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IMPROVED ALGORITHMS AND IMPLEMENTATIONS IN THE MULTI-WZ THEORY

A Dissertation

Submitted to

the Temple University Graduate Board

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in Partial Fulfillment

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DOCTOR OF PHILOSOPHY

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by

Akalu Tefera

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ABSTRACT

IMPROVED ALGORITHMS AND IMPLEMENTATIONS IN THE MULTI-WZ THEORY

by Akalu Tefera

Doctor of Philosophy

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Advisor: Professor Doron Zeilberger

In this dissertation we find improved algorithms and implementations that completely automate the continuous version of the multi-WZ method.

In the first chapter we give a brief review of the multi-WZ method. In Chapter 2 we describe complete automation of the continuous multi-WZ method. In Chapter 3, using our Maple packages, we give automated proofs to several mathematical problems. In Chapter 4, we discuss future directions of our research.

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Last, but not least, I must indicate my sincere gratitude to my best friend Mayumi Sakamoto for her unconditional and continuous support, encouragement and love.

DEDICATION

With all of my love and sorrow, I dedicate this work to my late parents Sentayehu Mammo and Tefera Demie. They were truly loving and caring parents.

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CHAPTER 1
INTRODUCTION

Identities in mathematics are usually hard to prove and often require lengthy and tedious verification. One of the most exciting discoveries in recent years, due to *Herb Wilf* and *Doron Zeilberger* [WZ92], is that every *proper-hypergeometric* multi-integral or sum identity with a *fixed* number of integration and summation signs possesses a computer-constructible proof.

In general, the "objects" of study in the WZ theory are expressions of the kind

$$\sum_{\mathbf{k}} \int F(\mathbf{n}, \mathbf{k}, \mathbf{x})\, d\mathbf{x}$$

and identities between them. In the above general integral-sum, $\mathbf{n}$ and $\mathbf{k}$ are *discrete* multi-variables, while $\mathbf{x}$ is a *continuous* multi-variable, and F is *hypergeometric* in all its arguments.

Presently the computer implementation of the WZ method is done by considering two special cases of the general integral-sum. One is the case of the pure *multi-sum*, i.e. $\mathbf{x}$ is empty, and the other is the case of the pure *multi-integral*, i.e. $\mathbf{k}$ is empty. These two cases are called *the discrete multi-WZ* and *the continuous multi-WZ*, respectively.

1.1 The Discrete Multi-WZ

This case includes expressions of the kind

$$\sum_{k_1} \cdots \sum_{k_r} F(\mathbf{n}, k_1, \ldots, k_r)$$

and identities between them.

Some examples of multi-sum identities that can be proved automatically by using the multi-WZ method are the Apéry-Shmidt-Strehl identity

$$\sum_k \sum_j \binom{n}{k}\binom{n+k}{k}\binom{k}{j}^3 = \sum_k \binom{n}{k}^2 \binom{n+k}{k}^2,$$

where $n \geq 0$ is an integer, the Andrews-Paule sum

$$\sum_{i=0}^{n}\sum_{j=0}^{n} \binom{i+j}{i}^2 \binom{4n-2i-2j}{2n-2i} = (2n+1)\binom{2n}{n},$$

and various other multi-sum binomial coefficient identities.

There are several efficient implementations that handle binomial summation identities. To mention a few: Zeilberger's [Z90] **EKHAD**[1], C. Krattenthaler's [K95] HYP[2], B. Gauthier's [G99] **HYPERG**[3], and Schorn's [PS95] `fastZeil`[4]. For multi-sum binomial identities K. Wegschaider's [W97] `MultiSum`[5] is a nice Mathematica implementation of Wilf and Zeilberger's [WZ92] extension of Sister Celine's technique.

1.2 The Continuous Multi-WZ

This case includes expressions of the kind

$$\int \cdots \int F(\mathbf{n}, x_1, \ldots, x_k)\, dx_1 \cdots dx_k$$

and identities between them. In the above integral expression, $\mathbf{n}$ is a *discrete* multi-variable and F is *hypergeometric* in all its arguments. Some examples

[1]available from `http://www.math.temple.edu/~zeilberg/`

[2]available from `http://radon.mat.univie.ac.at/People/kratt/`

[3]available from `http://www-igm.univ-mlv.fr/~gauthier/`

[4]available from `http://www.risc.uni-linz.ac.at/research/combinat/risc/software/`

[5]available from `http://www.risc.uni-linz.ac.at/research/combinat/risc/software/`

of multi-integral identities that can be automatically proved by the multi-WZ method, for *any fixed* given dimension, are the celebrated Mehta-Dyson integral

$$\frac{1}{(2\pi)^{k/2}}\int_{(-\infty,\infty)^k} e^{-\sum_{j=1}^{k} x_j{}^2/2} \prod_{1\le i<j\le k} (x_i-x_j)^{2c}\, dx_1\cdots dx_k = \prod_{j=1}^{k} \frac{(c\,j)!}{c!},$$

where $k > 0$ is an integer and c is a non-negative integer, the Selberg's integral

$$\int_{[0,1]^k} \left\{ \prod_{i=1}^{k} x_i^a\,(1-x_i)^b \prod_{1\le i<j\le k} (x_i-x_j)^{2c} \right\} dx_1\cdots dx_k =$$

$$\prod_{j=1}^{k} \frac{(a+(j-1)\,c)!\,(b+(j-1)\,c)!\,(j\,c)!}{(a+b+(k+j-2)\,c+1)!\,c!},$$

where a, b and c are non-negative integers, and constant term expressions such as the celebrated Dyson's ex-conjecture

$$\mathrm{CT}\left(\prod_{\substack{1\le i,j\le k\\ i\ne j}} \left(1-\frac{x_i}{x_j}\right)^a \right) = \frac{(k\,a)!}{a!^k},$$

where a is a non-negative integer and

$$\begin{aligned} \mathrm{CT}(f(x_1,\ldots,x_k)) &:= \text{constant term of } f(x_1,x_2,\ldots,x_k) \\ &:= \text{coeff. of } x_1^0\,x_2^0\cdots x_k^0 \text{ of the Laurent polynomial } f \\ &= \frac{1}{(2\,\pi\,i)^k}\int_C\cdots\int_C \frac{f(x_1,x_2,\ldots,x_k)}{\prod_{i=1}^{k} x_i}\, dx_1\cdots dx_k. \end{aligned}$$

where C is a circle around the origin.

See [AAR98] for a superb exposition of these and others very important identities and of numerous applications. At present, the continuous multi-WZ method [WZ92] is capable of mechanically proving these identities only for fixed k. In principle it should work for any specific value of k, but in practice it only works for $k < N$, where the value of N depends on a given identity and implementation

.

Doron Zeilberger wrote a Maple implementation, **TRIPLE_INTEGRAL**[6], that performs the algorithm described in [WZ92] for the case of *three* continuous variables.

[6]available from `http://www.math.temple.edu/~zeilberg/`

But `TRIPLE_INTEGRAL` does not completely automate the method, for instance, it requires the user to guess and input the denominators of the rational certificates of the required recurrence-differential (WZ) equation. One of the goals of this thesis was to improve and generalize Zeilberger's `TRIPLE_INTEGRAL` for *any* specific number of continuous variables so that it completely automates the continuous multi-WZ method. To this end, we wrote Maple implementations, `Mint` and `SMint`[7] which improve and generalize Zeilberger's `TRIPLE_INTEGRAL` for *any* specific number of continuous variables. In Chapter 2 we describe complete automation of the continuous multi-WZ method. In Chapter 3, using our Maple packages, we give automated proofs of various mathematical problems. In Chapter 4 we discuss future directions of our research.

[7]they are available from `http://www.math.temple.edu/~akalu/`

CHAPTER 2

COMPLETE AUTOMATION OF THE CONTINUOUS MULTI-WZ METHOD

We will describe Maple implementations, `Mint` and `SMint`, of the continuous version of the multi-WZ method. We will also give several examples of how these packages can be used to systematically generate proofs of identities (or recurrences) which involve multiple integrals of *proper-hypergeometric* functions.

2.1 Notations and Basic Definitions

Numbers. We denote the set of integers by $\mathbf{Z}$, the set of positive integers by $\mathbf{N}$, and the set of negative integers by $-\mathbf{N}$.

Operators. Let n be a *discrete* variable, x be a *continuous* variable, $\mathbf{x}$ be a *continuous* multi-variable, and F be a function. We use the following operator notations.

$$\begin{aligned} E_n F(n;\mathbf{x}) &:= F(n+1;\mathbf{x}), \\ D_x F &:= \frac{\partial}{\partial x} F, \\ \Delta_n F(n;\mathbf{x}) &:= F(n+1;\mathbf{x}) - F(n;\mathbf{x}). \end{aligned}$$

Annihilators. An operator P is said to *annihilate* a function F if $P\,F = 0$.

For example, $E_n - E_k - I$ annihilates $F(n,k) = \binom{n}{k+1}$, and $D_x^2 + 4\,I$ annihilates $F(x) = \sin(2\,x)$.

Rising Factorial. The *rising factorial* symbol $(a)_n$ for $n \in \mathbf{Z}$, is defined as

$$(a)_n := \begin{cases} \prod_{i=0}^{n-1}(a+i) & \text{if } n \in \mathbf{N} \\ 1 & \text{for } n = 0 \\ \frac{1}{\prod_{i=1}^{-n}(a-i)} & \text{if } n \in -\mathbf{N} \text{ and } a \notin \{1, 2, \ldots, -n\} \end{cases}.$$

Elementary Symmetric Functions. The *elementary symmetric* function of $x_1, \ldots, x_n$ of *order* r, denoted by e_r, is defined as

$$e_r = \sum_{1 \le i_1 < \cdots < i_r \le n} \prod_{j=1}^{r} x_{i_j}.$$

Hypergeometric functions. Let $F := F(\mathbf{n}, \mathbf{x})$ be a function of m discrete variables $\mathbf{n} = (n_1, \ldots, n_m)$ and k continuous variables $\mathbf{x} = (x_1, \ldots, x_k)$. We say that

- F is *hypergeometric* if

  (i). $\frac{E_{n_i} F}{F} = \text{Rational}_{1,i}(\mathbf{n}, \mathbf{x}), (i = 1, \ldots, m)$,

  (ii). $\frac{D_{x_i}(F)}{F} = \text{Rational}_{2,i}(\mathbf{n}, \mathbf{x}), (i = 1, \ldots, k)$.

- F is *proper-hypergeometric* if it can be reduced to the form:

$$P(\mathbf{n}, \mathbf{x}) e^{R_0(\mathbf{x})} \prod_{i=1}^{m} R_i(\mathbf{x})^{n_i} \prod_{j=1}^{J} S_j(\mathbf{x})^{c_j} \prod_{l=1}^{L} (e_1^{(l)} n_1 + \cdots + e_m^{(l)} n_m + f_l)!^{g_l}$$

  where

  (i). $P(\mathbf{n}, \mathbf{x})$ is a polynomial,

  (ii). R_0, R_i, S_j are rational functions in $\mathbf{x}$,

  (iii). c_j and f_l are complex numbers (in general, commuting indeterminates),

  (iv). $e_1^{(l)}, \ldots, e_m^{(l)}$ and g_l $(l = 1, \ldots, L)$ are integers.

Example : $F(n,x) = x^n$ is a *hypergeometric* function, since $\frac{E_n F}{F} = x$ and $\frac{D_x F}{F} = \frac{n}{x}$ are rational functions.

Many important identities, for instance, those mentioned in section 1.2 involve multiple integrals of proper-hypergeometric functions.

2.2 Description of the Algorithm

The heart of the algorithm is the following *fundamental theorem of the (continuous) multi-WZ method.*

Theorem 2.1 (The fundamental theorem) *Let $F(n; x_1, \ldots, x_k)$ be a proper--hypergeometric function in all its arguments where n is a discrete variable and $x_1, \ldots, x_k$ are continuous variables. There exists a non-zero linear ordinary recurrence operator with polynomial coefficients $P(E_n, n)$ and a k-tuple of rational functions $[R_1, \ldots, R_k]$ in the variables n, $x_1, \ldots, x_k$ such that*

$$P(E_n, n)F = \sum_{j=1}^{k} D_{x_j}(R_j F) \qquad \textit{(WZ equation).}$$

Note that the polynomial coefficients of $P(E_n, n)$ are free of $x_1, \ldots, x_k$.

Suppose we have to prove a multiple integral identity of the form

$$\int_{\gamma_1} \cdots \int_{\gamma_k} F(n; x_1, \ldots, x_k)\, dx_1 \cdots dx_k = 1, \qquad n \in \mathbf{Z}_{\geq 0}, \tag{2.1}$$

where γ_i, $i = 1, \ldots, k$, are circles around the origin.

The general method to prove (2.1) is to find a recurrence equation satisfied by

$$f(n) := \int_{\gamma_1} \cdots \int_{\gamma_k} F(n; x_1, \ldots, x_k)\, dx_1 \cdots dx_k.$$

By the fundamental theorem, F satisfies a WZ equation:

$$\sum_{i=0}^{L} a_i(n)\, F(n+i; x_1, \ldots, x_k) = \sum_{j=1}^{k} D_{x_j}\left(R_j\, F\right), \tag{2.2}$$

where $a_i(n)$ are polynomials in n and $a_0(n) = 1$.

By integrating both sides of (2.2) with respect to $x_1, \ldots, x_k$, we get

$$\sum_{i=0}^{L} a_i(n)\, f(n+i) = 0.$$

The identity (2.1) follows once the initial conditions $f(n) = 1$, $n = 0, 1, \ldots, L-1$ are checked. Computational experience shows that the algorithm which is described below is usually successful in getting a WZ equation with $L = 1$.

For the proof of the fundamental theorem and its beautiful theoretical aspects see [WZ92].

The algorithm that implements the continuous multi-WZ method is summarized as follows.

- INPUT:
  - A *proper-hypergeometric* function $F(n; \mathbf{x})$, where $\mathbf{x} = (x_1, \ldots, x_k)$;
  - Variable names: $n, x_1, \ldots, x_k$;
  - Maximal order of the required recurrence operator in n, i.e., degree with respect to E_n (optional, default = 6).

- OUTPUT:

  A non-zero recurrence-differential equation (WZ-equation):

$$\sum_{i=0}^{\text{order}} a_i(n)\, F(n+i, \mathbf{x}) = \sum_{j=1}^{k} D_{x_j}(R_j\, F)$$

  if it exists; 0 otherwise.

- DESCRIPTION:

  1. Set up the WZ (rational) equation :

$$\sum_{i=0}^{\text{order}} a_i(n) \frac{E_n^i F}{F} - \sum_{j=1}^{k} \frac{1}{F} D_{x_j} \left[\frac{p_j(n, \mathbf{x})}{q_j(n, \mathbf{x})} F \right] = 0$$

$i=0$ $j=1$

with unknowns $a_i(n)$ and $p_j(n, \mathbf{x})$, where

$$p_j(n, \mathbf{x}) = \sum_{\mathbf{l}=\mathbf{0}}^{\mathbf{M}_j} b_{j,\mathbf{l}}(n)\mathbf{x}^{\mathbf{l}},$$

$\mathbf{x}^{\mathbf{l}} = \prod_{i=1}^{k} x_i^{l_i}$, $\mathbf{l} = (l_1, \ldots, l_k)$, $\mathbf{M}_j = (M_{1,j}, \ldots, M_{k,j})$, and replace $q_j(n, \mathbf{x})$ by 1.

2. Clear the denominators in (1) and equate the coefficients of the monomials in $\mathbf{x}$ to get a homogeneous system of linear equations in the unknowns a_i and $b_{j,\mathbf{l}}$.

3. Solve the resulting system.

4. If a non-zero solution is obtained, then stop. If not, increase degrees of $p_j(n, \mathbf{x})$, replace $q_j(n, \mathbf{x})$ by next "best" conceivable value by looking at the factors of the denominators of $D_{x_j} \log(F)$, and then go back to step 3.

Remarks:

(i). The fundamental theorem of WZ *guarantees* the algorithm's eventual success. One just needs to keep increasing the order of the recurrence operator.

(ii). Computational experience shows that, for many real life examples (as can be seen in Chapter 3), the computer time is not prohibitive.

(iii). If $F(n, \mathbf{x})$ is *symmetric* in $x = (x_1, \ldots, x_k)$, then the efficiency of the algorithm can be improved by looking for a rational function

$$R(u; v_1, \ldots, v_{k-1}) := \frac{p(u; v_1, \ldots, v_{k-1})}{q(u; v_1, \ldots, v_{k-1})}$$

instead of a $k-$*tuple* of rational functions. To this end, set the WZ (rational) equation

$$\sum_{i=0}^{order} a_i(n) \frac{E_n^i F}{F} - \sum_{j=1}^{k} \frac{1}{F} D_{x_j} \left[\frac{p(x_j; \hat{\mathbf{x}}_j)}{q(x_j; \hat{\mathbf{x}}_j)} F \right] = 0,$$

where

$$\hat{\mathbf{x}}_j := \begin{cases} (x_2, \ldots, x_{k-1}) & \text{for } j = 1 \\ (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{k-1}) & \text{for } 1 < j < k-1 \\ (x_1, \ldots, x_{k-2}) & \text{for } j = k-1 \end{cases}$$

and

$$p(u; v_1, \ldots, v_{k-1}) := \sum_S b_{\mathbf{i},j}(n) u^j \prod_{m=1}^{k-1} e_m(v_1, \ldots, v_{k-1})^{i_m},$$

where,

$$\mathbf{i} = (i_1, \ldots, i_{k-1}),$$

$$S = \{(i_1, \ldots, i_{k-1}, j) \in \mathbf{Z}^k_{\geq 0} : i_1 + 2\, i_2 + \cdots + (k-1)\, i_{k-1} \leq M_1, j \leq M_2\}.$$

2.3 Description of Mint and SMint

`Mint` and `SMint` are packages of Maple programs that implement the above algorithm. `Mint` stands for *Multiple integrals* and `SMint` for *Symmetric Multiple integrals*. Basically, both packages contain the same procedures, but `SMint` is applicable for proper-hypergeometric functions $F(n, \mathbf{x})$ which are *symmetric* in $\mathbf{x}$. Below we give a description of the main procedures that are contained in the package `Mint`.

There are two versions of `Mint`. The first one is for Maple V release 2 or 3, which is stored by the name `Mint3`, the second one is for Maple V release 4 and above, which is stored by the name `Mint5`. Both of them are available from

`http://www.math.temple.edu/~akalu/maplepack/`

They are loaded into the Maple session by typing `read 'Mint3'` or `read 'Mint5'`.

```
> read `Mint5`;

        Mint: a Maple package for Multiple Integration
             of proper-hypergeometric functions by
        the continuous version of the multi-WZ method.

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 Please report all bugs and comments to: akalu@math.temple.edu

                For a list of procedures, type:
                     ?Mint or  help(Mint)

           For help with a specific procedure, type:
           ?procedure name  or  help(procedure name)

    CAUTION: this version of Mint is for Maple V Release 5

>
```

2.3.1 findrecl

The function **findrecl** finds a WZ-equation that is satisfied by a given proper-hypergeometric function. **findrecl** can be called in several ways:

- findrecl(*integrand, n, intnvars, auxiliary_vars, recurrence_order, denom-_poly*); tries to find a WZ-equation that is satisfied by the proper-hypergeometric *integrand*. The program searches for a recurrence operator in n of order is equal to *recurrence_order* and looks for a k-tuple of rational functions (the certificate), where k is equal to the number of integration variables. Here the user inputs and guesses the denominators *denom_poly* of the rational functions.

- `findrecl(`*integrand, n, intnvars, auxiliary_vars, recurrence_order*`);` tries to find a non-zero WZ-equation for *integrand*, but searching for possible denominators of the rational certificates is done automatically.

- `findrecl(`*integrand, n, intnvars, auxiliary_vars, denom_poly*`);` is like both of the above **findrecl**(*arguments*), but looks for a non-zero recurrence operator whose order in E_n is at most 6 (the default maximum order) and rational certificates with denominators *denom_poly*.

- `findrecl(`*integrand, n, intnvars, auxiliary_vars*`);` and `findrecl(`*integrand, n, intnvars*`);` look for a non-zero recurrence operator whose order in E_n is at most 6.

The following example illustrates how the function is used.

Example: Prove

$$\frac{1}{(2\,\pi\,i)^3}\int_C\int_C\int_C \frac{(x+y+z)^n}{x^{m+1}\,y^{k+1}\,z^{n-m-k+1}}\,dx\,dy\,dz = \frac{n!}{m!\,k!\,(n-m-k)!}$$

where C a circle surrounding the origin.

Proof:

Let

$$\begin{aligned} F := F(n,m,k,x,y,z) &:= \frac{(x+y+z)^n\,m!\,k!\,(n-m-k)!}{(2\,\pi\,i)^3\,n!\,x^{m+1}\,y^{k+1}\,z^{n-m-k+1}}, \\ M(n,m,k) &:= \int_C\int_C\int_C F\,dx\,dy\,dz \end{aligned}$$

We want to show that $M(n,m,k) = 1$ for all m, n and k in $\mathbf{Z}_{\geq 0}$, where $m \leq n$ and $k \leq n$.

`findrec1(F,n,[x,y,z],[m,k],1);` gives the output:

$$\mathrm{F\_}[n,x,y,z]-\mathrm{F\_}[n+1,x,y,z]=\mathrm{D\_}[x,-\frac{x}{n+1}\mathrm{F\_}[n,x,y,z]]+$$
$$\mathrm{D\_}[y,-\tfrac{y}{n+1}\mathrm{F\_}[n,x,y,z]]+\mathrm{D\_}[z,\tfrac{x+y}{n+1}\mathrm{F\_}[n,x,y,z]]$$

and the result follows by triple contour integration with respect to x, y and z and the fact that $M(0,0,0)=1$. □

Remark:

(i). In the computer output, F_[*variables*] means F(*variables*) and D_[x, G] means $\mathrm{D}_x(G)=\frac{\partial}{\partial x}G$. We used this notation for the sake of convenience and to avoid conflict with the Maple built in global variables.

(ii). Without *recurrence_order* specification, **findrec1** searches for a non-zero WZ-equation, starting from order zero all the way up to order 6 (the default). Depending on the *integrand*, this may take some time before `findrec1` gets the required WZ-equation. Therefore, we recommend that the user inputs the expected order of the WZ-equation.

2.3.2 findrec2 and findrec3

The function `findrec2` is similar to `findrec1` but is targeted to find a non-zero recurrence operator in $\mathbf{n}=(n_1,\ldots,n_m)$, $m>1$.

The function **findrec3** tries to find a non-zero WZ-equation for the *integrand* by using a given *ansatz* (a list of monomials in forward-shift variables). It can be called in either of the following formats.

`findrec3(`*integrand, n, forward_shift_vars, intnvars, auxiliary_vars, ansatz, denom_poly*`);`

`findrec3(`*integrand, n, forward_shift_vars, intnvars, auxiliary_vars, ansatz*`);`

Examples: Let $F := F(n, m, x, y) := \frac{1}{(1 - x - x^2 - y - y^2)\, x^{(m+1)}\, y^{(n+1)}}$.
Then `findrec2(F,[n,m],[x,y],[],[1,2]);` outputs in 7 seconds[1]

$$
\begin{aligned}
&(4\,m + 8 + 4\,n)\mathrm{F\_}[n, m, x, y] - (5\,m + 10)\mathrm{F\_}[n, m + 2, x, y] + \\
&\quad (2\,n + 2)\mathrm{F\_}[n + 1, m, x, y] + (8 + 2\,n + 4\,m)\mathrm{F\_}[n, m + 1, x, y] + \\
&\quad (n + 1)\mathrm{F\_}[n + 1, m + 1, x, y] = \mathrm{D\_}[x, -\frac{-5 + 4\,x^2 + 4\,x}{x}\mathrm{F\_}[n, m, x, y]] + \\
&\quad \mathrm{D\_}[y, -\frac{(2\,y + 1)(1 + 2\,x)}{x}\mathrm{F\_}[n, m, x, y]]
\end{aligned}
$$

Alternatively, `findrec3(F,[n,m],[N,M],[x,y],[],[N,M,N*M,M^2]);` outputs the above WZ-equation in 4 seconds. Even on this simple example the result is obtained noticeably quicker when the *orders* are input with *ansatz*.

2.3.3 checkrec

The function **checkrec** takes a WZ-equation which involves the symbols F_ or D_, and returns **true** if the given function satisfies the equation, **false** otherwise.

Example: If **rec** is the above WZ-equation then the call `checkrec(rec,F);` outputs **true**.

2.3.4 sumtointn and msumtointn

Using Egorychev's [Eg84] method, every binomial coefficient sum or multi-sum can be expressed as a contour or multi-contour integral.

Example: Consider

$$S(n) = \sum_{k=0}^{n} (-1)^k \binom{2\,n + 1}{2\,k}.$$

---

[1] This and others computer running time mentioned in the paper are based on the computations which were done in Maple V release 3 on HDS work station.

An integral representation of $\binom{a}{b}$:

$$\binom{a}{b} = \mathrm{CT}\left((1+z)^a z^{-b}\right) = \frac{1}{2\pi i}\int_{|z|=r} (1+z)^a z^{-b-1}\, dz, 0 < r < \infty.$$

Thus,

$$\begin{aligned} S(n) &= \sum_{k=0}^{\infty} (-1)^k \mathrm{CT}_z \left((1+z)^{2n+1} z^{2k}\right) \\ &= \mathrm{CT}\left(\sum_{k=0}^{\infty} (-1)^k (1+z)^{2n+1} z^{2k}\right) \\ &= \mathrm{CT}\left(\frac{w^2 (1+w)^{2n+1}}{1+w^2}\right). \end{aligned}$$

Therefore by interfacing Egorychev's method with the continuous multi-WZ method we get an alternative approach to sums. One of the advantages is that we get new *companion identities* quite different from the ones one gets by the direct approach. For example, Dixon's identity

$$\sum_k (-1)^k \binom{a+b}{a+k}\binom{a+c}{c+k}\binom{b+c}{b+k} = \frac{(a+b+c)!}{a!\, b!\, c!}$$

can be written as the multi-contour integral

$$\frac{1}{(2\pi i)^2}\int_C\int_C \frac{(z_1+1)^{a+c}(z_2+1)^{b+c}(1-z_1 z_2)^{a+b}}{{z_1}^{2a+1}\, {z_2}^{a+c+1}} = (-1)^a \frac{(a+b+c)!}{a!\, b!\, c!}$$

By using Egorychev's approach, the functions `sumtointn` and `msumtointn` find a constant term (CT) expression for a given sum.

sumtointn:

`sumtointn` finds a constant term expression for a single-sum of the form

$$S(n) = \sum_k \left[\prod_{i=1}^{N} \binom{a_i}{b_i}\right] y^k,$$

i.e., it looks for a function $F(z_1, \ldots, z_t)$ such that $S(n) = CT(F)$ and outputs:

$$F, [z_1, \ldots, z_t].$$

The calling syntax is:

`sumtointn(`*bino_coeffs*, x, k , `[`m,n`]);`

where *bino_coeffs* is the (finite) product of binomial coefficients in the summand, x takes the remaining part of the summand, k is the summation index and `[`m,n`]` is the range of the summation index.

For the sake of convenience, `binomial(`a,b`)` should be input as `[`a,b`]`.

Example: To find a CT expression for Dixon's sum

$$\sum_k (-1)^k \binom{2n}{k}^3,$$

we make the call: `sumtointn([2*n,k]^3,(-1)^k,k,[0,infinity]);` and get

$$\left(\frac{(z_1+1)(z_2+1)(1-z_1 z_2)}{z_1 z_2}\right)^{2n}, [z_1, z_2]$$

This means

$$\sum_k (-1)^k \binom{2n}{k}^3 = \mathrm{CT}\left(\left(\frac{(z_1+1)(z_2+1)(1-z_1 z_2)}{z_1 z_2}\right)^{2n}\right).$$

msumtointn:

`msumtointn` finds a constant term expression for a multi-sum of the form

$$S(n) = \sum_{k_1,\ldots,k_m} \left[\prod_{i=1}^{N} \binom{a_i}{b_i}\right] \mathbf{y}^{\mathbf{k}}.$$

The calling syntax is:

`msumtointn(`*bino_coeffs*, x, `[`k_1,...,k_l`],[[`m_1,n_1`],...,[`m_l,n_l`]]);`,

where $k_1, \ldots, k_l$ are the summation indices and `[`m_i,n_i`]` $(i = 1, \ldots, l)$ are the corresponding ranges of the summation indices.

Example: Let

$$S(n) := \sum_k \sum_j \binom{n}{k}\binom{n+k}{k}\binom{k}{j}^3.$$

Then the call:

```
msumtointn([n,k]*[n+k,k]*[k,j]^3,1,[k,j],[[0,infinity],
[0,infinity]]);
```

outputs:

$$\frac{((z_1+1)(2\,z_2+z_2\,z_1+z_1+1))^n}{(z_1\,z_2)^n(z_2-1)},[z_1,z_2]$$

2.3.5 ssum and msum

By interfacing Egorychev's method with the continuous multi-WZ method, the functions **ssum** and **msum** find a non-zero WZ-equation for the constant term expression of a given sum, i.e., if

$$\begin{aligned}\sum_{\mathbf{k}}\mathrm{SUMMAND}(n,\mathbf{k}) &= \mathrm{CT}(F(z_1,\ldots,z_t)) \\ &= \int_C\cdots\int_C\frac{F}{(2\pi i)^t\prod_{j=1}^t z_i}\,dz_1\cdots dz_t,\end{aligned}$$

where $\mathrm{SUMMAND}(n,\mathbf{k}) = \left[\prod_{i=1}^N\binom{a_i}{b_i}\right]\eta^{\mathbf{k}}$, then **ssum** and **msum** find a non-zero WZ-equation for $\dfrac{F}{(2\pi i)^t\prod_{j=1}^t z_i}$.

ssum:

ssum finds a non-zero WZ-equation for the constant term expression of a given single sum. It can be called in either of the following forms.

ssum(*bino_coeffs, x, sumvar, mainvar, sum_bound, order*);

ssum(*bino_coeffs, x, sumvar, mainvar, sum_bound*);

Example: Let

$$S(a,b,c) := \sum_k(-1)^k\binom{a+b}{a+k}\binom{a+c}{c+k}\binom{b+c}{b+k}.$$

Then

ssum([a + b, a + k] * [a + c, c + k] * [b + c, b + k], (−1)^k, k, a, [−a, infinity], 1);

outputs:

$$(a+1+b+c)\mathrm{F\_}[a, z_1, z_2] - (a+1)\mathrm{F\_}[a+1, z_1, z_2] =$$
$$\mathrm{D\_}\left[z_1, \frac{(z_1+1)(2\,z_2\,z_1-1)\mathrm{F\_}[a, z_1, z_2]}{2\,z_2\,z_1}\right] + \mathrm{D\_}\left[z_2, -\frac{(z_2+1)\mathrm{F\_}[a, z_1, z_2]}{2\,z_1}\right],$$
$$\mathrm{F\_}[a, z_1, z_2] = \frac{(-1)^{(a+1)} z_1^{(c-a)} z_2^{(b-a)} \left(\frac{z_1+1}{z_1}\right)^{(a+c)} \left(\frac{z_2+1}{z_2}\right)^{(b+c)} (1-z_2\,z_1)^{(a+b)}}{4\pi^2\,z_2\,z_1}$$

msum:

msum finds a non-zero WZ-equation for the constant term expression of a given multi-sum. It can be called in either of the following forms.

msum(*bino_coeffs*, *x*, *sumvars*, *mainvar*, *sum_bound*, *order*);

msum(*bino_coeffs*, *x*, *sumvars*, *mainvar*, *sum_bound*);

Example: Let

$$S(n) := \sum_i \sum_j (-1)^{i+j} \binom{i+j}{i} \binom{n}{i} \binom{n}{j}.$$

Then the call

```
msum([i+j,i]*[n,i]*[n,j],(-1)^(i+j),[i,j],n,[[0,infinity],
[0,infinity]],1);
```

outputs:

$$\mathrm{F\_}[n, z_1] - \mathrm{F\_}[n+1, z_1] = \mathrm{D\_}[z_1, 0], \mathrm{F\_}[n, z_1] = \frac{1}{2\,\pi\,i\,z_1}$$

2.4 The WZ Proof Procedure

In this section we would overview the WZ proof procedure by considering two important special cases.

2.4.1 Type I Identities

Suppose we have to prove an identity of the form

$$\int \cdots \int F(n; x_1, \ldots, x_r)\, dx_1 \cdots dx_r = \text{Answer}(n), \qquad n \in \mathbf{Z}_{\geq 0},$$

where Answer$(n) \neq 0$ for all $n \in \mathbf{Z}_{\geq 0}$, and F is a *proper-hypergeometric* function in $n, x_1, \ldots, x_r$.

- Divide through by Answer(n) to get

  $$\int \cdots \int \frac{F(n; x_1, \ldots, x_r)}{\text{Answer}(n)}\, dx_1 \cdots dx_r = 1.$$

  Let

  $$G(n; x_1, \ldots, x_r) = \frac{F(n; x_1, \ldots, x_r)}{\text{Answer}(n)}.$$

  Hence the given problem reduces to proving

  $$\int \cdots \int G(n; x_1, \ldots, x_r)\, dx_1 \cdots dx_r = 1. \tag{2.3}$$

- Define

  $$f(n) = \int \cdots \int G(n; x_1, \ldots, x_r)\, dx_1 \cdots dx_r.$$

  We want to prove

  $$f(n) = 1 \qquad \forall n \in \mathbf{Z}_{\geq 0}$$

  i.e.,

  $$\Delta_n f(n) = 0, f(0) = 1. \tag{2.4}$$

- A good way to certify (2.4) would be to display a *WZ equation*:

$$\Delta_n G = \sum_{i=1}^{r} \frac{\partial}{\partial x_i} H_i, \tag{2.5}$$

where $H_i = R_i F$, and R_i is a rational function of $n, x_1, \ldots, x_r$. For then we would simply integrate with respect to $x_i, (i = 1, \ldots, r)$ to get

$$\int \cdots \int \Delta_n G(n; x_1, \ldots, x_r)\, dx_1 \cdots dx_r = \sum_{i=1}^{r} \int \cdots \int \frac{\partial}{\partial x_i} H_i = 0$$

Hence

$$\Delta_n f(n) = 0.$$

- Hence (2.5), together with the trivially verifiable case $n = 0$, implies (2.3).

Remark:

Note that the creative and central part of the WZ proof procedure is the production of (2.5) (the WZ equation), but this is done by the computer using the packages `Mint` or `SMint`. All we have to do is to input G to `Mint` and then `Mint` will deliver us (2.5).

2.4.2 Type II Identities

Suppose we have to prove an identity of the form

$$\int \cdots \int F(n; x_1, \ldots, x_r)\, dx_1 \cdots dx_r = \int \cdots \int G(n; y_1, \ldots, y_s)\, dy_1 \cdots dy_s \tag{2.6}$$

Let us call the left side $lhs(n)$ and the right side $rhs(n)$.

- Find WZ-equations for F and G:

$$P(E_n, n)\, F = \sum D_{x_i} (R_i F)\,,$$

$$\overline{P}(E_n, n)\, G = \sum D_{y_i} \left(\overline{R}_i G\right).$$

- From the WZ-equations, get the operators $A(E_n, n)$ and $\overline{A}(E_n, n)$.

- If, as is usually the case, A and $\overline{A}$ are identical, this proves (2.6) once the trivially evaluable initial conditions $lhs(n) = rhs(n)$, $n = 0, 1, \ldots, order(A) - 1$, are checked. In the rare event that A and $\overline{A}$ are different, one can use the Euclidean algorithm (adapted to the non-commutative ring of linear recurrence operators with polynomial coefficients) to find a "minimal" operator $B(E_n, n)$ such that A and $\overline{A}$ are left multiples of it. It follows that both $lhs(n)$ and $rhs(n)$ are annihilated by $B(E_n, n)$ if it is true up to $max(order(A), order(\overline{A}))$.

In the next chapter we give a number of examples of identities of the above types and their computerized proofs.

CHAPTER 3

EXAMPLES OF COMPUTER GENERATED PROOFS

In this chapter we give various examples to show how one can systematically use the packages **Mint** and **SMint** to generate proofs of identities (or recurrences) which involve multiple integrals of *proper-hypergeometric* functions.

3.1 An Identity Equivalent to the Pfaff-Saalschütz Identity

Theorem 3.1

$$\frac{(1+x)^k(1+y)^l}{(1-x\,y)^{k+l+1}} = \sum_{m,n\geq 0} \binom{k+n}{m}\binom{l+m}{n} x^m\, y^n \qquad k, l \in \mathbf{Z}_{\geq 0}.$$

Proof:

To prove the above identity, we use the constant term (CT) approach.
Fix $m,\ n \in \mathbf{Z}_{\geq 0}$ and let $f(k,l,x,y) := \frac{1}{\binom{k+n}{m}\binom{l+m}{n}} \frac{(1+x)^k(1+y)^l}{(1-x\,y)^{(k+l+1)}x^m\,y^n}$. Then $\mathrm{CT}(f(k,l,x,y))$ is given by

$$\mathrm{CT}(f(k,l,x,y)) \;\; := \;\; \frac{1}{(2\pi i)^2}\int_C\int_C \frac{f(k,l,x,y)}{x\,y}\,dx\,dy,$$

where $C = \{z : |z| = r\}$ and $0 < r < 1$.

Thus, we want to show $\mathrm{CT}(f(k,l,x,y)) = 1$.
Let $F := F(k,l,x,y) := \frac{f(k,l,x,y)}{x\,y}$ and $a(k,l) := \frac{1}{(2\pi i)^2}\int_C\int_C F\,dx\,dy$.

We want to prove that $a(k,l) = 1$ for all k, l in $\mathbf{Z}_{\geq 0}$. But, by symmetry of k and l, it suffices to show $a(k,l) = 1$ for all k in $\mathbf{Z}_{\geq 0}$, i.e, $\Delta_k a(k,l) = 0$ for all k in $\mathbf{Z}_{\geq 0}$ and $a(0,l) = 1$.

By applying `findrecl`[1] we obtain the WZ-equation for F:

$$\Delta_k F = D_x (R_1 F) + D_y (R_2 F),$$

where

$$R_1 = \frac{x(1+x)}{(k+n+1)(1-xy)} \text{ and } R_2 = -\frac{xy(1+y)}{(k+n+1)(1-xy)}.$$

Hence, by contour integration with respect to x and y, we get $\Delta_k a(k,l) = 0$. To complete the proof, we evaluate $a(0,l)$. To this end, let $b(l) := a(0,l)$ and $G := F(0,l,x,y)$. Then G satisfies the WZ-equation[2]:

$$\Delta_l G = D_x (R_1 G) + D_y (R_2 G),$$

where

$$R_1 = -\frac{n\,x(1+y)}{(l+1)(m+l+1)} \text{ and } R_2 = \frac{y(1+y)}{l+1}.$$

Hence, by contour integration with respect to x and y, we get $\Delta_l b(l) = 0$. Since $b(0) = 1$, it follows that $b(l) = 1$ for all l in $\mathbf{Z}_{\geq 0}$. □

3.2 An Identity Equivalent to the Dixon's Identity

Theorem 3.2

$$\frac{1}{(2\pi i)^3} \int_C \int_C \int_C \frac{(z_1 - z_2)^{a+b}(z_3 - z_1)^{a+c}(z_2 - z_3)^{b+c}}{{z_1}^{2a+1}{z_2}^{2b+1}{z_3}^{2c+1}}\, dz_1\, dz_2\, dz_3 = (-1)^{a+b+c}\frac{(a+b+c)!}{a!\,b!\,c!},$$

where a, b and c are in $\mathbf{Z}_{\geq 0}$ and C is a circle around the origin.

[1] `findrecl(F,k,[x,y],[l,m,n]);` in 46 seconds

[2] `findrecl(G,l,[x,y],[m,n]);` in 2 seconds

Proof:

Let

$$F(a,b,c,z_1,z_2,z_3) := \frac{a!\,b!\,c!\,(-1)^{a+b+c}(z_1-z_2)^{a+b}(z_3-z_1)^{a+c}(z_2-z_3)^{b+c}}{(2\pi i)^3(a+b+c)!{z_1}^{2a+1}{z_2}^{2b+1}{z_3}^{2c+1}}$$

and

$$I(a,b,c) := \int_C\int_C\int_C F(a,b,c,z_1,z_2,z_3)\,dz_1\,dz_2\,dz_3.$$

We want to show that $I(a,b,c)=1$ for all a, b and c in $\mathbf{Z}_{\geq 0}$.

The function **findrec1** outputs the following WZ-equation[3]:

$$\Delta_a F = D_{z_1}(R_1 F) + D_{z_2}(R_2 F) + D_{z_3}(R_3 F),$$

where

$$R_1 = \frac{(-z_2z_3+2z_1z_3+2\,z_2\,z_1)}{2\,(a+b+c+1)z_1},\ R_2 = \frac{z_2\,(z_3+2\,z_1)}{2\,(a+b+c+1)z_1} \text{ and}$$

$$R_3 = \frac{z_3(z_2+2\,z_1)}{2\,(a+b+c+1)z_1}. \tag{3.1}$$

Then by triple contour integration with respect to z_1, z_2 and z_3, we get

$$\Delta_a I(a,b,c) = 0.$$

By the symmetry of the problem with respect to a, b and c, we get from (3.1)

$$\Delta_b I(a,b,c) = \Delta_c I(a,b,c) = 0.$$

Since $I(0,0,0)=1$, it follows that $I(a,b,c)=1$ for all a, b and c in $\mathbf{Z}_{\geq 0}$. □

3.3 The 3-Dimensional Dyson's Ex-conjecture

Theorem 3.3

$$CT\left(\prod_{\substack{1\leq i,j\leq 3\\ i\neq j}}\left(1-\frac{x_i}{x_j}\right)^a\right) = \frac{(3\,a)!}{a!^3} \qquad a\in\mathbf{Z}_{\geq 0}.$$

---

[3] `findrec1(F, a, [z1,z2,z3], [b,c], 1);` in 65 seconds.

Proof:

Let

$$T(a) := \frac{1}{(2\,\pi\,i)^3} \int_C \int_C \int_C \frac{1}{x_1\,x_2\,x_3} \frac{a!^3}{(3\,a)!} \prod_{\substack{1\le i,j\le 3 \\ i\ne j}} \left(1 - \frac{x_i}{x_j}\right)^a dx_1\,dx_2\,dx_3,$$

where C is a circle surrounding the origin.

We want to show that $T(a) = 1$ for all a in $\mathbf{Z}_{\geq 0}$. To this end, let

$$F := \frac{a!^3}{(2\,\pi\,i)^3\,x_1\,x_2\,x_3\,(3\,a)!} \prod_{\substack{1\le i,j\le 3 \\ i\ne j}} \left(1 - \frac{x_i}{x_j}\right)^a$$

Then the function `findrec1` delivers the following WZ-equation[4].

$$\Delta_a F = D_{x_1}(R_1\,F) + D_{x_2}(R_2\,F) + D_{x_3}(R_3\,F),$$

where $R_1 = R(x_1; x_2, x_3)$, $R_2 = R(x_2; x_1, x_3)$, $R_3 = R(x_3; x_1, x_2)$ and

$$R(x; y, z) = \frac{a\,y^2 - 2\,a\,y\,z + a\,z^2 - 2\,y\,z + z^2 + y}{6\,x\,(3\,a+1)(3\,a+2)} - \frac{3\,a\,y^3 - 6\,a\,y^2\,z - 6\,a\,y\,z^2 + 3\,a\,z^3 - 4\,y\,z^3 + 2\,y^3 - 4\,y^2\,z + 2\,z^3}{6\,y\,z\,(3\,a+1)(3\,a+2)}$$

The result follows by triple contour integration with respect to x_1, x_2 and x_3 and the fact that $T(0) = 1$. □

3.4 The Habsieger-Zeilberger G_2-Case of Macdonald's Conjecture

Theorem 3.4 *Let $g(m, n, x, y, z)$ be*

$$\left[\left(1-\frac{x}{y}\right)\left(1-\frac{y}{z}\right)\left(1-\frac{z}{x}\right)\right]^m \left[\left(1-\frac{x\,y}{z^2}\right)\left(1-\frac{x\,z}{y^2}\right)\left(1-\frac{y\,z}{x^2}\right)\right]^n$$
$$\left[\left(1-\frac{y}{x}\right)\left(1-\frac{z}{y}\right)\left(1-\frac{x}{z}\right)\right]^m \left[\left(1-\frac{z^2}{x\,y}\right)\left(1-\frac{y^2}{x\,z}\right)\left(1-\frac{x^2}{y\,z}\right)\right]^n$$

[4] `findrec1(F,a,[x1,x2,x3],[],1)` in 173 seconds.

and $h(m,n) := \dfrac{(3\,m+3\,n)!\,(3\,n)!\,(2\,m)!\,(2\,n)!}{(2\,m+3\,n)!\,(m+2\,n)!\,(m+n)!\,m!\,n!^2}$. *Then*

$$CT(g(m,n,x,y,z)) = h(m,n)$$

Proof:

Let

$$F := F(m,n,x,y,z) := \frac{g(m,n,x,y,z)}{(2\,\pi\,i)^3\,x\,y\,z\,h(m,n)}$$

and

$$G(m,n) := \int_C \int_C \int_C F\,dx\,dy\,dz.$$

We want to show that $G(m,n) = 1$ for all m and n in $\mathbf{Z}_{\geq 0}$.

The function **findrec1** delivers the following WZ-equation[5].

$$\Delta_m F = D_x\,(R_1\,F) + D_y\,(R_2\,F) + D_z\,(R_3\,F)\,,$$

where $R_1 = R(x;y,z)$, $R_2 = R(y;x,z)$, $R_3 = R(z;x,y)$ and

$$R(u;v,w) = \frac{1}{12\,u\,(2\,m+1)(3\,m+3n+2)(3\,m+3\,n+1)^2\,v^2\,w}\big(6\,v^4\,w^2\,m+$$
$$10\,u^3\,v^3\,m^2 + 18\,u^3\,v^3\,n^2 + 13\,u^3\,w^3\,m - 4\,u^3\,v\,w^2 + 4\,v^2\,w^3\,u - 4\,u\,v\,w^4 +$$
$$-4\,v^4\,u\,w + 4\,v^3\,w^2\,u - 4\,u^3\,v^2\,w + 12\,v^4\,w^2\,n^2 + 4\,v^2\,w\,m^2 + 18\,u^3\,w^3\,n +$$
$$5\,u^2\,w^4\,n + 3\,u^2\,w^4\,m + 6\,u^2\,w^4\,n^2 + 12\,v^2\,w^4\,n + 18\,u^3\,w^3\,n^2 + 2\,v^2\,w^4 +$$
$$v^4\,u^2 + 2\,v^4\,w^2 + 4\,u^3\,v^3 + 4\,u^3\,w^3 + u^2\,w^4 + 10\,u^3\,w^3\,m^2 + 5\,v^4\,u^2\,n +$$
$$10\,v^4\,w^2\,n + 10\,v^2\,w^4\,n + 6\,v^2\,w^4\,m + 2\,u^2\,w^4\,m^2 + 4\,v^2\,w^4\,m^2 + 3\,v^4\,u^2\,m +$$
$$2\,v^4\,u^2\,m^2 + 27\,u^2\,v^3\,m\,n + 6\,v^2\,w^3\,u\,n^2 + 19\,v^2\,w^3\,u\,m\,n + 12\,v^2\,w^3\,u\,m^2 +$$
$$11\,v^2\,w^3\,u\,n + 14\,v^2\,w^3\,u\,m + 6\,v^3\,w^2\,u\,n + 19\,v^3\,w^2\,u\,m\,n + 11\,v^3\,w^2\,u\,n +$$
$$14\,v^3\,w^2\,u\,m - 6\,u^3\,v\,w^2\,n - 27\,u^3\,w^3\,m\,n + 19\,u^3\,v\,w^2\,m\,n + 18\,u^3\,v^3\,n -$$

---

[5] `findrec1(F,m,[x,y,z],[n],1);` in 3062 seconds, and `findrec1(F,m,[x,y,z],[n],1,[x^2*y^2*z^2, x^2*y^2*z^2, x^2*y^2*z^2]);` in 1786 seconds.

$$12\,u^3\,v^2\,w\,m^2 - 11\,u^3\,v^2\,w\,n - 14\,u^3\,v^2\,w\,m + 19\,u^3\,v^2\,w\,m\,n - 12\,u^3\,v^2\,w\,m^2 - 11\,u^3\,v^2\,w\,n - 14\,u^3\,v^2\,w\,m + 4\,v^4\,w^2\,m\,n - 27\,v^4\,u\,w\,m\,n - 18\,v^4\,u\,w\,n^2 + 7\,u^2\,w^4\,m\,n - 18\,u\,v\,w^4\,n^2 - 10\,u\,v\,w^4\,m^2 - 18\,u\,v\,w^4\,n - 13\,u\,v\,w^4\,m - 27\,u\,v\,w^4\,m\,n + 14\,v^2\,w^4\,m\,n - 10\,v^4\,u\,w\,m - 18\,v^4\,u\,w\,n - 13\,v^4\,u\,w\,m - 6\,u^3\,v^2\,w\,n^2 + 7\,v^4\,u^2\,m\,n + 6\,v^4\,u^2\,n^2 + 13\,u^3\,v^3\,m + 12\,v^3\,w^2\,u\,m\Big).$$

By triple contour integration with respect to x, y and z, we get $\Delta_m G(m,n) = 0$. To complete the proof, set $G(n) := G(0,n)$ and $F := F(0,n,x,y,z)$, and then we show $G(n) = 1$ for all n in $\mathbf{Z}_{\geq 0}$.

F satisfies the WZ-equation[6]:

$$\Delta_n F = D_x\,(R_1\,F) + D_y\,(R_2\,F) + D_z\,(R_3\,F)\,,$$

where $R_1 = R(x;y,z)$, $R_2 = R(y;x,z)$, $R_3 = R(z;x,y)$ and

$$R(u;v,w) = \frac{1}{108\,n\,(3\,n+2)(3\,n+1)u^3\,w^3\,v^3}\Big(- 4\,u^4\,v^6\,n - 5\,u^4\,v^6\,n - 4\,u^4\,w^6\,n + 10\,v^3\,u^7\,n^2 + 8\,v^3\,u^7\,n + 54\,u^3\,w^5\,v^2\,n^2 - 20\,u\,w^6\,v^3\,n^2 - 16\,u\,w^6\,v^3\,n + 36\,u^3\,w^5\,v^2\,n + 90\,u^5\,w\,v^4\,n + 84\,u^5\,w\,v^4\,n^2 + 24\,u^5\,w\,v^4 + 10\,w^3\,u^7\,n^2 + 8\,w^3\,u^7\,n + 9\,u^2\,v\,w^7\,n + 36\,u^3\,v^5\,w^2\,n + 9\,u^2\,v\,w^7\,n^2 + 54\,u^3\,v^5\,w^2\,n^2 - 16\,u\,v^6\,w^3\,n + 90\,u^5\,v\,w^4\,n + 84\,u^5\,v\,w^4\,n^2 + 9\,v^5\,w^5\,n + 9\,u^3\,v^5\,w^2\,n^2 + 24\,u^5\,v\,w^4 - 20\,u\,v^6\,w^3\,n^2 + 9\,u^2\,v^7\,w\,n + 126\,u^2\,v^4\,w^4\,n + 138\,u^2\,v^4\,w^4\,n^2 - 5\,u^4\,w^6\,n^2 + 24\,u^2\,v^4\,w^4 + 9\,v^5\,w^5\,n^2\Big).$$

By triple contour integration with respect to x, y and z, we get $\Delta_n G(n) = 0$. Since $G(0) = 1$, it follows that $G(n) = 1$ for all n in $\mathbf{Z}_{\geq 0}$. □

Remark: The above identity was conjectured in [Ma82] and [Mo82] and proved independently, simultaneously and humanly by Laurent Habsiger [H86]

[6] `findrec1(F,n,[x,y,z],[],1);` in 9506 seconds, `findrec1(F,n,[x,y,z],[],1,[x^3*y^3*z^3,x^3*y^3*z^3,x^3* y^3*z^3]);` in 3615 seconds.

and Doron Zeilberger [Z87]. Then the first computer proof for the two variable version of the identity was given by Shalosh B. Ekahd [Ek91], but the method used was ad-hoc.

3.5 The 3-Dimensional Mehta-Dyson Integral

Theorem 3.5

$$\frac{1}{(2\,\pi)^{3/2}}\int_{(-\infty,\infty)^3} e^{-\sum_{j=1}^{3} x_j{}^2/2} \prod_{1\le i<j\le 3} (x_i - x_j)^{2n}\, dx_1\, dx_2\, dx_3 = \prod_{j=1}^{3} \frac{(n\,j)!}{n!}$$

Proof:

Let

$$F := F(n, x_1, x_2, x_3) := \frac{e^{-\sum_{j=1}^{3} x_j{}^2/2} \prod_{1\le i<j\le 3}(x_i - x_j)^{2n}}{(2\,\pi)^{3/2}} \prod_{j=1}^{3} \frac{n!}{(n\,j)!}$$

and $\mathrm{MD}(n) := \int_{(-\infty,\infty)^3} F\, dx_1\, dx_2\, dx_3$. Then F satisfies the WZ-equation[7]:

$$\Delta_n F = D_{x_1}(R_1\, F) + D_{x_2}(R_2\, F) + D_{x_3}(R_3\, F)\,,$$

where $R_1 = R(x_1; x_2, x_3)$, $R_2 = R(x_2; x_1, x_3)$, $R_3 = R(x_3; x_1, x_2)$ and

$$R(u; v, w) = -\frac{1}{36\,(3\,n+2)(3\,n+1)(2\,n+1)}\big(709\,v^2\,w + 304\,u^2\,w\,n^2 + 304\,u^2\,v\,n^2 -$$
$$4\,u^2\,w^3 + 709\,v\,w^2 - 393\,u\,w^2 + 24\,u + 1002\,v + 1002\,w - 4\,v^4\,w^3 + 567\,u^2\,w -$$
$$574\,v^3 - 574\,w^3 - 28\,u^3\,v^2\,n + 786\,u^2\,v\,n + 786\,u^2\,w\,n + 270\,u^3 + 480\,n^2\,w\,v^2 -$$
$$64\,n^2\,u\,v^2 - 562\,n\,w^2\,u - 562\,n\,u\,v^2 - 32\,v^3\,w^2\,n + 2112\,n\,u + 2946\,n\,v +$$
$$92\,u^2\,v\,w^2 - 64\,n^2\,w^2\,u + 1346\,n\,w\,v^2 + 1346\,n\,w^2\,v + 567\,u^2\,v + 2946\,n\,w +$$
$$135\,u\,v^4 - 173\,v^4\,w + 108\,u^2\,v\,w^2\,n - 18\,u^2\,v\,w^4 + 22\,u^3\,v\,w^3 - 10\,u\,v^2\,w^4 -$$
$$28\,u^3\,w^2\,n + 36\,u\,w^4\,n - 2\,u^2\,v^2\,w^3 + 20\,u^3\,v^2\,w^2 - 2\,u^2\,v^3\,w^2 + 24\,u\,v^3\,w\,n +$$
$$8\,u^2\,w^3\,n + 135\,u\,w^4 - 139\,u^3\,w^2 - 4\,w^4\,v^3 + 36\,u\,v^4\,n + u\,v\,w^3 + 24\,u\,v\,w^3\,n -$$

[7] `findrec1(F,n,[x1,x2,x3],[],1);` in 416 seconds.

$32\,v^2\,w^3\,n + u\,v^3\,w - 173\,v\,w^4 - 68\,v\,w^4\,n - 68\,v^4\,w\,n + 384\,u\,n^3 + 1992\,u\,n^2 +$

$2076\,w\,n^2 + 384\,w\,n^3 + 2076\,v\,n^2 + 688\,u^3\,n + 384\,v\,n^3 - 640\,w^3\,n + 208\,u^3\,n^2 -$

$160\,w^3\,n^2 - 18\,u^2\,v^4\,w + 4\,u\,v^3\,w^3 + 22\,u^3\,v^3\,w + 64\,u\,v^2\,w^2\,n + 48\,u\,v^2\,w^2 +$

$92\,u^2\,v^2\,w - 176\,u\,v\,w\,n^2 - 292\,u\,v\,w\,n + 128\,u^3\,v\,w\,n + 262\,u^3\,v\,w - 19\,v^3\,w^2 +$

$480\,n^2\,w^2\,v - 264\,u\,v\,w - 10\,w^2\,u\,v^4 - 160\,v^3\,n^2 - 19\,v^2\,w^3 - 640\,v^3\,n +$

$108\,u^3\,v^3\,w\,n - 393\,u\,v^2 - 139\,u^3\,v^2 - 4\,u^2\,v^3 + 8\,u^2\,v^3\,n\Big)$

By triple integration with respect to x_1, x_2 and x_3, we get $\mathrm{MD}(n+1)-\mathrm{MD}(n) = 0$. Since $\mathrm{MD}(0) = 1$, it follows that $\mathrm{MD}(n) = 1$ for all n in $\mathbf{Z}_{\geq 0}$. □

3.6 Problem Number 10777 of the MONTHLY

Using `Mint` we give a complete solution of the following problem [P00].

Problem: For nonnegative integers m and n, evaluate

$$\int_0^\infty \frac{d^m}{dx^m}\left(\frac{1}{1+x^2}\right)\frac{d^n}{dx^n}\left(\frac{1}{1+x^2}\right)\,dx$$

Solution:

Let $a(m,n) := \int_0^\infty \frac{d^m}{dx^m}\left(\frac{1}{1+x^2}\right)\frac{d^n}{dx^n}\left(\frac{1}{1+x^2}\right)\,dx$. We will show that

$$a(m,n) = \begin{cases} (-1)^n \frac{(n+m)!\,\pi}{4}\left(\frac{i}{2}\right)^{n+m} & \text{if } n+m \text{ is even} \\ (-1)^{\frac{m+n+1}{2}} \sum_{k=0}^{\frac{n-1}{2}} (m+2\,k)!(n-2\,k-1)! - \\ \frac{(-1)^{\frac{m+n+1}{2}}(m+n)!}{2^{m+n}}\left(1+\sum_{k=0}^{\frac{m+n-1}{2}-1}\frac{2^{2\,k+1}(3\,k+5)}{(2\,k+3)(k+2)}\right) & \text{if } n+m \text{ is odd} \end{cases}$$

Case 1: $m+n$ **even.**

We have

$$a(m,n) = \frac{1}{2}\int_{-\infty}^\infty \frac{d^m}{dx^m}\left(\frac{1}{1+x^2}\right)\frac{d^n}{dx^n}\left(\frac{1}{1+x^2}\right)\,dx.$$

By integration by parts we get

$$a(m,n) = \frac{(-1)^n}{2}\int_{-\infty}^{\infty}\frac{1}{1+x^2}\frac{d^{n+m}}{dx^{n+m}}\left(\frac{1}{1+x^2}\right)dx = (-1)^n a(0,m+n).$$

Let $b(n) := a(0,n)$, where n is a nonnegative integer. Then the Cauchy integral formula yields

$$\begin{aligned} b(n) &= \frac{n!}{4\pi i}\int_{-\infty}^{\infty}\int_{|z-x|=\frac{1}{2}}\frac{1}{(1+x^2)(1+z^2)(z-x)^{n+1}}\,dz\,dx \\ &= \frac{n!}{4\pi i}\int_{-\infty}^{\infty}\int_{|z|=\frac{1}{2}}\frac{1}{(1+x^2)(1+(z+x)^2)z^{n+1}}\,dz\,dx \end{aligned}$$

Now set $F(n;x,z) := \dfrac{n!}{(1+x^2)(1+(z+x)^2)\,z^{n+1}}$. Then $F(n;x,z)$ satisfies the WZ-equation[8]:

$$4\,F(n+2;x,z) + (n+2)(n+1)\,F(n;x,z) = D_x\,(F\,R_1) + D_z\,(F\,R_2) \qquad \text{(WZ)}$$

where

$$R_1 = \frac{(n+1)(1+x^2)(2\,n\,x+4\,x+3\,n\,z+7\,z)}{z^2\,(n+3)}$$

and

$$R_2 = -\frac{(n+1)(n\,z^2+4\,n+2\,x^2+2\,x\,z+3\,z^2+10)}{z\,(n+3)}.$$

Integrating (WZ) with respect to x and z, we get

$$4\,b(n+2) + (n+2)(n+1)\,b(n) = 0$$

Since $b(0) = \pi/2$, $b(1) = 0$, it follows that

$$b(n) = \frac{n!\,\pi}{4}\left(\frac{i}{2}\right)^n(1+(-1)^n) \qquad \text{integer } n \geq 0.$$

Consequently,

$$a(m,n) = (-1)^n\frac{(n+m)!\,\pi}{4}\left(\frac{i}{2}\right)^{n+m}.$$

---

[8] `findrecl(n!/(1+x^2)/(1+(z+x)^2)/z^(n+1),n,[x,z],[],2);` in 2 seconds.

Case 2: $m + n$ **odd.**

Without loss of generality, assume m is even and n is odd. Then by integration by parts, we get

$$a(m,n) = (-1)^{\frac{m}{2}+\frac{n+1}{2}} \sum_{k=0}^{\frac{n-1}{2}} (m+2k)!\,(n-2k-1)! - a(0, n+m)$$

Let $b(n) := a(0,n)$, where n is a nonnegative integer, and $F(n; x, z)$ be as above. Then Cauchy integral formula yields

$$b(n) = \frac{n!}{2\pi i} \int_0^\infty \int_{|z|=\frac{1}{2}} \frac{1}{(1+x^2)(1+(z+x)^2) z^{n+1}}\, dz\, dx.$$

Then by integrating (WZ) with respect to x and z, we get

$$4\, b(n+2) + (n+2)(n+1)\, b(n) = -\frac{(n+1)!\,(3n+7)(1+(-1)^{n+1})\, i^{n+1}}{2\,(n+3)}.$$

Solving the above recurrence equation for n odd, we get

$$b(n) = \frac{(-1)^{\frac{n+1}{2}} n!}{2^n} \left(1 + \sum_{k=0}^{\frac{n-1}{2}-1} \frac{2^{2k+1}(3k+5)}{(2k+3)(k+2)} \right).$$

Hence,

$$a(m,n) = (-1)^{\frac{m+n+1}{2}} \sum_{k=0}^{\frac{n-1}{2}} (m+2k)!(n-2k-1)! - \frac{(-1)^{\frac{m+n+1}{2}} (m+n)!}{2^{m+n}} \left(1 + \sum_{k=0}^{\frac{m+n-1}{2}-1} \frac{2^{2k+1}(3k+5)}{(2k+3)(k+2)} \right).$$

□

3.7 The Probability of Returning Home After n Steps

In a biased simple random walk, the probability of returning home after n steps is given by the constant term of $f(n,x,y) := (a\,x + b\,x^{-1} + c\,y + d\,y^{-1})^n$ with respect to x and y, where $n \in \mathbf{N}$ and a, b, c and d are arbitrary non-zero constants.

Now $S(n) := \mathrm{CT}(f) := \frac{1}{2\pi i}\int_C\int_C F(n,x,y)\,dx\,dy$, where $F(n,x,y) := \frac{f(n,x,y)}{x\,y}$ and C is a circle around the origin.

Then F satisfies the WZ-equation[9]:

$$(n+4)^2\,F(n+4,x,y) - 8\,(n+3)^2(d\,c+b\,a)\,F(n+2,x,y) +$$
$$16\,(b\,a-d\,c)^2(n+3)(n+1)\,F(n,x,y) = D_x\,(R_1\,F) + D_y\,(R_2\,F)\,,$$

where

$$R_1 = \frac{P(n,a,b,c,d,x,y)}{8\,y^3x^3d^3n\,bc^2},\, R_2 = \frac{Q(n,a,b,c,d,x,y)}{8\,nd^3x^3y^3(n+1)bc^2}$$

and $P(n,a,b,c,d,x,y)$, $Q(n,a,b,c,d,x,y)$ are polynomials in n,a,b,c,d,y [10].

By double contour integration with respect to x and y, we get

$$(n+4)^2\,S(n+4)-8\,(n+3)^2(d\,c+b\,a)\,S(n+2)+16\,(b\,a-d\,c)^2(n+3)(n+1)\,S(n) = 0.$$

Hence, finding the $\mathrm{CT}(f)$ boils down to solving the above recurrence equation.

□

3.8 Tefera's Identity

Doron Zeilberger asked to find the closed form evaluation of the following k-dimensional integral.

$$A_k(m,n) := \int_{[0,+\infty)^k} (e_2(\mathbf{x}))^m(e_1(\mathbf{x}))^n e^{-e_1(\mathbf{x})}\,d\mathbf{x}$$

First by using **Mint** we found recurrence equations for $A_k(m,n)$ w.r.t. n for $k = 2,3,4,5,6$ (note that for $k=1$, trivially $A_k(m,n)=0$).

Let $F_k(m,n;\mathbf{x}) := (e_2(\mathbf{x}))^m(e_1(\mathbf{x}))^n e^{-e_1(\mathbf{x})}$. The output generated by **Mint** using the procedure **findrec1** is summarized in Table 3.1.

[9] `findrec1(F,n,[x,y],[a,b,c,d],4);` in 331 seconds.

[10] P and Q are available from `http://www.math.temple.edu/~akalu/multiwz/PQ.ps`

Table 3.1: WZ-equations for specific k

k	WZ-equation	CPU time (seconds)
2	$(E_n - (2m+n+2)\,I)\,F_2 \;=\; -\sum_{i=1}^{2} D_{x_i}\,[R(x_i;\hat{\mathbf{x}}_i)F_2]$, $R(u;v)=u$	0.19
3	$(E_n - (2m+n+3)\,I)\,F_3 \;=\; -\sum_{i=1}^{3} D_{x_i}\,[R(x_i;\hat{\mathbf{x}}_i)F_3]$, $R(u;v_1,v_2)=u$	0.62
4	$(E_n - (2m+n+4)\,I)\,F_4 \;=\; -\sum_{i=1}^{4} D_{x_i}\,[R(x_i;\hat{\mathbf{x}}_i)F_4]$, $R(u;v_1,\ldots,v_3)=u$	2.54
5	$(E_n - (2m+n+5)\,I)\,F_5 \;=\; -\sum_{i=1}^{5} D_{x_i}\,[R(x_i;\hat{\mathbf{x}}_i)F_5]$, $R(u;v_1,\ldots,v_4)=u$	11.37
6	$(E_n - (2m+n+6)\,I)\,F_6 \;=\; -\sum_{i=1}^{6} D_{x_i}\,[R(x_i;\hat{\mathbf{x}}_i)F_6]$, $R(u;v_1,\ldots,v_5)=u$	61.61

It is a matter of time before the following conjecture emerges.

$$(E_n - (2m+n+k)\,I)\,F_k(m,n;\mathbf{x}) = -\sum_{i=1}^{k} D_{x_i}[R(x_i;\hat{\mathbf{x}}_i)F_k(m,n;\mathbf{x})]. \qquad (3.2)$$

where

$$R(u;v_1,\ldots,v_{k-1}) = u$$

Hence, by integrating both sides of (3.2) w.r.t. $x_1,\ldots,x_k$ over $[0,\infty)^k$, we obtain,

$$A_k(m+1,n) - (2m+n+k)A_k(m,n) = 0 \qquad (3.3)$$

Now we evaluate the boundary value $A_k(m) := A_k(m,0)$. Let $F_k(m;\mathbf{x}) := F_k(m,0;\mathbf{x})$. The results produced by **Mint** for $k = 2,\ldots,6$ are summarized in Table 3.2.

Hence, we conjecture:

$$(k\,E_m - (k-1)(2m+k)(m+1)\,I)\,F_k(m;\mathbf{x}) = -\sum_{i=1}^{k} D_{x_i}\,[R(x_i;\hat{\mathbf{x}}_i)F_k(m;\mathbf{x})]\,, \qquad (3.4)$$

Table 3.2: WZ-equations for the boundary case

k	WZ-equation	CPU time (seconds)
2	$(2E_m - (2m+2)(m+1)I)\ F_2 = -\sum_{i=1}^{2} D_{x_i}\ [R(x_i; \hat{\mathbf{x}}_i)\ F_2]$, $R(u; v) = (m+1+v)\ u$	0.21
3	$(3E_m - 2(2m+3)(m+1)I)\ F_3$ $= -\sum_{i=1}^{3} D_{x_i}\ [R(x_i; \hat{\mathbf{x}}_i) F_3]$, $R(u; v_1, v_2) = (2(m+1) + e_1(v_1, v_2))u + e_2(v_1, v_2)$	0.47
4	$(4E_m - 3(2m+4)(m+1)I)\ F_4$ $= -\sum_{i=1}^{4} D_{x_i}\ [R(x_i; \hat{\mathbf{x}}_i)\ F_4]$, $R(u; v_1, v_2, v_3) = (3(m+1) + e_1(v_1, v_2, v_3))u + e_2(v_1, v_2, v_3)$	1.31
5	$(5E_n - 4(2m+5)(m+1)I)\ F_5$ $= -\sum_{i=1}^{5} D_{x_i}\ [R(x_i; \hat{\mathbf{x}}_i)\ F_5]$, $R(u; v_1, \ldots, v_4) = (4(m+1)+e_1(v_1, \ldots, v_4))u+e_2(v_1, \ldots, v_4)$	5.64
6	$(6E_n - 5(2m+6)(m+1)I)\ F_6$ $= -\sum_{i=1}^{6} D_{x_i}\ [R(x_i; \hat{\mathbf{x}}_i) F_6]$, $R(u; v_1, \ldots, v_5) = (5(m+1)+e_1(v_1, \ldots, v_5))u+e_2(v_1, \ldots, v_5)$	22.16

where

$$R(u; v_1, \ldots, v_{k-1}) := ((k-1)(m+1) + e_1(v_1, \ldots, v_{k-1}))u + e_2(v_1, \ldots, v_{k-1})$$

Hence, by integrating both sides of (3.4) w.r.t. $x_1, \ldots, x_k$ over $[0, \infty)^k$, we obtain,

$$k\,A_k(m+1) - (k-1)(2m+k)(m+1)\,A_k(m) = k\,A_{k-1}(m+1) \qquad (3.5)$$

Therefore, using (3.3) and (3.5) we get:

$$A_k(m,n) = \frac{m!\,(2m+n+k-1)!(k/2)_m}{(2m+k-1)!}\left(\frac{2(k-1)}{k}\right)^m B_k(m)$$

for any positive integer k, and for all non-negative integers m and n, where,

$$B_k(m) - B_k(m-1) = \frac{(k(k-2))^m((k-1)/2)_m}{(k-1)^{2m}(k/2)_m} B_{k-1}(m) \qquad k \geq 2,$$

$B_1(m) = 0$, $m \geq 0$, and $B_k(0) = 1$, $k \geq 2$.

The above conjecture was proved [T99] by a close collaboration with **Mint**. Our proof may hence be termed *computer assisted*. Thus we have the following theorem.

Theorem 3.6

$$\int_{[0,+\infty)^k} (e_2(\mathbf{x}))^m (e_1(\mathbf{x}))^n e^{-e_1(\mathbf{x})}\, d\mathbf{x} =$$

$$\frac{m!\,(2\,m+n+k-1)!(k/2)_m}{(2\,m+k-1)!} \left(\frac{2\,(k-1)}{k}\right)^m B_k(m)$$

for any positive integer k, and for all non-negative integers m and n, where,

$$B_k(m) - B_k(m-1) = \frac{(k(k-2))^m((k-1)/2)_m}{(k-1)^{2m}(k/2)_m} B_{k-1}(m) \qquad k \geq 2,$$

$B_1(m) = 0$, $m \geq 0$, *and* $B_k(0) = 1$, $k \geq 2$.

For the complete proof of the above theorem, see [T99].

3.9 Binomial (Multi)Sum Identities

The following examples illustrate how one can use the functions **ssum** and **msum** to prove sum and multi-sum identities.

3.9.1 A Generalized Vandermonde Identity

$$\sum_i \sum_j \binom{r}{i}\binom{s}{j}\binom{t}{n-i-j} = \binom{r+s+t}{n}$$

Proof:

The call

```
msum([r, i] * [s, j] * [t, n - i - j], 1, [i, j], r, [[0, infinity], [0, infinity]], 1);
```

outputs, in 8 seconds,

$$(t+r+1+s)F(r,z_1)+(n-t-r-s-1)F(r+1,z_1)=D_{z_1}\left((z_1+1)F(r,z_1)\right).$$

where $F(r,z_1)=\dfrac{(z_1+1)^{t+r+s}}{2\,\pi\,i{z_1}^{r+s+t-n+1}}$. This means

$$\sum_i\sum_j\binom{r}{i}\binom{s}{j}\binom{t}{n-i-j}=\int_C F(r,z_1)\,dz_1,$$

where C is a circle around the origin.

Now, let $S(r;s,t,n):=\int_C F(r,z_1)dz_1$. Then, by contour integration with respect to z_1, we get

$$(t+r+1+s)S(r;s,t,n)+(n-t-r-s-1)S(r+1;s,t,n)=0$$

Since $S(0;s,t,n)=\binom{s+t}{s+t-n}$, it follows that

$$S(r;s,t,n)=\binom{s+r+t}{n}. \qquad \square$$

3.9.2 The Sum of Carlitz

In the problem section of the American Mathematical Monthly, L. Carlitz [C68] asked for a proof the following statement.

Let $S_n=\sum_{i+j+k=n}\binom{i+j}{i}\binom{i+k}{k}\binom{k+i}{k}, \qquad n\in\mathbf{Z}_{\geq 0}$. Show that

$$S_n-S_{n-1}=\binom{2\,n}{n}.$$

Proof:

Observe that the above recurrence equation is equivalent to

$$\sum_i\sum_j\binom{i+j}{i}\binom{n-i}{j}\binom{n-j}{n-i-j}=\sum_{k=0}^{n}\binom{2\,k}{k}.$$

Invoking `Mint` with

```
msum([i+j,i]*[n-i,j]*[n-j,n-i-j],1,[i,j],n,[[0,infinity],
 [0,infinity]],2);
```

we get, in 37 seconds,

$$(4\,n+6)\,F(n,z_1,z_2)-(8+5\,n)F(1+n,z_1,z_2)+(n+2)F(n+2,z_1,z_2)=$$

$$D_{z_1}\,(R_1\,F)+D_{z_2}\,(R_2\,F)\,,$$

where

$$\begin{aligned}
R_1 &= \frac{(z_2\,z_1+z_2+1)}{(z_2\,z_1+2\,z_2+1)(n+2)z_2}\big(z_1^3\,z_2\,n+2\,z_1^3\,z_2+4\,z_2\,z_1^2+2\,n\,z_2\,z_1^2+\\
&\quad 4\,z_1^2\,n+5\,z_1^2-5\,z_2\,z_1-4\,n\,z_2\,z_1+6\,n\,z_1+6\,z_1-10\,z_2-8\,n\,z_2-\\
&\quad n-2\big)\\
R_2 &= \frac{(z_2\,z_1+z_2+1)}{(z_2\,z_1+2\,z_2+1)(n+2)z_2}\big(2\,z_2^2\,z_1^2+n\,z_2^2\,z_1^2+4\,z_2^2\,z_1+2\,n\,z_2^2\,z_1-\\
&\quad 3\,n\,z_2\,z_1-3\,z_2\,z_1+3\,n\,z_2^2+3\,z_2^2-4\,z_2-2\,n\,z_2-n-2\big)\\
F(n,z_1,z_2) &= \frac{(z_1+1)^n(z_2\,z_1+2\,z_2+1)^{n+1}}{(2\,\pi\,i)^2 z_2^{n+1}(z_1+1)^n(-z_2^2\,z_1-z_2^2+z_2\,z_1-z_2+z_2\,z_1^2+z_1)}
\end{aligned}$$

Let $S(n) := \int_C \int_C F(n,z_1,z_2)\,dz_1\,dz_2$, where C is a circle around the origin. By integration with respect to z_1 and z_2, we get

$$(4\,n+6)\,S(n)-(8+5\,n)\,S(n+1)+(n+2)\,S(n+2)=0.$$

Checking that the above recurrence equation is satisfied by $rhs(n) := \sum_{k=0}^{n} \binom{2\,k}{k}$ and comparing the initial values for $n=0$ and $n=1$, completes the proof. □

Remark: The above Carlitz problem is also proved automatically in [WZ92] by using the discrete (double sum) WZ method and in [W97].

CHAPTER 4
FUTURE DIRECTIONS

Our goals are:

1. To write an efficient and fast Maple implementation of the q-multi-WZ method,

2. To improve the discrete-continuous multi-WZ algorithm and to write fast and efficient Maple programs to implement it.

4.1 The q-Case

4.1.1 Notations and Basic Definitions

Constant Term. For any Laurent polynomial $f(x_1, \ldots, x_n)$, $\mathrm{CT}_{x_1,\ldots,x_m}(f)$, $m \leq n$, denotes the constant term with respect to $x_1, \ldots, x_m$.

Operators. In addition to the notations and definitions introduced in 2.1 we use the operator notation:

$$Q_x F(x, \mathbf{y}) \;=\; F(q\,x, \mathbf{y}).$$

q-Rising Factorial. The *q-rising factorial*, denoted by $(a; q)_n$, is defined as

$$(a; q)_n = \prod_{i=0}^{n-1} (1 - q^i\, a)$$

and $(a;q)_\infty$ denotes the product $\prod_{i=0}^{\infty}(1-q^i a)$.

Note that $(a;q)_n = \dfrac{(a;q)_\infty}{(a\,q^n;q)_\infty}$.

q-Hypergeometric functions. Let $F := F(\mathbf{n},\mathbf{x})$ be a function of m discrete variables $\mathbf{n} = (n_1,\ldots,n_m)$ and k continuous variables $\mathbf{x} = (x_1,\ldots,x_k)$. We say F is *q-hypergeometric* if

(i). $\frac{E_{n_i} F}{F} = \text{Rational}_{1,i}(q, q^{n_1},\ldots,q^{n_m},\mathbf{n},\mathbf{x})$, $(i = 1,\ldots,m)$

(ii). $\frac{Q_{x_i} F}{F} = \text{Rational}_{2,i}(q, q^{n_1},\ldots,q^{n_m},\mathbf{n},\mathbf{x})$, $(i = 1,\ldots,k)$

Example: $F(n,x) = (x;q)_n$ is *q-hypergeometric*. To see this, we only need to observe that

$$\begin{aligned}\frac{E_n F}{F} &= \frac{F(n+1,x)}{F(n,x)} \\ &= 1 - q^n x \\ \frac{Q_x F}{F} &= \frac{F(n,q\,x)}{F(n,x)} \\ &= \frac{1}{1-x}.\end{aligned}$$

q-Proper-hypergeometric functions. A function $F(n_1,\ldots,n_m,x_1,\ldots,x_r)$ is *q-proper-hypergeometric* if it is the product of the following types of expressions.

(qPH-I) Polynomials $P(q,q^{n_1},\ldots,q^{n_m},x_1,\ldots,x_r)$,

(qHP-II) $(c\,{x_1}^{\alpha_1}\cdots{x_r}^{\alpha_r}\,q^{\beta_1 n_1}\cdots q^{\beta_m n_m})_\infty{}^{\gamma}$, where the α_i and β_i and γ are integers, and c is any indeterminate constant or parameter,

(qPH-III) $q^{\sum_{i,j} a_{i,j}\, n_i\, n_j + \sum_i b_i\, n_i}$, where the $a_{i,j}$ and the b_i are integers,

(qPH-IV) ${z_1}^{n_1}\cdots{z_m}^{n_m}$

Many important identities, for instance, *constant term identities* of the following type involve q-proper-hypergeometric functions.

The q-Dyson identity, conjectured by Andrews and proved by Zeilberger and Bressoud [ZB85],

$$\mathrm{CT}_{x_1,\ldots,x_n} \prod_{1\le i<j\le n} \left(\frac{x_i}{x_j};q\right)_{a_i} \left(\frac{q\,x_j}{x_i};q\right)_{a_j} = \frac{(q;q)_{a_1+\cdots+a_n}}{(q;q)_{a_1}\cdots(q)_{a_n}} \tag{4.1}$$

and the Askey-Wilson identity

$$\mathrm{CT}_z \frac{(z^2;q)_\infty(z^{-2})_\infty}{(a\,z;q)_\infty(a/z;q)_\infty(b\,z;q)_\infty(b/z;q)_\infty(c\,z;q)_\infty(d\,z;q)_\infty}$$

$$= \frac{2}{(q;q)_\infty}\frac{(a\,b\,c\,d;q)_\infty}{(a\,b;q)_\infty(a\,c;q)_\infty(a\,d;q)_\infty(b\,d;q)_\infty(c\,d;q)_\infty} \tag{4.2}$$

where $|a|, |b|, |c|, |d| < 1$.

4.1.2 Description of the Algorithm

The heart of the algorithm is the following *fundamental theorem of the (continuous) q-multi-WZ method.*

Theorem 4.1 (The q-fundamental theorem) *Let $F(n;x_1,\ldots,x_r)$ be a proper-hypergeometric function in all its arguments, where n is a discrete variable and $x_1,\ldots,x_r$ are continuous variables. There exists a linear ordinary recurrence operator with polynomial coefficients $P(E_n,q^n,q)$ and a r-tuple rational of functions $[R_1,\ldots,R_r]$ in $(q,q^n,x_1,\ldots,x_r)$, such that*

$$P(E_n,q^n,q)\,F = \sum_{j=1}^{r}\left(Q_{x_j}-I\right)(R_j\,F) \qquad \text{(q-WZ equation)}$$

For the proof of the theorem and its beautiful theoretical aspects see [WZ92]. The algorithm that implements the continuous q-multi-WZ method is summarized as follows.

- INPUT:
  - A *q-proper-hypergeometric* function F,

– Variable names: $n, x_1, \ldots, x_r$,

– Maximal order of the required recurrence operator in n, i.e., degree with respect to E_n (optional; default = 6).

- OUTPUT:

A recurrence operator $\sum_{i=0}^{\text{order}} a_i(q, q^n)\, E_n^i$ where $a_i(q, q^n)$ are polynomials in q, q^n and a r-tuple of *rational* functions (the *certificate*) $[R_1, \ldots, R_r]$ such that

$$\sum_{i=0}^{\text{order}} a_i(q, q^n)\, F(n+i; \mathbf{x}) - \sum_{j=1}^{r} \left(Q_{x_j} - I\right)(R_j\, F) = 0$$

if it exists; 0 otherwise.

- DESCRIPTION:

1. Set up the q-multi-WZ (rational) equation:

$$\sum_{i=0}^{\text{order}} a_i(q, q^n)\frac{E_n^i F}{F} - \sum_{j=1}^{k} \frac{1}{F}(Q_{x_j} - I)\left[\frac{s_j(q, q^n, \mathbf{x})}{t_j(q, q^n, \mathbf{x})}\, F\right] = 0.$$

with

$$s_j(q, q^n, \mathbf{x}) = \sum_{\mathbf{l}=\mathbf{0}}^{\mathbf{M}_j} b_{j,\mathbf{l}}(q, q^n)\, \mathbf{x}^{\mathbf{l}}$$

where, $\mathbf{x}^{\mathbf{l}} = \prod_{i=1}^{k} x_i^{l_i}$, $\mathbf{l} = (l_1, \ldots, l_k)$, $\mathbf{M}_j = (M_{1,j}, \ldots, M_{k,j})$, replace $t_j(q, q^n, \mathbf{x})$ by "best" conceivable value,

2. Clear the denominators in (1) and equate the coefficients of the monomials in $\mathbf{x}$ to get a homogeneous system of linear equations in the unknowns a_i and $b_{j,\mathbf{l}}$.

3. Solve the resulting system,

4. If a non-zero solution obtained, stop. If not, increase degrees of $s_j(q, q^n, \mathbf{x})$, replace $t_j(q, q^n, \mathbf{x})$ by next "best" conceivable value by looking at the factors of the denominators of $\frac{(Q_{x_j} - I)F}{F}$, and then go to step 3.

qMint is a package of Maple programs that implement the above algorithm. The present version of **qMint** is not very efficient when the number of continuous variables $x_1, \ldots, x_r$ gets large (say $r \geq 3$), since the system of homogeneous linear equations to be solved gets very huge and Maple needs a lot of computer time to solve it. Currently we are trying to make the program more efficient and in the near future we will give a complete description and set of applications of the package and show trough examples how one can use it to generate proofs of q-identities involving q-hypergeometric functions. In particular, the package can be used to prove *constant term identities*, such as (4.1) and (4.2).

4.2 The Concrete Multi-WZ Case

This is the case which integrates both the *CONtinuous* and the *disCRETE* multi-WZ cases. This case includes identities which involve both $\sum$ and $\int$ sign, i.e identities of general form

$$\sum_{\mathbf{k}} \int F(\mathbf{n}, \mathbf{k}, \mathbf{x}, \mathbf{y})\, d\mathbf{y} = \sum_{\mathbf{k}'} \int G(\mathbf{n}', \mathbf{k}', \mathbf{x}', \mathbf{y}') d\mathbf{y}'$$

or

$$\sum_{\mathbf{k}} \int F(\mathbf{n}, \mathbf{k}, \mathbf{x}, \mathbf{y})\, d\mathbf{y} = \text{answer}(\mathbf{n}, \mathbf{x})$$

where $\mathbf{k}$ and $\mathbf{y}$ are not empty. In the near future we hope to write a Maple implementation of the concrete multi-WZ case and try to produce examples that show how one can use it to prove identities of the above types.

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