# INFORMATION TO USERS

This manuscript has been reproduced from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

**The quality of this reproduction is dependent upon the quality of the copy submitted.** Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps.

ProQuest Information and Learning
300 North Zeeb Road, Ann Arbor, MI 48106-1346 USA
800-521-0600

# Temple University
## Doctoral Dissertation
## Submitted to the Graduate Board

*Title of Dissertation:* $W^{2,P}$ Estimates for the linearized Monge-Ampere Equation
(Please type)

*Author:* Federico Tournier
(Please type)

*Date of Defense:* July 15, 2002
(Please type)

Dissertation Examining Committee:(please type)

Professor Cristian E. Gutierrez
Dissertation Advisory Committee Chairperson

Professor Shiferaw Berhanu

Professor Eric Grinberg

Professor Peter Riseborough

PROF. E.L. GRINBERG
Examining Committee Chairperson

Read and Approved By:(Signatures)

S. Berhanu

If Member of the Dissertation Examining Committee

Date Submitted to Graduate Board: _____

Accepted by the Graduate Board of Temple University in partial fulfillment of the requirements for the degree of **Doctor of Philosophy.**

Date _____

Mark A. Schmidt
**(Dean of the Graduate School)**

# $W^{2,p}$–estimates for the linearized Monge–Ampère equation

---

A Dissertation
Submitted to
the Temple University Graduate Board

---

in Partial Fulfillment
of the Requirements for the Degree of
DOCTOR OF PHILOSOPHY

---

by
Federico Tournier
July, 2002

ProQuest Information and Learning Company
300 North Zeeb Road
P.O. Box 1346
Ann Arbor, MI 48106-1346

# ABSTRACT

$W^{2,p}$–estimates for the linearized Monge–Ampère equation

Federico Tournier

DOCTOR OF PHILOSOPHY

Temple University, July, 2002

Professor Cristian E. Gutiérrez, Chair

In this dissertation we study interior a priori estimates in $L^p$ for second derivatives of solutions $u$ to the linearized Monge–Ampère equation $\text{trace}(\Phi D^2 u) = f$ where $\Phi$ is the matrix of cofactors of $D^2\phi$ and $\phi$ is a $C^2$ convex function on a convex normalized domain $\Omega$. We assume $\lambda \leq \det D^2\phi \leq \Lambda$ and $\phi = 0$ on $\partial\Omega$. We show the existence of a number $p$ depending only on $\lambda, \Lambda, n$ and on the distance between $\Omega'$ and $\Omega$ such that $u \in W^{2,p}$.

# ACKNOWLEDGEMENTS

I would like to express my gratitude to my advisor, Professor Cristian Gutiérrez. He suggested this problem to me and offered me constant support throughout its development. I would like to thank him for his patience during our lengthy discussions, for his suggestions and valuable insights, and most of all for his friendship.

I am very grateful to Professor Shiferaw Berhanu. I have benefited greatly from his guidance and advise.

I also would like to acknowledge all the other members of the Temple faculty who have helped me in many ways: Professors Gerardo Mendoza, Boris Dastkosvky, Yury Grabovsky and Eric Grinberg.

Finally, I would like to thank Professor Juan Gil, my fellow student David Hartenstine, and my friend Daniel Birmajer who have made my time at Temple an enjoyable experience.

To my parents, my wife Leticia, and my son Luis.

# TABLE OF CONTENTS

# CHAPTER 1

# Introduction

The $L^p$ estimates for the second derivatives of solutions to second order elliptic nondivergence equations of the form $\text{trace}(A(x)D^2u(x)) = f(x)$, where the matrix $A(x)$ is uniformly elliptic in a domain $\Omega \subset \mathbb{R}^n$, that is, $\lambda|\xi|^2 \leq \langle A(x)\xi, \xi\rangle \leq \Lambda|\xi|^2$ for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$, were derived in the 50's as a consequence of the Calderón-Zygmund theory of singular integrals. More precisely, if $A(x)$ is continuous in $\Omega$, then for any domain $\Omega' \Subset \Omega$ and any $1 < p < \infty$ we have

$$\|D^2u\|_{L^p(\Omega')} \leq C(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}), \tag{1.1}$$

where $C$ is a constant depending only on $n, p, \lambda, \Lambda, \text{dist}(\Omega', \partial\Omega)$ and the modulus of continuity of the matrix $A(x)$, see [**GT83**, Chapter 9].

If the coefficient matrix $A$ is uniformly elliptic and only measurable, then it was discovered by Lin [**L86**] that the inequality

$$\|D^2u\|_{L^p(\Omega')} \leq C(\|u\|_{L^\infty(\Omega)} + \|f\|_{L^n(\Omega)}), \tag{1.2}$$

holds true for some $p > 0$, possibly small, and $C > 0$ depending only on $n, \lambda, \Lambda, \text{dist}(\Omega', \partial\Omega)$, and it is false for any $p$. An estimate similar to (1.2) was discovered by Evans [**E85**] for fully nonlinear uniformly elliptic equations of the form $F(D^2u) = 0$.

Recently, Caffarelli proved that if $u$ is convex in $\Omega$ convex normalized, $f$ is continuous with $0 < \lambda \leq f \leq \Lambda$ in $\Omega$, and $u$ solves the Monge–Ampère

equation $\det D^2 u = f$ with $u = 0$ on $\partial\Omega$, then $\|D^2 u\|_{L^p(\Omega')} \leq C$ for all $p > 0$ with a constant $C$ depending only on $n, p, \lambda, \Lambda, \text{dist}(\Omega', \partial\Omega)$, and the modulus of continuity of $f$, see [**C**] and [**G01**, Chapter 6].

In this thesis we consider the linearized Monge–Ampère operator. We stress that this operator is in general not uniformly elliptic in $\Omega$. More precisely, let $\Omega \subseteq \mathbb{R}^n$ be a strictly convex domain and $\phi \in C^2(\Omega)$ be a convex function such that $\lambda \leq \det D^2\phi \leq \Lambda$ in $\Omega$. The linearized Monge–Ampère equation is

$$L_\Phi u = \text{trace}(\Phi D^2 u) = f \tag{1.3}$$

where $\Phi = (\det D^2\phi)(D^2\phi)^{-1}$ is the matrix of cofactors of $D^2\phi$. A theory for this equation has been developed in [**CG97**] where, in particular, Harnack's inequality is proved. Applications of this theory include the solution of a problem in affine differential geometry, see [**TW00**].

The *purpose in this thesis is to study the $L^p$ integrability of second derivatives of solutions* to the equation (1.3). *The main result of the thesis*, Theorem 6.1, is that there exist $p > 0$ and $C > 0$ depending only on $n, \lambda, \Lambda$,and $\text{dist}(\Omega', \Omega)$ such that

$$\|D^2 u\|_{L^p(\Omega')} \leq C(\|u\|_{L^\infty(\Omega)} + \|f\|_{L^n(\Omega)}) \tag{1.4}$$

for all solutions $u \in C^2(\Omega)$ of equation (1.3). It is important to notice that the matrix $\Phi$ is locally uniformly elliptic but the constant $C$ in (1.4) is independent of the size of its eigenvalues. In other words, we have no control on the size of each eigenvalue of $\Phi$, we only assume that their product is controlled by $\lambda$ and $\Lambda$. In addition, the estimate (1.4) is false for all $p > 0$. Indeed, we show examples where given $p > 1$, there exist $\lambda(p)$ and $\Lambda(p)$ and a sequence $\phi_n \in C^\infty(\Omega)$ such that $\phi = 0$ on $\partial\Omega$ and $\lambda \leq \det D^2\phi_n \leq \Lambda$ for each $n$ and such that $\|D^2\phi_n\|_{L^p(B_\delta(0))} \to \infty$ as $n \to \infty$.

One important element in the analysis of equation (1.3) is the fact that $L_\Phi \phi \approx 1$ which means that the function $\phi$ takes the role that quadratic polynomials take on a uniformly elliptic equation. The analogous sets to Euclidean

balls in this setting are the level sets of the function $\phi$ which are called sections. The main idea to prove our result is to separate the problem into two main steps. The first step is to estimate the measure of the set of points where a solution $u$ can be approximated by quadratic polynomials in which the quadratic term is defined by the quasidistance given by the sections of $\phi$; a covering theorem with sections is crucial in this step. The second step consists of estimating the measure of the set of points where the quasidistance is dominated by Euclidean distance; this step deals only with the convex function $\phi$.

A main difficulty in proving our theorem was to realize the right notion for the sets $G_M(u)$ given in Definition 3.2. If one defines $G_M(u)$ directly with Euclidean distance, then this does not takes into account the behavior of $\phi$ in different directions. The difficulty is that then, one has to deal with the norms of $T$ and $T^{-1}$ normalizing $S_\phi(x,t)$ and then, the eccentricity of $S_\phi(x,t)$ becomes an obstacle. Once we define the sets $G_M(u)$ using the quasidistance given by the sections, Definition 3.1, the difficulty comes from deciding the size of the neighborhood in which we want the inequality in the definition of $G_M(u)$ to hold. To estimate $D^2u(x_0)$ it is enough to have $|u(x) - u(x_0) - \nabla(u)(x_0)(x - x_0)| \leq Md(x,x_0)^2$ for all $x$ in a neighborhood of $x_0$ but it is important to specify this neighborhood since eventually we have to compare $G_M(u)$ with $G_{\lambda M}(u)$; in particular this becomes important in the hypothesis of Lemma 4.2.

Our proof uses the method of Caffarelli [**C**] clarified by Gutiérrez in [**G01**]; in particular, the $W^{2,p}$ estimates for the Monge–Ampére equation are a good guideline for our proof; some of these ideas are also used in the $W^{2,p}$ estimates for non-linear elliptic equations in Caffarelli and Cabré's book [**CC95**] from where some of the notation is borrowed.

The method of proof differs considerably from the classical $W^{2,p}$ estimates as in Gilbarg and Trudinger [**GT83**] as well as the estimates of Lin and Evans. Gilbarg and Trudinger proof uses perturbation arguments and is based on the inequality $\|D^2u\|_{L^p(\Omega)} \leq C\|\Delta u\|_{L^p(\Omega)}$ for $u \in W_0^{2,p}(\Omega)$. The ellipticity of the

operator $L$ and the modulus of continuity of the coefficients of $L$ are essential in that proof; these two elements are not present in our setting.

A hypothesis requires that $\lambda \leq \det D^2\phi \leq \Lambda$ and our method does not seem to give information about the relation between the best exponent $p$ for which our theorem holds with the constants $\lambda$ and $\Lambda$. A conjecture is that if $|\det D^2\phi - 1| \leq \epsilon$ then $p(\epsilon)$ tends to $\infty$ as $\epsilon$ tends to 0. This conjecture does not follow from our proof. It is true that if $|\det D^2\phi - 1| \leq \epsilon$ with $\phi = 0$ on $\partial\Omega$ normalized then $\|D^2\phi\|_{L^p(\Omega_{1/2})} \leq C$ where $p(\epsilon)$ tends to $\infty$ as $\epsilon$ tends to 0. In some sense this says that the "ellipticity becomes better" when $\det D^2\phi$ is closer to one but it does not seem to give information on the modulus of continuity of the coefficients of our equation.

The organization of the thesis is the following. In Chapter 2 we introduce all the preliminary material. Chapter 3 contains the main definitions and the second step mentioned above is proved. Chapter 4 contains the proof of the density lemmas need later. In Chapter 5 we prove the main result for the homogeneous case, and Chapter 6 contains the non-homogeneous case. Chapter 7 contains the proof of an inequality used in Chapter 3, and Chapter 8 contains the examples mentioned in this Introduction.

# CHAPTER 2

# Preliminary results

In this chapter we list the results about sections and normalization that are relevant for what follows.

Given a function $\phi : \Omega \to \mathbb{R}$, $\partial\phi$ denotes the subdifferential of $\phi$. The Monge-Ampère measure associated with $\phi$ is

$$\mu(E) = M\phi(E) = |\partial\phi(E)|, \tag{2.1}$$

for all Borel subsets $E \subset \Omega$. In case $\phi$ is convex and $C^2(\Omega)$, we have

$$|\partial\phi(E)| = \int_E \det D^2\phi(y)\, dy. \tag{2.2}$$

A *convex normalized domain* is a strictly convex domain $\Omega \subseteq \mathbb{R}^n$ such that $B_{\alpha_n}(0) \subseteq \Omega \subseteq B_1(0)$. If $S$ is any convex set with nonempty interior, there exists an ellipsoid $E$ such that $\alpha_n E \subseteq S \subseteq E$ and hence, there exists an affine transformation $T$ such that $B_{\alpha_n}(0) \subseteq T(S) \subseteq B_1(0)$.

A *section of the convex function* $\phi \in C^1(\Omega)$ is defined by

$$S_\phi(\bar{x}, t) = \{x \in \Omega : \phi(x) < \phi(\bar{x}) + \nabla\phi(\bar{x}) \cdot (x - \bar{x}) + t\}, \tag{2.3}$$

where $\nabla$ denotes the standard gradient.

Given $0 < \alpha < 1$ we define the sections at the minimum to be the sets

$$\Omega_\alpha = \{x \in \Omega : \phi(x) < (1 - \alpha)\min_\Omega \phi\}$$

The next five results about sections hold under the assumption that $\phi \in C^2(\Omega)$ is a convex function such that $\lambda \leq \det(D^2\phi) \leq \Lambda$ and $\phi = 0$ on $\partial\Omega$ where $\Omega$ is a convex normalized domain. These results can be found in Chapters 3 and 5 of [**G01**].

**Lemma 2.1** *For each $0 < \alpha < 1$ there exists $\eta$ depending only on $\alpha$, $\lambda$ and $\Lambda$ such that for all $x \in \Omega_\alpha$ and for all $t \leq \eta$, we have $S_\phi(x,t) \Subset \Omega$.*

Lemma 2.1 is crucial in two ways for our result. First, because as we mentioned in the introduction, it allows for sections to take the role of Euclidean balls and certain very important sets are defined using sections. Second, because the exponent $p$ appearing in our main theorem depends on the constant $\eta$ of this lemma.

It is also important to remark that Lemma 2.1 holds under weaker assumptions on the function $\phi$ but certain regularity of $\phi$ is required on $\partial\Omega$ as examples of Pogorelov show; see [**G01**, Chapter 5]. We will only use this lemma as stated.

The following four lemmas hold for sections which a priori are known to be compactly contained in $\Omega$.

**Lemma 2.2 (Engulfing property)** *There exists $\theta$ such that if $x \in S_\phi(y,t)$ then $S_\phi(y,t) \subseteq S_\phi(x,\theta t)$.*

The following Lemma is used for an iteration argument in Chapters 5 and 6.

**Lemma 2.3** *Suppose $\alpha < \beta$ and $x \in \Omega_\alpha$ then $S_\phi(x, C_0(\beta-\alpha)^\gamma) \subseteq \Omega_\beta$ for some $C_0$ and $\gamma$ depending only on $\lambda$ and $\Lambda$.*

**Lemma 2.4** *There exists constants $C_1$ and $C_2$ depending only on $n, \lambda$ and $\Lambda$ such that*

$$C_1\, t^{n/2} \leq |S_\phi(x,t)| \leq C_2\, t^{n/2}.$$

Lemma 2.4 says that the measure of any section depends essentially on the parameter $t$ and is comparable to the measure of an Euclidean ball of radius

$\sqrt{t}$. However, a section may look like an ellipsoid in which the ratio between the longest axes and the shortest axes goes to infinity as the parameter $t$ goes to 0. In other words, the eccentricity of a section is not bounded by constants depending only on $\lambda, \Lambda$, and $n$.

The following lemma is a simple consequence of Lemma 2.4:

**Lemma 2.5** *Given $0 < \alpha < 1$ there exists $0 < \epsilon < 1$ depending only on $\lambda$,$\Lambda$ and $n$ such that $(1-\epsilon)\mu(S_\phi(x,t)) \leq \mu(S_\phi(x,\alpha t))$ for any section $S_\phi(x,t) \Subset \Omega$.*

We will also need the following observation about normalization. Let $S_\phi(\bar{x},t)$ be a section and let $T$ normalize $S_\phi(\bar{x},t)$. Let $\tilde{\phi}(y) = \frac{1}{t}\phi(T^{-1}y)$. We then have $T(S_\phi(\bar{x},t)) = S_{\tilde{\phi}}(T\bar{x},1)$ and $\det D^2\tilde{\phi}(y) = \dfrac{1}{t|\det T|^{\frac{2}{n}}}\det D^2\phi(T^{-1}y)$. Since $B_{\alpha_n}(0) \subseteq T(S_\phi(\bar{x},t)) \subseteq B_1(0)$ ,it follows that $\dfrac{C_1}{t} \leq |\det T|^{\frac{2}{n}} \leq \dfrac{C_2}{t}$ where $C_1$ and $C_2$ depend on $\lambda$,$\Lambda$ and $n$. Hence, $\lambda' \leq \det D^2\tilde{\phi}(y) \leq \Lambda'$ on $S_{\tilde{\phi}}(T\bar{x},1)$. Letting $\psi(y) = \tilde{\phi}(y) - \tilde{\phi}(\bar{y}) - \nabla\tilde{\phi}(\bar{y})\cdot(y-\bar{y}) - 1$ where $T\bar{x} = \bar{y}$ we have $\psi = 0$ on $\partial S_{\tilde{\phi}}(\bar{y},1)$ and $\lambda' \leq \det D^2\psi \leq \Lambda'$. In particular, Lemma 2.1 holds for $\psi$.

Next, we state a covering lemma for sections, which is in Chapter 6 of [**G01**].

**Lemma 2.6** *Let $A \subseteq \Omega$ and suppose that for each $x \in A$ a section $S_\phi(x,t) \Subset \Omega$ is given such that $t$ is bounded by a fixed number $\eta$. Then, there exists a countable subfamily $\{S_\phi(x_k,t_k)\}_k$ with the following properties:*

1. *$A \subseteq \bigcup_k S_\phi(x_k,t_k)$, and*
2. *for $0 < \epsilon < \epsilon_0(n,\lambda,\Lambda)$ we have that the family $\{S_\phi(x_k,(1-\epsilon)t_k)\}_k$ has bounded overlaps, more precisely,*

$$\sum_k \chi_{S_\phi(x_k,(1-\epsilon)t_k)}(x) \leq C \log\frac{1}{\epsilon}$$

*where $C$ depends also on $n$,$\lambda$,and $\Lambda$ and $\chi_E$ denotes the characteristic function of the set $E$.*

The next lemma is of fundamental importance in setting up an iteration argument, and it complements the previous lemma. Its proof is in [**CG97**].

**Lemma 2.7** *Let $\mathcal{O} \subset \Omega$ and $0 < \epsilon < 1$. Suppose that for each $x \in \mathcal{O}$, there exists a section $S_\phi(x, t_x) \subset \Omega$ with $t_x \leq \beta$ and*

$$\frac{\mu(S_\phi(x, t_x) \cap \mathcal{O})}{\mu(S_\phi(x, t_x))} = \epsilon.$$

*Then, there exists a countable subfamily of sections $\{S_\phi(x_k, t_k)\}_k$ and $0 < \delta(\epsilon) < 1$ such that $\mathcal{O} \subseteq \bigcup_k S_\phi(x_k, t_k)$ and $\mu(\mathcal{O}) \leq \delta(\epsilon)\, \mu(\bigcup_k S_\phi(x_k, t_k))$.*

We now define and state a theorem for the maximal function that will be used in Chapter 6.

**Definition 2.1** *For $x \in \Omega_{\alpha_0}$, define*

$$\mathcal{M}_\mu(f)(x) = \sup_{t \leq \eta_0/2} \frac{1}{\mu(S_\phi(x, t))} \int_{S_\phi(x,t)} |f(y)| d\mu(y).$$

**Theorem 2.1** *There exists a constant $C$ depending on $\lambda$ and $\Lambda$ such that*

$$\mu(\{x \in \Omega_{\alpha_0} : \mathcal{M}_\mu(f)(x) > \beta\}) \leq \frac{C}{\beta} \int_\Omega |f(y)| d\mu(y).$$

PROOF: Let $A_\beta = \{x \in \Omega_{\alpha_0} : M_\mu(f)(x) > \beta\}$. For each $x \in A_\beta$, there exists $t_x \leq \eta_0/2$ such that

$$\frac{1}{\mu(S_\phi(x, t_x))} \int_{S_\phi(x,t_x)} |f(y)| d\mu(y) \geq \beta.$$

Consider the family $S_\phi(x, 2t_x))$. By Lemma 2.6, there exists a countable subfamily $\{S_\phi(x_k, 2t_k)\}_k$ such that $A_\beta \subseteq \bigcup S_\phi(x_k, 2t_k)$ and such that $\sum_k \chi_{s_\phi(x_k,(1-\epsilon)2t_k)}(x) \leq C \log \frac{1}{\epsilon}$ for every $\epsilon \leq \epsilon_0$.

We have

$$\mu(A_\beta) \leq \sum_k \mu(S_\phi(x_k, 2t_k)) \leq C \sum_k \mu(S_\phi(x_k, (1-\epsilon)2t_k)).$$

Also,

$$\begin{aligned}\beta &\leq \frac{1}{\mu(S_\phi(x_k,t_k))}\int_{S_\phi(x_k,t_k)}|f(y)|\,d\mu(y)\\ &\leq C\frac{1}{\mu(S_\phi(x_k,(1-\epsilon)2t_k))}\int_{S_\phi(x_k,(1-\epsilon)2t_k)}|f(y)|\,d\mu(y).\end{aligned}$$

Therefore,

$$\beta\,\mu(S_\phi(x_k,(1-\epsilon)2t_k)) \leq C\int_{S_\phi(x_k,(1-\epsilon)2t_k)}|f(y)|\,d\mu(y).$$

Hence,

$$\begin{aligned}\mu(A_\beta) &\leq \frac{1}{\beta}C\sum_k \beta\mu(S_\phi(x_k,(1-\epsilon)2t_k))\\ &\leq \frac{C}{\beta}\sum_k\int_{S_\phi(x_k,(1-\epsilon)2t_k)}|f(y)|\,d\mu(y)\\ &= \frac{C}{\beta}\sum_k\int_\Omega \chi_{S_\phi(x_k,(1-\epsilon)2t_k)}(y)|f(y)|\,d\mu(y)\\ &= \frac{C}{\beta}\int_\Omega\sum_k \chi_{S_\phi(x_k,(1-\epsilon)2t_k)}(y)|f(y)|\,d\mu(y)\\ &\leq \frac{C\,\log\frac{1}{\epsilon}}{\beta}\int_\Omega|f(y)|\,d\mu(y).\end{aligned}$$

□

# CHAPTER 3

# Main definitions

In this chapter we define the quasidistance given by the sections of $\phi$ and using this quasidistance we define the sets where the solution $u$ is touched from above and below by certain polynomials. We also prove in this chapter a lemma for solutions to the Monge–Ampère equation which is the second step mentioned in the introduction.

First, we begin with a simple lemma that is used repeatedly in our calculations.

**Lemma 3.1** *Let $\Omega$ be a bounded domain, $\phi \in C^2(\Omega)$ convex with $\det D^2\phi(x) > 0$ for $x \in \Omega$; and $u \in C^2(\Omega)$. Let $w = u + \phi$, $\Gamma(w)$ the convex envelope of $w$, and the contact set $\mathcal{C} = \{x \in \Omega : w(x) = \Gamma(w)(x)\}$. Suppose $E \subset \mathcal{C}$ and $F \subset \Omega$ be measurable sets such that $\partial\phi(F) \subset \partial w(E)$.*

*Then*

$$M\phi(F) \leq \frac{1}{n^n} \int_E \left( \left( \frac{L_\phi u(x)}{\det D^2\phi(x)} + n \right)^+ \right)^n \det D^2\phi(x)\, dx. \qquad (3.1)$$

PROOF: We have

$$\begin{aligned} M\phi(F) \le Mw(E) &\le \int_E \det D^2 w(x)\,dx \\ &= \int_E \frac{\det \Phi(x) \det D^2 w(x)}{\det \Phi(x)}\,dx \\ &\le \frac{1}{n^n}\int_E \frac{1}{\det \Phi(x)} \left(\text{trace}\left(\Phi(x) D^2 w(x)\right)\right)^n dx \\ &= \frac{1}{n^n}\int_E \frac{1}{(\det D^2\phi(x))^{n-1}} \left(\text{trace}\left(\Phi(x)(D^2 u(x) + D^2\phi(x))\right)\right)^n dx \\ &= \frac{1}{n^n}\int_E \frac{1}{(\det D^2\phi(x))^{n-1}} \left(L_\phi u(x) + n \det D^2\phi(x)\right)^n dx, \end{aligned}$$

and (3.1) follows. □

## 3.1 A quasi-metric

Let $\Omega$ be a convex normalized domain in $\mathbb{R}^n$, $\phi \in C^2(\Omega)$ convex, $\phi = 0$ on $\partial\Omega$, $0 < \lambda \le \det D^2\phi \le \Lambda$ in $\Omega$.

**Definition 3.1** *Let $S_\phi(x_0, \eta_0) \Subset \Omega$. Given $x \in S_\phi(x_0, \eta_0)$ we define*

$$d(x, x_0)^2 = \inf\{t : x \in S_\phi(x_0, t)\}. \tag{3.2}$$

**Remark 3.1** If $\phi \in C^1(\Omega)$, then $d(x, x_0)^2 = \phi(x) - \phi(x_0) - \nabla\phi(x_0)\cdot(x - x_0)$.

**Lemma 3.2** *The function $d$ in Definition 3.1 satisfies:*

*(a) $d(x, y)^2 \le \theta\,(d(x, z)^2 + d(y, z)^2)$ whenever are all defined.*

*(b) If $S_\phi(\bar{x}, \eta_0) \Subset \Omega$, then $d(x, \bar{x})^2$ is a convex function of $x$ on the set $S_\phi(\bar{x}, \eta_0)$.*

PROOF: (a) Let $x \in S_\phi(z, t_1), y \in S_\phi(z, t_2)$ and $t = \max\{t_1, t_2\}$. Since $y \in S_\phi(z, t)$, we have by the engulfing property $S_\phi(z, t) \subset S_\phi(y, \theta t)$; and $x \in S_\phi(z, t_1) \subset S_\phi(z, t) \subset S_\phi(y, \theta t)$. Hence $d(x, y)^2 \le \theta t \le \theta(t_1 + t_2)$, and so

$$\begin{aligned} d(x, y)^2 &\le \theta \inf\{t_1 + t_2 : x \in S_\phi(z, t_1), y \in S_\phi(z, t_2)\} \\ &= \theta\left(\inf\{t_1 : x \in S_\phi(z, t_1)\} + \inf\{t_2 : y \in S_\phi(z, t_2)\}\right) \\ &= \theta\left(d(x, z)^2 + d(y, z)^2\right). \end{aligned}$$

Notice that if $x, y \in S_\phi(z, \delta)$ and $\delta\theta \leq \eta_0$, then $d(x,y)^2$ is defined because $x \in S_\phi(y, \theta\delta)$.

(b) Let $x_1, x_2 \in S_\phi(\bar{x}, \eta_0)$ and $0 < \lambda < 1$. We have

$$d(\lambda x_1 + (1-\lambda)x_2, \bar{x})^2 = \inf\{t : \lambda x_1 + (1-\lambda)x_2 \in S_\phi(\bar{x}, t)\}.$$

Suppose $x_1 \in S_\phi(\bar{x}, t_1)$ and $x_2 \in S_\phi(\bar{x}, t_2)$, and let $\psi(x) = \phi(x) - \phi(\bar{x}) - \nabla\phi(\bar{x}) \cdot (x - \bar{x})$. The function $\psi$ is convex and so

$$\psi(\lambda x_1 + (1-\lambda)x_2) \leq \lambda\psi(x_1) + (1-\lambda)\psi(x_2) < \lambda t_1 + (1-\lambda)t_2.$$

Then $\lambda x_1 + (1-\lambda)x_2 \in S_\phi(\bar{x}, \lambda t_1 + (1-\lambda)t_2)$ and

$$\inf\{t : \lambda x_1 + (1-\lambda)x_2 \in S_\phi(\bar{x}, t)\} \leq \lambda t_1 + (1-\lambda)t_2.$$

Since $t_1$ and $t_2$ are arbitrary, we get

$$\begin{aligned} &\inf\{t : \lambda x_1 + (1-\lambda)x_2 \in S_\phi(\bar{x}, t)\} \\ &\leq \lambda \inf\{t : x_1 \in S_\phi(\bar{x}, t)\} + (1-\lambda)\inf\{t : x_2 \in S_\phi(\bar{x}, t)\}; \end{aligned}$$

that is,

$$d(\lambda x_1 + (1-\lambda)x_2, \bar{x})^2 \leq \lambda\, d(x_1, \bar{x})^2 + (1-\lambda)\, d(x_2, \bar{x})^2.$$

□

## 3.2 Spaces

*For the following definitions we fix $0 < \alpha_0 < 1$ once for all.*

**Definition 3.2** *Given $u \in C^2(\Omega)$, $0 < \alpha_0 < 1$, $M > 1$, $\theta$ the engulfing constant, and $\theta^2/M \leq \eta(\alpha_0)$, we define*

$$\begin{aligned} G_M^+(u) = \{\bar{x} \in \Omega_{\alpha_0} : u(x) \leq u(\bar{x}) + \nabla u(\bar{x}) \cdot (x - \bar{x}) + M\, d(x, \bar{x})^2, \\ \forall x \in S_\phi(\bar{x}, \theta^2/M)\}; \\ G_M^-(u) = \{\bar{x} \in \Omega_{\alpha_0} : u(x) \geq u(\bar{x}) + \nabla u(\bar{x}) \cdot (x - \bar{x}) - M\, d(x, \bar{x})^2, \\ \forall x \in S_\phi(\bar{x}, \theta^2/M)\}; \end{aligned}$$

*and*

$$G_M(u) = G_M^+(u) \cap G_M^-(u).$$

## 3.3 A Lemma for the Monge–Ampère equation

**Lemma 3.3** *Let $\Omega$ be a normalized convex domain, $\phi \in C^2(\Omega)$ with $\phi = 0$ on $\partial\Omega$, and $0 \le \det D^2\phi \le \Lambda$ in $\Omega$. Let $0 < \alpha_0 < \alpha_1 < 1$ and*

$$\mathcal{G}^+_M(\phi) = \{\bar{x} \in \Omega_{\alpha_0} : \phi(x) \le \phi(\bar{x}) + \nabla\phi(\bar{x}) \cdot (x - \bar{x}) + M\,|x - \bar{x}|^2, \quad \forall x \in \Omega_{\alpha_1}\}.$$

*Then*

$$\left|\Omega_{\alpha_0} \setminus \mathcal{G}^+_{M(t)}(\phi)\right| \le \left(1 - \frac{1}{(t+1)^n}\right) |\Omega_{\alpha_0}|,$$

*where $M(t) = \left(\frac{t+1}{t}\right) \Lambda^{1/n} C(\alpha_1)$, and for all $0 < t < 1$.*

PROOF: Let $w$ be the solution to $\det D^2 w = 1$ in $\Omega$ with $w = 0$ on $\partial\Omega$. It follows from Pogorelov's estimate, [**G01**, inequality (4.2.6)], that

$$\mathcal{G}^+_{C(\alpha_1)}(w) = \Omega_{\alpha_0}.$$

We have that $\det D^2(\Lambda^{1/n} w) = \Lambda \ge \det D^2\phi$. Then by the comparison principle, [**G01**, Theorem 1.4.6], $\Lambda^{1/n} w \le \phi$ in $\Omega$, and so $-w/2 \ge -\phi/(2\Lambda^{1/n}) \ge 0$. Consequently, $\frac{w}{2} \ge w - \frac{\phi}{2\Lambda^{1/n}} \ge w$ in $\Omega$.

Let $v_t(x) = t\left(w(x) - \frac{\phi(x)}{2\Lambda^{1/n}}\right) + (1-t)\frac{w(x)}{2}$. Then

$$w \le w - \frac{\phi}{2\Lambda^{1/n}} \le v_t \le \frac{w}{2},$$

for $0 < t < 1$. Let $\Gamma(v_t)$ be the convex envelope of $v_t$ in $\Omega$, and $C_t$ be the set of contact points, $C_t = \{x \in \Omega : \Gamma(v_t)(x) = v_t(x)\}$. Then $w \le \Gamma(v_t) \le v_t \le w/2$, and so $\nabla(w/2)(\Omega) \subset \partial\Gamma(v_t)(\Omega)$, by [**G01**, Lemma 1.4.1]. Therefore

$$\begin{aligned}
|\nabla(w/2)(\Omega)| \le |\partial\Gamma(v_t)(\Omega)| &= |\partial\Gamma(v_t)(C_t)| \qquad \text{by Lemma 7.1} \\
&\le \int_{C_t} \left( \left( \left( M\left(\frac{t+1}{2} w\right)\right)^{1/n} - \left( M\left(\frac{t}{2\Lambda^{1/n}}\phi\right)\right)^{1/n} \right)^+ \right)^n dx \\
&\le \int_{C_t} \left( \frac{t+1}{2} - \frac{t}{2\Lambda^{1/n}} (M\phi)^{1/n} \right)^n dx \\
&\le |C_t| \left(\frac{t+1}{2}\right)^n,
\end{aligned}$$

by Theorem 7.1 since $v_t(x) = \left(\frac{t+1}{2}\right) w(x) - t\frac{\phi(x)}{2\Lambda^{1/n}}$. Hence we obtain

$$\frac{1}{(t+1)^n}|\Omega| \leq |\mathcal{C}_t|.$$

This implies $|\Omega_{\alpha_0} \setminus \mathcal{C}_t| \leq \left(1 - \frac{1}{(t+1)^n}\right)|\Omega_{\alpha_0}|$, and since $\mathcal{C}_t \cap \Omega_{\alpha_0} \subset \Omega_{\alpha_0} \cap \mathcal{G}^+_{M(t)}(\phi)$, we obtain the desired estimate. □

Notice that if $0 < \alpha_0 < \alpha_1 < 1$, then $\operatorname{dist}(\Omega_{\alpha_0}, \partial\Omega_{\alpha_1}) \geq C(\alpha_0, \alpha_1)$. Hence if $\bar{x} \in \Omega_{\alpha_0}$, then the Euclidean ball $B_{\sqrt{t}/\beta}(\bar{x}) \subset \Omega_{\alpha_1}$ for $t \leq (\beta C(\alpha_0, \alpha_1))^2$.

**Lemma 3.4** *Let*

$$A_\beta = \{x \in \Omega_{\alpha_0} : B_{\sqrt{t}/\beta}(x) \subset S_\phi(x,t), \quad \forall t \leq \eta_0\}, \tag{3.3}$$

*where* $\eta_0 \leq \min\{\eta(\alpha_0), (\beta C(\alpha_0, \alpha_1))^2\}$. *Then we have*

$$\mathcal{G}^+_{\beta^2}(\phi) \subset A_\beta.$$

PROOF: Let $\bar{x} \in \mathcal{G}^+_{\beta^2}(\phi)$, and let $x \in B_{\sqrt{t}/\beta}(\bar{x})$. We have $\phi(x) \leq \phi(\bar{x}) + \nabla\phi(\bar{x}) \cdot (x - \bar{x}) + \beta^2|x - \bar{x}|^2$ for each $x \in \Omega_{\alpha_1}$. By the choice of $\eta_0$, we have $B_{\sqrt{t}/\beta}(\bar{x}) \subset \Omega_{\alpha_1}$. Then $\phi(x) < \phi(\bar{x}) + \nabla\phi(\bar{x}) \cdot (x - \bar{x}) + t$, that is, $x \in S_\phi(\bar{x}, t)$ and so $\bar{x} \in A_\beta$. □

**Remark 3.2** From Lemma 3.4, $\Omega_{\alpha_0} \setminus A_\beta \subset \Omega_{\alpha_0} \setminus \mathcal{G}^+_{\beta^2}(\phi)$, and therefore

$$\mu(\Omega_{\alpha_0} \setminus A_\beta) \leq \mu(\Omega_{\alpha_0} \setminus \mathcal{G}^+_{\beta^2}(\phi)) \leq \Lambda|\Omega_{\alpha_0} \setminus \mathcal{G}^+_{\beta^2}(\phi)| \leq \frac{C(n, \Lambda, \alpha_0)}{\beta^2}, \tag{3.4}$$

by Lemma 3.3.

**Remark 3.3** If $\bar{x} \in A_\beta$, then $d(x, \bar{x})^2 \leq \beta^2|x - \bar{x}|^2$ for all $x \in S_\phi(\bar{x}, \eta_0)$.

Indeed, we have $B_{\sqrt{t}/\beta}(\bar{x}) \subset S_\phi(\bar{x}, t)$ for all $t \leq \eta_0$. If $x \in S_\phi(\bar{x}, \eta_0)$, then

$$\{t : x \in B_{\sqrt{t}/\beta}(\bar{x})\} \subset \{t : x \in S_\phi(\bar{x}, t)\},$$

and so, $\inf\{t : x \in S_\phi(\bar{x}, t)\} \leq \inf\{t : x \in B_{\sqrt{t}/\beta}(\bar{x})\}$. Consequently, $d(x, \bar{x})^2 \leq \beta^2|x - \bar{x}|^2$.

**Remark 3.4** We have

$$A_\beta \cap G_\beta(u) \subset \{x \in \Omega_{\alpha_0} : |D_{ij}u(x)| \leq \beta^3, \text{ for } i,j = 1, \cdots, n\},$$

for each $\beta$ such that $\theta^2/\beta \leq \eta(\alpha_0)$.

In fact, if $\bar{x} \in A_\beta \cap G_\beta(u)$, then

$$-\beta\, d(x,\bar{x})^2 \leq u(x) - u(\bar{x}) - \nabla u(\bar{x}) \cdot (x - \bar{x}) \leq \beta\, d(x,\bar{x})^2$$

for all $x \in S_\phi(\bar{x}, \theta^2/\beta)$. Since $\bar{x} \in A_\beta$, by the previous remark $d(x,\bar{x})^2 \leq \beta^2|x - \bar{x}|^2$ we get

$$-\beta^3|x - \bar{x}|^2 \leq u(x) - u(\bar{x}) - \nabla u(\bar{x}) \cdot (x - \bar{x}) \leq \beta^3|x - \bar{x}|^2$$

for each $x \in S_\phi(\bar{x}, \theta^2/\beta)$, and so $|D_{ij}u(\bar{x})| \leq \beta^3$.

If $0 < \alpha \leq \alpha_0$, then from Remark 3.4 we have that

$$\begin{aligned} \{x \in \Omega_\alpha : |D_{ij}u(x)| > \beta^3\} &= \Omega_\alpha \setminus \{x \in \Omega_\alpha : |D_{ij}u(x)| \leq \beta^3\} \\ &\subset \Omega_\alpha \setminus (A_\beta \cap G_\beta(u)), \qquad \text{where } \theta^2/\beta \leq \eta(\alpha_0) \\ &\subset (\Omega_\alpha \setminus A_\beta) \cup (\Omega_\alpha \setminus G_\beta(u)). \end{aligned} \tag{3.5}$$

# CHAPTER 4

# Density Lemmas

In this chapter we prove two density lemmas that will be used in Chapters 5 and 6 to prove the decay of $\mu(\Omega_\alpha \setminus G_\beta(u))$ for $\beta$ large.

**Lemma 4.1** *Let $\Omega$ be a normalized convex domain, $\phi \in C^2(\Omega)$, $\phi = 0$ on $\partial\Omega$, and $\lambda \leq \det D^2\phi \leq \Lambda$. Suppose $u \in C^2(\Omega)$, $0 \leq u \leq 1$, and $L_\phi u = 0$ in $\Omega$.*

*Then for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that if $M \geq \dfrac{1}{\delta(\epsilon)}$, then we have*

$$\mu(G^-_{M/t_0}(u) \cap S_\phi(x_0, t_0)) > (1-\epsilon)\,\mu(S_\phi(x_0, t_0)), \tag{4.1}$$

*for all sections $S_\phi(x_0, t_0) \subset \Omega_{\alpha_0}$ with $t_0 \leq \eta_0/\theta$.*

PROOF: Let $T$ normalize the section $S_\phi(x_0, t_0)$. Let

$$\tilde{\phi}(y) = \frac{1}{t_0}\phi(T^{-1}y), \qquad y_0 = Tx_0,$$
$$\psi(y) = \tilde{\phi}(y) - \tilde{\phi}(y_0) - \nabla\tilde{\phi}(y_0)\cdot(y - y_0) - 1,$$
$$\text{and} \qquad \tilde{u}(y) = u(T^{-1}y).$$

We have $T(S_\phi(x_0, t_0)) = S_{\tilde{\phi}}(y_0, 1) = \tilde{\Omega}$ is normalized, $\psi = 0$ on $\partial\tilde{\Omega}$, and $\lambda' \leq \det D^2\psi \leq \Lambda'$ in $\tilde{\Omega}$ by the observation after lemma 2.5. For each $0 < \alpha < 1$ there exists $\eta(\alpha)$ such that if $\bar{y} \in \tilde{\Omega}_\alpha$, then $S_\psi(\bar{y}, \eta(\alpha)) \subset\subset \tilde{\Omega}$ by Lemma 2.1. Therefore if $\bar{y} \in \tilde{\Omega}_\alpha$, then $\psi(\bar{y}) + \nabla\psi(\bar{y})\cdot(y - \bar{y}) + \eta(\alpha) < 0$ for all $y \in \overline{\tilde{\Omega}}$.

Let $w_\alpha(y) = \eta(\alpha)\tilde{u}(y) + \psi(y)$, and let $\gamma_\alpha$ be the convex envelope of $w_\alpha$ in $\tilde{\Omega}$, and

$$\mathcal{C}_\alpha = \{y \in \tilde{\Omega} : w_\alpha(y) = \gamma_\alpha(y),$$
$$\text{and } \exists\ \ell_y \text{ supporting hyperplane to } \gamma_\alpha \text{ at } y \text{ with } \ell_y < 0 \text{ in } \tilde{\Omega}\}.$$

**Claim 1.** $\nabla\psi(\tilde{\Omega}_\alpha) \subset \nabla w_\alpha(\mathcal{C}_\alpha)$.

To prove this claim, let $\bar{y} \in \tilde{\Omega}_\alpha$ and

$$\begin{aligned}\delta &= \max\{\psi(\bar{y}) + \nabla\psi(\bar{y}) \cdot (y - \bar{y}) + \eta(\alpha) - w_\alpha(y) : y \in \overline{\tilde{\Omega}}\} \\ &= \psi(\bar{y}) + \nabla\psi(\bar{y}) \cdot (y^* - \bar{y}) + \eta(\alpha) - w_\alpha(y^*),\end{aligned}$$

for some $y^* \in \overline{\tilde{\Omega}}$. Notice that $y^* \in \tilde{\Omega}$. Because since $\bar{y} \in \overline{\tilde{\Omega}}$, we have $\delta \geq \psi(\bar{y}) + \eta(\alpha) - w_\alpha(\bar{y}) = \eta(\alpha)(1 - \tilde{u}(\bar{y})) \geq 0$. And if it were $y^* \in \partial\tilde{\Omega}$, then we would have $\delta < -w_\alpha(y^*) \leq 0$, which is contradictory. Therefore

$$\psi(\bar{y}) + \nabla\psi(\bar{y}) \cdot (y^* - \bar{y}) + \eta(\alpha) - w_\alpha(y^*) \geq \psi(\bar{y}) + \nabla\psi(\bar{y}) \cdot (y - \bar{y}) + \eta(\alpha) - w_\alpha(y)$$

for all $y \in \tilde{\Omega}$, that is,

$$w_\alpha(y) \geq w_\alpha(y^*) + \nabla\psi(\bar{y}) \cdot (y - y^*),$$

for all $y \in \tilde{\Omega}$. Since $w_\alpha(y^*) + \nabla\psi(\bar{y}) \cdot (y - y^*)$ is convex as a function of $y$, we have that

$$w_\alpha(y) \geq \gamma_\alpha(y) \geq w_\alpha(y^*) + \nabla\psi(\bar{y}) \cdot (y - y^*),$$

and so $w_\alpha(y^*) = \gamma_\alpha(y^*)$. Also $w_\alpha(y^*) + \nabla\psi(\bar{y}) \cdot (y - y^*) = \psi(\bar{y}) + \nabla\psi(\bar{y}) \cdot (y - \bar{y}) + \eta(\alpha) - \delta < -\delta \leq 0$ in $\tilde{\Omega}$, and therefore $\nabla\psi(\bar{y}) \in \nabla w_\alpha(\mathcal{C}_\alpha)$. This proves claim 1.

We have $D^2\tilde{\phi}(y) = \frac{1}{t_0}\,(T^{-1})^t\, D^2\phi(T^{-1}y)\, T^{-1}$, so

$$(D^2\tilde{\phi}(y))^{-1} = t_0\, T\, (D^2\phi(T^{-1}y))^{-1}\, T^t.$$

Also $D^2\tilde{u}(y) = (T^{-1})^t\, D^2u(T^{-1}y)\, T^{-1}$. If $\tilde{\Phi}(y)$ is the matrix of cofactors of $D^2\tilde{\phi}(y)$, i.e., $\tilde{\Phi}(y) = (D^2\tilde{\phi}(y))^{-1} \det D^2\tilde{\phi}(y)$, then

$$L_{\tilde{\phi}}\tilde{u}(y) = \operatorname{trace}(\tilde{\Phi}(y)\, D^2\tilde{u}(y)) = 0$$

in $\tilde{\Omega}$. Applying Lemma 3.1 with $u \rightsquigarrow \eta(\alpha)\tilde{u}$, $\phi \rightsquigarrow \tilde{\psi}$, $E = \mathcal{C}_\alpha$, $F = \tilde{\Omega}_\alpha$, we get

$$\int_{\tilde{\Omega}_\alpha} \det D^2\tilde{\phi}(y)\,dy \leq \int_{\mathcal{C}_\alpha} \det D^2\tilde{\phi}(y)\,dy.$$

We also have

$$\det D^2\tilde{\phi}(y) = \frac{1}{t_0^n\,|\det T|^2} \det D^2\phi(T^{-1}y),$$

and hence

$$\int_{\tilde{\Omega}_\alpha} \det D^2\phi(T^{-1}y)\,dy \leq \int_{\mathcal{C}_\alpha} \det D^2\phi(T^{-1}y)\,dy.$$

We have $\tilde{\Omega}_\alpha = T(S_\phi(x_0, \alpha t_0))$ and for now suppose

$$\mathcal{C}_\alpha \subset T(A),$$

*where $A$ is a set that will be determined.* Then

$$\int_{S_\phi(x_0,\alpha t_0)} \det D^2\phi(x)\,dx \leq \int_A \det D^2\phi(x)\,dx,$$

that is,

$$\mu(S_\phi(x_0, \alpha t_0)) \leq \mu(A). \tag{4.2}$$

**Claim 2.** $\mathcal{C}_\alpha \subset T\left(S_\phi(x_0, t_0) \cap G^-_{1/(\eta(\alpha)t_0)}(u)\right)$, for all $\alpha$ sufficiently close to one.

Proof of Claim 2. Let $\bar{y} \in \mathcal{C}_\alpha$. Then $\bar{y} = T\bar{x}$ for some $\bar{x} \in S_\phi(x_0, t_0)$ and

$$\eta(\alpha)\tilde{u}(y) + \psi(y) \geq \ell(y), \text{ for all } y \in T(S_\phi(x_0, t_0)) \tag{4.3}$$

with equality at $y = \bar{y}$, for some $\ell$ affine with $\ell < 0$ in $T(S_\phi(x_0, t_0))$. We now claim that

$$\phi(x) \leq \phi(\bar{x}) + \nabla\phi(\bar{x}) \cdot (x - \bar{x}) + d(x, \bar{x})^2, \tag{4.4}$$

for all $x \in S_\phi(\bar{x}, \eta_0)$. Indeed, let $x \in S_\phi(\bar{x}, t)$. Then $\phi(x) - (\phi(\bar{x}) + \nabla\phi(\bar{x}) \cdot (x - \bar{x})) \leq t$, and hence

$$\phi(x) - (\phi(\bar{x}) + \nabla\phi(\bar{x}) \cdot (x - \bar{x})) \leq \inf\{t : x \in S_\phi(\bar{x}, t)\} = d(x, \bar{x})^2.$$

This proves (4.4).

Recall that $\tilde{\phi}(y) = \frac{1}{t_0}\phi(T^{-1}y)$ where $T$ normalizes $S_\phi(x_0, t_0)$. Also $S_\phi(x_0, t_0) \subset \Omega_{\alpha_0}$ and $\bar{x} \in S_\phi(x_0, t_0)$, so $\bar{x} \in \Omega_{\alpha_0}$, and $S_\phi(\bar{x}, \eta_0) \Subset \Omega$. And from (4.4), if $x \in S_\phi(\bar{x}, \eta_0)$, then $\phi(x) \leq \phi(\bar{x}) + \nabla\phi(\bar{x}) \cdot (x - \bar{x}) + d(x, \bar{x})^2$. Consequently,

$$\tilde{\phi}(y) \leq \tilde{\phi}(\bar{y}) + \nabla\tilde{\phi}(\bar{y}) \cdot (y - \bar{y}) + \frac{1}{t_0} d(T^{-1}y, T^{-1}\bar{y})^2, \tag{4.5}$$

for all $y \in T(S_\phi(\bar{x}, \eta_0))$. On the other hand, by definition of $\tilde{\phi}$ and $\psi$, the inequality (4.5) is equivalent to

$$\psi(y) \leq \psi(\bar{y}) + \nabla\psi(\bar{y}) \cdot (y - \bar{y}) + \frac{1}{t_0} d(T^{-1}y, T^{-1}\bar{y})^2 = \tilde{\ell}_{\bar{y}}(y) + \frac{1}{t_0} d(T^{-1}y, T^{-1}\bar{y})^2$$

for all $y \in T(S_\phi(\bar{x}, \eta_0))$, with equality at $\bar{y} = T\bar{x}$. Notice that by engulfing and since $t_0 \leq \eta_0/\theta$, we have $S_\phi(x_0, t_0) \subset S_\phi(\bar{x}, \theta t_0) \subset S_\phi(\bar{x}, \eta_0)$, and hence by (4.3) we get

$$\eta(\alpha)\tilde{u}(y) \geq \ell(y) - \tilde{\ell}_{\bar{y}}(y) - \frac{1}{t_0} d(T^{-1}y, T^{-1}\bar{y})^2,$$

for all $y \in T(S_\phi(x_0, t_0))$ with equality for $y = \bar{y}$. Let

$$g(y) = \ell(y) - \tilde{\ell}_{\bar{y}}(y) - \frac{1}{t_0} d(T^{-1}y, T^{-1}\bar{y})^2,$$

for $y \in T(S_\phi(\bar{x}, \eta_0))$, and

$$B = \{y \in T(S_\phi(\bar{x}, \eta_0)) : g(y) \geq 0\}.$$

If $y \in \partial T(S_\phi(x_0, t_0))$ then $\ell(y) < 0$ and so $g(y) < -\tilde{\ell}_{\bar{y}}(y) - \frac{1}{t_0} d(T^{-1}y, T^{-1}\bar{y})^2 \leq -\psi(y) = 0$, and therefore $B \cap \partial T(S_\phi(x_0, t_0)) = \emptyset$, and since, by Lemma 3.2(b), $g$ is concave, it follows that $B$ is connected. Hence $B \subset T(S_\phi(x_0, t_0))$. This implies that

$$\eta(\alpha)\,\tilde{u}(y) \geq g(y), \tag{4.6}$$

for each $y \in T(S_\phi(\bar{x}, \eta_0))$ with equality at $y = \bar{y}$. Because we have that $\eta(\alpha)\tilde{u}(y) \geq g(y)$ for each $y \in T(S_\phi(x_0, t_0))$, and if $y \in T(S_\phi(\bar{x}, \eta_0)) \backslash T(S_\phi(x_0, t_0))$, then $y \notin B$ and so $g(y) \leq 0 \leq \eta(\alpha)\tilde{u}(y)$, that is, (4.6) follows. Since $\ell$ is a supporting hyperplane to $\eta(\alpha)\,\tilde{u}(y) + \psi(y)$ at $\bar{y}$, we then get

$$\tilde{u}(y) \geq \tilde{u}(\bar{y}) + \nabla\tilde{u}(\bar{y}) \cdot (y - \bar{y}) - \frac{1}{t_0\eta(\alpha)} d(T^{-1}y, T^{-1}\bar{y})^2,$$

for all $y \in T(S_\phi(\bar{x}, \eta_0))$.

Hence we have proved that for each $0 < \alpha < 1$ we have

$$C_\alpha \subset \{\bar{y} \in T(S_\phi(x_0, t_0)) : \tilde{u}(y) \geq \tilde{u}(\bar{y}) + \nabla\tilde{u}(\bar{y}) \cdot (y - \bar{y}) - \frac{1}{t_0\eta(\alpha)} d(T^{-1}y, T^{-1}\bar{y})^2, \forall y \in T(S_\phi(\bar{x}, \eta_0)), \text{where } \bar{y} = T\bar{x}\}.$$

Since $\tilde{u}(y) = u(T^{-1}y)$, it follows that

$$C_\alpha \subset T\left(S_\phi(x_0, t_0) \cap G^-_{1/t_0\eta(\alpha)}(u)\right),$$

as long as $t_0\,\eta(\alpha)\,\theta^2 \leq \eta_0$ (as required to define the set $G$ with the definition 3.2) and for $t_0\,\theta \leq \eta_0$. Notice that since $\eta(\alpha) \to 0$ as $\alpha \to 1$, the condition $t_0\,\eta(\alpha)\,\theta^2 \leq \eta_0$ is always satisfied for $\alpha$ close to one. This completes the proof of claim 2.

To complete the proof of the lemma, we notice that from Lemma 2.5, given $\epsilon > 0$ there exists $\alpha = \alpha(\epsilon)$ sufficiently close to one such that

$$(1 - \epsilon)\mu(S_\phi(x_0, t_0)) \leq \mu(S_\phi(x_0, \alpha t_0)),$$

for all sections $S_\phi(x_0, t_0)$ compactly contained in $\Omega$. This inequality combined with (4.2) and Claim 2 yields the desired result with $M \geq \frac{1}{\delta(\epsilon)} = \frac{1}{\eta(\alpha(\epsilon))}$. □

**Lemma 4.2** *Let $\Omega$ be a normalized convex domain, $\phi \in C^2(\Omega)$, $\phi = 0$ on $\partial\Omega$, and $\lambda \leq \det D^2\phi \leq \Lambda$. Suppose $u \in C^2(\Omega)$, and $L_\phi u = 0$ in $\Omega$.*

*Then for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that if $M \geq \frac{1}{\delta(\epsilon)}$ and $S_\phi(x_0, t_0) \cap G_\lambda(u) \neq \emptyset$, then we have*

$$\mu(G^-_{\lambda M}(u) \cap S_\phi(x_0, t_0)) > (1 - \epsilon)\,\mu(S_\phi(x_0, t_0)), \tag{4.7}$$

*where $t_0 < 1/\lambda$, $S_\phi(x_0, t_0) \subset \Omega_{\alpha_0}$, and $\theta^3/\lambda \leq \eta_0$.*

PROOF: Let $S_\phi(x_0,t_0) \subset \Omega_{\alpha_0}$, $t_0 < \frac{1}{\lambda} \leq \frac{\eta_0}{\theta^3}$, and let $\bar{x} \in S_\phi(x_0,t_0) \cap G_\lambda(u)$. Let $T$ normalize $S_\phi(x_0,t_0)$, and set

$$\tilde{\phi}(y) = \frac{1}{t_0}\phi(T^{-1}y), \qquad y_0 = Tx_0,$$
$$\psi(y) = \tilde{\phi}(y) - \tilde{\phi}(y_0) - \nabla\tilde{\phi}(y_0)\cdot(y-y_0) - 1,$$
$$\text{and} \qquad \tilde{u}(y) = \frac{1}{t_0}u(T^{-1}y).$$

We have as in Lemma 4.1 that $T(S_\phi(x_0,t_0)) = S_{\tilde{\phi}}(y_0,1) = \tilde{\Omega}$ is normalized, $\psi = 0$ on $\partial\tilde{\Omega}$, and $\lambda' \leq \det D^2\psi \leq \Lambda'$ in $\tilde{\Omega}$.

Let $\bar{y} = T\bar{x}$. Since $\bar{x} \in S_\phi(x_0,t_0) \cap G_\lambda(u)$, we have

$$-\lambda\, d(x,\bar{x})^2 \leq u(x) - u(\bar{x}) - \nabla u(\bar{x})\cdot(x-\bar{x}) \leq \lambda\, d(x,\bar{x})^2,$$

for all $x \in S_\phi(\bar{x},\theta^2/\lambda)$. Hence changing variables we get

$$-\lambda\frac{d(T^{-1}y,T^{-1}\bar{y})^2}{t_0} \leq \tilde{u}(y)-\tilde{u}(\bar{y})-\nabla\tilde{u}(\bar{y})\cdot(y-\bar{y}) \leq \lambda\frac{d(T^{-1}y,T^{-1}\bar{y})^2}{t_0}, \quad (4.8)$$

for all $y \in T(S_\phi(\bar{x},\theta^2/\lambda))$.

Since $\bar{x} \in S_\phi(x_0,t_0)$, we have $S_\phi(x_0,t_0) \subset S_\phi(\bar{x},\theta t_0)$ by the engulfing property. So, if $x \in S_\phi(x_0,t_0)$, then $d(x,\bar{x})^2 \leq \theta t_0$, and consequently $d(T^{-1}y,T^{-1}\bar{y})^2 \leq \theta t_0$ for all $y \in T(S_\phi(x_0,t_0))$. Also since $t_0 < 1/\lambda$, we have $S_\phi(x_0,t_0) \subset S_\phi(\bar{x},\theta^2/\lambda)$. Then

$$-\lambda\theta \leq \tilde{u}(y) - \tilde{u}(\bar{y}) - \nabla\tilde{u}(\bar{y})\cdot(y-\bar{y}) \leq \lambda\theta,$$

and if we set

$$v(y) = \frac{\tilde{u}(y) - \tilde{u}(\bar{y}) - \nabla\tilde{u}(\bar{y})\cdot(y-\bar{y}) + \lambda\theta}{2\lambda\theta},$$

then $0 \leq v \leq 1$ in $\tilde{\Omega}$.

Let $0 < \alpha < 1$. There exists[1] $\eta(\alpha) > 0$ such that if $\bar{y} \in \tilde{\Omega}_\alpha$, then

$$\psi(\bar{y}) + \nabla\psi(\bar{y})\cdot(y-\bar{y}) + \eta(\alpha)(\theta+1/2) < 0,$$

[1]Notice this $\eta(\alpha)$ is the $\eta(\alpha)$ from lemma 2.1 divided by $\theta + \frac{1}{2}$

for all $y \in \overline{\tilde{\Omega}}$.

Define

$$w_\alpha(y) = \eta(\alpha)v(y) + \psi(y).$$

We have $\psi \le w_\alpha \le \eta(\alpha) + \psi$ in $\tilde{\Omega}$. Let $\gamma_\alpha$ be the convex envelope of $w_\alpha$ in $\tilde{\Omega}$, and

$$\begin{aligned} \mathcal{C}_\alpha = \{\bar{y} \in \tilde{\Omega} : w_\alpha(\bar{y}) = \gamma_\alpha(\bar{y}), \\ \text{and } \exists\ \ell_{\bar{y}} \text{ supporting hyperplane to } \gamma_\alpha \text{ at } \bar{y}, \\ \text{with } \ell_{\bar{y}} < -\eta(\alpha)(\theta - 1/2) \text{ in } \tilde{\Omega}\}. \end{aligned}$$

**Claim 1.** $\nabla\psi(\tilde{\Omega}_\alpha) \subset \nabla w_\alpha(\mathcal{C}_\alpha)$.

Proof of Claim 2. Let $\bar{y} \in \tilde{\Omega}_\alpha$, and

$$\delta = \max\{\psi(\bar{y}) + \nabla\psi(\bar{y}) \cdot (y - \bar{y}) + \eta(\alpha) - w_\alpha(y) : y \in \overline{\tilde{\Omega}}\}.$$

We have $\delta \ge \psi(\bar{y}) - w_\alpha(\bar{y}) = \eta(\alpha)(1 - v(\bar{y})) \ge 0$, and if $y \in \partial\tilde{\Omega}$, then we have

$$\psi(\bar{y}) + \nabla\psi(\bar{y}) \cdot (y - \bar{y}) + \eta(\alpha) - w_\alpha(y) < 0.$$

Therefore there exists $y^* \in \tilde{\Omega}$ such that

$$\begin{aligned} \delta &= \psi(\bar{y}) + \nabla\psi(\bar{y}) \cdot (y^* - \bar{y}) + \eta(\alpha) - w_\alpha(y^*) \\ &\ge \psi(\bar{y}) + \nabla\psi(\bar{y}) \cdot (y - \bar{y}) + \eta(\alpha) - w_\alpha(y), \end{aligned}$$

for all $y \in \tilde{\Omega}$. Hence

$$w_\alpha(y) \ge w_\alpha(y^*) + \nabla\psi(\bar{y}) \cdot (y - y^*),$$

for all $y \in \tilde{\Omega}$. Also notice that

$$\begin{aligned} w_\alpha(y^*) + \nabla\psi(\bar{y}) \cdot (y - y^*) &= \psi(\bar{y}) + \nabla\psi(\bar{y}) \cdot (y - \bar{y}) + \eta(\alpha) - \delta \\ &\le \psi(\bar{y}) + \nabla\psi(\bar{y}) \cdot (y - \bar{y}) + \eta(\alpha) < -\eta(\alpha)(\theta - 1/2), \end{aligned}$$

for all $y \in \tilde{\Omega}$. This proves Claim 1.

Proceeding as in Lemma 4.1, we have

$$\int_{\tilde{\Omega}_\alpha} \det D^2\phi(T^{-1}y)\,dy \le \int_{\mathcal{C}_\alpha} \det D^2\phi(T^{-1}y)\,dy.$$

We have $\tilde{\Omega}_\alpha = T(S_\phi(x_0, \alpha t_0))$. If we assume that

$$\mathcal{C}_\alpha \subset T(A), \tag{4.9}$$

where $A$ is a set to be determined, then we have

$$\int_{S_\phi(x_0,\alpha t_0)} \det D^2\phi(x)\,dx \le \int_A \det D^2\phi(x)\,dx,$$

that is,

$$\mu(S_\phi(x_0, \alpha t_0)) \le \mu(A). \tag{4.10}$$

We next determine $A$.

**Claim 2.** The inequality (4.9) holds with $A = S_\phi(x_0, t_0) \cap G^-_{\lambda M}(u)$, for $M$ sufficiently large, $M \ge \dfrac{2\theta}{\eta(\alpha)}$.

Proof of Claim 2. Let $\tilde{y} \in \mathcal{C}_\alpha$. There exists $\ell$ affine such that $\eta(\alpha)v(y) + \psi(y) \ge \ell(y)$ for all $y \in T(S_\phi(x_0, t_0))$, and with equality at $\tilde{y}$, and $\ell < -\eta(\alpha)(\theta - 1/2)$ in $T(S_\phi(x_0, t_0))$. Here $\tilde{y} = T\tilde{x}$ for $\tilde{x} \in S_\phi(x_0, t_0)$, $S_\phi(\tilde{x}, \eta_0) \Subset \Omega$, and we have as in Lemma 4.1 (see proof of (4.5)) that

$$\psi(y) \le \tilde{\ell}_{\tilde{y}}(y) + \frac{1}{t_0} d(T^{-1}y, T^{-1}\tilde{y})^2,$$

for all $y \in T(S_\phi(\tilde{x}, \eta_0))$ with equality at $\tilde{y}$, and where $\tilde{\ell}_{\tilde{y}}$ is affine given by

$$\tilde{\ell}_{\tilde{y}}(y) = \psi(\tilde{y}) + \nabla\psi(\tilde{y}) \cdot (y - \tilde{y}).$$

Therefore

$$\eta(\alpha)v(y) \ge \ell(y) - \tilde{\ell}_{\tilde{y}}(y) - \frac{1}{t_0} d(T^{-1}y, T^{-1}\tilde{y})^2, \tag{4.11}$$

for each $y \in T(S_\phi(x_0, t_0))$, and with equality at $y = \tilde{y}$, because $S_\phi(x_0, t_0) \subset S_\phi(\tilde{x}, \theta t_0) \subset S_\phi(\tilde{x}, \eta_0)$ for $\theta t_0 < \eta_0$.

Define

$$g(y) = \ell(y) - \tilde{\ell}_{\tilde{y}}(y) - \frac{1}{t_0} d(T^{-1}y, T^{-1}\tilde{y})^2.$$

We have from (4.8)

$$g(\tilde{y}) = \eta(\alpha)v(\tilde{y}) \geq \eta(\alpha) \left( \frac{\lambda\theta - \frac{\lambda d(T^{-1}\tilde{y}, T^{-1}\bar{y})^2}{t_0}}{2\lambda\theta} \right).$$

*Our goal is to extend the inequality* (4.11) *to the set* $T(S_\phi(\tilde{x}, \theta^2/(\lambda M)))$, *for some* $M$ *sufficiently large.* Notice that since $\bar{x}, \tilde{x} \in S_\phi(x_0, t_0)$ and $t_0 < 1/\lambda$, by engulfing

$$S_\phi(x_0, t_0) \subset S_\phi(\bar{x}, \theta t_0) \subset S_\phi(\bar{x}, \theta/\lambda),$$

and

$$S_\phi(x_0, t_0) \subset S_\phi(\tilde{x}, \theta t_0) \subset S_\phi(\tilde{x}, \theta/\lambda).$$

Hence $\bar{x} \in S_\phi(\tilde{x}, \theta/\lambda)$, and again by engulfing $S_\phi(\tilde{x}, \theta/\lambda) \subset S_\phi(\bar{x}, \theta^2/\lambda)$ and therefore

$$S_\phi(\tilde{x}, \theta^2/(\lambda M)) \subset S_\phi(\tilde{x}, \theta/\lambda) \subset S_\phi(\bar{x}, \theta^2/\lambda) \tag{4.12}$$

for $M \geq \theta$. Therefore to achieve our goal, it is enough to extend the inequality (4.11) to $T(S_\phi(\bar{x}, \theta^2/\lambda))$; notice that $S_\phi(\bar{x}, \theta^2/\lambda) \Subset \Omega$.

Let

$$B = \{y \in T(S_\phi(\bar{x}, \theta^2/\lambda)) : g(y) \geq \eta(\alpha) \left( \frac{\lambda\theta - \frac{\lambda d(T^{-1}y, T^{-1}\bar{y})^2}{t_0}}{2\lambda\theta} \right)\}.$$

We claim that

$$B \subset T(S_\phi(x_0, t_0)). \tag{4.13}$$

We first prove that

$$B \subset \{y \in T(S_\phi(\bar{x}, \theta^2/\lambda)) :$$
$$0 \geq -\ell(y) + \tilde{\ell}_{\tilde{y}}(y) + \left(\frac{1}{t_0}\left(1 - \frac{\eta(\alpha)}{2}\right)\right) d(T^{-1}y, T^{-1}\tilde{y})^2 + \frac{\eta(\alpha)}{2}(1-\theta)\}$$
$$= \tilde{B}. \tag{4.14}$$

If $y \in B$, then

$$0 \geq -\ell(y) + \tilde{\ell}_{\tilde{y}}(y) + \frac{1}{t_0} d(T^{-1}y, T^{-1}\tilde{y})^2 - \frac{\eta(\alpha)}{2\theta}\frac{1}{t_0} d(T^{-1}y, T^{-1}\bar{y})^2 + \frac{\eta(\alpha)}{2}.$$

On the other hand, from Lemma 3.2(a)

$$\frac{\eta(\alpha)}{2\theta}\frac{d(T^{-1}y,T^{-1}\bar{y})^2}{t_0} \le \frac{\eta(\alpha)}{2}\left(\frac{d(T^{-1}y,T^{-1}\tilde{y})^2}{t_0}+\frac{d(T^{-1}\bar{y},T^{-1}\tilde{y})^2}{t_0}\right)$$

Since $d(T^{-1}\bar{y},T^{-1}\tilde{y})^2 \le \theta t_0$, we get that $y \in \tilde{B}$, and (4.14) follows.

Since $\eta(\alpha)/2 < 1$, we have that the function

$$-\ell(y)+\tilde{\ell}_{\tilde{y}}(y)+\left(\frac{1}{t_0}\left(1-\frac{\eta(\alpha)}{2}\right)\right)d(T^{-1}y,T^{-1}\tilde{y})^2+\frac{\eta(\alpha)}{2}(1-\theta)$$

is convex and hence $\tilde{B}$ is connected. Notice that we have $S_\phi(\bar{x},\theta^2/\lambda) \subset S_\phi(\tilde{x},\theta^3/\lambda) \subset S_\phi(\tilde{x},\eta_0)$ for $\theta^3/\lambda \le \eta_0$, and so $d(T^{-1}y,T^{-1}\tilde{y})^2$ is defined for $y \in T(S_\phi(\bar{x},\theta^2/\lambda))$.

Second, we shall prove that

$$\tilde{B} \subset T(S_\phi(x_0,t_0)). \tag{4.15}$$

We begin showing that

$$\tilde{B} \cap \partial T(S_\phi(x_0,t_0)) = \emptyset. \tag{4.16}$$

Suppose $y \in \partial T(S_\phi(x_0,t_0))$ then we claim that

$$-\ell(y)+\tilde{\ell}_{\tilde{y}}(y)+\left(\frac{1}{t_0}\left(1-\frac{\eta(\alpha)}{2}\right)\right)d(T^{-1}y,T^{-1}\tilde{y})^2+\frac{\eta(\alpha)}{2}(1-\theta) > 0. \tag{4.17}$$

Recall that $\ell < -\eta(\alpha)(\theta-1/2)$ in $\overline{T(S_\phi(x_0,t_0))}$, and $\tilde{\ell}_{\tilde{y}}(y)+\frac{1}{t_0}d(T^{-1}y,T^{-1}\tilde{y})^2 \ge \psi(y) = 0$. Since $S_\phi(x_0,t_0) \subset S_\phi(\tilde{x},\theta t_0)$, we have $d(T^{-1}y,T^{-1}\tilde{y})^2 \le \theta t_0$. Hence (4.17) follows, and therefore (4.16) holds. Since $\tilde{y} \in \tilde{B} \cap T(S_\phi(x_0,t_0))$ and $\tilde{B}$ is connected, we get (4.15).

Combining (4.14) with (4.15), we get (4.13) as desired.

Recall that $\eta(\alpha)v(y) \ge g(y)$ in $T(S_\phi(x_0,t_0))$ by (4.11), and if

$$y \in T(S_\phi(\bar{x},\theta^2/\lambda)) \setminus T(S_\phi(x_0,t_0)),$$

then $y \notin B$ by (4.13), that is,

$$g(y) < \eta(\alpha)\left(\frac{\lambda\theta-\frac{\lambda d(T^{-1}y,T^{-1}\bar{y})^2}{t_0}}{2\lambda\theta}\right).$$

On the other hand, by definition of $G_\lambda(u)$ we have from (4.8) that

$$\tilde{u}(y) - \tilde{u}(\bar{y}) - \nabla\tilde{u}(\bar{y}) \cdot (y - \bar{y}) \geq -\lambda \frac{d(T^{-1}y, T^{-1}\bar{y})^2}{t_0},$$

for all $y \in T(S_\phi(\bar{x}, \theta^2/\lambda))$, and hence

$$\frac{\eta(\alpha)}{2\lambda\theta}\left(\lambda\theta - \frac{d(T^{-1}y, T^{-1}\bar{y})^2}{t_0}\right) \leq \eta(\alpha)v(y),$$

so $g(y) < \eta(\alpha)v(y)$. Therefore, we have that $g(y) \leq \eta(\alpha)v(y)$ for all $y \in T(S_\phi(\bar{x}, \theta^2/\lambda))$ with equality at $y = \tilde{y}$. In particular, by (4.12) we have that $g(y) \leq \eta(\alpha)v(y)$ for all $y \in T(S_\phi(\tilde{x}, \theta^2/(\lambda M)))$ because $S_\phi(\tilde{x}, \theta^2/(\lambda M)) \subset S_\phi(\bar{x}, \theta^2/\lambda)$ with equality at $\tilde{y} = T\tilde{x}$, for $M \geq \theta$.

*Therefore* (4.11) *holds for all* $y \in T(S_\phi(\tilde{x}, \theta^2/(\lambda M)))$.

Notice that from the definitions of $\ell$ and $\tilde{\ell}_{\tilde{y}}$ we have $g(y) = a + p \cdot (y - \tilde{y}) - \frac{1}{t_0} d(T^{-1}y, T^{-1}\tilde{y})^2$ with $a = \eta(\alpha)v(\tilde{y})$ and $p = \eta(\alpha)\nabla v(\tilde{y})$. By definition of $v$ we then get that

$$\tilde{u}(y) \geq \tilde{u}(\tilde{y}) + \nabla\tilde{u}(\tilde{y}) \cdot (y - \tilde{y}) - \frac{2\lambda\theta}{t_0\eta(\alpha)} d(T^{-1}y, T^{-1}\tilde{y})^2,$$

for all $y \in T(S_\phi(\tilde{x}, \theta^2/(\lambda M))$ for all $M \geq \theta$. Therefore we have proved that

$$\begin{aligned}
&C_\alpha \\
&\subset \{\tilde{y} \in T(S_\phi(x_0, t_0)) : \tilde{u}(y) \geq \tilde{u}(\tilde{y}) + \nabla\tilde{u}(\tilde{y}) \cdot (y - \tilde{y}) - \frac{2\lambda\theta}{t_0\eta(\alpha)} d(T^{-1}y, T^{-1}\tilde{y})^2 \\
&\quad \forall y \in T(S_\phi(\tilde{x}, \theta^2/(\lambda M))) \text{ where } \tilde{y} = T\tilde{x}\} = F,
\end{aligned}$$

for each $M \geq \theta$. Now

$$\begin{aligned}
F &= T\{\tilde{x} \in S_\phi(x_0, t_0) : u(x) \geq u(\tilde{x}) + \nabla u(\tilde{x}) \cdot (x - \tilde{x}) - \frac{2\lambda\theta}{\eta(\alpha)} d(x, \tilde{x})^2 \\
&\qquad \forall x \in S_\phi(\tilde{x}, \theta^2/(\lambda M)))\} \\
&= T(S_\phi(x_0, t_0) \cap G^-_{\lambda M}(u)),
\end{aligned}$$

picking $M = \frac{2\theta}{\eta(\alpha)}$, (the requirement in the definition of $G$ is satisfied since $S_\phi(x_0, t_0) \subset \Omega_{\alpha_0}$ and $\theta^2/(\lambda M) = \eta(\alpha)\theta/(2\lambda) \leq \eta(\alpha_0)$ since $\lambda$ is sufficiently large). So Claim 2 is proved.

Then from (4.10) and Claim 2 we obtain

$$\mu(S_\phi(x_0, \alpha t_0)) \leq \mu(S_\phi(x_0, t_0) \cap G^-_{\lambda M}(u)).$$

From Lemma 2.5, given $\epsilon > 0$ there exists $0 < \alpha = \alpha(\epsilon) < 1$ such that

$$(1 - \epsilon)\mu(S_\phi(x_0, t_0)) \leq \mu(S_\phi(x_0, \alpha t_0)).$$

The lemma now follows taking $\delta(\epsilon) = \eta(\alpha(\epsilon))/(2\theta)$. □

# CHAPTER 5

# The homogeneous case

Let $u \in C^2(\Omega)$ satisfying $L_\phi u = 0$ in $\Omega$ with $|u| \leq 1$ in $\Omega$. If $0 < \alpha_0 < 1$, recall that

$$\Omega_{\alpha_0} = \{x \in \Omega : \phi(x) < (1 - \alpha_0) \min_{\Omega} \phi\}.$$

Let $0 < \epsilon < 1/2$ and $M$ the largest of the constants in Lemmas 4.1 and 4.2. Applying Lemma 4.1 to the functions $\frac{u+1}{2}$ and $\frac{-u+1}{2}$, and noticing that $G_M^-((u+1)/2) \subset G_{2M}^-(u)$ and $G_M^-((-u+1)/2) \subset G_{2M}^+(u)$, we get

$$\mu(S_\phi(x_0, t_0) \cap G_{2M/t_0}^-(u)) > (1 - \epsilon)\, \mu(S_\phi(x_0, t_0)),$$

and

$$\mu(S_\phi(x_0, t_0) \cap G_{2M/t_0}^+(u)) > (1 - \epsilon)\, \mu(S_\phi(x_0, t_0)),$$

as long as $S_\phi(x_0, t_0) \subset \Omega_{\alpha_0}$ and $t_0\theta^2/M \leq \eta_0$. Now

$$\begin{aligned}
\mu(S_\phi(x_0, t_0) \setminus G_{2M/t_0}) &= \mu\left(S_\phi(x_0, t_0) \setminus \left(G_{2M/t_0}^+ \cap G_{2M/t_0}^-\right)\right) \\
&= \mu((S_\phi(x_0, t_0) \setminus G_{2M/t_0}^+) \cup (S_\phi(x_0, t_0) \setminus G_{2M/t_0}^-)) \\
&\leq \mu(S_\phi(x_0, t_0) \setminus G_{2M/t_0}^+) + \mu(S_\phi(x_0, t_0) \setminus G_{2M/t_0}^-) \\
&\leq 2\epsilon\mu(S_\phi(x_0, t_0)).
\end{aligned}$$

Let $\alpha_2 < \alpha_1 < \alpha_0$. There exists $\eta_2$ such that if $x \in \Omega_{\alpha_2}$ and $t \leq \eta_2$ then $S_\phi(x, t) \subset \Omega_{\alpha_1}$; and there exists $\eta_1$ such that if $x \in \Omega_{\alpha_1}$ and $t \leq \eta_1$ then $S_\phi(x, t) \subset \Omega_{\alpha_0}$.

Let $\theta^3/\lambda \leq \eta_2/2$ and let $x_0 \in \Omega_{\alpha_2} \setminus G_{\lambda M}(u)$.

Define

$$g(t) = \frac{\mu((\Omega_{\alpha_2} \setminus G_{\lambda M}(u)) \cap S_\phi(x_0,t))}{\mu(S_\phi(x_0,t))}.$$

We have $\lim_{t\to 0} g(t) = 1$. Also, if $2/\lambda \leq t < \eta_1$, then $S_\phi(x_0,t) \subset \Omega_{\alpha_0}$ and

$$\begin{aligned}\mu((\Omega_{\alpha_2} \setminus G_{\lambda M}(u)) \cap S_\phi(x_0,t)) &\leq \mu(S_\phi(x_0,t) \setminus G_{\lambda M}(u)) \\ &\leq \mu(S_\phi(x_0,t) \setminus G_{2M/t}(u)) \\ &\leq 2\epsilon\,\mu(S_\phi(x_0,t)),\end{aligned}$$

since $G_{2M/t}(u) \subset G_{\lambda M}(u)$. Therefore $g(t) \leq 2\epsilon$ for $t \in [2/\lambda, \eta_1)$. Then by continuity of $g$, there exists $t_{x_0} \leq 2/\lambda \leq \theta^3/\lambda \leq \eta_2/2$ such that $g(t_{x_0}) = 2\epsilon$.

Then we have proved that if $0 < \epsilon < 1/2$ and $M$ is the constant in Lemma 4.1, then for $\alpha_2 < \alpha_1 < \alpha_0$ and for any $\lambda > 0$ such that $\theta^3/\lambda \leq \eta_2/2$ and for any $x_0 \in \Omega_{\alpha_2} \setminus G_{\lambda M}(u)$ there exists $t_{x_0} \leq 2/\lambda$ such that

$$\mu((\Omega_{\alpha_2} \setminus G_{\lambda M}(u)) \cap S_\phi(x_0,t_{x_0})) = 2\epsilon\,\mu(S_\phi(x_0,t_{x_0})). \tag{5.1}$$

We now claim that (5.1) implies that

$$S_\phi(x_0,t_{x_0}) \subset \Omega_{\alpha_1} \setminus G_\lambda(u). \tag{5.2}$$

Suppose that (5.2) is false. Since $x_0 \in \Omega_{\alpha_2}$ and $t_{x_0} \leq 2/\lambda \leq \eta_2$, we have that $S_\phi(x_0,t_{x_0}) \subset \Omega_{\alpha_1}$. Therefore, if $S_\phi(x_0,t_{x_0}) \not\subseteq \Omega_{\alpha_1} \setminus G_\lambda(u)$, then $S_\phi(x_0,t_{x_0}) \cap G_\lambda(u) \neq \emptyset$ and hence from Lemma 4.2 applied to $u$ and $-u$, we have for $M$ sufficiently large

$$(1-\epsilon)\,\mu(S_\phi(x_0,t_{x_0})) < \mu(S_\phi(x_0,t_{x_0}) \cap G^-_{\lambda M}(u)),$$

and

$$(1-\epsilon)\,\mu(S_\phi(x_0,t_{x_0})) < \mu(S_\phi(x_0,t_{x_0}) \cap G^+_{\lambda M}(u));$$

notice that $G^-_{\lambda M}(-u) = G^+_{\lambda M}(u)$ and $G_\lambda(-u) = G_\lambda(u)$. Therefore

$$\mu(S_\phi(x_0,t_{x_0}) \setminus G_{\lambda M}(u)) < 2\epsilon\mu(S_\phi(x_0,t_{x_0}),$$

and so

$$\mu((\Omega_{\alpha_2} \setminus G_{\lambda M}(u)) \cap S_\phi(x_0, t_{x_0})) \leq \mu(S_\phi(x_0, t_{x_0}) \setminus G_{\lambda M}(u)) < 2\epsilon\mu(S_\phi(x_0, t_{x_0}),$$

contradicting (5.1). This proves (5.2).

Then from Lemma 2.7 we get that

$$\mu(\Omega_{\alpha_2} \setminus G_{\lambda M}(u)) \leq \delta(\epsilon)\mu(\Omega_{\alpha_1} \setminus G_\lambda(u)), \tag{5.3}$$

where $\alpha_2 < \alpha_1 < \alpha_0$, $\theta^3/\lambda \leq \eta_2/2$, and $M = M(\epsilon)$ large, and for all $\epsilon$ sufficiently small.

## 5.1 Power decay

Recalling Lemma 2.3, we can apply (5.3) with $\lambda = M$ and

$$\frac{\theta^3}{M} = \frac{C_0(\alpha_1 - \alpha_2)^\gamma}{2} = \frac{\eta_2}{2}$$

that is,

$$\alpha_2 = \alpha_1 - \left(\frac{2\theta^3}{C_0 M}\right)^{1/\gamma}.$$

Then

$$\mu(\Omega_{\alpha_2} \setminus G_{M^2}(u)) \leq \delta(\epsilon)\mu(\Omega_{\alpha_1} \setminus G_M(u)).$$

Next apply (5.3) with $\lambda = M^2$ and $\frac{\theta^3}{M} = \frac{C_0(\alpha_2 - \alpha_3)^\gamma}{2} = \frac{\eta_3}{2}$, that is,

$$\alpha_3 = \alpha_2 - \left(\frac{2\theta^3}{C_0 M^2}\right)^{1/\gamma} = \alpha_1 - \left(\frac{2\theta^3}{C_0 M}\right)^{1/\gamma} - \left(\frac{2\theta^3}{C_0 M^2}\right)^{1/\gamma}.$$

Then

$$\mu(\Omega_{\alpha_3} \setminus G_{M^3}(u)) \leq \delta(\epsilon)\mu(\Omega_{\alpha_2} \setminus G_{M^2}(u)) \leq (\delta(\epsilon))^2 \mu(\Omega_{\alpha_1} \setminus G_M(u)).$$

Continuing in this way, if

$$\alpha_k = \alpha_1 - \sum_{j=1}^{k-1} \left(\frac{2\theta^3}{C_0 M^j}\right)^{1/\gamma},$$

then

$$\mu(\Omega_{\alpha_k} \setminus G_{M^k}(u)) \leq (\delta(\epsilon))^{k-1}\, \mu(\Omega_{\alpha_1} \setminus G_M(u)).$$

Now $\sum_{j=1}^{\infty} \left(\frac{1}{M^{1/\gamma}}\right)^j = \frac{1}{M^{1/\gamma}-1}$, and so

$$\mu(\Omega_{\alpha_1 - \frac{c}{M^{1/\gamma}-1}} \setminus G_{M^k}(u)) \leq (\delta(\epsilon))^{k-1}\, \mu(\Omega_{\alpha_1} \setminus G_M(u)),$$

for $k = 1, 2, \cdots$.

We then obtain the following theorem:

**Theorem 5.1** *Let $\Omega$ be a normalized convex domain, $\phi \in C^2(\Omega)$ convex, $\phi = 0$ on $\partial\Omega$, $\lambda \leq \det D^2\phi \leq \Lambda$, $u \in C^2(\Omega)$, $|u| \leq 1$ in $\Omega$, solution to $L_\phi u = 0$ in $\Omega$. Then for each $\epsilon > 0$ and $0 < \alpha_0 < 1$ there exist a constant $M = M(\epsilon, \alpha_0, \lambda, \Lambda)$ such that for all $u$ and $\phi$ as above we have*

$$\mu(\Omega_{\alpha_0/2} \setminus G_{M^k}(u)) \leq (\delta(\epsilon))^{k-1}\, \mu(\Omega_{\alpha_0} \setminus G_M(u)),$$

*for $k = 1, 2, \cdots$.*

This statement implies the following. Let $t \geq M$ and pick $k$ such that $M^k \leq t \leq M^{k+1}$. Then $G_{M^k}(u) \subset G_t(u) \subset G_{M^{k+1}}(u)$ and $k \leq \log_M t \leq k+1$. So

$$\begin{aligned}
\mu(\Omega_{\alpha_0/2} \setminus G_t(u)) &\leq \mu(\Omega_{\alpha_0/2} \setminus G_{M^k}(u)) \\
&\leq (\delta(\epsilon))^{k-1}\, \mu(\Omega_{\alpha_0} \setminus G_M(u)) \\
&\leq (\delta(\epsilon))^{k-1}\, \mu(\Omega_{\alpha_0}) = \frac{1}{\delta(\epsilon)^2}\, (\delta(\epsilon))^{k+1}\, \mu(\Omega_{\alpha_0}) \\
&\leq \frac{1}{\delta(\epsilon)^2}\, (\delta(\epsilon))^{\log_M t}\, \mu(\Omega_{\alpha_0}) = \frac{1}{\delta(\epsilon)^2}\, t^{\frac{\ln \delta(\epsilon)}{\ln M}}\, \mu(\Omega_{\alpha_0}).
\end{aligned}$$

Thus, combining this estimate with Remark 3.2 and the inclusion (3.5), we obtain our main theorem in the homogeneous case:

**Theorem 5.2** *Let $\Omega$ be a normalized convex domain, $\phi \in C^2(\Omega)$, $\phi = 0$ on $\partial\Omega$, and $\lambda \leq \det D^2\phi \leq \Lambda$. Suppose $u \in C^2(\Omega)$, and $L_\Phi u = 0$ in $\Omega$. Then,*

*given* $0 < \alpha_0 < 1$ *there exist positive constants* $p$ *and* $C$ *depending only on* $\lambda$, $\Lambda$, *and* $\alpha_0$ *such that*

$$\|D^2 u\|_{L^p(\Omega_{\alpha_0/2})} \leq C\|u\|_{L^\infty(\Omega)}.$$

PROOF: Let $v = \dfrac{u}{\|u\|_{L^\infty(\Omega)}}$. We can apply the above estimate to $v$ to get $\|D^2 v\|_{L^p(\Omega_{\alpha_0/2}, d\mu)} \leq C$ for some $p$ and $C$ depending on $\lambda, \Lambda$, and $\alpha_0$ which implies the theorem, since in our case Lebesgue measure is comparable to the Monge–Ampère measure $\mu$. □

# CHAPTER 6

# The nonhomogeneous case

**Lemma 6.1** *Let $\Omega$ be a normalized convex domain, $\phi \in C^2(\Omega)$, $\phi = 0$ on $\partial\Omega$, and $\lambda \le \det D^2\phi \le \Lambda$. Suppose $u \in C^2(\Omega)$, $0 \le u \le 1$, and $L_\phi u = f$ in $\Omega$. Assume that $\|f / \det D^2\phi\|_{L^n(\Omega)} \le 1$.*

*Then for each $\epsilon > 0$ there exist positive constants $\eta(\epsilon, \lambda, \Lambda)$ and $C(\lambda, \Lambda, n)$ such that for each $\eta \le \eta(\epsilon, \lambda, \Lambda)$ and for each section $S_\phi(x_0, t_0) \subset \Omega_{\alpha_0}$, we have*

$$\mu(G^-_{1/(\eta t_0)}(u) \cap S_\phi(x_0, t_0)) > \Big((1-\epsilon)^{1/n} - \eta\, t_0^{1/2}\, C(\lambda, \Lambda, n)\Big)^n\, \mu(S_\phi(x_0, t_0)). \tag{6.1}$$

PROOF: We use the notation from the proof of Lemma 4.1. The proof proceeds in the same way, in particular, Claims 1 and 2 hold; the only difference is to handle the integral $\int_{\tilde\Omega_\alpha} \det D^2\tilde\phi(y)\, dy$.

Let $\tilde\Phi(y) = \det D^2\tilde\phi(y)\, (D^2\tilde\phi(y))^{-1}$. We have

$$D^2\tilde\phi(y) = \frac{1}{t_0}\, (T^{-1})^t\, D^2\phi(T^{-1}y)\, T^{-1},$$

and $D^2\tilde u(y) = (T^{-1})^t\, D^2u(T^{-1}y)\, T^{-1}$. So

$$\det D^2\tilde\phi(y) = \frac{1}{t_0^n\, |\det T|^2}\, \det D^2\phi(T^{-1}y),$$

and $(D^2\tilde\phi(y))^{-1} = t_0\, T\, (D^2\phi(T^{-1}y))^{-1}\, T^t$. Hence

$$\operatorname{trace}(\tilde\Phi(y)\, D^2\tilde u(y)) = \frac{1}{t_0^{n-1}\, |\det T|^2}\, f(T^{-1}y),$$

and

$$\det \tilde{\Phi}(y) = \left(\frac{1}{t_0^n \,|\det T|^2}\right)^{n-1} (\det D^2\phi(T^{-1}y))^{n-1}.$$

Applying Lemma 3.1 with $u \rightsquigarrow \eta(\alpha)\tilde{u}$, $\phi \rightsquigarrow \tilde{\psi}$, $E = \mathcal{C}_\alpha$, $F = \tilde{\Omega}_\alpha$, and using Minkowski's inequality we get

$$\begin{aligned}
&\left(\int_{\tilde{\Omega}_\alpha} \det D^2\tilde{\phi}(y)\,dy\right)^{1/n} \\
&\leq \frac{\eta(\alpha)}{n}\frac{1}{|\det T|^{2/n}}\left(\int_{\mathcal{C}_\alpha}\frac{|f(T^{-1}y)|^n}{(\det D^2\phi(T^{-1}y))^{n-1}}\,dy\right)^{1/n} \\
&\qquad + \left(\int_{\mathcal{C}_\alpha}\det D^2\tilde{\phi}(y)\,dy\right)^{1/n}.
\end{aligned}$$

Since by Claim 2, $\mathcal{C}_\alpha \subset T(S_\phi(x_0,t_0)\cap G^-_{1/(\eta(\alpha)t_0)}(u))$, we get

$$\begin{aligned}
&\left(\int_{T(S_\phi(x_0,\alpha\, t_0))} \det D^2\tilde{\phi}(y)\,dy\right)^{1/n} \\
&\leq \frac{\eta(\alpha)}{n}\frac{1}{|\det T|^{2/n}}\left(\int_{T(S_\phi(x_0,t_0)\cap G^-_{1/(\eta(\alpha)t_0)}(u))}\frac{|f(T^{-1}y)|^n}{(\det D^2\phi(T^{-1}y))^{n-1}}\,dy\right)^{1/n} \\
&\qquad + \left(\int_{T(S_\phi(x_0,t_0)\cap G^-_{1/(\eta(\alpha)t_0)}(u))}\det D^2\tilde{\phi}(y)\,dy\right)^{1/n}.
\end{aligned}$$

Changing variables yields

$$\begin{aligned}
&\left(\int_{S_\phi(x_0,\alpha\, t_0)} \det D^2\phi(x)\,dx\right)^{1/n} \qquad (6.2)\\
&\leq \frac{\eta(\alpha)}{n}\,t_0\left(\int_{S_\phi(x_0,t_0)\cap G^-_{1/(\eta(\alpha)t_0)}(u)}\frac{|f(x)|^n}{(\det D^2\phi(x))^{n-1}}\,dx\right)^{1/n} \\
&\qquad + \left(\int_{S_\phi(x_0,t_0)\cap G^-_{1/(\eta(\alpha)t_0)}(u)}\det D^2\phi(x)\,dx\right)^{1/n}.
\end{aligned}$$

That is,

$$\begin{aligned}
&\mu(S_\phi(x_0, \alpha\, t_0))^{1/n} \\
&\leq \frac{\eta(\alpha)}{n}\, t_0\, \|f/(\det D^2\phi)\|_{L^n(\Omega)} + \mu(S_\phi(x_0,t_0) \cap G^-_{1/(\eta(\alpha)t_0)}(u))^{1/n} \\
&\leq \frac{\eta(\alpha)}{n}\, t_0 + \mu(S_\phi(x_0,t_0) \cap G^-_{1/(\eta(\alpha)t_0)}(u))^{1/n} \\
&\leq \frac{\eta(\alpha)}{n}\, t_0^{1/2}\, C(n,\lambda,\Lambda)\, \mu(S_\phi(x_0,t_0))^{1/n} + \mu(S_\phi(x_0,t_0) \cap G^-_{1/(\eta(\alpha)t_0)}(u))^{1/n}.
\end{aligned}$$

Notice that this inequality holds for any $\eta \leq \eta(\alpha)$ Given $0 < \epsilon < 1/2$ pick $\alpha = \alpha(\epsilon)$ such that

$$(1-\epsilon)\, \mu(S_\phi(x_0,t_0)) \leq \mu(S_\phi(x_0, \alpha\, t_0)),$$

which combined with the previous inequality yields the lemma for any $\eta \leq \eta(\alpha(\epsilon))$. □

**Lemma 6.2** *Let $\Omega$ be a normalized convex domain, $\phi \in C^2(\Omega)$, $\phi = 0$ on $\partial\Omega$, and $\lambda \leq \det D^2\phi \leq \Lambda$. Suppose $u \in C^2(\Omega)$, and $L_\phi u = f$ in $\Omega$.*

*Then for each $0 < \epsilon < 1/2$ there exist positive constants $\eta(\epsilon,\lambda,\Lambda)$ and $C(\epsilon,\lambda,\Lambda)$ such that if $S_\phi(x_0,t_0) \subset \Omega_{\alpha_0}$, $t_0 < 1/\lambda < \eta_0/\theta^3$, $\bar{x} \in S_\phi(x_0,t_0) \cap G_\lambda(u)$, and $\mathcal{M}_\mu((f/\det D^2\phi))^n)(\bar{x}) \leq (2\theta n\lambda)^n$, then*

$$\mu(G^-_{2\lambda\theta/\eta}(u) \cap S_\phi(x_0,t_0)) > \left((1-\epsilon)^{1/n} - C(\epsilon,\lambda,\Lambda)\eta\right)^n \mu(S_\phi(x_0,t_0), \quad (6.3)$$

*for $\eta \leq \eta(\epsilon,\lambda,\Lambda)$.*

PROOF: We use the notation from Lemma 4.2 and notice that Claims 1

and 2 hold. As in the proof of (6.2) we get

$$
\begin{aligned}
&\mu(S_\phi(x_0,\alpha\, t_0))^{1/n}\\
&\le \frac{\eta(\alpha)}{n}\,\frac{1}{2\lambda\theta}\left(\int_{S_\phi(x_0,t_0)}\frac{f(x)^n}{(\det D^2\phi(x))^{n-1}}\,dx\right)^{1/n}\\
&\qquad+\mu(S_\phi(x_0,t_0)\cap G^-_{2\lambda\theta/\eta(\alpha)}(u))^{1/n}\\
&\le \frac{\eta(\alpha)}{n}\,\frac{1}{2\lambda\theta}\,\mu(S_\phi(\bar x,\theta\, t_0))^{1/n}\\
&\qquad\times\left(\frac{1}{\mu(S_\phi(\bar x,\theta\, t_0))}\int_{S_\phi(\bar x,\theta\, t_0)}\det D^2\phi(x)\,\frac{f(x)^n}{(\det D^2\phi(x))^n}\,dx\right)^{1/n}\\
&\qquad+\mu(S_\phi(x_0,t_0)\cap G^-_{2\lambda\theta/\eta(\alpha)}(u))^{1/n}\\
&\le \frac{\eta(\alpha)}{n}\,\frac{1}{2\lambda\theta}\,\mu(S_\phi(\bar x,\theta\, t_0))^{1/n}\,\mathcal{M}_\mu((f/\det D^2\phi)^n)(\bar x)^{1/n}\\
&\qquad+\mu(S_\phi(x_0,t_0)\cap G^-_{2\lambda\theta/\eta(\alpha)}(u))^{1/n}\\
&\le C(\lambda,\Lambda)\,\eta(\alpha)\,\mu(S_\phi(x_0,t_0))^{1/n}+\mu(S_\phi(x_0,t_0)\cap G^-_{2\lambda\theta/\eta(\alpha)}(u))^{1/n},
\end{aligned}
$$

notice that in the above inequality, $\eta(\alpha)$ can be replaced by any $\eta\le\eta(\alpha)$. Therefore picking $\alpha=\alpha(\epsilon)$ as in the proof of the previous lemma we get

$$
\mu(S_\phi(x_0,t_0))\,\left((1-\epsilon)^{1/n}-C(n,\lambda,\Lambda)\eta\right)^n\le\mu(S_\phi(x_0,t_0)\cap G^-_{2\lambda\theta/\eta}(u)),
$$

for each $\eta\le\eta(\alpha)$. □

## 6.1 Power decay for the non homogeneous case

Let $\Omega$ be a normalized convex domain, $\phi\in C^2(\Omega)$ convex, $\phi=0$ on $\partial\Omega$, $\lambda\le\det D^2\phi\le\Lambda$, $u\in C^2(\Omega)$, $|u|\le 1$ in $\Omega$, $L_\phi u=f$ in $\Omega$ and suppose that $\|f/\det D^2\phi\|_{L^n(\Omega,\mu)}\le 1$. Given $0<\epsilon<1/4$ we pick $\eta$ sufficiently small depending on $\epsilon$ such that from Lemma 6.1 we get with $M=1/\eta$ and for $S_\phi(x_0,t_0)\subset\Omega_{\alpha_0}$ that

$$
\frac{3}{4}(1-\epsilon)\,\mu(S_\phi(x_0,t_0))\le\mu(S_\phi(x_0,t_0)\cap G^-_{M/t_0}(u))
$$

and

$$\frac{3}{4}(1-\epsilon)\,\mu(S_\phi(x_0,t_0)) \le \mu(S_\phi(x_0,t_0)\cap G^+_{M/t_0}(u)).$$

Hence

$$\begin{aligned}\mu(S_\phi(x_0,t_0)\setminus G_{M/t_0}(u)) &= \mu(S_\phi(x_0,t_0)\setminus (G^+_{M/t_0}(u)\cap G^-_{M/t_0}(u)))\\ &\le \mu(S_\phi(x_0,t_0)\setminus G^+_{M/t_0}(u)) + \mu(S_\phi(x_0,t_0)\setminus G^-_{M/t_0}(u))\\ &\le (2-\frac{3}{2}(1-\epsilon))\mu(S_\phi(x_0,t_0)) = \delta(\epsilon)\,\mu(S_\phi(x_0,t_0)).\end{aligned}$$

Proceeding as in Section 5 we get that given $0<\epsilon<1/4$ there exist $M = M(\epsilon,\lambda,\Lambda)$ as in Lemmas 6.1 and 6.2 such that for $\alpha_2<\alpha_1<\alpha_0$ and for $\theta^3/\lambda \le \eta_2/2$ and for any $x_0\in\Omega_{\alpha_2}\setminus G_{\lambda M}(u)$ there exists $t_{x_0}\le 1/\lambda$ such that

$$\mu((\Omega_{\alpha_2}\setminus G_{\lambda M}(u))\cap S_\phi(x_0,t_{x_0})) = \delta(\epsilon)\,\mu(S_\phi(x_0,t_{x_0})). \tag{6.4}$$

We now claim that if $x_0\in\Omega_{\alpha_2}\setminus G_{\lambda M}(u)$, $\theta^3/\lambda\le\eta_2/2$ and (6.4) holds, then

$$S_\phi(x_0,t_{x_0}) \subset (\Omega_{\alpha_1}\setminus G_\lambda(u))\cup\{x\in\Omega_{\alpha_0} : \mathcal{M}_\mu((f/\det D^2\phi)^n)(x) > (2\theta n\lambda)^n\}. \tag{6.5}$$

Otherwise and since $x_0\in\Omega_{\alpha_2}$ and $t_{x_0}\le 1/\lambda<\eta_2$, we have that $S_\phi(x_0,t_{x_0})\subset\Omega_{\alpha_1}$ and there exists $\bar{x}\in S_\phi(x_0,t_{x_0})\cap G_\lambda(u)$ such that $\mathcal{M}_\mu((f/\det D^2\phi)^n)(\bar{x})\le(2\theta n\lambda)^n$. Then by Lemma 6.2 we get that

$$\frac{3}{4}(1-\epsilon)\,\mu(S_\phi(x_0,t_{x_0})) < \mu(S_\phi(x_0,t_{x_0})\cap G^-_{\lambda M}(u)),$$

and

$$\frac{3}{4}(1-\epsilon)\,\mu(S_\phi(x_0,t_{x_0})) < \mu(S_\phi(x_0,t_{x_0})\cap G^+_{\lambda M}(u)).$$

Hence

$$\mu((\Omega_{\alpha_2}\setminus G_{\lambda M}(u))\cap S_\phi(x_0,t_{x_0})) \le \mu(S_\phi(x_0,t_{x_0})\setminus G_{\lambda M}(u)) < \delta(\epsilon)\,\mu(S_\phi(x_0,t_{x_0})),$$

a contradiction with (6.4). So (6.5) is proved.

Hence by Lemma 2.7, there exist $0 < \tilde{\delta}(\epsilon) < 1$ and a family of sections $S_\phi(x_k, t_k)$ whose union covers $\Omega_{\alpha_2} \setminus G_{\lambda M}(u)$ satisfying (6.4) and

$$\begin{aligned}\mu(\Omega_{\alpha_2} \setminus G_{\lambda M}(u)) &\le \tilde{\delta}(\epsilon)\, \mu(\cup_{k=1}^{\infty} S_\phi(x_k, t_k)) \\ &\le \tilde{\delta}(\epsilon)\, \mu(\Omega_{\alpha_1} \setminus G_\lambda(u)) \\ &\quad + \tilde{\delta}(\epsilon)\, \mu\{x \in \Omega_{\alpha_0} : \mathcal{M}_\mu((f/\det D^2\phi)^n)(x) > (2\theta n\lambda)^n\},\end{aligned} \tag{6.6}$$

by (6.5).

We now proceed as in Subsection 5.1. Fix $\alpha_1 < \alpha_0$ and let $\lambda = M$, and set $\alpha_2 = \alpha_1 - \left(\frac{\theta^3}{C_0 M}\right)^{1/\gamma}$ where $\gamma$ and $C_0$ are the constants in Lemma 2.1. Then $\frac{\theta^3}{\lambda} = \frac{\theta^3}{M} = C_0(\alpha_1 - \alpha_0)^\gamma = \eta_2$, and so from (6.6) we get

$$\begin{aligned}\mu(\Omega_{\alpha_2} \setminus G_{M^2}(u)) &\le \tilde{\delta}(\epsilon)\, \mu(\Omega_{\alpha_1} \setminus G_M(u)) \\ &\quad + \tilde{\delta}(\epsilon)\, \mu\{x \in \Omega_{\alpha_0} : \mathcal{M}_\mu((f/\det D^2\phi)^n)(x) > (2\theta nM)^n\}.\end{aligned}$$

Next let $\lambda = M^2$ and $\alpha_3 = \alpha_2 - \left(\frac{\theta^3}{C_0 M^2}\right)^{1/\gamma}$, so $\frac{\theta^3}{\lambda} = \frac{\theta^3}{M^2} = C_0(\alpha_2 - \alpha_3)^\gamma = \eta_3$. Then

$$\begin{aligned}\mu(\Omega_{\alpha_3} \setminus G_{M^3}(u)) &\le \tilde{\delta}(\epsilon)\, \mu(\Omega_{\alpha_2} \setminus G_{M^2}(u)) \\ &\quad + \tilde{\delta}(\epsilon)\, \mu\{x \in \Omega_{\alpha_0} : \mathcal{M}_\mu((f/\det D^2\phi)^n)(x) > (2\theta nM^2)^n\} \\ &\le \tilde{\delta}(\epsilon)^2\, \mu(\Omega_{\alpha_1} \setminus G_M(u)) \\ &\quad + \tilde{\delta}(\epsilon)^2\, \mu\{x \in \Omega_{\alpha_0} : \mathcal{M}_\mu((f/\det D^2\phi)^n)(x) > (2\theta nM)^n\} \\ &\quad + \tilde{\delta}(\epsilon)\, \mu\{x \in \Omega_{\alpha_0} : \mathcal{M}_\mu((f/\det D^2\phi)^n)(x) > (2\theta nM^2)^n\}.\end{aligned}$$

Continuing in this way we let $\lambda = M^k$ and $\alpha_{k+1} = \alpha_k - \left(\frac{\theta^3}{C_0 M^k}\right)^{1/\gamma}$. Then $\frac{\theta^3}{\lambda} = \frac{\theta^3}{M^k} = C_0(\alpha_k - \alpha_{k+1})^\gamma = \eta_k$, and

$$\begin{aligned}&\mu(\Omega_{\alpha_{k+1}} \setminus G_{M^{k+1}}(u)) \\ &\le \tilde{\delta}^k\, \mu(\Omega_{\alpha_1} \setminus G_M(u)) \\ &\quad + \sum_{j=1}^{k} \tilde{\delta}^j\, \mu\{x \in \Omega_{\alpha_0} : \mathcal{M}_\mu((f/\det D^2\phi)^n)(x) > C\,(M^n)^{k+1-j}\}.\end{aligned}$$

Since $\mathcal{M}_\mu$ is of weak type 1–1, Theorem 2.1, we get

$$\mu\{x \in \Omega_{\alpha_0} : \mathcal{M}_\mu((f/\det D^2\phi)^n)(x) > C\,(M^n)^{k+1-j}\}$$
$$\leq \frac{C_n}{(M^n)^{k+1-j}} \int_\Omega \left|\frac{f}{\det D^2\phi}\right|^n d\mu(x) \leq \frac{C_n}{(M^n)^{k+1-j}}.$$

Set $m_0 = \max\{\tilde{\delta}, M^{-n}\}$. Then we have

$$\sum_{j=1}^{k} \tilde{\delta}^j\, \mu\{x \in \Omega_{\alpha_0} : \mathcal{M}_\mu((f/\det D^2\phi)^n)(x) > C\,(M^n)^{k+1-j}\}$$
$$\leq C_n \sum_{j=1}^{k} m_0^{k+1} = C_n\, k\, m_0^{k+1}.$$

Hence

$$\mu(\Omega_{\alpha_{k+1}} \setminus G_{M^{k+1}}(u)) \leq m_0^k\, \mu(\Omega_{\alpha_1} \setminus G_M(u)) + C_n\, k\, m_0^{k+1} \leq C\, m_0^{k+1}\,(1+k),$$

where the constant $C$ depends on $\mu(\Omega)$. Writing $m_1 = \sqrt{m_0}$ and since $m_0 < 1$, we have that $m_0^{k+1}\,(1+k) \leq \tilde{C}\, m_1^{k+1}$. So

$$\mu(\Omega_{\alpha_{k+1}} \setminus G_{M^{k+1}}(u)) \leq \tilde{C}\, m_1^{k+1}.$$

On the other hand, $\alpha_{k+1} = \alpha_1 - \sum_{j=1}^{k} \left(\frac{\theta^3}{C_0 M^j}\right)^{1/\gamma} \geq \alpha_1 - \sum_{j=1}^{\infty} \left(\frac{\theta^3}{C_0 M^j}\right)^{1/\gamma} =$ $\alpha_1 - \left(\frac{\theta^3}{C_0}\right)^{1/\gamma} \frac{1}{M^{1/\gamma} - 1} \geq \alpha_0/2$, for $M$ sufficiently large depending on $\alpha_1$, and therefore

$$\mu(\Omega_{\alpha_0/2} \setminus G_{M^{k+1}}(u)) \leq \tilde{C}\, m_1^{k+1},$$

for $k = 1, 2, \cdots$.

If $t \geq M$, pick $k$ such that $M^{k+1} \leq t < M^{k+2}$, then $k+1 \leq \log_M t < k+2$ and

$$\mu(\Omega_{\alpha_0/2} \setminus G_t(u)) \leq \mu(\Omega_{\alpha_0/2} \setminus G_{M^{k+1}}(u)) \leq \tilde{C}\, m_1^{k+1} \leq \frac{C}{m_1}\, t^{\log_M m_1}.$$

Combining this inequality with Remark 3.2 and (3.5) we get the estimate

$$\mu(\{x \in \Omega_{\alpha_0/2} : |D_{ij}u(x)| > \beta\}) \leq \frac{C}{\beta^\tau}, \tag{6.7}$$

for some $\tau > 0$ and for all $\beta$ large. Therefore, we obtain the main theorem in the nonhomogeneous case:

**Theorem 6.1** *Let $\Omega$ be a normalized convex domain, $\phi \in C^2(\Omega)$ convex, $\phi = 0$ on $\partial\Omega$, $\lambda \leq \det D^2\phi \leq \Lambda$, $u \in C^2(\Omega)$, and $L_\phi u = f$ in $\Omega$. Then, given $0 < \alpha_0 < 1$ there exist positive constants $p$ and $C$ depending only on $\lambda, \lambda$ and $\alpha_0$ such that*

$$\|D^2 u\|_{L^p(\Omega_{\alpha_0/2})} \leq C(\|u\|_{L^\infty(\Omega)} + \|f\|_{L^n(\Omega)}).$$

PROOF: Let $v = \dfrac{u}{\|u\|_{L^\infty(\Omega)} + \|\frac{f}{\det D^2\phi}\|_{L^n(\Omega,d\mu)}}$. Applying (6.7) to $v$, we obtain $\|D^2 v\|_{L^p(\Omega_{\alpha_0/2},d\mu)} \leq C$ for some $p$ and $C$ depending on $\lambda, \Lambda$, and $\alpha_0$. This implies that

$$\|D^2 u\|_{L^p(\Omega_{\alpha_0/2},d\mu)} \leq C\left(\|u\|_{L^\infty(\Omega)} + \|f\|_{L^n(\Omega,d\mu)}\right).$$

The theorem then follows since by assumption Lebesgue measure is comparable to the Monge–Ampère $\mu$. □

## 6.2 Remark

Suppose $\lambda \leq M\phi = f \leq \Lambda$ in $\Omega$ with $\phi = 0$ on $\partial\Omega$. If $f \in C^\alpha(\Omega)$, then from [C, Theorem 2] we have that $\phi \in C^{2,\alpha}(\Omega)$. Then $\lambda' Id \leq D^2\phi(x) \leq \Lambda' Id$ for all $x \in \Omega' \Subset \Omega$ and with $\lambda'$ and $\Lambda'$ positive constants depending only on $n, \lambda, \Lambda, \mathrm{dist}(\Omega', \partial\Omega)$ and the $C^\alpha$-norm of $f$. Suppose $L_\phi u = 0$ in $\Omega$. Then by classical Schauder's estimates $u \in C^{2,\alpha}(\Omega'')$ for $\Omega'' \Subset \Omega'$, in particular, $\|D^2 u\|_{L^p(\Omega'')} \leq C$, for all $p > 0$ where the constant $C$ depends on $n, \lambda, \Lambda, \mathrm{dist}(\Omega'', \partial\Omega')$ and the $C^\alpha$-norm of $f$.

# CHAPTER 7

# A convexity inequality

Let $u, f \in C(\Omega)$, $f \geq 0$, we say that $Mu \leq f$ in the viscosity sense if $u$ is a viscosity supersolution to $Mu = f$ in $\Omega$.

**Theorem 7.1** *Let $u, v \in C(\bar{\Omega})$ convex, with $Mu \leq 1+\epsilon$ in the viscosity sense, and $v$ generalized solution to $Mv = \alpha$ in $\Omega$, with $\alpha$ a positive constant. Let $\Gamma(u-v)$ be the convex envelope of $u - v$ in $\Omega$. Then*

$$|\partial\Gamma(u-v)(E)| \leq \int_{\mathcal{C}\cap E} \left(\left((1+\epsilon)^{1/n} - \alpha^{1/n}\right)^+\right)^n dx, \tag{7.1}$$

*for each Borel set $E \subset \Omega$, where $\mathcal{C} = \{x \in \Omega : \Gamma(u-v)(x) = (u-v)(x)\}$. If $v \in C^2(\Omega)$ is convex then*

$$|\partial\Gamma(u-v)(E)| \leq \int_{\mathcal{C}\cap E} \left(\left((1+\epsilon)^{1/n} - \det D^2 v(x)^{1/n}\right)^+\right)^n dx. \tag{7.2}$$

We use the following lemma.

**Lemma 7.1** *If $u \in C(\bar{\Omega})$, $\Gamma(u)$ is the convex envelope of $u$ in $\Omega$, and $\mu$ is the Monge–Ampère measure associated with $\Gamma(u)$, then $\mu$ has support in the contact set $\mathcal{C} = \{x \in \Omega : \Gamma(u)(x) = u(x)\}$.*

PROOF: Let $B \subset \Omega$ be a Borel set. We have $\mu(B) = \mu(B \cap \mathcal{C}) + \mu(B \cap (\Omega \setminus \mathcal{C}))$. If $p \in \partial\Gamma(u)(\Omega \setminus \mathcal{C})$, then there exists $x_0 \in \Omega \setminus \mathcal{C}$ with $p \in \partial\Gamma(u)(x_0)$, and so $\ell(x) = \Gamma(u)(x_0) + p \cdot (x - x_0)$ is a supporting hyperplane to $\Gamma(u)$ at

$x_0$ in $\Omega$. By [**G01**, Lemma 6.6.2] there exist at most $n+1$ points $x_i \in \mathcal{C}$ such that $x_0 = \sum_{i=1}^{n+1} \lambda_i x_i$ with $\lambda_i \geq 0$, $\sum_{i=1}^{n+1} \lambda_i = 1$, and $\Gamma(u)(x_i) = u(x_i) = \ell(x_i)$. So $\Gamma(u)(z) = \ell(z)$ for each $z$ in the simplex generated by $\{x_i\}_{i=1}^{n+1}$, and in particular for some $z \neq x_0$; that is, $p \in \partial\Gamma(u)(z)$. Then by Aleksandrov [**G01**, Lemma 1.1.12], $|\partial\Gamma(u)(\Omega \setminus \mathcal{C})| = 0$. □

PROOF: [of Theorem 7.1] We proceed in a sequence of steps.

**Step 1.** Let $B = B_R(y)$ be a ball with $\bar{B} \subset \Omega$, and $u \in C(\Omega)$. There exists a sequence $g_k \in C^\infty(\partial B)$ such that $|g_k(x) - u(x)| \leq 1/k$ for each $x \in \partial B$. Let $u_k$ be the convex solution to $\det D^2 u_k = 1 + \epsilon$ in $B$, $u_k = g_k$ in $\partial B$, and $u_k \in C^\infty(B) \cap C(\bar{B})$. A variant of [**G01**, Lemma 1.6.1] gives that there exist a subsequence of $u_k$, also denoted by $u_k$, and $u_\infty \in C(\bar{B})$ convex such that $u_k \to u_\infty$ uniformly on compact subsets of $B$, with $Mu_\infty = 1 + \epsilon$ in the generalized sense in $B$ and $u_\infty = u$ on $\partial B$. Since $\det D^2 u_k = Mu$ in the generalized sense, by the comparison principle [**G01**, Theorem 1.4.6] we have that

$$\min_B(u_k - u_\infty) = \min_{\partial B}(g_k - u), \quad \text{and} \quad \max_B(u_k - u_\infty) = \max_{\partial B}(g_k - u).$$

Therefore $|u_k - u_\infty| \leq 1/k$ in $\bar{B}$, and so $u_k$ converges uniformly to $u_\infty$ in $\bar{B}$.

**Step 1'.** Let $v$ be a convex generalized solution to $Mv = \alpha$ in $\Omega$. There exist $h_k \in C^\infty(\partial B)$ such that $|h_k(x) - v(x)| \leq 1/k$ in $\partial B$. Let $v_k$ be the solution to $\det D^2 v_k = \alpha$ in $B$ with $v_k = h_k$ on $\partial B$. We have that $v_k \in C^2(B)$. Since $Mv = \det D^2 v_k = \alpha$ in $B$, by the comparison principle $|v_k(x) - v(x)| \leq 1/k$ for all $x \in B$. Therefore $v_k \to v$ uniformly in $\bar{B}$.

**Step 2.** If $Mu \leq 1 + \epsilon$ in the viscosity sense in a ball $B = B_R(y)$ with $u \in C(\bar{B})$, then $u_\infty \leq u$ in $\bar{B}$. Let $\delta > 0$ and $\phi(x) = \delta\,|x - y|^2$. Expanding the determinant we obtain

$$\det D^2(u_k + \phi) \geq \det D^2 u_k + (2\,\delta)^n > 1 + \epsilon \geq Mu.$$

Then from Lemma 1.7.2 we get that

$$\min_B(u - u_k - \phi) = \min_{\partial B}(u - g_k - \delta R^2) \geq -1/k - \delta\,R^2.$$

Hence $u(x) \geq u_k(x) + \delta\,|x-y|^2 - \frac{1}{k} - \delta\,R^2$ for $x \in \bar{B}$. Letting $\delta \to 0$ and $k \to \infty$ we get $u \geq u_\infty$.

**Step 3.** We have $\Gamma_B(u_k - v_k) \to \Gamma_B(u_\infty - v)$ uniformly in $\bar{B}$; here $\Gamma_B$ denotes the convex envelope in $B$. This follows from Steps 1-1'.

**Step 4.** Since $u \geq u_\infty$ in $\bar{B}$ and $u = u_\infty$ on $\partial B$, we have that

$$|\partial\Gamma_\Omega(u-v)(B)| \leq |\partial\Gamma_B(u_\infty - v)(B)|,$$

where $\Gamma_\Omega$ and $\Gamma_B$ denote the convex envelopes in $\Omega$ and in $B$ respectively, for any $v \in C(\bar{\Omega})$. In fact, from Lemma 7.1

$$|\partial\Gamma_\Omega(u-v)(B)| = |\partial\Gamma_\Omega(u-v)(B \cap C)|,$$

where $C = \{x \in \Omega : \Gamma_\Omega(u-v)(x) = (u-v)(x)\}$. Let $p \in \Gamma_\Omega(u-v)(B \cap C)$. Then $p \in \partial(u_\infty - v)(B)$. Because, there exists $y \in B \cap C$ such that $\Gamma_\Omega(u-v)(x) \geq \Gamma_\Omega(u-v)(y) + p \cdot (x-y)$ for all $x \in \Omega$. Since $\Gamma_\Omega(u-v)(x) \leq (u-v)(x)$ for all $x \in \Omega$, we obtain $(u-v)(x) \geq (u-v)(y) + p \cdot (x-y)$ for all $x \in \Omega$, i.e., $p \in \partial(u-v)(x)$. Since $u = u_\infty$ on $\partial\Omega$, we get from [**G01**, Lemma 1.4.1] that $p \in \partial(u_\infty - v)(B)$. We now need the following: *if $h \in C(\overline{D})$ with $D$ strictly convex, then $\Gamma_D(h)(x) = h(x)$ for all $x \in \partial D$ (this follows arguing as in the first part of the proof of [**G01**, Theorem 1.5.2]; or noticing that $\Gamma_D(h)(x) = \Gamma_{\text{affine}}(h)(x)$ for $x \in D$, where $\Gamma_{\text{affine}}(h)(x) = \sup\{a(x) :$ $a$ is affine and $a(x) \leq h(x)$ for all $x \in \partial D\}$, and then applying the first part of [**G01**, Theorem 1.5.2] directly.)* Applying this fact with $h = u_\infty - v$ and $D = B$, we get that $\Gamma_B(u_\infty - v) = u_\infty - v$ in $\partial B$ and since $\Gamma_B(u_\infty - v) \leq u_\infty - v$ in $B$, it follows once again from [**G01**, Lemma 1.4.1] that $\partial(u_\infty - v)(B) \subset \partial\Gamma_B(u_\infty - v)(B)$, and Step 4 is complete.

**Step 5.** We claim that

$$|\partial\Gamma_B(u_\infty - v)(B)| \leq \liminf_{k\to\infty} |\partial\Gamma_B(u_k - v_k)(B)|.$$

From Step 3 and [**G01**, Lemma 1.2.2 (ii)], we have

$$|\partial\Gamma_B(u_\infty - v)(K)| \leq \liminf_{k\to\infty} |\partial\Gamma_B(u_k - v_k)(U)|,$$

for each $K \subset U \subset \bar{U} \subset B$ with $K$ compact and $U$ open. Since the Monge–Ampère measure is a Borel measure finite on compact sets, it is regular and so

$$|\partial\Gamma_B(u_\infty - v)(B)| = \sup_{K\subset B, K\text{cpt}} |\partial\Gamma_B(u_\infty - v)(K)|,$$

and the claim follows.

**Step 6.** We have

$$|\partial\Gamma_B(u_k - v_k)(B)| \leq \int_{\mathcal{C}_k} \det D^2(u_k - v_k)(x)\, dx,$$

where $\mathcal{C}_k = \{x \in B : \Gamma_B(u_k - v_k)(x) = (u_k - v_k)(x)\}$, from [**G01**, Theorem 1.4.5]. For $x \in \mathcal{C}_k$ we have

$$\left(\det D^2 v_k(x)\right)^{1/n} + \left(\det D^2(u_k - v_k)(x)\right)^{1/n} \leq \left(\det D^2 u_k(x)\right)^{1/n},$$

and hence

$$\begin{aligned}\int_{\mathcal{C}_k} \det D^2(u_k - v_k)(x)\, dx &\leq \int_{\mathcal{C}_k} \left((1+\epsilon)^{1/n} - (\det D^2 v_k(x))^{1/n}\right)^n dx \\ &\leq \int_B \left(\left((1+\epsilon)^{1/n} - \alpha^{1/n}\right)^+\right)^n dx \qquad (7.3)\end{aligned}$$

for each ball $B \subset \Omega$. Combining Steps 4–5 and (7.3), we get that

$$|\partial\Gamma_\Omega(u - v)(B)| \leq \left(\left((1+\epsilon)^{1/n} - \alpha^{1/n}\right)^+\right)^n |B|,$$

for each ball $B \subset \Omega$. Therefore, the measure $|\partial\Gamma_\Omega(u - v)(\cdot)|$ is absolutely continuous with respect to Lebesgue measure, and so there exists $h \in L^1_{loc}(\Omega)$ such that

$$|\partial\Gamma_\Omega(u - v)(E)| = \int_E h(x)\, dx,$$

for each Borel set $E \subset \Omega$, and

$$h(x) \leq \left(\left((1+\epsilon)^{1/n} - \alpha^{1/n}\right)^+\right)^n, \qquad \text{a.e. in } \Omega.$$

Then (7.1) follows from Lemma 7.1.

The proof of (7.2) follows omitting Step 1' and taking $v_k = v$. □

# CHAPTER 8

# Counterexamples

In this chapter we show with an example that the estimate (1.4) is not valid for all $p$. We consider a slight variant of the two dimensional example from [**W95**, p. 842]. Let $\alpha \geq 4$ and

$$\phi(x,y) = \begin{cases} x^{\alpha} + \beta\, y^2\, x^{2-\alpha}, & \text{if } |y| \leq |x|^{\alpha-1} \\ \gamma\, x^2\, y^{(\alpha-2)/(\alpha-1)} + \delta\, y^{\alpha/(\alpha-1)}, & \text{if } |y| > |x|^{\alpha-1}, \end{cases}$$

where $\beta, \gamma$ and $\delta$ are constants that will be chosen such that $\phi$ and $D\phi$ are continuous across the curve $|y| = |x|^{\alpha-1}$, $\phi$ is convex in the regions $|y| > |x|^{\alpha-1}$ and $|y| < |x|^{\alpha-1}$, and $C_1(\alpha) \leq M\phi \leq C_2(\alpha)$ in the generalized sense, where $C_i(\alpha)$ are positive constants depending only on $\alpha$.

We have

$$\phi_x(x,y) = \begin{cases} \alpha\, x^{\alpha-1} + \beta\,(2-\alpha)\, y^2\, x^{1-\alpha}, & \text{if } |y| < |x|^{\alpha-1} \\ 2\gamma\, x\, y^{(\alpha-2)/(\alpha-1)}, & \text{if } |y| > |x|^{\alpha-1}; \end{cases}$$

$$\phi_y(x,y) = \begin{cases} 2\beta\, y\, x^{2-\alpha}, & \text{if } |y| < |x|^{\alpha-1} \\ \gamma\, \dfrac{\alpha-2}{\alpha-1}\, x^2\, y^{-1/(\alpha-1)} + \delta\, \dfrac{\alpha}{\alpha-1}\, y^{1/(\alpha-1)}, & \text{if } |y| > |x|^{\alpha-1}; \end{cases}$$

$$\phi_{xx}(x,y) = \begin{cases} \alpha\,(\alpha-1)\, x^{\alpha-2} + \beta\,(2-\alpha)\,(1-\alpha)\, y^2\, x^{-\alpha}, & \text{if } |y| < |x|^{\alpha-1} \\ 2\gamma\, y^{(\alpha-2)/(\alpha-1)}, & \text{if } |y| > |x|^{\alpha-1}; \end{cases}$$

$$\phi_{xy}(x,y) = \begin{cases} 2\beta(2-\alpha)\, y\, x^{1-\alpha}, & \text{if } |y| < |x|^{\alpha-1} \\ 2\gamma \dfrac{\alpha-2}{\alpha-1} y^{-1/(\alpha-1)}, & \text{if } |y| > |x|^{\alpha-1}; \end{cases}$$

$\phi_{yy}(x,y)$

$$= \begin{cases} 2\beta\, x^{2-\alpha}, & \text{if } |y| < |x|^{\alpha-1} \\ -\gamma \dfrac{\alpha-2}{\alpha-1} \dfrac{1}{\alpha-1} x^2\, y^{-\alpha/(\alpha-1)} + \delta \dfrac{\alpha}{\alpha-1} \dfrac{1}{\alpha-1} y^{(2-\alpha)/(\alpha-1)}, & \text{if } |y| > |x|^{\alpha-1}. \end{cases}$$

Therefore

$$2\beta(\alpha(\alpha-1) \geq \det D^2\phi \geq 2\beta(\alpha(\alpha-1) - \beta(\alpha-2)(\alpha-3)) \tag{8.1}$$

for $|y| < |x|^{\alpha-1}$; and

$$\frac{2\gamma\alpha\delta}{(\alpha-1)^2} \geq \det D^2\phi \geq \frac{2\gamma}{(\alpha-1)^2}(\delta\alpha - \gamma(\alpha-2)(2\alpha-3)) \tag{8.2}$$

for $|y| > |x|^{\alpha-1}$.

We now study the continuity of $\phi$ and $D\phi$ across $|y| = |x|^{\alpha-1}$. For the continuity of $\phi$ we need

$$x^\alpha + \beta x^{2\alpha-2} x^{2-\alpha} = \gamma x^2 x^{\alpha-2} + \delta x^\alpha,$$

so

$$1 + \beta = \gamma + \delta. \tag{8.3}$$

For the continuity of $\phi_x$ we need

$$\alpha x^{\alpha-1} + \beta(2-\alpha) x^{2\alpha-2} x^{1-\alpha} = 2\gamma x x^{\alpha-2},$$

so

$$\alpha + \beta(2-\alpha) = 2\gamma. \tag{8.4}$$

For the continuity of $\phi_y$ we need

$$2\beta x^{\alpha-1} x^{\alpha-2} = \gamma \frac{\alpha-2}{\alpha-1} x^2 x^{-1} + \frac{\delta\alpha}{\alpha-1} x,$$

so

$$2\beta = \gamma\frac{\alpha-2}{\alpha-1} + \frac{\delta\alpha}{\alpha-1}. \tag{8.5}$$

Therefore $\beta, \delta, \gamma$ solve the system of equations

$$\begin{aligned} \beta - \delta - \gamma &= 1 \\ (2-\alpha)\beta - 2\gamma &= -\alpha \\ 2(\alpha-1)\beta - \alpha\delta - (\alpha-2)\gamma &= 0. \end{aligned}$$

The solutions to this system are given by

$$(\beta, \delta, \gamma) = \left(1 - \frac{2}{\alpha}, 0, 2 - \frac{2}{\alpha}\right) + \delta\left(\frac{2}{\alpha}, 1, \frac{2}{\alpha} - 1\right),$$

where $\delta$ is arbitrary.

We next impose conditions on $\beta, \delta, \gamma$ such that $\phi$ is convex. We have for $|y| < |x|^{\alpha-1}$ that

$$\phi_{xx} = (\alpha-1)(\alpha x^{\alpha-2} + \beta(\alpha-2)y^2x^{-\alpha}) > 0$$

if $\beta > 0$. If $|y| > |x|^{\alpha-1}$, then

$$\phi_{xx} = 2\gamma y^{(\alpha-2)/(\alpha-1)} > 0$$

for $\gamma > 0$. Also, we have for $|y| < |x|^{\alpha-1}$ that

$$\phi_{yy} = 2\beta x^{2-\alpha} > 0$$

for $\beta > 0$. If $|y| > |x|^{\alpha-1}$, then

$$\begin{aligned} \phi_{yy} &= \frac{1}{(\alpha-1)^2}(\delta\alpha y^{(2-\alpha)/(\alpha-1)} - \gamma(\alpha-2)x^2y^{-\alpha/(\alpha-1)}) \\ &\geq \frac{1}{(\alpha-1)^2}y^{(2-\alpha)/(\alpha-1)}(\delta\alpha - \gamma(\alpha-2)). \end{aligned}$$

Since $\gamma = 2 - \frac{2}{\alpha} + \delta\left(\frac{2}{\alpha} - 1\right)$, we get that $\delta\alpha - \gamma(\alpha-2) > 0$, and so $\phi_{xx} > 0$, as long as

$$\delta > \frac{2(\alpha-2)(\alpha-1)}{\alpha^2 + (\alpha-2)^2}.$$

We need $\det D^2\phi > 0$. The right hand side of (8.1) is positive if $\beta < \frac{\alpha(\alpha-1)}{(\alpha-2)(\alpha-3)}$; and since $\beta = 1 - \frac{2}{\alpha} + \delta\frac{2}{\alpha}$, we must have

$$\delta < \frac{\alpha^2(\alpha-1) - (\alpha-2)^2(\alpha-3)}{2(\alpha-2)(\alpha-3)}.$$

On the other hand, the right hand side of (8.2) is positive if $\delta\alpha - \gamma(\alpha-2)(2\alpha-3) > 0$, and since $\gamma = 2 - \frac{2}{\alpha} + \delta(\frac{2}{\alpha} - 1)$, we get

$$\delta > \frac{2(\alpha-2)(\alpha-1)(2\alpha-3)}{\alpha^2 + (\alpha-2)^2(2\alpha-3)}.$$

Also $\gamma > 0$ amounts

$$\delta < \frac{2(\alpha-1)}{\alpha-2}.$$

We have

$$\delta_2(\alpha) = \min\left\{\frac{2(\alpha-1)}{\alpha-2}, \frac{\alpha^2(\alpha-1) - (\alpha-2)^2(\alpha-3)}{2(\alpha-2)(\alpha-3)}\right\} = \frac{2(\alpha-1)}{\alpha-2},$$

$$\begin{aligned}\delta_1(\alpha) &= \max\left\{\frac{2(\alpha-2)(\alpha-1)}{\alpha^2+(\alpha-2)^2}, \frac{2(\alpha-2)(\alpha-1)(2\alpha-3)}{\alpha^2+(\alpha-2)^2(2\alpha-3)}\right\} \\ &= \frac{2(\alpha-2)(\alpha-1)(2\alpha-3)}{\alpha^2+(\alpha-2)^2(2\alpha-3)},\end{aligned}$$

and it is easy to check that $\delta_1(\alpha) < \delta_2(\alpha)$. Therefore each $\delta$ satisfying

$$\delta_1(\alpha) < \delta < \delta_2(\alpha)$$

determines $\beta$ and $\gamma$ with the desired properties and therefore $\phi$.

We have that $D^2\phi \notin L^p$ for $p > \frac{\alpha}{\alpha-2}$.

**Lemma 8.1** *Given $p > 1$ there exist positive constants $\lambda \leq \Lambda$ depending only on $p$ and a strictly convex bounded domain $\Omega$ and a convex function $\psi \in C^2(\Omega)$ with $\lambda \leq \det D^2\psi \leq \Lambda$ in $\Omega$ and with $\psi = 1$ on $\partial\Omega$ such that the inequality $\|D^2\psi\|_{L^p} \leq C$ is impossible with a constant $C$ depending only on $n, p, \lambda$ and $\Lambda$.*

PROOF: Let $\phi$ be the function constructed above for $\alpha \geq 4$ sufficiently large such that $p > \frac{\alpha}{\alpha - 2}$, $\Omega = \{z : \phi(z) < 1\}$, $f(z) = \det D^2\phi(z)$, and $\lambda = C_1(\alpha)$, $\Lambda = C_2(\alpha)$. Consider $\eta_\epsilon(z)$ a nonnegative smooth mollifier and let $f_\epsilon = f \star \eta_\epsilon$. We have that $\lambda \leq f_\epsilon \leq \Lambda$ in $\Omega$. Let $\phi_\epsilon$ be the solution to $\det D^2\phi_\epsilon = f_\epsilon$ in $\Omega$ and $\phi_\epsilon = 1$ on $\partial\Omega$. We have that $f_\epsilon \to f$ weakly. Then by [**G01**, Lemma 1.6.1], $\{\phi_\epsilon\}$ contains a subsequence $\phi_{\epsilon'} \to \phi$. If $\|D^2\phi_{\epsilon'}\|_{L^p} \leq C$ with a constant depending only on $p, n, \lambda$ and $\Lambda$, then $D^2\phi_{\epsilon'}$ would contain a subsequence $D^2\phi_{\epsilon''}$ converging in $L^p$ to some function $v$. But then $D^2\phi = v$, a contradiction. □

# REFERENCES


[C] L. A. Caffarelli. *Interior $W^{2,p}$ estimates for solutions to the Monge-Ampère equation.* Ann. of Math. **131**, 135–150, 1990.

[**C90**] Caffarelli, L.A. 1990. *A localization property of viscosity solutions to the Monge-Ampère equation and their strict convexity.* Ann. of Math. **131**, 129–134.

[**C91**] Caffarelli, L.A. 1991. *Some regularity properties of solutions of Monge-Ampère equation.* Comm. Pure Appl. Math. **44**, 965–969.

[**CC95**] Cafarelli, L.A. and Cabré, X. 1995. *Fully nonlinear elliptic equations* American Mathematical Society Colloquium Publications, volume 43.

[**CG96**] Caffarelli, L.A. and Gutiérrez, C.E. 1996. *Real analysis related to the Monge-Ampère equation.* Trans. A. M. S. **348(3)**, 1075–1092.

[**CG97**] Caffarelli L. A. and Gutiérrez C. E. 1997. *Properties of the Solutions of the Linearized Monge–Ampère Equation.* Amer. J. Math. **119(2)**, 423–465.

[**E85**] Evans, L. C. 1985. *Some estimates for nondivergence structure, second order elliptic equations.* Trans. A. M. S. **287(2)**, 701–712.

[**GT83**] D. Gilbarg and N. S. Trudinger 1983. *Elliptic Partial Differential Equations of Second Order.* Springer–Verlag, New York.

[**G01**] C. E. Gutiérrez 2001. *The Monge-Ampère Equation.* Birkhaüser, Boston.

[**GH00**] C. E. Gutiérrez and Qingbo Huang 2000. *Geometric Properties of the Sections of Solutions to the Monge–Ampère Equation.* Trans. A. M. S. **352**, 4381–4396.

[**L86**] Fang-Hua Lin 1986. *Second derivative $L^p$-estimates for elliptic equations of nondivergent type.* Proc. Amer. Math. Soc. **96(3)**, 447–451.

[**P78**] Pogorelov, A.V. 1978. *The Minkowski Multidimensional Problem.* V.H. Winston and Sons, Washington.

[**Sc93**] Schneider, R. 1993. *Convex Bodies: The Brunn-Minkowski Theory.* Cambridge University Press, Cambridge.

[**TW00**] Trudinger, N. S. and Xu-Jia Wang 2000. *The Bernstein problem for affine maximal hypersurfaces.* Invent. Math. **140(2)**, 399–422.

[**W95**] Xu-Jia Wang 1995. *Some counterexamples to the regularity of Monge–Ampère equations.* Proc. Amer. Math. Soc. **123(3)**, 841–845.