)^(1/2))*q^2)^(2/3)+2*2^(2/3)*q^2)/((27-50*q
+27*q^2+3*3^(1/2)*(146*q^2+27-100*q-100*q^3+27*q^4)^(1/2))*
q^2)^(1/3))^(1/2)*2^(2/3)*q^2+36*3^(1/2)*q*((27-50*q+27*q^2
+3*3^(1/2)*(146*q^2+27-100*q-100*q^3+27*q^4)^(1/2))*q^2)^(1
/3)+18*3^(1/2)*q^3*((27-50*q+27*q^2+3*3^(1/2)*(146*q^2+27-1
00*q-100*q^3+27*q^4)^(1/2))*q^2)^(1/3)-36*3^(1/2)*q^2*((27-
50*q+27*q^2+3*3^(1/2)*(146*q^2+27-100*q-100*q^3+27*q^4)^(1/
2))*q^2)^(1/3))/((27-50*q+27*q^2+3*3^(1/2)*(146*q^2+27-100*
q-100*q^3+27*q^4)^(1/2))*q^2)^(1/3)/((3*q^2*((27-50*q+27*q^
2+3*3^(1/2)*(146*q^2+27-100*q-100*q^3+27*q^4)^(1/2))*q^2)^(
1/3)-4*q*((27-50*q+27*q^2+3*3^(1/2)*(146*q^2+27-100*q-100*q
^3+27*q^4)^(1/2))*q^2)^(1/3)+2^(1/3)*((27-50*q+27*q^2+3*3^(
1/2)*(146*q^2+27-100*q-100*q^3+27*q^4)^(1/2))*q^2)^(2/3)+2*
2^(2/3)*q^2)/((27-50*q+27*q^2+3*3^(1/2)*(146*q^2+27-100*q-1
00*q^3+27*q^4)^(1/2))*q^2)^(1/3))^(1/2))^(1/2)
```

$zp1$ has the second smallest absolute value and is the $\tilde{z}(p)$ in our proof.

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