Embedding problems in Symplectic Topology

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Plan of lecture:

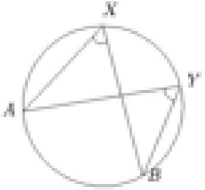
A symplectic structure is a rather elusive geometric structure that can be put on an even dimensional space. This talk will describe some basic properties of this structure, explain some fundamental results and briefly discuss some open problems.

- (I): What is symplectic topology?
- (II) Some fundamental results on symplectic embeddings
- (III) Open Questions

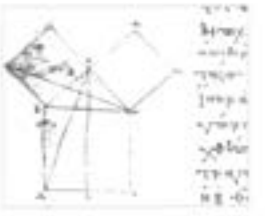
Geometry I: — Euclidean

Geometry is the study of the structure of space. (many possibilities)

We are all familiar with plane Euclidean Geometry with its lines, angles, distance measure and circles:

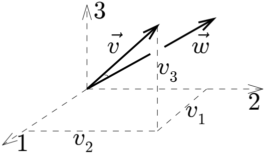


lengths, angles, circles
in plane Euclidean geometry



Geometry in higher dimensions – via linear algebra

Position in two dimensions is given by two numbers (or coordinates).
Position in three dimensions is given by three numbers: $\vec{v} = (v_1, v_2, v_3)$.
Position in n dimensions is given by n numbers $(v_1, \ldots, v_n)$ and we measure lengths and angles using the dot product.



$$\begin{aligned} \vec{v} \bullet \vec{w} &= v_1 w_1 + v_2 w_2 + v_3 w_3 \\ &= |\vec{v}|\,|\vec{w}|\,\cos(\theta) \end{aligned}$$

Pythagoras theorem. The length $|\vec{v}|$ of the vector $\vec{v}$ is $\sqrt{v_1^2 + v_2^2 + v_3^2 + \cdots + v_n^2)} = \sqrt{\vec{v} \cdot \vec{v}}$.

The dot product $\vec{v} \cdot \vec{v}$ is symmetric, i.e. $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$.

Symplectic Geometry

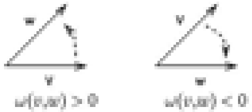
In Symplectic Geometry the dot product $\vec{v} \cdot \vec{w}$ is replaced by an anti- (or skew-) symmetric form $\omega_0(\vec{v}, \vec{w})$, i.e. $\omega_0(\vec{v}, \vec{w}) = -\omega_0(\vec{w}, \vec{v})$, so that $\omega_0(\vec{v}, \vec{v}) = 0$. Thus there is no notion of length, but there is a notion of the (signed) area of 2-dim objects.

Basic example: on $\mathbb{R}^2$, we have $\omega_0 = dxdy$ (in calculus notation). It is an area form often written $dx \wedge dy$

$$\omega_0\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}\right) = \det \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} = v_1 w_2 - v_2 w_1.$$

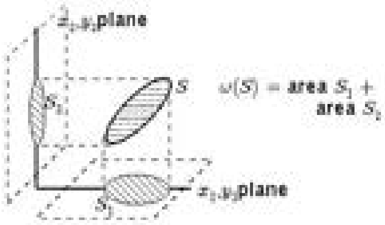
$\omega_0(\vec{v}, \vec{w})$ is the area of the rectangle spanned by these two vectors.

Its sign depends on orientation.

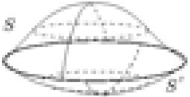


The standard (linear) symplectic form ω_0

on $\mathbb{R}^4$ we have $\omega_0 := dx_1 \wedge dy_1 + dx_2 \wedge dy_2$, a sum of area forms



Here S is a piece of surface that you project in two different ways and then add the areas.

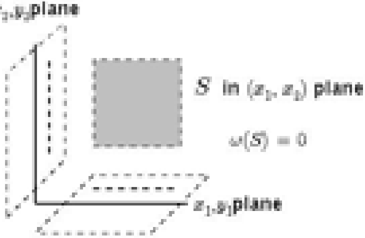


$$\int_S \omega = \int_{S'} \omega$$

Since ω_0 is closed, i.e. $d\omega_0 = 0$, we get flabby measurements: by Stokes' theorem, the area of a surface S can be written as an integral $\int_S \omega_0$; it does not change as S moves as long as the boundary remains fixed.

The standard (linear) symplectic form ω_0

In $\mathbb{R}^4$ the x_1, x_2- plane in $\mathbb{R}^4$ is Lagrangian: any surface S in this plane projects to a line in both the x_1, y_1 and x_2, y_2-planes and so has zero symplectic area. Thus directions are not all the same — the geometry is anisotropic.



It's the same in higher dimensions: in $\mathbb{R}^6$ we have $\omega_0 := dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + dx_3 \wedge dy_3$, again a sum of area forms, with coordinates grouped in pairs.

In physics, the pair x_i, y_i represents the position and velocity of a particle in one direction – so a particle moving in 3-space gives 6 coordinates. The symplectic form gives an important (but geometrically somewhat obscure) measurement of the mutual entanglement of position and velocity.

Many symplectomorphisms:

In general, a symplectic structure on a $2n$-dimensional manifold M is a closed, nondegenerate 2-form ω.

Every function $H : M \to \mathbb{R}$ generates a flow $\phi_t^H, t \in \mathbb{R}$ (1-parameter group of motions of space) that preserves ω: $(\phi_t^H)^*(\omega) = \omega$. Such transformations are called symplectomorphisms.
The flow is a solution of Hamilton's differential equations — generated by the vector field X_H given by $\omega(X_H, \cdot) = dH(\cdot)$– so that in $\mathbb{R}^{2n}$ we have $\frac{\partial x}{\partial t} = \frac{\partial H}{\partial y}, \frac{\partial y}{\partial t} = -\frac{\partial H}{\partial x}$.

Example: if $H = \frac{1}{2}(x^2 + y^2)$ on $(\mathbb{R}^2, dx \wedge dy)$, we find

$$dH = xdx + ydy \implies X_H = y\partial_x - x\partial_y$$

giving a clockwise rotation, preserving the circles $H = const$.

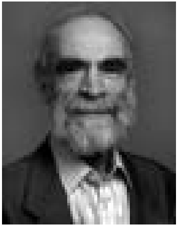
But there are many, much more twisty symplectomorphisms Because there are so many, symplectic geometry is very flexible.

Nineteenth century symplectic geometry

In classical mechanics, the flow ϕ_t^H describes the time evolution of a mechanical system with energy function H; thus energy is conserved as the system evolves with time. This picture is largely due to William Rowan Hamilton (also the inventor of quaternions); Sonia Kovalevsky used this approach in her celebrated study of spinning tops.

Late twentieth century symplectic geometry

Very deep and subtle connections between symplectic and complex geometry were first noticed and exploited by Misha Gromov (on left) in 1985, giving rise to a main tool in the modern theory *J*-holomorphic curves.



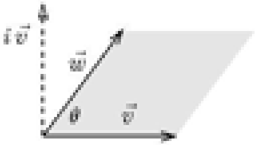
His ideas were extended by Andreas Floer (on right) in the late 1980s, who used them to develop an influential Morse theory for spaces of loop and paths in a symplectic manifold, now called Floer homology. These new concepts are the basis for the relevance of symplectic geometry to string theory and mirror symmetry.

Relation of Symplectic to Complex geometry:

Symplectic and Euclidean geometry are related via complex numbers: Identify $\mathbb{R}^2$ with $\mathbb{C}$ where $z = x + iy$, so $\mathbb{R}^{2n} \equiv \mathbb{C}^n$. Then

$$\underbrace{\omega_0(\vec{v}, \vec{w})}_{\text{symplectic form}} = \|\vec{v}\| \, \|\vec{w}\| \sin(\theta) = \underbrace{i\vec{v} \cdot \vec{w}}_{\text{dot product}}$$

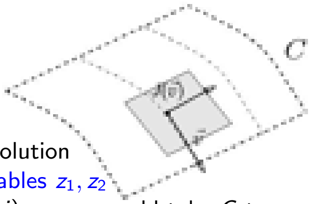
where θ is the angle between $\vec{v}, \vec{w}$ and $i = \sqrt{-1}$. (see left-hand picture)



Gromov discovered that it makes sense to consider perturbations J of the operation "multiply tangent vectors by i ". So J is an operator $\vec{v} \mapsto J(\vec{v})$ on tangent vectors such that $J^2 = -\mathrm{Id}$ and that is positively related to ω_0 by: $\omega_0(\vec{v}, J\vec{v}) \geq 0$. J is called an almost complex structure.

J-holomorphic curves:

The analogs of geodesics are J-holomorphic curves – these are one dimensional complex curves (so two real dimensions)
i.e. they are surfaces C such as a sphere or torus and such that, for every tangent vector $\vec{v}$, the vector $J(\vec{v})$ is also tangent to C.



- In $\mathbb{R}^4 = \mathbb{C}^2$ (complex plane) every solution of a polynomial in the complex variables z_1, z_2 is a J-holomorphic curve (with $J = i$). eg we could take C to consist of the solutions to $z_2 = z_1^3 + z_1 + 1$.
- But J is much more flexible that the complex structure i.
- So there are many more J-holomorphic curves than complex curves; – they are most useful when they persist under perturbations of J.

Relation of Symplectic to Complex geometry:

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$$\omega_0(\vec{v}, \vec{w}) = \|\vec{v}\| \, \|\vec{w}\| \sin(\theta) = i\vec{v} \cdot \vec{w}$$

where θ is the angle between $\vec{v}, \vec{w}$ and $i = \sqrt{-1}$.

Since multiplication by $i = \sqrt{-1}$ rotates the complex plane by 90^0, the above identity also implies that the two gradients are mutually perpendicular:

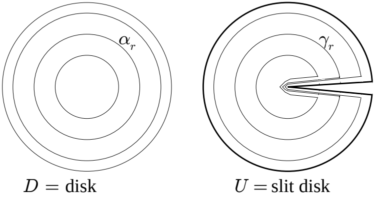
Euclidean gradient $\mathrm{grad}\, H = i$ (Symplectic gradient) $= i\, X_H$.

Special Features of Symplectic Geometry:

- The theory has two faces, with two kinds of submanifolds;
  - symplectic – the restriction of ω is nongenerate, relation to complex surfaces
  - Lagrangian – the restriction of ω vanishes; relation to dynamics, totally real objects
- tight connection with physics (from Hamiltonian dynamics in 19th century to dualities/mirror symmetry in string theory today);
- Darboux's theorem – all symplectic forms are locally diffeomorphic;
- the group of symplectomorphisms ϕ is infinite dimensional (since it contains all ϕ_t^H) and $\mathcal{C}^0$- (i.e. uniformly) closed among all diffeomorphisms; i.e. although the equation $\phi^*(\omega) = \omega$ involves first derivatives, there is a notion of symplectic capacity c such that
  - $\phi^*(\omega) = \omega$ iff ϕ preserves c;
  - the condition ϕ preserves c does not involve derivatives. more later
- Thus symplectic geometry is in many respects a topological theory, involving a very interesting interplay between flexibility and rigidity.

In 2 dimensions, a symplectic form is an area form. Hence every closed oriented surface (eg a sphere or torus) has a natural symplectic structure, unique up to a scaling factor. Moreover there are many area preserving diffeomorphisms.

(Moser - 1965) *If a closed disc $D \subset \mathbb{R}^2$ is diffeomorphic to a closed region U of the same total area, there is an area preserving diffeomorphism $\phi : D \xrightarrow{\cong} U$.*



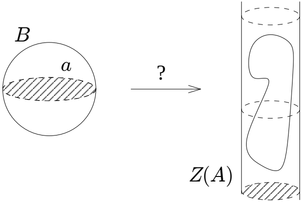
The situation is quite different in higher dimensions.

The Nonsqueezing theorem

Consider the problem of embedding the ball

$$B^4(a) = \{(z_1, z_2) : \pi(|z_1|^2 + |z_2|^2) \le a\} \subset \mathbb{C}^2 = \mathbb{R}^4$$

into the cylinder $Z(A) = \{(z_1, z_2) : \pi|z_1|^2 \le A\} \subset \mathbb{R}^4$, where we write $z_1 = x_1 + iy_1 \in \mathbb{C} = \mathbb{R}^2$:



Nonsqueezing Theorem (Gromov: 1985) *There is a symplectic embedding $B(a) \hookrightarrow Z(A)$ if and only if $a \le A$.*

The volume preserving map $(z_1, z_2) \mapsto (\lambda z_1, \frac{1}{\lambda} z_2)$, $\lambda := \sqrt{\frac{A}{a}}$ does squeeze the ball into the cylinder.

Symplectic measurements:

Although the Nonsqueezing Theorem may seem just like a curiosity, it turns out to be a cornerstone of the modern theory.

Gromov, Ekeland–Hofer: Given an open $U \subset \mathbb{R}^{2n}$ define the symplectic capacity by

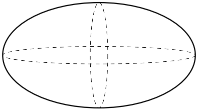
$$c(U) = \sup\{a : B(a) \text{ embeds symplectically in } U\}.$$

- $c(U)$ is a symplectic invariant;
- it is essentially 2-dimensional, eg $Z(A)$ is a set of infinite volume with finite capacity
- any orientation preserving diffeomorphism ϕ that preserves c (i.e. $c(\phi(U)) = c(U)$ for all U) is an (anti)-symplectomorphism, i.e. $\phi^*(\omega) = \pm\,\omega$.

Many other very interesting symplectic measurements have been developed by Hofer, Polterovich, Hutchings, among many others ...

Embedding 4-dimensional ellipsoids

Let $E(a,b)$ be the ellipsoid $\{(z_1,z_2) : \pi\big(\frac{|z_1|^2}{a} + \frac{|z_2|^2}{b}\big) \le 1\}$.



$\pi(|z_1|^2/a + |z_2|^2/b) \le 1$

the ellipsoid $E_{a,b}$

Hofer conjectured around 2010 that $\mathrm{int} E(a,b)$ *embeds symplectically in* $\mathrm{int} E(c,d)$ *if and only if* $\mathcal{N}(a,b) \le \mathcal{N}(c,d)$. Here $\mathcal{N}(a,b)$ is the set of all numbers $ka + \ell b,\ \ k,\ell \ge 0$, arranged with multiplicities in increasing order. So,

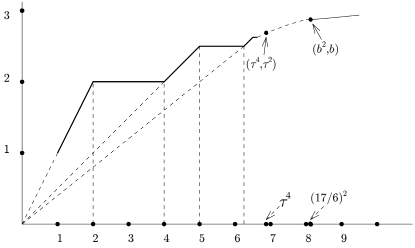
$\mathcal{N}(2,2) = (0, \underbrace{2,2}, \underbrace{4,4,4}, \underbrace{6,6,6,6}, \underbrace{8,8,8,8,8}, \dots)$, and

$\mathcal{N}(1,4) = (0, \underbrace{1,2}, \underbrace{3,4,4}, \underbrace{5,5,6,6}, \underbrace{7,7,8,8,8}, \dots$ Thus $\mathcal{N}(1,4) \le \mathcal{N}(2,2)$ because the first sequence is termwise no larger than the second.

This conjecture now proved by McDuff (2012). An illustration of what it means:

The "ellipsoid into ball" embedding capacity

For $a \geq 1$ define $c(a) := \inf\{\mu : E(1,a) \text{ embeds sympl. in } B(\mu)\}$.
This function was calculated by McDuff–Schlenk (2012).



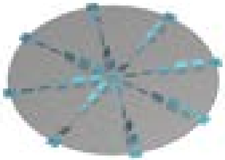
- For $a < \tau^4 \approx 6.7$ (where $\tau = \frac{1+\sqrt{5}}{2}$) there is an infinite staircase (with numerics based on the Fibonacci numbers),
- for $a \geq 8\frac{1}{36} = \left(\frac{17}{6}\right)^2$, $c(a) = \sqrt{a}$ – no obstruction except for volume
- $\tau^4 < a < 8\frac{1}{36}$ is a transitional region;
- there are rather few results or plausible guesses as to behavior in dim > 4.
  The obvious analog of Hofer's conjecture is false (Guth).

III. Some open questions

(I) Which manifolds are symplectic?

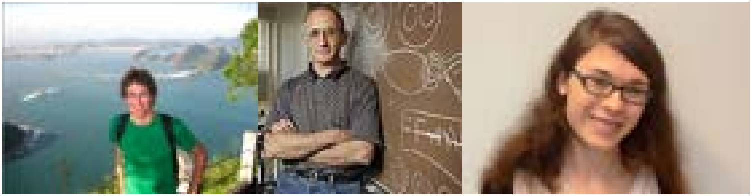
- Necessary topological conditions: the manifold must admit an almost complex structure (a higher dim. analog of an orientation) and (if closed, i.e. compact without boundary) a cohomology class $a = [\omega]$ with $a^n > 0$ (the de Rham class represented by ω);
- (Gromov) if M is connected and open (i.e. noncompact or with boundary) this is enough;
- (Taubes) there are closed 4-dim manifolds satisfying these conditions with no symplectic structure; e.g. the connected sum of three copies of $\mathbb{C}P^2$, the complex projective plane.
- no such examples are known in higher dim — but almost surely exist.
- The analogous questions in contact geometry (symplectic geometry's odd-dimensional twin) now have some answers.

Contact and symplectic geometry I:



- A contact structure on a manifold Y is a nowhere integrable hyperplane field ξ given by the kernel (or vanishing locus) of a 1-form α. Nonintegrability means that the planes continually twist, and that the curvature of the plane field (given by the 2-form $d(t\alpha)$) defines a symplectic structure in a little neighbourhood $Y \times (-\varepsilon, \varepsilon)$ of Y.
- Conversely, every convex codimension 1 hypersurface in a symplectic manifold (e.g. a sphere S^{2n-1} in $\mathbb{R}^{2n}$) has a natural contact structure.
- Some aspects of symplectic and contact geometry are the same but some are very different:
    - in both geometries, every function H on the manifold generates a flow, so there are many structure-preserving diffeomorphisms;
    - the contact structure in Euclidean space can be rescaled by a map of the form $(x, y, z) \mapsto (\lambda x, \lambda y, \lambda^2 z)$; so the nonsqueezing phenomena are very different.

Contact and symplectic geometry II:



- In 2014 Borman, Eliashberg and Murphy showed that *every hyperplane field that satisfies a mild topological condition is homotopic to a contact structure* that is unique up to contactomorphism.
- All contact structures constructed in this way are overtwisted.
- But contact structures that bound symplectic manifolds are never overtwisted.
- the existence and classification of tight (i.e. not overtwisted) contact structures is not at all understood in dimensions > 3.

A very few References:

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