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Hecke-Weil Correspondence On Conjugate Groups

A Dissertation
Submitted to
the Temple University Graduate Board

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in Partial Fulfillment
of the Requirements for the Degree of
DOCTOR OF PHILOSOPHY

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by
Daniel T. Russo
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ABSTRACT

Hecke-Weil Correspondence On Conjugate Groups

Daniel T. Russo
DOCTOR OF PHILOSOPHY

Temple University, August, 2002

Professor Marvin I. Knopp, Chair

In this thesis, we study the problem of establishing Hecke-Weil correspondence on conjugates, by the matrix $A \in SL(2,\mathbb{R})$, of groups for which Hecke-Weil correspondence is known.

We begin with certain sets of conjugates of the Hecke groups $H(\lambda)$. The main result here is a generalization of the classical Hecke correspondence. (With $A = I$, we recover Erich Hecke's result in [5, 6].) The procedure we employ follows Hecke's original correspondence. By considering particular cases within each set of conjugates, a generalized Hecke group arises naturally; we denote this by $H(\lambda, \mu)$. In addition to establishing Hecke correspondence on $H(\lambda, \mu)$, we use a result of Marvin Knopp and Morris Newman [11] to prove that $H(\lambda, \mu)$ is discrete $\Leftrightarrow \lambda\sqrt{\mu} \geq 2$ or $\lambda\sqrt{\mu} = 2\cos\frac{\pi}{q}$, $q \in \mathbf{Z}, q \geq 3$, thereby generalizing Hecke's result in [5, 6] (where $\mu = 1$) from a second point of view.

We close with the "modular analogue" $H\left(\frac{R}{S}, \frac{S^2}{R^2}\right)$, which contains the congruence subgroup $\Gamma(R,S) := \Gamma^0(R) \cap \Gamma_0(S)$. Note that $\Gamma(1,S) = \Gamma_0(S)$. After giving a brief description of $\Gamma(R,S)$, including some information about the cusps and its index in the modular group, we use the framework of the generalized Hecke group to establish a correspondence theorem on $\Gamma(R,S)$, an extension of André Weil's 1967 result on $\Gamma_0(S)$. The methodology is from Weil [20]. In [15], Morris Newman points out that $\Gamma(R,S)$ is conjugate to $\Gamma_0(RS)$. Therefore, our result follows the theme of establishing a Hecke-Weil correspondence on conjugates of groups for which there exists one.

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CHAPTER 1

INTRODUCTION

1.1 Outline

The *modular group*, denoted $\Gamma(1)$, is the set of linear fractional transformations: $Mz = \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{Z}$ such that $ad - bc = 1$. Here $z \in \mathbb{H}$, the upper half-plane. Note that $\Gamma(1)$ preserves the upper half-plane, the real line, and the lower half-plane. By considering a matrix group acting on $\mathbb{H}$ by way of linear fractional transformations, we can write $\Gamma(1) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid ad - bc = 1, a, b, c, d \in \mathbb{Z}\}$. With this action, $\pm\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ yields the same linear fractional transformation.

Let $S(1)$ be the translation: $z \mapsto z + 1$, and T be the inversion: $z \mapsto \frac{-1}{z}$. In matrix form, we write $S(1) = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. It is a standard result that $\Gamma(1)$ is generated by $S(1)$ and T. See, for example, [10]. A similar group is Jacobi's theta group: $\Gamma_\theta = \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(1) \middle| \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \text{ or } \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \text{ modulo } 2\right\}$, which can be shown to be generated by the translation $z \mapsto z + 2$ and the inversion $z \mapsto \frac{-1}{z}$, i.e., $S(2)$ and T. See, for example, [10].

Define $S(\lambda) = \left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right)$ as the translation: $z \mapsto z + \lambda$, $\lambda \in \mathbb{R}^+$. In the mid-1930's, Erich Hecke studied the broader class of subgroups of $SL(2, \mathbb{R})$ generated by $S(\lambda)$ and T. We call these groups the *Hecke groups of width* λ and we denote them by $H(\lambda)$. In his celebrated work in [5, 6], Hecke proves the now-classical correspondence theorem, which we discuss in the next section. He further shows that $H(\lambda)$ is discrete $\Leftrightarrow \lambda \geq 2$ or $\lambda = 2\cos\frac{\pi}{q}$, $q \in \mathbb{Z}$, $q \geq 3$. Note that $q = 3$ gives $H(1) = \Gamma(1)$, the modular group. Of course, $H(2) = \Gamma_\theta$, Jacobi's theta group.

We now turn our attention to a more general class of groups. Let $T(\mu)$ be the inversion $z \to \frac{-1}{\mu z}$, $\mu \in \mathbb{R}$. In matrix form, we take as a canonical representation: $\left(\begin{smallmatrix} 0 & -1 \\ \mu & 0 \end{smallmatrix}\right)$, although we make frequent use of the representation $\left(\begin{smallmatrix} 0 & -x \\ y & 0 \end{smallmatrix}\right)$, $\frac{y}{x} = \mu$. We call the groups generated by $S(\lambda)$ and $T(\mu)$ the *Hecke groups of width λ and inversion μ* and we denote them by $H(\lambda, \mu)$. Note that this group is a generalization of the classical Hecke group.

In this thesis, we first study the problem of establishing Hecke correspondence on conjugates, by a matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $SL(2, \mathbb{R})$, of the classical Hecke groups. We give a solution in Chapter 2 for the cases $c = 0$ and $d = 0$ which generalizes Hecke's original result. By considering particular conjugates, we will see that $H(\lambda, \mu)$ arises naturally. Algebraic and geometric aspects of $H(\lambda, \mu)$ are studied in Chapter 3.

The results of Chapter 2 and Chapter 3 can be interpreted as a generalization of Hecke's theorem in two different ways. First, we see that Theorem 2.1 is a generalization of Hecke's theorem from the immediate fact that setting $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ (the trivial conjugate) yields Hecke's theorem. More to the point, Theorem 2.1 includes a more general transformation law in terms of conjugates and a more general functional equation involving twisted Dirichlet series. The second point of view can be seen through the lens of Theorem 2.2 and Chapter 3, whereby Hecke's correspondence theorem and discreteness condition are given on the generalized Hecke group.

Consider the following important congruence subgroups of the modular group:

$$\Gamma_0(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(1) \middle| \, c \equiv 0 \bmod N \right\},$$
$$\Gamma^0(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(1) \middle| \, b \equiv 0 \bmod N \right\}.$$

In Chapter 4, we study the congruence subgroup $\Gamma(R,S) := \Gamma^0(R) \cap \Gamma_0(S)$ of the modular group. The group $\Gamma(R,S)$ will turn out to be a subgroup of the "modular analogue" $H\left(\frac{R}{S}, \frac{S^2}{R^2}\right)$, a generalized Hecke group. Furthermore, we prove Newman's remark in [15] that $\Gamma(R,S)$ and $\Gamma_0(RS)$ are conjugate, and use it to show that certain extended groups of $\Gamma(R,S)$ and $\Gamma_0(RS)$ are also conjugate. By using the context of the generalized Hecke group, and following the techniques of Weil, we prove an extension to the group $\Gamma(R,S)$ of Weil's correspondence theorem on $\Gamma_0(S)$, thereby continuing in the theme of matrix conjugation in Hecke-Weil correspondence theory. (Note that $\Gamma(1,S) = \Gamma_0(S)$.) This is the content of the final chapter.

1.2 Definitions and Notations

Let Γ be a discrete subgroup of $SL(2,\mathbb{R})$ and let k be a real number. Suppose F is holomorphic in $\mathbb{H}$ and has a Fourier expansion at infinity:

$$F(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z/\lambda}, \lambda > 0. \tag{1.1}$$

If F satisfies the transformation law

$$F(Mz) = v(M)(cz+d)^k F(z), \ \forall M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma, \tag{1.2}$$

then F is called an *entire automorphic (modular, if* $\Gamma \subset \Gamma(1)$*) form on* Γ*, of weight* k*, with multiplier system (M.S.)* v. Here, $v(M)$ is assumed to be a complex number of modulus one. By the open mapping theorem, v is independent of z. We note that it suffices that (1.2) hold for the generators of the group Γ. Moreover, if F has the expansion (1.1), then (1.2) is trivially satisfied for $M = S(\lambda)$, with $v(S(\lambda)) = 1$. Following the notation of Marvin Knopp in [10], we fix the branch of $(cz+d)^k$ by adopting the following convention: $-\pi \leq \arg z < \pi$. It is straightforward to show that if there exists $F(z) \not\equiv 0$ satisfying (1.2), then for any $M_1, M_2 \in \Gamma$, and for any $z \in \mathbb{H}$, we have the following *consistency condition* (C.C.):

$$v(M_1M_2)(c_3z+d_3)^k = v(M_1)v(M_2)(c_1M_2z+d_1)^k(c_2z+d_2)^k, \tag{1.3}$$

where $M_1 = \left(\begin{smallmatrix} * & * \\ c_1 & d_1 \end{smallmatrix}\right)$, $M_2 = \left(\begin{smallmatrix} * & * \\ c_2 & d_2 \end{smallmatrix}\right)$, $M_3 = M_1 M_2 = \left(\begin{smallmatrix} * & * \\ c_3 & d_3 \end{smallmatrix}\right)$. Note that if $k \in \mathbb{Z}$, then v is a character on Γ. We say that v is a *multiplier system (M.S.) of weight k on the group* Γ provided $|v(M)| = 1$ $\forall M \in \Gamma$ and v satisfies the consistency condition (C.C.) (1.3). Note that v is a function on the matrix group associated to Γ rather than on the linear fractional transformation group.

It is customary to use the **slash operator** "$|$", defined by

$$F|_k^v M \equiv (F|_k^v M)(z) := \bar{v}(M)(cz+d)^{-k} F(Mz). \tag{1.4}$$

Then the transformation law (1.2) can be rewritten as

$$F|_k^v M = F \ \forall M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma. \tag{1.5}$$

1.3 Hecke-Weil Correspondence

By a Hecke-Weil correspondence theorem, one means a direct theorem and a converse theorem between entire automorphic forms and Dirichlet series satisfying a prescribed functional equation. This association can be traced back to 1859 in the short memoir *Über die Anzahl der Primzahlen unter einer gegebenen Grösse* by Bernhard Riemann where he states the now-famous Riemann Hypothesis. See [19]. There, he dealt with the Riemann Zeta function: $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ (the prototype for Dirichlet series), and the holomorphic function $\Theta(z) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 z}$, $z \in \mathbb{H}$, an entire modular form on Γ_θ. In particular, Riemann showed that the functional equation

$$\pi^{\frac{-s}{2}}\Gamma(\tfrac{s}{2})\zeta(s) = \pi^{\frac{-(1-s)}{2}}\Gamma(\tfrac{1-s}{2})\zeta(1-s)$$

is a consequence of the fact that $\Theta(z)$ is an entire modular form of weight $1/2$ and M.S. v_θ on the group $\Gamma_\theta = \langle S(2), T\rangle \equiv \left\langle \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\right\rangle$. That is, $\Theta(z)$ satisfies the transformation law for the inversion T:

$$\Theta(\tfrac{-1}{z}) = v_\theta(T) z^{1/2}\, \Theta(z), \ v_\theta(T) = e^{-\pi i/4}.$$

The transformation law for the translation $S(2)$ follows by definition of $\Theta(z)$.

In the mid-1930's, Hecke clarified the connection by way of the Mellin transform. In [5, 6], he proved a correspondence theorem between Dirichlet series satisfying a functional equation (and certain analytic conditions) and entire automorphic forms on the Hecke groups. Roughly speaking, this says that each Dirichlet series $L(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ satisfying

$$\left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s)L(s) = (-1)^{k/2} \left(\frac{2\pi}{\lambda}\right)^{-(k-s)} \Gamma(k-s)L(k-s)$$

corresponds to an entire modular form of weight k and M.S. v on $H(\lambda)$, i.e., a function $F(z) = \sum_{n=0}^{\infty} a_n e^{2\pi inz/\lambda}$, holomorphic in $\mathbb{H}$, and satisfying the transformation law:

$$F\left(\frac{-1}{z}\right) = v(T)z^k F(z).$$

This is known as the Hecke correspondence.

In 1967, Weil established a correspondence theorem of a similar type on the group $\Gamma_0(N)$. This group is more complicated in the sense that, in general, it has many more generators than $H(\lambda)$. Even more problematic, $\Gamma_0(N)$ lacks the inversion $z \mapsto \frac{-1}{z}$. Hecke's proof exploits the fact that the Hecke group has two simple generators, one of which is an inversion. To make up for this, Weil considers modular forms of integral weight on the extended group $\Gamma_0^*(N)$ defined as the group generated by $\Gamma_0(N)$ and the inversion $z \mapsto \frac{-1}{Nz}$. As pointed out in [4], the mapping $F \mapsto F|\,T(N)$ preserves the finite-dimensional space of entire modular forms on $\Gamma_0(N)$ of fixed even integral weight (and M.S.), and furthermore, as an operator on the subspace of cusp forms on $\Gamma_0(N)$ of even integral weight (and M.S.), $T(N)$ is self-adjoint with respect to the Petersson inner product, and thus, is normal. Therefore, a basis can be chosen such that for each basis element F, we have $F|\,T(N) = \lambda_F\, F$, where $\lambda_F \in \mathbb{C}$. Slashing both sides of this equality shows that $\lambda_F = \pm e^{\pi ik/2} = \pm 1$ (since the weight k is even). On page 7, we show that this constant agrees with the value of the M.S. of even integral weight for any inversion $T(\mu)$. Therefore, Weil assumes that modular forms, initially defined on $\Gamma_0(N)$, are actually on $\Gamma_0^*(N)$. With the extended group, Hecke's theory is applicable to Weil. (Note that $\Gamma_0^*(N)$ contains a special case of $H(\lambda, \mu)$ with $\lambda = 1$ and $\mu = N \in \mathbb{Z}^+$.)

The second key ingredient and truly innovative concept in his proof is his introduction of the following "twisted" modular form and Dirichlet series by primitive Dirichlet characters of conductor relatively prime to N:

$$\begin{aligned} F(z;\chi) &= \sum_{n=1}^{\infty} a_n \chi(n) e^{2\pi i n z}, \\ L_F(s;\chi) &= \sum_{n=1}^{\infty} a_n \chi(n) n^{-s}. \end{aligned}$$

Recall that a *Dirichlet character modulo N* is a group homomorphism χ from the multiplicative group of integers into the unit circle $|z| = 1$ such that $\chi(a) = 0$ if $(a, N) \neq 1$ and $\chi(a + N) = \chi(a)$. A character modulo N is called *primitive* if N is the smallest positive integer such that $\chi(a + N) = \chi(a)$ for all integers a. If a character modulo N is not primitive, then there exists a smallest such integer N^*, a divisor of N, for which $\chi(a + N^*) = \chi(a)$ holds for all integers a. The number N^* is called the conductor of χ.

By obtaining functional equations for the usual Dirichlet series attached to the entire modular form, as well as for the infinite class of twisted Dirichlet series arising from an infinite set of Dirichlet characters, Weil is able to establish a correspondence theorem.

Lastly, we note that Weil restricts the M.S. to be of *Hecke type*, that is, if $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, then $v(M) = \epsilon(a)$ where ϵ is a Dirichlet character modulo N. He also assumes that the weight is an integer, and therefore, any M.S. is a character. Indeed, if $M_1 = \left(\begin{smallmatrix} a_1 & b_1 \\ Nc_1 & d_1 \end{smallmatrix}\right), M_2 = \left(\begin{smallmatrix} a_2 & b_2 \\ Nc_2 & d_2 \end{smallmatrix}\right) \in \Gamma_0(N)$, then $M_3 \equiv M_1 M_2 = \left(\begin{smallmatrix} a_1 a_2 + N b_1 c_2 & * \\ * & * \end{smallmatrix}\right)$, and thus, since ϵ is a character modulo N, we have $v(M_1 M_2) = \epsilon(a_1 a_2 + N b_1 c_2) = \epsilon(a_1 a_2) = \epsilon(a_1)\epsilon(a_2) = v(M_1)v(M_2)$.

To complete the M.S. for $\Gamma_0^*(N)$, we must determine $v(T(N))$. For weight $k \in \mathbb{Z}$, we actually determine the multiplier $v(T(\mu))$, with $\mu \in \mathbb{R}$. Note that for any weight $k \in \mathbb{R}$, the transformation law (1.2), with $M = -I$, implies $F(z) = F(-Iz) = v(-I)(-1)^k F(z)$, so that $v(-I)(-1)^k = 1$. By our argument convention, $\arg(-1) = -\pi$, and thus, $v(-I) = e^{\pi i k}$.

If $k \in \mathbb{Z}$, then v is a character and we have $[v(T(\mu))]^2 = v(T(\mu) \cdot T(\mu)) = v(-I) = e^{\pi i k}$. Therefore,

$$v(T(\mu)) = \pm e^{\pi i \frac{k}{2}} = \pm i^k = \begin{cases} \pm 1 & : \quad k \text{ even} \\ \pm i & : \quad k \text{ odd} \end{cases}$$

Note that $v(T(\mu))$ is independent of μ.

CHAPTER 2

HECKE CORRESPONDENCE ON $AH(\lambda)A^{-1}$

2.1 Conjugates of Hecke Groups

Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{R})$. For the cases $c = 0$ and $d = 0$, we calculate the generators of the conjugate group $A\langle S(\lambda), T(1)\rangle A^{-1}$ and show that, in each case, there exists a minimal translation in the group. This is seen in the following

Lemma 2.1 *Let* $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{R})$.

(i) If $c = 0$, *then* $A\langle S(\lambda), T(1)\rangle A^{-1} = \langle S(\lambda a^2), AT(1)A^{-1}\rangle$.

(ii) If $d = 0$, *then* $A\langle S(\lambda), T(1)\rangle A^{-1} = \langle S(\lambda b^2), AT(1)A^{-1}\rangle$.

Proof: Suppose $c = 0$. Then $d = a^{-1}$ and we have the following:

$$
\begin{aligned}
AS(\lambda)A^{-1} &= \left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a^{-1} & -b \\ 0 & a \end{smallmatrix}\right) \\
&= \left(\begin{smallmatrix} a & a\lambda+b \\ 0 & a^{-1} \end{smallmatrix}\right)\left(\begin{smallmatrix} a^{-1} & -b \\ 0 & a \end{smallmatrix}\right) \\
&= \left(\begin{smallmatrix} 1 & a^2\lambda \\ 0 & 1 \end{smallmatrix}\right),
\end{aligned}
$$

and

$$\begin{aligned} AT(1)A^{-1} &= \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a^{-1} & -b \\ 0 & a \end{pmatrix} \\ &= \begin{pmatrix} b & -a \\ a^{-1} & 0 \end{pmatrix}\begin{pmatrix} a^{-1} & -b \\ 0 & a \end{pmatrix} \\ &= \begin{pmatrix} \frac{b}{a} & -(a^2+b^2) \\ \frac{1}{a^2} & -\frac{b}{a} \end{pmatrix}. \end{aligned}$$

The first generator proves part (i). Now suppose $d = 0$. Then $c = -\frac{1}{b}$ and we have the following:

$$\begin{aligned} AS(\lambda)A^{-1} &= \begin{pmatrix} a & b \\ -\frac{1}{b} & 0 \end{pmatrix}\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -b \\ \frac{1}{b} & a \end{pmatrix} \\ &= \begin{pmatrix} a & a\lambda+b \\ -\frac{1}{b} & -\frac{\lambda}{b} \end{pmatrix}\begin{pmatrix} 0 & -b \\ \frac{1}{b} & a \end{pmatrix} \\ &= \begin{pmatrix} 1+\frac{a\lambda}{b} & a^2\lambda \\ -\frac{\lambda}{b^2} & 1-\frac{a\lambda}{b} \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} AT(1)A^{-1} &= \begin{pmatrix} a & b \\ -\frac{1}{b} & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -b \\ \frac{1}{b} & a \end{pmatrix} \\ &= \begin{pmatrix} b & -a \\ 0 & \frac{1}{b} \end{pmatrix}\begin{pmatrix} 0 & -b \\ \frac{1}{b} & a \end{pmatrix} \\ &= \begin{pmatrix} -\frac{a}{b} & -(a^2+b^2) \\ \frac{1}{b^2} & \frac{a}{b} \end{pmatrix}, \end{aligned}$$

neither of which is a translation. However, as we now prove, the group generated by these two elements contains the translation $S(\lambda b^2)$, which we prove is the minimal translation. Indeed, we have the following

Claim: $S(\lambda b^2) = (AT(1)A^{-1})(AS(\lambda)A^{-1})(AT(1)A^{-1})^{-1} = A\,(T(1)S(\lambda)T(1)^{-1})\,A^{-1}$.

To see this, we calculate:

$$\begin{aligned} T(1)S(\lambda)T(1)^{-1} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \\ 1 & \lambda \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix}, \end{aligned}$$

and therefore,

$$\begin{aligned} A\,(T(1)S(\lambda)T(1)^{-1})\,A^{-1} &= \left(\begin{smallmatrix} a & b \\ -\frac{1}{b} & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ -\lambda & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -b \\ \frac{1}{b} & a \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} a-b\lambda & b \\ -\frac{1}{b} & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -b \\ \frac{1}{b} & a \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} 1 & \lambda b^2 \\ 0 & 1 \end{smallmatrix}\right). \end{aligned}$$

From the first equality in the claim, we see that by multiplying on the right and left of $S(\lambda b^2)$ by $AT(1)A^{-1}$ and $(AT(1)A^{-1})^{-1}$, respectively, we have $A\,\langle S(\lambda), T(1)\rangle\,A^{-1} = \langle S(\lambda b^2), AT(1)A^{-1}\rangle$.

2.2 Hecke Correspondence

Consider the exponential series: $F(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z/\beta}, \beta > 0,\ a_n \in \mathbb{C}$ such that $a_n = \mathcal{O}(n^\gamma)$, for some $\gamma > 0$. We make the following

Definition 2.1 *Let* $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{R})$. *We define*

$$M_F(s;A) = \int_0^\infty \{F(Aiy) - a_0\} y^s \frac{dy}{y}. \tag{2.1}$$

Note first that this is the Mellin transform of $F - a_0$ *along the image under* A *of the non-negative imaginary axis.* This definition is motivated by a letter from James Hafner to Marvin Knopp in 1987 in which Hafner discusses the Mellin transform on a ray based at the origin. Note also that, in our case, (2.1) is the usual Mellin tranform of $F \circ A$.

The main idea here is to use $M_F(s;A)$ to study Hecke correspondence on conjugates, by the matrix A, of the classical Hecke groups $\langle S(\lambda), T(1)\rangle$. The function F will (eventually) be an entire automorphic form of weight k and multiplier system (M.S.) v on the conjugate group $A\,\langle S(\lambda), T(1)\rangle\,A^{-1}$. Throughout this chapter, k will be an arbitrary positive real number.

First, we provide the framework for the generalized Hecke correspondence on $A\,\langle S(\lambda), T(1)\rangle\,A^{-1}$.

Definition 2.2 *(Twisted Dirichlet Series) Let F and G be two functions, holomorphic in the upper half-plane $\mathbb{H}$, and representable in $\mathbb{H}$ by the exponential series: $F(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z/\lambda_1}$, $G(z) = \sum_{n=0}^{\infty} b_n e^{2\pi i n z/\lambda_2}$, $\lambda_1, \lambda_2 > 0$, a_n, b_n sequences in $\mathbb{C}$ s.t. $a_n = \mathcal{O}(n^\gamma)$ and $b_n = \mathcal{O}(n^\gamma)$, for some $\gamma > 0$. Let χ be a character from $(\mathbb{R}, +, 0)$ to $(\mathbb{C}, \cdot, 1)$. For $s = \sigma + it$, define*

$$\begin{aligned} L_F(s;\chi) &= \sum_{n=1}^{\infty} a_n \chi(n) n^{-s}, & L_G(s;\chi) &= \sum_{n=1}^{\infty} b_n \chi(n) n^{-s}, \\ \Omega_F(s;\chi) &= \left(\frac{\lambda_1}{2\pi}\right)^s \Gamma(s) L_F(s;\chi), & \Omega_G(s;\chi) &= \left(\frac{\lambda_2}{2\pi}\right)^s \Gamma(s) L_G(s;\chi). \end{aligned}$$

Here, χ should have sufficiently mild growth (such as polynomial growth) to ensure convergence of the above Dirichlet series. In what follows, we consider the case $\chi_u \equiv \chi_u(n) := e^{2\pi i n u}$ (for fixed $u \in \mathbb{R}$), a character from $(\mathbb{Z}, +, 0)$ to the complex unit circle.

Remark 2.1 *$a_n = \mathcal{O}(n^\gamma)$ and $b_n = \mathcal{O}(n^\gamma)$, for some $\gamma > 0$, implies $\Omega_F(s;\chi)$ and $\Omega_G(s;\chi)$ converge absolutely in the RHP: $\sigma > \gamma + 1$.*

To see this, note that $\left(\frac{\lambda_1}{2\pi}\right)^\sigma \Gamma(\sigma)|L_F(s;\chi)|$ converges in a RHP if and only if $\sum_{n=1}^{\infty} \frac{|a_n \chi(n)|}{n^\sigma} = \sum_{n=1}^{\infty} \frac{|a_n|}{n^\sigma}$ converges in a RHP. But $a_n = \mathcal{O}(n^\gamma)$ implies that the sum $\sum_{n=1}^{\infty} \frac{|a_n|}{n^\sigma} \leq K \sum_{n=1}^{\infty} \frac{1}{n^{\sigma-\gamma}}$, which converges in the RHP: $\sigma > \gamma + 1$. The same argument applies to $\Omega_G(s;\chi)$.

Note that by Lemma 2.1, if F and G are automorphic on the conjugate groups $A\langle S(\lambda_1), T(1)\rangle A^{-1}$ and $A\langle S(\lambda_2), T(1)\rangle A^{-1}$, respectively, then F and G have fourier expansions at $i\infty$ defined by exponential series (provided that $c = 0$ or $d = 0$). In the case $c = 0$, F has period $\lambda_1 a^2$ and G has period $\lambda_2 a^2$. In the case $d = 0$, F has period $\lambda_1 b^2$ and G has period $\lambda_2 b^2$. With this setup, we prove the following lemma.

Lemma 2.2 *The Mellin transforms of $F - a_0$ and $G - b_0$ along the image under A of the nonnegative imaginary axis are Dirichlet series of the type described in Definition 2.2.*

Proof: For the case $c = 0$, since F has period $\lambda_1 a^2$, and $Aiy = a^2 iy + ab$, we obtain:

$$\begin{aligned}
M_F(s;A) &= \int_0^\infty \{F(Aiy) - a_0\} y^s \frac{dy}{y} \\
&= \int_0^\infty \sum_{n=1}^\infty a_n e^{2\pi i n Aiy/\lambda_1 a^2}\ y^{s-1} dy \\
&= \int_0^\infty \sum_{n=1}^\infty a_n e^{2\pi i n (ab+a^2 iy)/\lambda_1 a^2}\ y^{s-1} dy \\
&= \int_0^\infty \sum_{n=1}^\infty a_n e^{2\pi i n ab/\lambda_1 a^2} e^{-2\pi n a^2 y/\lambda_1 a^2}\ y^{s-1} dy \\
&= \sum_{n=1}^\infty a_n e^{2\pi i n b/\lambda_1 a} \int_0^\infty e^{\frac{-2\pi n y}{\lambda_1}}\ y^{s-1} dy \\
&= \sum_{n=1}^\infty a_n e^{2\pi i n b/\lambda_1 a} \int_0^\infty e^{-t} t^{s-1}\ dt\ (\frac{\lambda_1}{2\pi n})^s \\
&= \left(\frac{\lambda_1}{2\pi}\right)^s \Gamma(s) L_F(s; \chi_{\frac{b}{\lambda_1 a}}) \\
&= a^{-2s} \left(\frac{\lambda_1 a^2}{2\pi}\right)^s \Gamma(s) L_F(s; \chi_{\frac{b}{\lambda_1 a}}) \\
&= a^{-2s}\ \Omega_F(s; \chi_{\frac{b}{\lambda_1 a}}),
\end{aligned}$$

the interchange of the integral and sum being justified by the absolute convergence of $a^{-2s}\ \Omega_F(s; \chi_{\frac{b}{\lambda_1 a}})$. Similarly, since G has period $\lambda_2 a^2$, we have:

$$\begin{aligned}
M_G(s;A) &= \left(\frac{\lambda_2}{2\pi}\right)^s \Gamma(s) L_G(s; \chi_{\frac{b}{\lambda_2 a}}) \\
&= a^{-2s}\ \Omega_G(s; \chi_{\frac{b}{\lambda_2 a}}).
\end{aligned}$$

For the case $d = 0$, since F has period $\lambda_1 b^2$, and $Aiy = b^2\frac{i}{y} - ab$, we obtain (as in the case $c = 0$) the following:

$$\begin{aligned}
M_F(s;A) &= \int_0^\infty \{F(Aiy) - a_0\} y^s \frac{dy}{y} \\
&= \int_0^\infty \sum_{n=1}^\infty a_n e^{2\pi i n Aiy/\lambda_1 b^2}\ y^{s-1} dy \\
&= \int_0^\infty \sum_{n=1}^\infty a_n e^{2\pi i n (b^2 \frac{i}{y}\ -\ ab)/\lambda_1 b^2}\ y^{s-1} dy \\
&= \int_0^\infty \sum_{n=1}^\infty a_n e^{-2\pi i n ab/\lambda_1 b^2} e^{-2\pi n (\frac{b^2}{y})/\lambda_1 b^2}\ y^{s-1} dy \\
&= \sum_{n=1}^\infty a_n e^{-2\pi i n a/\lambda_1 b} \int_0^\infty e^{\frac{-2\pi n}{\lambda_1 y}}\ y^{s-1} dy \\
&= \sum_{n=1}^\infty a_n e^{-2\pi i n a/\lambda_1 b} \int_0^\infty e^{\frac{-2\pi n y}{\lambda_1}}\ y^{-s-1} dy \\
&= \sum_{n=1}^\infty a_n e^{-2\pi i n a/\lambda_1 b} \int_0^\infty e^{-t} t^{-s-1}\ dt\ (\frac{\lambda_1}{2\pi n})^{-s} \\
&= \sum_{n=1}^\infty a_n e^{-2\pi i n a/\lambda_1 b} \left(\frac{\lambda_1}{2\pi n}\right)^{-s} \Gamma(-s),
\end{aligned}$$

and therefore,

$$\begin{aligned}
M_F(-s;A) &= \left(\frac{\lambda_1}{2\pi}\right)^s \Gamma(s) L_F(s;\chi_{\frac{-a}{\lambda_1 b}}) \\
&= b^{-2s}\left(\frac{\lambda_1 b^2}{2\pi}\right)^s \Gamma(s) L_F(s;\chi_{\frac{-a}{\lambda_1 b}}) \\
&= b^{-2s}\ \Omega_F(s;\chi_{\frac{-a}{\lambda_1 b}}).
\end{aligned}$$

As before, we note that the interchange of the integral and sum is justified by the absolute convergence of $b^{-2s}\ \Omega_F(s;\chi_{\frac{-a}{\lambda_1 b}})$. Similarly, since G has period

$\lambda_2 b^2$, we have

$$\begin{aligned} M_G(-s;A) &= \left(\frac{\lambda_2}{2\pi}\right)^s \Gamma(s) L_G(s;\chi_{\frac{-a}{\lambda_2 b}}) \\ &= b^{-2s}\left(\frac{\lambda_2 b^2}{2\pi}\right)^s \Gamma(s) L_G(s;\chi_{\frac{-a}{\lambda_2 b}}) \\ &= b^{-2s}\,\Omega_G(s;\chi_{\frac{-a}{\lambda_2 b}}). \end{aligned}$$

Theorem 2.1 *(Hecke Correspondence On Conjugate Groups)*
Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{R})$.

The case $c = 0$*: Let* $F(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z/\lambda_1 a^2}$ *and* $G(z) = \sum_{n=0}^{\infty} b_n e^{2\pi i n z/\lambda_2 a^2}$, $a_n = \mathcal{O}(n^\gamma), b_n = \mathcal{O}(n^\gamma)$*, for some* $\gamma > 0$*. TFAE:*

A_1.

(i) $\Omega_F(s;\chi_{\frac{b}{\lambda_1 a}})$ *and* $\Omega_G(s;\chi_{\frac{b}{\lambda_2 a}})$ *can be continued meromorphically to the entire s-plane with* $\Omega_F(s;\chi_{\frac{b}{\lambda_1 a}}) - a^{2s}\left(\frac{b_0 i^k}{s-k} - \frac{a_0}{s}\right)$ *and* $\Omega_G(s;\chi_{\frac{b}{\lambda_2 a}}) - a^{2s}\left(\frac{a_0 i^{-k}}{s-k} - \frac{b_0}{s}\right)$ *entire.*

(ii) $\Omega_F(s;\chi_{\frac{b}{\lambda_1 a}})$ *and* $\Omega_G(s;\chi_{\frac{b}{\lambda_2 a}})$ *satisfy the functional equation*

$$\Omega_F(k-s;\chi_{\frac{b}{\lambda_1 a}}) = i^k\, a^{-4s+2k}\,\Omega_G(s;\chi_{\frac{b}{\lambda_2 a}}). \tag{2.2}$$

(iii) $\Omega_F(s;\chi_{\frac{b}{\lambda_1 a}})$ *and* $\Omega_G(s;\chi_{\frac{b}{\lambda_2 a}})$ *remain bounded in every lacunary vertical strip (LVS):* $\sigma_1 \le Re(s) \le \sigma_2,\ |t| \ge t_0 > 0$.

B_1.

For $z \in \mathbb{H}$,

$$\left(F|_k A T(1) A^{-1}\right)(z) = G(z). \tag{2.3}$$

The case $d = 0$*: Let* $F(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z/\lambda_1 b^2}$ *and* $G(z) = \sum_{n=0}^{\infty} b_n e^{2\pi i n z/\lambda_2 b^2}$, $a_n = \mathcal{O}(n^\gamma), b_n = \mathcal{O}(n^\gamma)$*, for some* $\gamma > 0$*. TFAE:*

A_2.

(i) $\Omega_F(s;\chi_{\frac{-a}{\lambda_1 b}})$ *and* $\Omega_G(s;\chi_{\frac{-a}{\lambda_2 b}})$ *can be continued meromorphically to the entire s-plane with* $\Omega_F(s;\chi_{\frac{-a}{\lambda_1 b}}) - b^{2s}\left(\frac{b_0 i^k}{k-s} + \frac{a_0}{s}\right)$ *and* $\Omega_G(s;\chi_{\frac{-a}{\lambda_2 b}}) - b^{2s}\left(\frac{a_0 i^{-k}}{k-s} + \frac{b_0}{s}\right)$ *entire.*

(ii) $\Omega_F(s;\chi_{\frac{-a}{\lambda_1 b}})$ *and* $\Omega_G(s;\chi_{\frac{-a}{\lambda_2 b}})$ *satisfy the functional equation*

$$\Omega_F(k-s;\chi_{\frac{-a}{\lambda_1 b}}) = i^k\, b^{-4s+2k}\, \Omega_G(s;\chi_{\frac{-a}{\lambda_2 b}}). \tag{2.4}$$

(iii) $\Omega_F(s;\chi_{\frac{-a}{\lambda_1 b}})$ *and* $\Omega_G(s;\chi_{\frac{-a}{\lambda_2 b}})$ *remain bounded in every lacunary vertical strip (LVS):* $\sigma_1 \le Re(s) \le \sigma_2,\ |t| \ge t_0 > 0$.

B_2.

For $z \in \mathbb{H}$,

$$\left(F|_k AT(1)A^{-1}\right)(z) = G(z). \tag{2.5}$$

Proof: First, we prove the Dirichlet series conditions from the transformation law. We begin by calculating $AT(1)A^{-1}$ for general $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$:

$$\begin{aligned} AT(1)A^{-1} &= \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} b & -a \\ d & -c \end{smallmatrix}\right)\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} ac+bd & -(b^2+a^2) \\ c^2+d^2 & -(ac+bd) \end{smallmatrix}\right). \end{aligned}$$

Therefore, the transformation law $\left(F|_k AT(1)A^{-1}\right)(z) = G(z)$ is the following:

$$\begin{aligned} F(AT(1)A^{-1}z) &= \{(c^2+d^2)z - (ac+bd)\}^k\, G(z) \\ &= (c^2+d^2)^k \left\{z - \frac{ac+bd}{c^2+d^2}\right\}^k G(z). \end{aligned}$$

Put $z = Aiy$. This gives

$$\begin{aligned} F(A\frac{i}{y}) &= (c^2+d^2)^k \left\{ Aiy - \frac{ac+bd}{c^2+d^2} \right\}^k G(Aiy) \\ &= (c^2+d^2)^k \left\{ \frac{aiy+b}{ciy+d} - \frac{ac+bd}{c^2+d^2} \right\}^k G(Aiy) \\ &= (c^2+d^2)^k \left\{ \frac{acy^2+bd+iy}{c^2y^2+d^2} - \frac{ac+bd}{c^2+d^2} \right\}^k G(Aiy). \end{aligned}$$

Using that $ad - bc = 1$, we obtain the following:

$$\begin{aligned} F(A\frac{i}{y}) &= (c^2+d^2)^k \left\{ \frac{cd(y^2-1)+iy(c^2+d^2)}{(c^2+d^2)(c^2y^2+d^2)} \right\}^k G(Aiy) \\ &= \left\{ \frac{cd(y^2-1)+iy(c^2+d^2)}{c^2y^2+d^2} \right\}^k G(Aiy). \end{aligned}$$

Note 1: Setting $c = 0$, we obtain $F(A\frac{i}{y}) = i^k\ G(Aiy)\ y^k$. Setting $d = 0$, we obtain $F(A\frac{i}{y}) = i^k\ G(Aiy)\ y^{-k}$. Note that the expression above provides an approach for obtaining a functional equation in the generic case: *neither* $c = 0$ *nor* $d = 0$, although a classical functional equation does not appear likely from this.

We first prove the case $c = 0$. By the change of variable y goes to $1/y$, and the transformation law for the case $c = 0$ described in Note 1, we have

$$\begin{aligned} M_F(s;A) &= \int_0^\infty \{F(Aiy) - a_0\} y^s \frac{dy}{y} \\ &= \int_0^1 \{F(Aiy) - a_0\} y^s \frac{dy}{y} + \int_1^\infty \{F(Aiy) - a_0\} y^s \frac{dy}{y}. \\ &= \int_1^\infty \{F\left(A\frac{i}{y}\right) - a_0\} y^{-s} \frac{dy}{y} + \int_1^\infty \{F(Aiy) - a_0\} y^s \frac{dy}{y} \\ &= \int_1^\infty \{i^k\ G(Aiy)\ y^k - a_0\} y^{-s} \frac{dy}{y} + \int_1^\infty \{F(Aiy) - a_0\} y^s \frac{dy}{y}. \end{aligned}$$

Suppose Re$(s) > \max\ \{0, k, \gamma+1\} = \max\ \{k, \gamma+1\}$. Then the first integral can be rewritten as

$$i^k \int_1^\infty \{G(Aiy) - b_0\} y^{k-s} \tfrac{dy}{y} + b_0 i^k \int_1^\infty y^{k-s} \tfrac{dy}{y} - a_0 \int_1^\infty y^{-s} \tfrac{dy}{y}$$

$$= i^k \int_1^\infty \{G(Aiy) - b_0\} y^{k-s} \tfrac{dy}{y} + \left(\tfrac{b_0 i^k}{s-k} - \tfrac{a_0}{s}\right).$$

Therefore,

$$M_F(s;A) = \underbrace{i^k \int_1^\infty \{G(Aiy) - b_0\} y^{k-s} \frac{dy}{y} + \int_1^\infty \{F(Aiy) - a_0\} y^s \frac{dy}{y}}_{E_F(s;A)} + \underbrace{\left(\frac{b_0 i^k}{s-k} - \frac{a_0}{s}\right)}_{R_F(s;A)}.$$

By Lemma 2.2, case $c = 0$, this is equivalent to

$$\Omega_F(s; \chi_{\frac{b}{\lambda_1 a}}) = a^{2s} E_F(s;A) + a^{2s} R_F(s;A).$$

To get the analogue for $\Omega_G(s; \chi_{\frac{b}{\lambda_2 a}})$, we rewrite the transformation law (2.3) with $z \to AT(1)A^{-1}z$. In the case $c = 0$, the transformation law is

$$F(AT(1)A^{-1}z) = a^{-2k}[z - ab]^k\ G(z).$$

With $z \to AT(1)A^{-1}z$, we obtain

$$F(z) = a^{-2k}[AT(1)A^{-1}z - ab]^k\ G(AT(1)A^{-1}z),$$

i.e.,

$$\begin{aligned} G(AT(1)A^{-1}z) &= a^{2k}[AT(1)A^{-1}z - ab]^{-k} F(z) \\ &= a^{2k}\left[\frac{abz - a^2(b^2+a^2)}{z - ab} - ab\right]^{-k} F(z) \\ &= a^{2k}\left[\frac{-a^4}{z-ab}\right]^{-k} F(z) \\ &= a^{-2k}\,[-(z-ab)]^k\, F(z). \end{aligned}$$

To simplify this further, we need the following:

Note 2: $z^k(-z)^{-k} = i^{2k}, \ \forall z \in \mathbb{H}, \forall k \in \mathbb{R}$.

To see this, it suffices to show that the arguments on both sides of the equality are the same. Recall our convention: $-\pi \leq \arg z < \pi$. Suppose $z = re^{i\theta}$. Then $\arg(-z) = \theta - \pi$, and we have

$$\begin{aligned} z^k(-z)^{-k} &= e^{klogz}e^{-klog(-z)} \\ &= e^{k[log|z|+i\theta]-k[log|-z|+i(\theta-\pi)]} \\ &= e^{ki[\theta-(\theta-\pi)]} \\ &= e^{ki\pi}. \end{aligned}$$

On the other hand,

$$\begin{aligned} i^{2k} &= e^{2klogi} \\ &= e^{2k[i\pi/2]} \\ &= e^{ki\pi}. \end{aligned}$$

Therefore, $G(AT(1)A^{-1}z) = a^{-2k}i^{-2k}[z-ab]^k \ F(z)$. Putting $z = Aiy$, we have

$$\begin{aligned} G\left(A\frac{i}{y}\right) &= a^{-2k}i^{-2k}[Aiy - ab]^k \ F(Aiy) \\ &= a^{-2k}i^{-2k}[a^2iy + ab - ab]^k \ F(Aiy) \\ &= i^{-k}F(Aiy)y^k. \end{aligned}$$

In analogy with $M_F(s;A)$, with i^k replaced by i^{-k}, we obtain

$$M_G(s;A) = \underbrace{i^{-k}\int_1^\infty \{F(Aiy) - a_0\}y^{k-s}\frac{\mathrm{d}y}{y} + \int_1^\infty \{G(Aiy) - b_0\}y^s\frac{\mathrm{d}y}{y}}_{E_G(s;A)} + \underbrace{\left(\frac{a_0 i^{-k}}{s-k} - \frac{b_0}{s}\right)}_{R_G(s;A)}.$$

By Lemma 2.2, case $c = 0$, this is equivalent to

$$\Omega_G(s;\chi_{\frac{b}{\lambda_1 a}}) = a^{2s}E_G(s;A) + a^{2s}R_G(s;A).$$

Since $F(Aiy) - a_0$ and $G(Aiy) - b_0$ are absolutely convergent exponential series starting with $n = 1$, whose coefficients have polynomial growth, $F(Aiy) - a_0$

and $G(Aiy)-b_0 \to 0$ exponentially, as $y \to \infty$. An application of the Lebesgue Dominated Convergence theorem and Morera's theorem shows that $E_F(s;A)$ and $E_G(s;A)$ are entire in s and hence, $a^{2s}E_F(s;A)$ and $a^{2s}E_G(s;A)$ are entire in s. This proves part (i). Consequently, we can replace s by $k-s$ and obtain part (ii) immediately.

To prove part (iii), we need to show that $\Omega_F(s;\chi_{\frac{b}{\lambda a}}) = a^{2s}E_F(s;A)+a^{2s}R_F(s;A)$ and $\Omega_G(s;\chi_{\frac{b}{\lambda a}}) = a^{2s}E_G(s;A)+a^{2s}R_G(s;A)$ are bounded in LVS's. We prove it for Ω_F. The proof for Ω_G is analogous. Note that the LVS's avoid the simple poles at $s=0$ and $s=k$. Clearly then,

$$a^{2s}R_F(s;A) = a^{2s}\left(\frac{b_0 i^k}{s-k} - \frac{a_0}{s}\right) \to 0 \text{ as } |t| \to \infty.$$

Therefore, $a^{2s}R_F(s;A)$ is bounded in LVS's. For $a^{2s}E_F(s;A)$, we have

$$|a^{2s}E_F(s;A)| \le a^{2\sigma_2}\left(\int_1^\infty |G(Aiy)-b_0|\ y^{k-\sigma_1}\frac{dy}{y} + \int_1^\infty |F(Aiy)-a_0|\ y^{\sigma_2}\frac{dy}{y}\right).$$

Note that

$$\begin{aligned}
|G(Aiy)-b_0| &\le \sum_{n=1}^\infty |b_n||\chi_{\frac{b}{\lambda_2 a}}(n)||e^{2\pi i n z/\lambda_2 a^2}| \\
&= \sum_{n=1}^\infty |b_n| e^{-2\pi n y/\lambda_2 a^2} \\
&\le Ke^{-2\pi y/\lambda_2 a^2}\sum_{n=1}^\infty n^\gamma e^{-2\pi(n-1)y/\lambda_2 a^2} \\
&= Ke^{-2\pi y/\lambda_2 a^2}\sum_{n=1}^\infty n^\gamma e^{-2\pi(n-1)/\lambda_2 a^2} \text{(since y} \ge 1) \\
&= K^* e^{-2\pi y/\lambda_2 a^2}.
\end{aligned}$$

Similarly, $|F(Aiy)-a_0| \le L^* e^{-2\pi y/\lambda_1 a^2}$. Let $M^* = \max\{K^*, L^*\}$. Then

$$|a^{2s}E_F(s;A)| \le a^{2\sigma_2}M^*\left(\int_1^\infty e^{-2\pi y/\lambda_2 a^2}y^{k-\sigma_1}\frac{dy}{y} + \int_1^\infty e^{-2\pi y/\lambda_1 a^2}y^{\sigma_2}\frac{dy}{y}\right) \le M^{**},$$

and so $a^{2s}E_F(s;A)$ is bounded in LVS's. This completes the proof of $B_1 \Rightarrow A_1$.

Next, we assume the Dirichlet series conditions in A_1, including the functional equation (2.2). We will prove that the transformation law (2.3) in B_1 holds.

For any $r > \gamma+1$, where γ is from Definition 2.2, we obtain the inverse Mellin transform of $G - b_0$ *along the image under* A *of the non-negative imaginary axis* as follows:

$$\begin{aligned}
\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} M_G(s;A)y^{-s}ds &= \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty}\left(\frac{\lambda_2}{2\pi}\right)^s \Gamma(s)L_G(s;\chi_{\frac{b}{\lambda_2 a}})\, y^{-s}ds \\
&= \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty}\left(\frac{\lambda_2}{2\pi}\right)^s \Gamma(s)\sum_{n=1}^{\infty}\frac{b_n e^{2\pi i n b/\lambda_2 a}}{n^s}\, y^{-s}ds \\
&\overset{(*)}{=} \sum_{n=1}^{\infty} b_n e^{2\pi i n b/\lambda_2 a}\,\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty}\left(\frac{2\pi}{\lambda_2}ny\right)^{-s}\Gamma(s)ds \\
&= \sum_{n=1}^{\infty} b_n e^{2\pi i n b/\lambda_2 c}\, e^{-2\pi n y/\lambda_2} \\
&= G(Aiy) - b_0. \qquad (2.6)
\end{aligned}$$

To justify (*), we show the absolute convergence of

$$\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty}\left(\tfrac{\lambda_2}{2\pi}\right)^s\Gamma(s)\sum_{n=1}^{\infty}\tfrac{b_n e^{2\pi i n b/\lambda_2 a}}{n^s}\, y^{-s}ds = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} a^{-2s}\Omega_G(s;\chi_{\frac{b}{\lambda_2 a}})\, y^{-s}ds.$$

By Remark 2.1, $a^{-2s}\Omega_G(s;\chi_{\frac{b}{\lambda_2 a}})$ converges absolutely on Re(s)$= r$. Moreover, convergence is uniform on the compact set $\{r+it : \ |t| \le t_0\}$, for any t_0. Therefore, $\frac{1}{2\pi i}\int_{r-t_0 i}^{r+t_0 i} a^{-2s}\Omega_G(s;\chi_{\frac{b}{\lambda_2 a}})\, y^{-s}ds$ converges absolutely. To handle the growth of $\Gamma(s)$ on $\{r+it : \ |t| \ge t_0\}$, we use Stirling's formula on the vertical line Re(s)$= r$, as $|t| \to +\infty$:

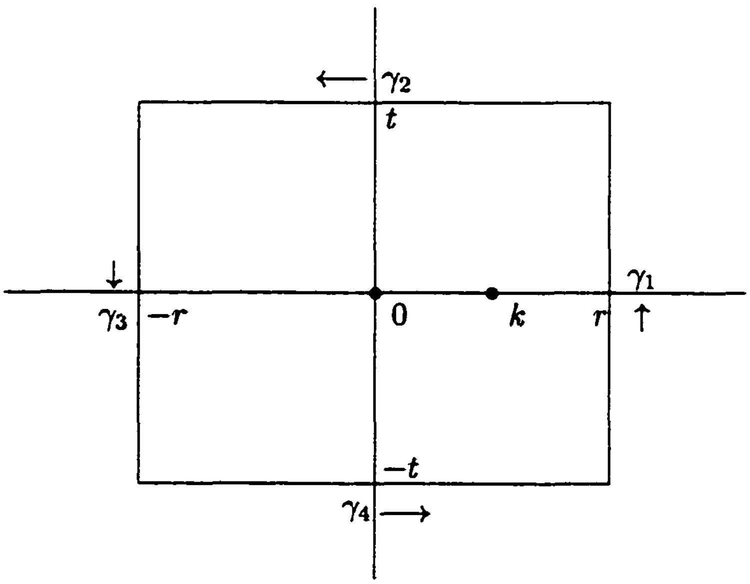
$$\begin{aligned}
\left|\left(\frac{\lambda_2}{2\pi}\right)^s \Gamma(s)\sum_{n=1}^{\infty}\frac{b_n e^{2\pi i n b/\lambda_2 a}}{n^s}\, y^{-s}\right| &\le \left(\frac{\lambda_2}{2\pi}\right)^r \sqrt{2\pi}\, e^{-r}|t|^{r-\frac{1}{2}}e^{-\pi|t|/2}\sum_{n=1}^{\infty}\frac{|b_n|}{n^r}\, y^{-r} \\
&\le \left(\frac{\lambda_2}{2\pi}\right)^r \sqrt{2\pi}\, e^{-r}|t|^{r-\frac{1}{2}}e^{-\pi|t|/2}K\sum_{n=1}^{\infty} n^{\gamma-r}\, y^{-r} \\
&\le \left(\frac{\lambda_2}{2\pi}\right)^r \sqrt{2\pi}\, e^{-r}|t|^{r-\frac{1}{2}}e^{-\pi|t|/2}\, K_{r,\gamma}\, y^{-r}.
\end{aligned}$$

Therefore,

$$\int\limits_{|t|\geq t_0} \left(\frac{\lambda_2}{2\pi}\right)^r \sqrt{2\pi}\, e^{-r}|t|^{r-\frac{1}{2}}e^{-\pi|t|/2}\, K_{r,\gamma}\, y^{-r}\mathrm{d}t \;\leq\; K^{*}_{r,\gamma} \int\limits_{|t|\geq t_0} |t|^{r-\frac{1}{2}}e^{-\pi|t|/2}\mathrm{d}t$$
$$\leq\; K^{**}_{r,\gamma}\, \Gamma(r+\frac{1}{2}),$$

and so $\frac{1}{2\pi i}\int\limits_{|t|\geq t_0} a^{-2s}\Omega_G(s;\chi_{\frac{b}{\lambda_2 a}})\; y^{-s}\mathrm{d}s$ converges absolutely. The same argument shows that $\frac{1}{2\pi i}\int\limits_{r-i\infty}^{r+i\infty} M_F(s;A)y^{-s}\mathrm{d}s = F(Aiy) - a_0$.

Let $r > k$. Consider the following contour:



Assume (for the moment) that $\frac{1}{2\pi i}\int\limits_{\gamma_2} M_G(s;A)y^{-s}\mathrm{d}s$, $\frac{1}{2\pi i}\int\limits_{\gamma_4} M_G(s;A)y^{-s}\mathrm{d}s \to 0$ as $|t| \to \infty$. Recall that by Lemma 2.2, case $c = 0$, $M_F(s;A) = a^{-2s}\,\Omega_F(s;\chi_{\frac{b}{\lambda_1 a}})$, which has principal part $\left(\frac{a_0 i^{-k}}{s-k} - \frac{b_0}{s}\right)$ by the assumption in A_1 part(i).

By (2.6) and the residue theorem, we have

$$\begin{aligned} G(Aiy)-b_0 &= \frac{1}{2\pi i}\int_{-r-i\infty}^{-r+i\infty} M_G(s;A)y^{-s}\mathrm{d}s + \sum_{s=0,k}\mathrm{Res}(M_G(s;A)y^{-s};s) \\ &= \frac{1}{2\pi i}\int_{-r-i\infty}^{-r+i\infty} M_G(s;A)y^{-s}\mathrm{d}s + [a_0 i^{-k}y^{-k}-b_0]. \end{aligned}$$

We assumed that $\Omega_F(k-s;\chi_{\frac{b}{\lambda_1 a}}) = i^k\, a^{-4s+2k}\,\Omega_G(s;\chi_{\frac{b}{\lambda_2 a}})$; equivalently, by the proof of Lemma 2.2 (the case $c=0$), $M_F(k-s;A) = i^k\, M_G(s;A)$. Therefore,

$$\begin{aligned} G(Aiy)-b_0 &= \frac{1}{2\pi i}\int_{-r-i\infty}^{-r+i\infty} M_G(s;A)y^{-s}\mathrm{d}s + [a_0 i^{-k}y^{-k}-b_0] \\ &= \frac{i^{-k}}{2\pi i}\int_{-r-i\infty}^{-r+i\infty} M_F(k-s;A)y^{-s}\mathrm{d}s + [a_0 i^{-k}y^{-k}-b_0] \\ &= \frac{i^{-k}}{2\pi i}\int_{k+r+i\infty}^{k+r-i\infty} M_F(s;A)y^{s-k}(-\mathrm{d}s) + [a_0 i^{-k}y^{-k}-b_0] \\ &= (iy)^{-k}\frac{1}{2\pi i}\int_{k+r-i\infty}^{k+r+i\infty} M_F(s;A)\left(\frac{1}{y}\right)^{-s}\mathrm{d}s + [a_0 i^{-k}y^{-k}-b_0] \\ &\overset{(**)}{=} (iy)^{-k}\left[F\left(A\frac{i}{y}\right)-a_0\right] + [a_0 i^{-k}y^{-k}-b_0] \\ &= (iy)^{-k}F\left(A\frac{i}{y}\right)-b_0, \end{aligned}$$

(**) following from the fact that (2.6) is still valid for $k+r$ since $k>0$. Therefore, for $z=Aiy$, we have $G(z) = (A^{-1}z)^{-k}F(AT(1)A^{-1}z)$. Since this holds for all $y>0$, and both sides of this relation are analytic in $\mathbb{H}$, the uniqueness theorem implies that this relation holds for all $z\in\mathbb{H}$. With $A = \left(\begin{smallmatrix} a & b \\ 0 & a^{-1}\end{smallmatrix}\right)\in SL(2,\mathbb{R})$, $A^{-1} = \left(\begin{smallmatrix} a^{-1} & -b \\ 0 & a\end{smallmatrix}\right)$ and $A^{-1}z = \frac{1}{a^2}z-\frac{b}{a}$.

Therefore,

$$\begin{aligned} F(AT(1)A^{-1}z) &= (A^{-1}z)^k G(z) \\ &= \left(\frac{1}{a^2}z-\frac{b}{a}\right)^k G(z). \end{aligned}$$

By recalling the calculations from the proof of Lemma 2.1, we see that (2.3) follows. To complete the proof of the case $c = 0$, we need to show that both $\frac{1}{2\pi i}\int\limits_{\gamma_2} M_G(s;A)y^{-s}ds$ and $\frac{1}{2\pi i}\int\limits_{\gamma_4} M_G(s;A)y^{-s}ds \to 0$ as $t \to \infty$. By the Lebesgue Dominated Convergence theorem, it suffices to prove the following

Claim: $M_G(s;A)y^{-s}ds \to 0$, uniformly in $|\sigma| \le r$, as $|t| \to \infty$.

Proof of Claim: The essential tools used here are the Phragmén-Lindelöf principle and Stirling's formula. First, recall the following from the proof of Lemma 2.2 (case $c = 0$):

$$M_G(s;A) = \left(\tfrac{\lambda_2}{2\pi}\right)^s \Gamma(s) L_G(s;\chi_{\frac{b}{\lambda_2 a}}) = a^{-2s}\Omega_G(s;\chi_{\frac{b}{\lambda_2 a}}),$$

and thus,

$$L_G(s;\chi_{\frac{b}{\lambda_2 a}}) = \left(\tfrac{2\pi}{\lambda_2}\right)^s \tfrac{1}{\Gamma(s)} M_G(s;A) = \left(\tfrac{2\pi}{\lambda_2 a^2}\right)^s \tfrac{1}{\Gamma(s)}\Omega_G(s;\chi_{\frac{b}{\lambda_2 a}}).$$

Applying the asymptotic version of Stirling's formula and using the assumption that $\Omega_G(s;\chi_{\frac{b}{\lambda_2 a}})$ is bounded in lacunary vertical strips, we obtain the following:

Inside $|\sigma| \le r$, $|t| \ge t_0$, t_0 sufficiently large:

$L_G(s;\chi_{\frac{b}{\lambda_2 a}}) = \mathcal{O}(|t|^{\frac{1}{2}-\sigma} e^{\pi|t|/2}) = \mathcal{O}(\exp\{e^{K|s|}\})$, for any constant $K > 0$.

On $\sigma = r$, since $r > \gamma + 1$, $|L_G(s;\chi_{\frac{b}{\lambda_2 a}})| \le \sum_{n=1}^{\infty} |b_n| n^{-\sigma} = \sum_{n=1}^{\infty} |b_n| n^{-r}$, a finite number independent of t, i.e., $L_G(s;\chi_{\frac{b}{\lambda_2 a}})$ is bounded on $\sigma = r$.

On $\sigma = -r$, we use the functional equation and Stirling's formula. Applying the functional equation: $\Omega_G(s;\chi_{\frac{b}{\lambda_2 a}}) = i^{-k}\, a^{4s-2k}\, \Omega_F(k-s;\chi_{\frac{b}{\lambda_1 a}})$, to the series $\Omega_G(s;\chi_{\frac{b}{\lambda_2 a}}) = \left(\frac{2\pi}{\lambda_2 a^2}\right)^{-s}\Gamma(s)L_G(s;\chi_{\frac{b}{\lambda_2 a}})$, gives the following:

$$\left(\tfrac{2\pi}{\lambda_2 a^2}\right)^{-s}\Gamma(s)L_G(s;\chi_{\frac{b}{\lambda_2 a}}) = i^{-k}\, a^{4s-2k}\left(\tfrac{2\pi}{\lambda_1 a^2}\right)^{s-k}\Gamma(k-s)L_F(k-s;\chi_{\frac{b}{\lambda_1 a}})$$

or

$$L_G(s;\chi_{\frac{b}{\lambda_2 a}}) = i^{-k}\left(\tfrac{2\pi}{\lambda_2}\right)^s \left(\tfrac{2\pi}{\lambda_1}\right)^{s-k} \tfrac{\Gamma(k-s)}{\Gamma(s)} L_F(k-s;\chi_{\frac{b}{\lambda_1 a}}).$$

Note that $r > \gamma+1$ and $k > 0 \Rightarrow k+r > \gamma+1$. Therefore, on $\sigma = -r$, Remark 2.1 implies $L_F(k-s;\chi_{\frac{b}{\lambda_1 a}}) = L_F(k+r-it;\chi_{\frac{b}{\lambda_1 a}})$ remains bounded for $t \in \mathbb{R}$.

Also, $\left|i^{-k}\left(\frac{2\pi}{\lambda_2}\right)^s\left(\frac{2\pi}{\lambda_1}\right)^{s-k}\right| = (\frac{2\pi}{\lambda_2})^{\sigma}(\frac{2\pi}{\lambda_1})^{\sigma-k}$ is independent of $t \in \mathbb{R}$, and therefore, bounded for $t \in \mathbb{R}$.

To handle the growth of the quotient of Γ functions, we use Stirling's formula:

$$\left|\frac{\Gamma(k-s)}{\Gamma(s)}\right| \sim \frac{\sqrt{2\pi}\, e^{-r-k}|t|^{k+r-\frac{1}{2}}e^{-\pi|t|/2}}{\sqrt{2\pi}\, e^{r}|t|^{-r-\frac{1}{2}}e^{-\pi|t|/2}}, \text{ as } |t| \to +\infty$$

$$\sim e^{-2r-k}|t|^{k+2r}, \text{ as } |t| \to +\infty,$$

and thus, on $\sigma = -r$, $L_G(s;\chi_{\frac{b}{\lambda a}}) = \mathcal{O}(|t|^{k+2r})$, as $|t| \to +\infty$.

Note that $L_G(s;\chi_{\frac{b}{\lambda_2 a}})$ is analytic in $|\sigma| \le r, |t| \ge t_0$. Therefore, the Phragmén-Lindelöf principle implies that $L_G(s;\chi_{\frac{b}{\lambda_2 a}}) = \mathcal{O}(|t|^{(\frac{k+2r}{-2r})\sigma+(\frac{k}{2}+r)}) = \mathcal{O}(|t|^k)$, uniformly on $|\sigma| \le r$, $|t| \ge t_0$, t_0 sufficiently large, the constant from $\mathcal{O}$ being independent of s. Recall that $M_G(s;A) = \left(\frac{\lambda_2}{2\pi}\right)^s \Gamma(s) L_G(s;\chi_{\frac{b}{\lambda_2 a}})$. Then, Stirling's formula applied to $\Gamma(s)$ on LVS's shows that $M_G(s;A) = \mathcal{O}(|t|^{\rho+k}e^{-\pi|t|/2})$, uniformly on $|\sigma| \le r$, as $|t| \to \infty$. (ρ depends only on r.)

Therefore, $|M_G(s;A)y^{-s}| \le R|t|^{\rho+k}e^{-\pi|t|/2}y^{-\sigma}$ (R a constant) $\to 0$, uniformly on $|\sigma| \le r$, as $|t| \to \infty$. This completes the proof of the claim, and thus, the proof of the case $c = 0$.

The case $d = 0$ is similar. The first difference becomes evident in the transformation law described in Note 1, where the factor y^{-k} emerges instead of y^k. Further differences arise as a result of the change of variable in the Mellin transform from s to $-s$. As before, we change the variable of integration from y to $1/y$ and apply the transformation law described in Note 1, case $d = 0$. We obtain the following:

$$\begin{aligned} M_F(-s;A) &= \int_0^\infty \{F(Aiy) - a_0\} y^{-s} \frac{dy}{y} \\ &= \int_0^1 \{F(Aiy) - a_0\} y^{-s} \frac{dy}{y} + \int_1^\infty \{F(Aiy) - a_0\} y^{-s} \frac{dy}{y}. \\ &= \int_1^\infty \{F\left(A\frac{i}{y}\right) - a_0\} y^{s} \frac{dy}{y} + \int_1^\infty \{F(Aiy) - a_0\} y^{-s} \frac{dy}{y} \\ &= \int_1^\infty \{i^k\, G(Aiy)\, y^{-k} - a_0\} y^{s} \frac{dy}{y} + \int_1^\infty \{F(Aiy) - a_0\} y^{-s} \frac{dy}{y}. \end{aligned}$$

Suppose $\mathrm{Re}(s) < \min\{-\gamma - 1, 0, k\} = -\gamma - 1$. Then the first integral can be rewritten as

$$\begin{aligned} &i^k \int_1^\infty \{G(Aiy) - b_0\} y^{s-k} \tfrac{dy}{y} + b_0 i^k \int_1^\infty y^{s-k} \tfrac{dy}{y} - a_0 \int_1^\infty y^{s} \tfrac{dy}{y} \\ &= i^k \int_1^\infty \{G(Aiy) - b_0\} y^{s-k} \tfrac{dy}{y} + \left(\tfrac{b_0 i^k}{k-s} + \tfrac{a_0}{s}\right). \end{aligned}$$

Therefore,

$$M_F(-s;A) = \underbrace{i^k \int_1^\infty \{G(Aiy) - b_0\} y^{s-k} \frac{dy}{y} + \int_1^\infty \{F(Aiy) - a_0\} y^{-s} \frac{dy}{y}}_{E_F(s;A)} + \underbrace{\left(\frac{b_0 i^k}{k-s} + \frac{a_0}{s}\right)}_{R_F(s;A)}.$$

By the proof of Lemma 2.2 (case $d = 0$), this is equivalent to

$$\Omega_F(s; \chi_{\frac{-a}{\lambda_1 b}}) = b^{2s} E_F(s;A) + b^{2s} R_F(s;A).$$

To get the analogue for $\Omega_G(s;\chi_{\frac{-a}{\lambda_2 b}})$, we rewrite the transformation law (2.5) with $z \to AT(1)A^{-1}z$. In the case $d = 0$, the transformation law is

$$F(AT(1)A^{-1}z) = b^{-2k}[z+ab]^k\ G(z).$$

With $z \to AT(1)A^{-1}z$, we obtain

$$F(z) = b^{-2k}[AT(1)A^{-1}z + ab]^k\ G(AT(1)A^{-1}z),$$

i.e.,

$$\begin{aligned} G(AT(1)A^{-1}z) &= b^{2k}[AT(1)A^{-1}z+ab]^{-k}F(z) \\ &= b^{2k}\left[\frac{-abz - b^2(a^2+b^2)}{z+ab} + ab\right]^{-k} F(z) \\ &= b^{2k}\left[\frac{-b^4}{z+ab}\right]^{-k} F(z) \\ &= b^{-2k}\left[-(z+ab)\right]^k F(z) \\ &= b^{-2k}i^{-2k}\left[z+ab\right]^k F(z), \end{aligned}$$

the last equality following from Note 2. Putting $z = Aiy$, we have

$$\begin{aligned} G\left(A\frac{i}{y}\right) &= b^{-2k}i^{-2k}[Aiy+ab]^k\ F(Aiy) \\ &= b^{-2k}i^{-2k}\left[b^2\frac{i}{y} - ab + ab\right]^k\ F(Aiy) \\ &= i^{-k}F(Aiy)y^{-k}. \end{aligned}$$

In analogy with $M_F(-s;A)$, with i^k replaced by i^{-k}, we obtain

$$M_G(-s;A) = \underbrace{i^{-k}\int_1^\infty \{F(Aiy)-a_0\}y^{s-k}\frac{dy}{y} + \int_1^\infty \{G(Aiy)-b_0\}y^{-s}\frac{dy}{y}}_{E_G(s;A)} + \underbrace{\left(\frac{a_0 i^{-k}}{k-s} + \frac{b_0}{s}\right)}_{R_G(s;A)}.$$

By the proof of Lemma 2.2 (case $d=0$), this is equivalent to

$$\Omega_G(s;\chi_{\frac{-a}{\lambda_2 b}}) = b^{2s}E_G(s;A) + b^{2s}R_G(s;A).$$

With the same reasoning given in the case $c=0$, we have that $E_F(s;A)$ and $E_G(s;A)$ are entire in s and hence, $b^{2s}E_F(s;A)$ and $b^{2s}E_G(s;A)$ are entire in s. This proves part (i). Consequently, we can replace s by $k-s$ and obtain part

(ii) immediately. The proof of part (iii) is virtually the same as that given in the case $c = 0$, the minor differences being in the reverse placement of the bounds σ_1 and σ_2 as a result of the new variable $-s$. This completes the proof of $B_2 \Rightarrow A_2$.

Next, we assume the Dirichlet series conditions in A_2, including the functional equation (2.4). We will prove that the transformation law (2.5) in B_2 holds. As in the case $c = 0$, we let $r > \gamma + 1$ and obtain the inverse Mellin transform of $G - b_0$ *along the image under* A *of the non-negative imaginary axis* as follows:

$$
\begin{aligned}
\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} M_G(-s;A)y^s\mathrm{d}s &= \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty}\left(\frac{\lambda_2}{2\pi}\right)^s \Gamma(s)L_G(s;\chi_{\frac{-a}{\lambda_2 b}})\, y^s\mathrm{d}s \\
&= \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty}\left(\frac{\lambda_2}{2\pi}\right)^s \Gamma(s)\sum_{n=1}^{\infty}\frac{b_n e^{-2\pi i n a/\lambda_2 b}}{n^s}\, y^s\mathrm{d}s \\
&\overset{(*)}{=} \sum_{n=1}^{\infty} b_n e^{-2\pi i n a/\lambda_2 b}\,\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty}\left(\frac{2\pi}{\lambda_2}\frac{n}{y}\right)^{-s}\Gamma(s)\mathrm{d}s \\
&= \sum_{n=1}^{\infty} b_n e^{-2\pi i n a/\lambda_2 b}\, e^{-2\pi n/y\lambda_2} \\
&= G(Aiy) - b_0. \qquad (2.7)
\end{aligned}
$$

The justification of (*) is virtually the same as that given in the case $c = 0$, the only differences occuring in the character twist (which plays no role in absolute value) and the y^s term instead of the y^{-s} term from the case $c = 0$, neither term affecting the analysis.

Note that $k > 0$ implies that (2.7) is still valid for $k + r$. We consider the same contour of integration as in the case $c = 0$, and note that the Phragmén-Lindelöf principle can be applied in a completely analogous way to show that $\frac{1}{2\pi i}\int_{\gamma_2} M_G(-s;A)y^s\mathrm{d}s$ and $\frac{1}{2\pi i}\int_{\gamma_4} M_G(-s;A)y^s\mathrm{d}s \to 0$ as $t \to \infty$. Recall that by the proof of Lemma 2.2 (case $d = 0$), $M_G(-s;A) = b^{-2s}\Omega_G(s;\chi_{\frac{-a}{\lambda_2 b}})$, which,

the proof of Lemma 2.2 (case $d = 0$), $M_G(-s;A) = b^{-2s}\Omega_G(s;\chi_{\frac{-a}{\lambda_2 b}})$, which,

in the variable $-s$, has principal part $\left(\frac{a_0 i^{-k}}{-s+k} - \frac{b_0}{-s}\right)$ by the assumption in A_2 part(i). Therefore, by (2.7) and the residue theorem, we have

$$\begin{aligned} G(Aiy) - b_0 &= \frac{1}{2\pi i}\int_{-r-i\infty}^{-r+i\infty} M_G(-s;A)y^s \mathrm{d}s + \sum_{s=0,k} \mathrm{Res}(M_G(-s;A)y^s;-s) \\ &= \frac{1}{2\pi i}\int_{-r-i\infty}^{-r+i\infty} M_G(-s;A)y^s \mathrm{d}s + \left[a_0 i^{-k}y^k - b_0\right]. \end{aligned}$$

We assumed that $\Omega_F(k-s;\chi_{\frac{-a}{\lambda_1 b}}) = i^k\, b^{-4s+2k}\, \Omega_G(s;\chi_{\frac{-a}{\lambda_2 b}})$; equivalently, by the proof of Lemma 2.2 (case $d=0$), $M_F(s-k,A) = i^k\, M_G(-s,A)$. Thus,

$$\begin{aligned} G(Aiy) - b_0 &= \frac{1}{2\pi i}\int_{-r-i\infty}^{-r+i\infty} M_G(-s;A)y^s \mathrm{d}s + \left[a_0 i^{-k}y^k - b_0\right] \\ &= \frac{i^{-k}}{2\pi i}\int_{-r-i\infty}^{-r+i\infty} M_F(s-k;A)y^s \mathrm{d}s + \left[a_0 i^{-k}y^k - b_0\right] \\ &= \frac{i^{-k}}{2\pi i}\int_{k+r+i\infty}^{k+r-i\infty} M_F(-s;A)y^{k-s}(-\mathrm{d}s) + \left[a_0 i^{-k}y^k - b_0\right] \\ &= \left(\frac{i}{y}\right)^{-k} \frac{1}{2\pi i}\int_{k+r-i\infty}^{k+r+i\infty} M_F(-s;A)\left(\frac{1}{y}\right)^s \mathrm{d}s + \left[a_0 i^{-k}y^k - b_0\right] \\ &\overset{(**)}{=} \left(\frac{i}{y}\right)^{-k}\left[F\left(A\frac{i}{y}\right) - a_0\right] + \left[a_0 i^{-k}y^k - b_0\right] \\ &= \left(\frac{i}{y}\right)^{-k} F\left(A\frac{i}{y}\right) - b_0, \end{aligned}$$

(**) following from the fact that (2.8) is still valid for $k+r$ since $k>0$. Therefore, for $z = Aiy$, we have $G(z) = (TA^{-1}z)^{-k}\, F(AT(1)A^{-1}z)$. Since this holds for all $y>0$, and both sides of this relation are analytic in $\mathbb{H}$, the uniqueness theorem implies that this relation holds for all $z \in \mathbb{H}$.

With $A = \left(\begin{smallmatrix} a & b \\ -1/b & 0 \end{smallmatrix}\right) \in SL(2,\mathbb{R})$, $TA^{-1} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -b \\ 1/b & a \end{smallmatrix}\right) = \left(\begin{smallmatrix} -1/b & -a \\ 0 & -b \end{smallmatrix}\right)$ and $TA^{-1}z = \frac{1}{b^2}z + \frac{a}{b}$.

Therefore,

$$\begin{aligned} F(AT(1)A^{-1}z) &= (TA^{-1}z)^k G(z) \\ &= \left(\frac{1}{b^2}z + \frac{a}{b}\right)^k G(z). \end{aligned}$$

Again, by recalling the calculations from the proof of Lemma 2.1, we see that (2.5) follows. This proves the case $d = 0$, and completes the proof of the Theorem.

As a special case, we set $F = G$. Then $\lambda_1 = \lambda_2 = \lambda$. We also allow for a multiplier system (M.S.) v of weight k on the conjugate group $A\langle S(\lambda), T(1)\rangle A^{-1}$. Then F is an entire automorphic form of weight k and M.S. v on $A\langle S(\lambda), T(1)\rangle A^{-1}$. Theorem 2.1 implies

Corollary 2.1 *(Automorphic version of Hecke Correspondence on Conjugate Groups) Let* $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{R})$, *and let* v *be a M.S. of weight* k *on the conjugate group* $A\langle S(\lambda), T(1)\rangle A^{-1}$. *Set* $v(AT(1)A^{-1}) = C$.

The case $c = 0$: *Let* $F(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z/\lambda a^2}, a_n = \mathcal{O}(n^\gamma), \gamma > 0$. *TFAE:*

A_1.

(i) $\Omega_F(s; \chi_{\frac{b}{\lambda a}})$ *can be continued meromorphically to the entire s-plane and*

$$\Omega_F(s; \chi_{\frac{b}{\lambda a}}) - a^{2s} a_0 \left(\frac{i^k C}{s-k} - \frac{1}{s}\right) \text{ is entire.}$$

(ii) $\Omega_F(s; \chi_{\frac{b}{\lambda a}})$ *satisfies the functional equation*

$$\Omega_F(k - s; \chi_{\frac{b}{\lambda a}}) = i^k C\, a^{-4s+2k}\, \Omega_F(s; \chi_{\frac{b}{\lambda a}}). \tag{2.8}$$

(iii) $\Omega_F(s; \chi_{\frac{b}{\lambda a}})$ *remains bounded in every lacunary vertical strip:*

$$\sigma_1 \le Re(s) \le \sigma_2,\ |t| \ge t_0 > 0.$$

B_1.

$$\left(F|_k^v AT(1)A^{-1}\right)(z) = F(z). \tag{2.9}$$

The case $d=0$: Let $F(z)=\sum_{n=0}^{\infty} a_n e^{2\pi i n z/\lambda b^2}, a_n=\mathcal{O}(n^\gamma), \gamma>0$. TFAE:

A_2.

(i) $\Omega_F(s;\chi_{\frac{-a}{\lambda b}})$ can be continued meromorphically to the entire s-plane and

$$\Omega_F(s;\chi_{\frac{-a}{\lambda b}}) - b^{2s}a_0\left(\frac{i^kC}{k-s}+\frac{1}{s}\right) \text{ is entire.}$$

(ii) $\Omega_F(s;\chi_{\frac{-a}{\lambda b}})$ satisfies the functional equation

$$\Omega_F(k-s;\chi_{\frac{-a}{\lambda b}}) = i^kC\ b^{-4s+2k}\ \Omega_F(s;\chi_{\frac{-a}{\lambda b}}). \tag{2.10}$$

(iii) $\Omega_F(s;\chi_{\frac{-a}{\lambda b}})$ remains bounded in every lacunary vertical strip:

$$\sigma_1 \le Re(s) \le \sigma_2,\ |t| \ge t_0 > 0.$$

B_2.

For $z\in\mathbb{H}$,

$$\left(F|_k^v AT(1)A^{-1}\right)(z) = F(z). \tag{2.11}$$

Note: In each case, replacing s by $k-s$ in the functional equation shows that $i^kC=\pm 1$. This implies that $C=\pm\ e^{-i\pi k/2}$. However, as we will see, $i^kC=\pm 1$ is a necessary condition for proving (2.8) and (2.10), and follows directly from (2.9) and (2.11) respectively.

Proof: It suffices to show that $(i^kC)^2=1$. To see this, refer to the expressions for $M_F(s;A)$ and $M_G(s;A)$ on pages 17 and 18 in the proof of Theorem 2.1. There, we see that multiplying $M_G(s;A)$ by i^k (and replacing s by $k-s$) gives $M_F(k-s;A)$ (which implies the functional equation). The essential difference here is the presence of i^kC as opposed to i^k. To get the functional equation here, we multiply by i^kC (and replace s by $k-s$); it is here that we need $(i^kC)^2=1$. (The same reasoning holds for the case $d=0$.) For the case $c=0$, this follows from the transformation law and the calculations in Lemma 2.1 for the case $c=0$. Indeed,

$$\begin{aligned}\left(F|_k^v AT(1)A^{-1}\right)(z)=F(z) \ &\Leftrightarrow\ \bar{v}(AT(1)A^{-1})\left(\frac{z}{a^2}-\frac{b}{a}\right)^{-k} F(AT(1)A^{-1}z)=F(z)\\ &\Leftrightarrow\ F(AT(1)A^{-1}z)=C\left(\frac{z}{a^2}-\frac{b}{a}\right)^{k} F(z)\\ &\Leftrightarrow\ F(AT(1)A^{-1}z)=Ca^{-2k}(z-ab)^kF(z).\end{aligned}$$

By replacing z by $AT(1)A^{-1}z$ and using the transformation law again, this holds $\Leftrightarrow$

$$\begin{aligned} F(z) &= Ca^{-2k}(AT(1)A^{-1}z - ab)^k F(AT(1)A^{-1}z) \\ &= Ca^{-2k}(AT(1)A^{-1}z - ab)^k Ca^{-2k}(z-ab)^k F(z). \end{aligned}$$

By the uniqueness theorem, we have

$$1 = C^2 a^{-4k}(AT(1)A^{-1}z - ab)^k (z-ab)^k \ \forall z \in \mathbb{H}.$$

Putting $z = Ai$, we obtain

$$\begin{aligned} 1 &= C^2 a^{-4k}(Ai - ab)^{2k} \\ &= C^2 a^{-4k}(a^2 i + ab - ab)^{2k} \\ &= C^2 a^{-4k}(a^2 i)^{2k} \\ &= C^2 i^{2k} \\ &= (i^k C)^2. \end{aligned}$$

For the case $d = 0$, this will follow from the transformation law and the calculations in Lemma 2.1 for the case $d = 0$. We have

$$\begin{aligned} \left(F|_k^v AT(1)A^{-1}\right)(z) = F(z) &\Leftrightarrow \bar{v}(AT(1)A^{-1})\left(\frac{z}{b^2} + \frac{a}{b}\right)^{-k} F(AT(1)A^{-1}z) = F(z) \\ &\Leftrightarrow F(AT(1)A^{-1}z) = C\left(\frac{z}{b^2} + \frac{a}{b}\right)^k F(z) \\ &\Leftrightarrow F(AT(1)A^{-1}z) = Cb^{-2k}(z+ab)^k F(z). \end{aligned}$$

By replacing z by $AT(1)A^{-1}z$ and using the transformation law again, this holds $\Leftrightarrow$

$$\begin{aligned} F(z) &= Cb^{-2k}(AT(1)A^{-1}z + ab)^k F(AT(1)A^{-1}z) \\ &= Cb^{-2k}(AT(1)A^{-1}z + ab)^k Cb^{-2k}(z+ab)^k F(z). \end{aligned}$$

By the uniqueness theorem, we have

$$1 = C^2 b^{-4k}(AT(1)A^{-1}z + ab)^k (z+ab)^k \ \forall z \in \mathbb{H}.$$

Putting $z = Ai$, we obtain

$$\begin{aligned} 1 &= C^2 b^{-4k}(Ai + ab)^{2k} \\ &= C^2 b^{-4k}(b^2 i - ab + ab)^{2k} \\ &= C^2 b^{-4k}(b^2 i)^{2k} \\ &= C^2 i^{2k} \\ &= (i^k C)^2. \end{aligned}$$

We note that the case $c = 0$ can also be approached by using the classic Hecke correspondence since if F is a form on $A\Gamma A^{-1}$, then $F \mid A$ is a form on Γ, which implies $F \circ A$ is a form on Γ for $A = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in GL(2, \mathbb{R})$. However, the approach taken here has a much broader perspective that illustrates, for certain sets of conjugates, the exact type of functional equation satisfied by the corresponding Dirichlet series. Furthermore, we study two special sets of conjugates, which give rise to very natural generalizations of the Hecke groups. This is the topic of the next section.

2.3 The Case $\chi = 1$

We explore special cases of λ such that χ is the identity, i.e., cases in which the twisted Dirichlet series reduces to the usual Dirichlet series associated to the exponential series. We do this for each case, $c = 0$, $d = 0$. Furthermore, we show that the corresponding group, in each case, is a natural generalization of the classic Hecke group. From Theorem 2.1, we see that in the case $c = 0$, χ is the identity $\Leftrightarrow b = t\lambda a,\ t \in \mathbb{Z}$. In the case $d = 0$, χ is the identity $\Leftrightarrow a = t\lambda b,\ t \in \mathbb{Z}$. In each case, we examine the corresponding groups.

We begin with the case $c = 0$. By Lemma 2.1, $A\Gamma A^{-1} \equiv A\,\langle S(\lambda), T(1)\rangle\, A^{-1}$ $= \langle S(\lambda a^2), AT(1)A^{-1}\rangle$. With $b = t\lambda a,\ t \in \mathbb{Z}$, the proof of Lemma 2.1 shows that

$$AT(1)A^{-1} = \begin{pmatrix} t\lambda & -a^2[(t\lambda)^2+1] \\ \frac{1}{a^2} & -t\lambda \end{pmatrix}.$$

Claim: $A\Gamma A^{-1} = \langle S(\lambda a^2), AT(1)A^{-1}\rangle = \langle S(\lambda a^2), T(\frac{1}{a^4})\rangle$.

Proof of Claim: It suffices to show that $A\Gamma A^{-1} \ni T(\frac{1}{a^4})$ and $AT(1)A^{-1} \in \langle S(\lambda a^2), T(\frac{1}{a^4})\rangle$. However, these follow by considering the following:

$$\begin{aligned}
\underbrace{S(\lambda a^2)^{-1}}_{t\ times}(AT(1)A^{-1})\underbrace{S(\lambda a^2)}_{t\ times} &= S(-t\lambda a^2)(AT(1)A^{-1})S(t\lambda a^2) \\
&= \begin{pmatrix} 1 & -t\lambda a^2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t\lambda & -a^2[(t\lambda)^2+1] \\ \frac{1}{a^2} & -t\lambda \end{pmatrix}\begin{pmatrix} 1 & t\lambda a^2 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -a^2 \\ 1/a^2 & -t\lambda \end{pmatrix}\begin{pmatrix} 1 & t\lambda a^2 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -a^2 \\ 1/a^2 & 0 \end{pmatrix} \\
&= T\left(\frac{1}{a^4}\right).
\end{aligned}$$

For the case $d = 0$, Lemma 2.1 implies that $A\Gamma A^{-1} \equiv A\langle S(\lambda), T(1)\rangle A^{-1}$ $= \langle S(\lambda b^2), AT(1)A^{-1}\rangle$. With $a = t\lambda b$, $t \in \mathbb{Z}$, the proof of Lemma 2.1 shows that

$$AT(1)A^{-1} = \begin{pmatrix} -t\lambda & -b^2((t\lambda)^2+1) \\ \frac{1}{b^2} & t\lambda \end{pmatrix}.$$

Claim: $A\Gamma A^{-1} = \langle S(\lambda b^2), AT(1)A^{-1}\rangle = \langle S(\lambda b^2), T(\frac{1}{b^4})\rangle$.

Proof of Claim: It suffices to show that $A\Gamma A^{-1} \ni T(\frac{1}{b^4})$ and $AT(1)A^{-1} \in \langle S(\lambda b^2), T(\frac{1}{b^4})\rangle$. However, these follow by considering the following:

$$\begin{aligned}
\underbrace{S(\lambda b^2)}_{t\ times}(AT(1)A^{-1})\underbrace{S(\lambda b^2)^{-1}}_{t\ times} &= S(t\lambda b^2)(AT(1)A^{-1})S(-t\lambda b^2) \\
&= \begin{pmatrix} 1 & t\lambda b^2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -t\lambda & -b^2[(t\lambda)^2+1] \\ \frac{1}{b^2} & t\lambda \end{pmatrix}\begin{pmatrix} 1 & -t\lambda b^2 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -b^2 \\ 1/b^2 & t\lambda \end{pmatrix}\begin{pmatrix} 1 & -t\lambda b^2 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -b^2 \\ 1/b^2 & 0 \end{pmatrix} \\
&= T\left(\frac{1}{b^4}\right).
\end{aligned}$$

An immediate consequence of the two claims is the following:

Remark: If $A = \left(\begin{smallmatrix} a & t\lambda a \\ 0 & 1/a \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} t\lambda a & a \\ -1/a & 0 \end{smallmatrix}\right)$, with $a \in \mathbb{R}$, $t \in \mathbb{Z}$, then the conjugate group $A\langle S(\lambda), T(1)\rangle A^{-1} = \langle S(\lambda a^2), T(\frac{1}{a^4})\rangle$. Let $\widetilde{\lambda}$, μ be arbitrary positive real numbers. If we set $\lambda = \widetilde{\lambda}\sqrt{\mu}$ and $a = \frac{1}{\sqrt[4]{\mu}}$, then this group can be written as $\left\langle S(\widetilde{\lambda}), T(\mu)\right\rangle$. Note that $a^{-4s+2k} = (a^2)^{-2s+k} = \mu^{s-k/2}$. For convenience, we switch the roles of $\widetilde{\lambda}$ and λ. The group $\langle S(\lambda), T(\mu)\rangle$ is a generalization of the classic Hecke group. We denote this group by $H(\lambda, \mu)$. With these variables, Theorem 2.1 implies

Theorem 2.2 *(Hecke Correspondence on $H(\lambda, \mu)$)*
Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{R})$. If $A = \left(\begin{smallmatrix} a & t\widetilde{\lambda} a \\ 0 & 1/a \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} t\widetilde{\lambda} a & a \\ -1/a & 0 \end{smallmatrix}\right)$, with $a \in \mathbb{R}$, $t \in \mathbb{Z}$, then the conjugate group $A\left\langle S(\widetilde{\lambda}), T(1)\right\rangle A^{-1} = H(\lambda, \mu)$ and we have the following:

Let $F(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z/\lambda_1}$ and $G(z) = \sum_{n=0}^{\infty} b_n e^{2\pi i n z/\lambda_2}$, $a_n = \mathcal{O}(n^\gamma)$, and $b_n = \mathcal{O}(n^\gamma)$, for some $\gamma > 0$. Set $\Omega_F(s) \equiv \Omega_F(s, 1) = \sum_{n=1}^{\infty} a_n n^{-s}$ to be the associated Dirichlet series. TFAE:

A.

(i) $\Omega_F(s)$ and $\Omega_G(s)$ can be continued meromorphically to the entire s-plane with $\Omega_F(s) - \mu^{-s/2}\left(\frac{b_0 i^k}{s-k} - \frac{a_0}{s}\right)$ and $\Omega_G(s) - \mu^{-s/2}\left(\frac{a_0 i^{-k}}{s-k} - \frac{b_0}{s}\right)$ entire.

(ii) $\Omega_F(s)$ and $\Omega_G(s)$ satisfy the functional equation

$$\Omega_F(k - s) = i^k \mu^{s-k/2} \Omega_G(s). \tag{2.12}$$

(iii) $\Omega_F(s)$ and $\Omega_G(s)$ remain bounded in every LVS:

$$\sigma_1 \leq Re(s) \leq \sigma_2, \ |t| \geq t_0 > 0.$$

B.

For $z \in \mathbb{H}$,

$$(F|_k T(\mu))(z) = G(z). \tag{2.13}$$

We state the following corollary for convenience in applications to Chapter 5.

Corollary 2.2 *(Automorphic version of Hecke Correspondence on $H(\lambda,\mu)$)* *Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{R})$, v be a M.S. of weight k on the conjugate group $A\left\langle S(\widetilde{\lambda}), T(1)\right\rangle A^{-1}$. Set $v(AT(1)A^{-1}) = C$. If $A = \left(\begin{smallmatrix} a & t\lambda a \\ 0 & 1/a \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} t\lambda a & a \\ -1/a & 0 \end{smallmatrix}\right)$, with $a \in \mathbb{R}$, $t \in \mathbb{Z}$, then $A\left\langle S(\widetilde{\lambda}), T(1)\right\rangle A^{-1} = H(\lambda,\mu)$ and we have the following: Let $F(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z/\lambda}$ such that $a_n = \mathcal{O}(n^\gamma)$ for some $\gamma > 0$. Set $\Omega_F(s) \equiv \Omega_F(s,1) = \sum_{n=1}^{\infty} a_n n^{-s}$ to be the associated Dirichlet series. TFAE:*

A.

(i) $\Omega_F(s)$ can be continued meromorphically to the entire s-plane and

$$\Omega_F(s) - \mu^{-s/2} a_0 \left(\frac{i^k C}{s-k} - \frac{1}{s}\right) \text{ is entire.}$$

(ii) $\Omega_F(s)$ satisfies the functional equation

$$\Omega_F(k-s) = i^k C\ \mu^{s-k/2}\ \Omega_F(s). \tag{2.14}$$

(iii) $\Omega_F(s)$ remains bounded in every LVS:

$$\sigma_1 \le Re(s) \le \sigma_2,\ \ |t| \ge t_0 > 0.$$

B.

For $z \in \mathbb{H}$,

$$\left(F|_k^v T(\mu)\right)(z) = F(z). \tag{2.15}$$

Remark As before, $i^k C = \pm 1$ ($C = \pm\ e^{-i\pi k/2}$). Recall that if $k \in \mathbb{Z}$, then the multiplier system v for the group $H(\lambda,\mu)$ and weight k is a character, and therefore, v reduces to a M.S. on $H(\lambda) = \langle S(\lambda), T(1)\rangle$. As we pointed out at the end of Chapter 1, $v(T(\mu))$ is independent of μ. Thus, if $k \in \mathbb{Z}$, then $C = v(AT(1)A^{-1}) = v(T(1)) = v(T(\mu))$. A priori, in Corollary 2.2, with $k \in \mathbb{R}$, $C = v(AT(1)A^{-1}) \ne v(T(1)) = v(T(\mu))$.

In the next chapter, we examine algebraic and geometric properties of $H(\lambda,\mu)$ including a condition on discreteness, a fundamental region, and a modular analogue.

CHAPTER 3

THE GENERALIZED HECKE GROUP $H(\lambda, \mu)$

Let λ, $\mu \in \mathbb{R}^+$. We call the group $H(\lambda, \mu) = \langle S(\lambda), T(\mu) \rangle$ the *Hecke group of width λ and inversion μ.* Note that $H(\lambda, 1)$ is a classic Hecke group. Recall that a linear fractional transformation group has an associated matrix representation modulo $\pm$ the identity. In addition, we will make frequent use of the fact that the inversion $T(\mu)$ can be represented as $\left(\begin{smallmatrix} 0 & -x \\ y & 0 \end{smallmatrix}\right)$, where $\frac{y}{x} = \mu$. However, we take $\left(\begin{smallmatrix} 0 & -1 \\ \mu & 0 \end{smallmatrix}\right)$ as the canonical form.

In this chapter, we give a condition for the discreteness of $H(\lambda, \mu)$ and discuss briefly a standard fundamental region associated to the group. Finally, we give a context, in terms of group structure, for studying a modular analogue of the generalized Hecke group. We will see that this modular analogue contains a very special subgroup of the modular group.

3.1 Discreteness

The main result is the following:

Proposition 3.1 *$H(\lambda,\mu)$ is discrete* $\Leftrightarrow \lambda\sqrt{\mu} \geq 2$ *or* $\lambda\sqrt{\mu} = 2\cos\frac{\pi}{q}$, $q \in \mathbb{Z}, q \geq 3$.

Remark 3.1 *The case $\lambda,\mu \in \mathbb{Z}^+$ can be proved directly using only properties of discreteness and limit sets.*

This can be seen by first recalling two facts in [14]. First, let $L(\Gamma)$ = the limit set of Γ. Then $\Gamma_1 \subset \Gamma_2 \Rightarrow L(\Gamma_1) \subset L(\Gamma_2)$. In particular, a subgroup of a discontinuous group is discontinuous. Second, $\Gamma_1 \subset \Gamma_2$ and $[\Gamma_2 : \Gamma_1] < \infty \Rightarrow L(\Gamma_1) = L(\Gamma_2)$. Thus, a group containing a discontinuous group as a subgroup of finite index is itself discontinuous.

With $\mu \in \mathbb{Z}^+$, let $\Gamma_0^*(\mu)$ be the group generated by $\Gamma_0(\mu)$ and $T(\mu)$. Since $T(\mu)$ is in the normalizer of $\Gamma_0(\mu)$ and of order two, $[\Gamma_0^*(\mu) : \Gamma_0(\mu)] = 2$. Therefore, the second fact implies that $\Gamma_0^*(\mu)$ is discrete. Since $\langle S(\lambda), T(\mu)\rangle$ is a subgroup of $\Gamma_0^*(\mu)$ for any $\lambda \in \mathbb{Z}$, the first fact completes the proof.

Proof of Proposition 3.1: As in [11], let $W(\lambda) = \left(\begin{smallmatrix} 1 & 0 \\ \lambda & 1 \end{smallmatrix}\right)$ and put $K(\lambda_1,\lambda_2) = \langle S(\lambda_1), W(\lambda_2)\rangle$. Claim: $K(\lambda,\lambda\mu) = \langle S(\lambda),\ T(\mu)S(\lambda)T(\mu)\rangle$.

Proof of Claim:

$$\begin{aligned} T(\mu)S(\lambda)T(\mu) &= \left(\begin{smallmatrix} 0 & -1 \\ \mu & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -1 \\ \mu & 0 \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} 0 & -1 \\ \mu & \lambda\mu \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -1 \\ \mu & 0 \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} -\mu & 0 \\ \lambda\mu^2 & -\mu \end{smallmatrix}\right). \end{aligned}$$

Therefore, as a linear fractional transformation,

$$\begin{aligned} (T(\mu)S(\lambda)T(\mu))^{-1} &= \left(\begin{smallmatrix} -\mu & 0 \\ -\lambda\mu^2 & -\mu \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} -1 & 0 \\ -\lambda\mu & -1 \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} 1 & 0 \\ \lambda\mu & 1 \end{smallmatrix}\right) \\ &= W(\lambda\mu). \end{aligned}$$

Since $\langle S(\lambda),\ T(\mu)S(\lambda)T(\mu)\rangle = \langle S(\lambda),\ (T(\mu)S(\lambda)T(\mu))^{-1}\rangle$, the claim follows.

By Lemma 1 in [11], $K(\lambda, \lambda\mu)$ is conjugate to $K(\sqrt{\lambda\lambda\mu}, \sqrt{\lambda\lambda\mu}) = K(\lambda\sqrt{\mu}, \lambda\sqrt{\mu})$ which is discrete by Theorem 1 of [11]

$$\Leftrightarrow \lambda\sqrt{\mu}\ \lambda\sqrt{\mu} \geq 4 \text{ or } \lambda\sqrt{\mu}\ \lambda\sqrt{\mu} = 4\cos^2\tfrac{\pi}{q}, q \in \mathbb{Z}, q \geq 3$$

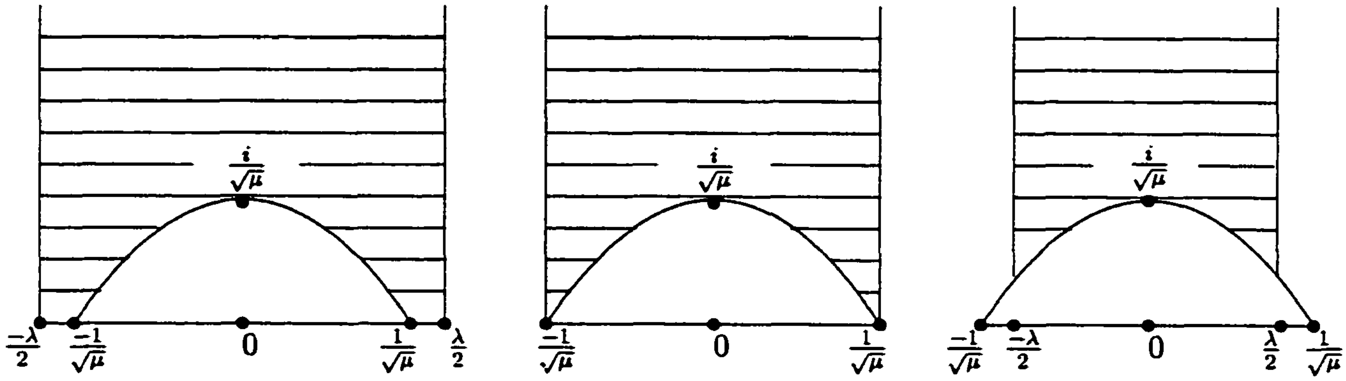
$$\Leftrightarrow \lambda\sqrt{\mu} \geq 2 \text{ or } \lambda\sqrt{\mu} = 2\cos\tfrac{\pi}{q}, q \in \mathbb{Z}, q \geq 3.$$

Therefore, $K(\lambda, \lambda\mu)$ is discrete $\Leftrightarrow \lambda\sqrt{\mu} \geq 2$ or $\lambda\sqrt{\mu} = 2\cos\frac{\pi}{q}, q \in \mathbb{Z}, q \geq 3$.

By Theorem 2 in [11], $K(\lambda, \lambda\mu)$ is of finite index(1 or 2) in $H(\lambda, \mu)$. It follows that $H(\lambda, \mu)$ is discrete $\Leftrightarrow \lambda\sqrt{\mu} \geq 2$ or $\lambda\sqrt{\mu} = 2\cos\frac{\pi}{q}, q \in \mathbb{Z}, q \geq 3$.

3.2 A Fundamental Region

In this section, we describe (the closure of) a fundamental region (F.R.) for $H(\lambda, \mu)$, and calculate its hyperbolic area (H-area). We have three cases:



The case $\lambda\sqrt{\mu} > 2$. H-area $= \infty$. Cusp: $i\infty$.

The case $\lambda\sqrt{\mu} = 2$. H-area $= \pi$. Cusps: $i\infty$, $\frac{1}{\sqrt{\mu}}$.

The case $\lambda\sqrt{\mu} = 2\ cos\frac{\pi}{q}$. H-area $= \frac{(q-2)\pi}{q}$. Cusp: $i\infty$.

Note: The semicircles of radius $\frac{1}{\sqrt{\mu}}$ are meant to be orthogonal to the real axis.

Remarks:

1. Since $S(\lambda)$ is the minimal translation in $H(\lambda, \mu)$, the width is λ and the lines $Re(z) = \pm \frac{\lambda}{2}$ are its fixed circles. The fixed circle of the elliptic transformation $T(\mu)$ is the semicircle of radius $\frac{1}{\sqrt{\mu}}$; the elliptic fixed point of $T(\mu)$ is $\frac{i}{\sqrt{\mu}}$. Moreover, the fixed semicircle is orthogonal to the family of rays through the origin, as noted above (see [14]).

2. For each case, the H-area is identical to the H-area of the corresponding classic Hecke group. This fact is trivial since, for each case, a generalized Hecke group is conjugate to a classic Hecke group, in that respective case (see the remark preceding Theorem 2.2), and therefore, for each respective case, a F.R. of a generalized Hecke group is just the image under a real linear fractional transformation of a F.R. of a classic Hecke group. However, for completeness, we calculate the H-area of the F.R. explicitly using the Gauss-Bonnet theorem (see [14]).

For the case $\lambda\sqrt{\mu} > 2$, we have free sides, and therefore, infinite H-area (see [14]). For the case $\lambda\sqrt{\mu} = 2$, we note that since the semicircle of radius $\frac{1}{\sqrt{\mu}}$ is orthogonal to the real axis, the angle made between the vertical lines and the semicircle is 0. Therefore, Gauss-Bonnet implies that the H-area is π. For the case $\lambda\sqrt{\mu} < 2$, i.e., the case $\lambda\sqrt{\mu} = 2\cos\frac{\pi}{q}$, $q \in \mathbf{Z}$, $q \geq 3$, Gauss-Bonnet implies that the H-area $= \pi - 2\theta$, where θ is the angle made between the vertical lines and the semicircle. By elementary geometry, $\cos\theta = \frac{\frac{\lambda}{2}}{\frac{1}{\sqrt{\mu}}} = \frac{\lambda\sqrt{\mu}}{2} = \cos\frac{\pi}{q}$, $q \in \mathbf{Z}$, $q \geq 3$. Since θ is acute, $\theta = \frac{\pi}{q}$. Therefore, the H-area is $\pi - \frac{2\pi}{q} = \frac{(q-2)\pi}{q}$.

Note: For each case, $\lambda\sqrt{\mu} > 2$, $\lambda\sqrt{\mu} = 2$, and $\lambda\sqrt{\mu} < 2$, by varying μ, we see that there exists a F.R. for $H(\lambda, \mu)$ of every positive width, having infinite H-area, H-area $= \pi$, and H-area $= \frac{(q-2)\pi}{q}$, respectively.

3.3 A Modular Analogue

Throughout, we will assume that $t \in \mathbb{R}^+$.

Notation 3.1 $\Gamma_t := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{R}) \mid \left(\begin{smallmatrix} a & bt \\ \frac{c}{t} & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z}) \right\}$.

A straightforward calculation shows that Γ_t is a group.

Remark 1:

$$\begin{pmatrix} \sqrt{t} & 0 \\ 0 & \frac{1}{\sqrt{t}} \end{pmatrix} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \begin{pmatrix} \frac{1}{\sqrt{t}} & 0 \\ 0 & \sqrt{t} \end{pmatrix} = \begin{pmatrix} a\sqrt{t} & b\sqrt{t} \\ \frac{c}{\sqrt{t}} & \frac{d}{\sqrt{t}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{t}} & 0 \\ 0 & \sqrt{t} \end{pmatrix} = \left(\begin{smallmatrix} a & bt \\ \frac{c}{t} & d \end{smallmatrix}\right),$$

so that $\Gamma_t = \left\{ M \in SL(2,\mathbb{R}) \mid AMA^{-1} \in SL(2,\mathbb{Z}),\ A = \left(\begin{smallmatrix} \sqrt{t} & 0 \\ 0 & \frac{1}{\sqrt{t}} \end{smallmatrix}\right) \right\}$.

Remark 2: $H(\frac{1}{t}, t^2) = \left\langle \left(\begin{smallmatrix} 1 & \frac{1}{t} \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & \frac{-1}{t} \\ t & 0 \end{smallmatrix}\right) \right\rangle \subset \Gamma_t$.

In fact, we have

Proposition 3.2 $H(\frac{1}{t}, t^2) = \left\langle \left(\begin{smallmatrix} 1 & \frac{1}{t} \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & \frac{-1}{t} \\ t & 0 \end{smallmatrix}\right) \right\rangle = \Gamma_t$.

Proof: Let $A = \begin{pmatrix} \sqrt{t} & 0 \\ 0 & \frac{1}{\sqrt{t}} \end{pmatrix}$. $A\Gamma_t A^{-1} = \langle S(1), T(1) \rangle \Leftrightarrow \Gamma_t = A^{-1} \langle S(1), T(1) \rangle A$. But the proof of Lemma 2.1 (for the case $c = 0$) with $a = \frac{1}{t}, \lambda = 1$, and $b = 0$, shows that

$$A^{-1}S(1)A = \left(\begin{smallmatrix} 1 & \frac{1}{t} \\ 0 & 1 \end{smallmatrix}\right) \text{ and } A^{-1}T(1)A = \left(\begin{smallmatrix} 0 & \frac{-1}{t} \\ t & 0 \end{smallmatrix}\right),$$

i.e.,

$$A^{-1} \langle S(1), T(1) \rangle A = H(\tfrac{1}{t}, t^2).$$

Therefore, it suffices to show that $A\Gamma_t A^{-1} = SL(2,\mathbb{Z})$. By the note on the previous page, we see that by the equivalent definition of Γ_t, if $M \in \Gamma_t$, then $AMA^{-1} \in SL(2,\mathbb{Z})$ and hence, $A\Gamma_t A^{-1} \subset SL(2,\mathbb{Z})$. Conversely, let $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$. We need to show the following:

$$\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in A\Gamma_t A^{-1} = \left\{ \left(\begin{smallmatrix} a & bt \\ \frac{c}{t} & d \end{smallmatrix}\right) \mid \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_t \right\}.$$

We have

$$\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) = \left(\begin{smallmatrix}\alpha & \frac{\beta}{t}t\\ \frac{\gamma t}{t} & \delta\end{smallmatrix}\right) \in A\Gamma_t A^{-1} \Leftrightarrow \left(\begin{smallmatrix}\alpha & \frac{\beta}{t}\\ \gamma t & \delta\end{smallmatrix}\right) \in \Gamma_t.$$

By definition of Γ_t,

$$\left(\begin{smallmatrix}\alpha & \frac{\beta}{t}\\ \gamma t & \delta\end{smallmatrix}\right) \in \Gamma_t \text{ since } \left(\begin{smallmatrix}\alpha & \frac{\beta}{t}t\\ \frac{\gamma t}{t} & \delta\end{smallmatrix}\right) = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in SL(2,\mathbb{Z}).$$

Therefore, $A\Gamma_t A^{-1} = SL(2,\mathbb{Z})$ and the proof of Proposition 3.2 is complete.

Corollary 3.1 Γ_t *is discrete.*

Proof: This follows from Proposition 3.1 since $\frac{1}{t}\sqrt{t^2} = 1 = 2cos\frac{\pi}{3}$.

Lemma 3.1 *Let* $M = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in H(\frac{1}{t}, t^2)$, $M^* = \left(\begin{smallmatrix}a & bt\\ \frac{c}{t} & d\end{smallmatrix}\right) \in \Gamma(1)$. *Let* v *be a M.S. of weight* k *on* $\Gamma(1)$. *Set* $v^*(M) = v(M^*)$. *Then* v^* *is a M.S. of weight* k *on* $H(\frac{1}{t}, t^2)$.

Proof: Claim: $* : M \mapsto M^*$ is a group homomorphism from $H(\frac{1}{t}, t^2)$ to $\Gamma(1)$ (under the usual matrix multiplication).

Proof of Claim:

$$\begin{aligned}
*(M_1M_2) &= (M_1M_2)^* \\
&= \left[\left(\begin{smallmatrix}a_1 & b_1\\ c_1 & d_1\end{smallmatrix}\right)\left(\begin{smallmatrix}a_2 & b_2\\ c_2 & d_2\end{smallmatrix}\right)\right]^* \\
&= \begin{pmatrix}a_1a_2+b_1c_2 & a_1b_2+b_1d_2\\ c_1a_2+d_1c_2 & c_1b_2+d_1d_2\end{pmatrix}^* \\
&= \begin{pmatrix}a_1a_2+b_1c_2 & (a_1b_2+b_1d_2)t\\ \frac{c_1a_2+d_1c_2}{t} & c_1b_2+d_1d_2\end{pmatrix}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
*(M_1) * (M_2) &= M_1^*M_2^* \\
&= \left(\begin{smallmatrix}a_1 & b_1t\\ \frac{c_1}{t} & d_1\end{smallmatrix}\right)\left(\begin{smallmatrix}a_2 & b_2t\\ \frac{c_2}{t} & d_2\end{smallmatrix}\right) \\
&= \begin{pmatrix}a_1a_2+b_1c_2 & (a_1b_2+b_1d_2)t\\ \frac{c_1a_2+d_1c_2}{t} & c_1b_2+d_1d_2\end{pmatrix}.
\end{aligned}$$

We need to show that the consistency condition (C.C.) holds for v^*. That is, for any $M_1, M_2 \in H(\frac{1}{t}, t^2)$, and for any $z \in \mathbb{H}$, we need to show the following:

$$v^*(M_1M_2)((c_1a_2 + d_1c_2)z + c_1b_2 + d_1d_2)^k = v^*(M_1)v^*(M_2)(c_1M_2z + d_1)^k(c_2z + d_2)^k.$$

Since v is a M.S. of weight k on $\Gamma(1), v$ satisfies its own (C.C.). In particular, for any $M_1^*, M_2^* \in \Gamma(1)$, and for any $z \in \mathbb{H}$, we have the following:

$$v(M_1^*M_2^*)(\tfrac{(c_1a_2+d_1c_2)}{t}z + c_1b_2 + d_1d_2)^k = v(M_1^*)v(M_2^*)(\tfrac{c_1}{t}M_2^*z + d_1)^k(\tfrac{c_2}{t}z + d_2)^k.$$

Note that $t \in \mathbb{R}^+ \Rightarrow tz \in \mathbb{H}$. Therefore, substituting tz for z, and using the definition of V^* with the claim above, we obtain

$$v^*(M_1M_2)((c_1a_2 + d_1c_2)z + c_1b_2 + d_1d_2)^k = v^*(M_1)v^*(M_2)(\tfrac{c_1}{t}M_2^*tz + d_1)^k(c_2z + d_2)^k.$$

By comparing this with the desired (C.C.) for v^*, we see that it suffices to show that $\frac{c_1}{t}M_2^*tz = c_1M_2z$. But

$$\begin{aligned}
\frac{c_1}{t}M_2^*tz &= \frac{c_1}{t}\left(\frac{a_2tz + b_2t}{\frac{c_2}{t}tz + d_2}\right) \\
&= c_1\left(\frac{a_2z + b_2}{c_2z + d_2}\right) \\
&= c_1M_2z.
\end{aligned}$$

Finally, since v is a M.S. on $\Gamma(1)$, $|v^*(M)| = |v(M^*)| = 1 \ \forall M \in H(\frac{1}{t}, t^2)$. This completes the proof of Lemma 3.1.

We are now in position to prove

Proposition 3.3 *If $F(z)$ is a MF of weight k and M.S. v on $\Gamma(1)$, then $F(tz)$ is a MF of weight k and M.S. v^* on $H(\frac{1}{t}, t^2)$.*

Proof: Let $G(z) = F(tz)$. The analytic properties of F remain unchanged upon multiplication by t. In particular, we know by the previous section that there exists a F.R. for $H(\frac{1}{t}, t^2)$ with a single cusp at $i\infty$. Therefore, since F is meromorphic at the cusp $i\infty$ for the modular group $\Gamma(1)$, G is meromorphic at the cusp $i\infty$ for the modular analogue $H(\frac{1}{t}, t^2)$. Furthermore, we know that the F.R. for $H(\frac{1}{t}, t^2)$ is just the image under a real linear fractional transformation of the F.R. for $\Gamma(1)$. Consequently, $F(tz)$ has (at most) finitely many poles in the F.R. for $H(\frac{1}{t}, t^2)$ since $F(z)$ has (at most) finitely many poles in the F.R. for $\Gamma(1)$. For the transformation law, let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in H(\frac{1}{t}, t^2)$ and $M^* = \left(\begin{smallmatrix} a & bt \\ \frac{c}{t} & d \end{smallmatrix}\right) \in SL(2, \mathbb{Z})$. Then

$$\begin{aligned} G(Mz) = G\left(\frac{az+b}{cz+d}\right) = F\left(\frac{taz+tb}{cz+d}\right) &= F\left(\frac{a(tz)+tb}{\frac{c}{t}(tz)\ +d}\right) \\ &= F(M^* tz) \\ &= v(M^*)\ \left(\frac{c}{t}(tz)+d\right)^k\ F(tz) \\ &= v^*(M)\ (cz+d)^k\ G(z). \end{aligned}$$

Alternatively, one can show that for $A = \left(\begin{smallmatrix} \sqrt{t} & 0 \\ 0 & \frac{1}{\sqrt{t}} \end{smallmatrix}\right)$, $(F|_k^v A)\,|_k^{v^*}(A^{-1}MA) = (F|_k^v A)$, i.e., $\bar{v}(A)t^{k/2}F(tz)$ satisfies the transformation law, which implies that $F(tz)$ does as well.

We now turn our attention to congruence subgroups of the modular analogue $H(\frac{1}{t}, t^2)$. These subgroups will be the basis for the remainder of this thesis.

Notation 3.2 $\Delta_t := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{Z}) \mid \left(\begin{smallmatrix} a & bt \\ \frac{c}{t} & d \end{smallmatrix}\right) \in SL(2, \mathbb{Z}) \right\}$.

Note that for Δ_t to be nontrivial, t must be in $\mathbb{Q}$. A straightforward calculation shows that Δ_t is a subgroup of Γ_t and that $\Delta_1 = \Gamma_1 = \Gamma(1)$. By Remark 1, $\Delta_t = ASL(2, \mathbb{Z})A^{-1}$, $A = \left(\begin{smallmatrix} \sqrt{t} & 0 \\ 0 & \frac{1}{\sqrt{t}} \end{smallmatrix}\right)$.

Remark 3: $\Gamma_0(S) \cap \Gamma^0(R) = \Delta_{\frac{S}{R}}$, $\frac{S}{R} \in \mathbb{Q}, (S, R) = 1$.

This gives the following:

Corollary 3.2 $H\left(\frac{R}{S}, \frac{S^2}{R^2}\right) \supset \Gamma_0(S) \cap \Gamma^0(R)$, $\frac{S}{R} \in \mathbb{Q}, (S, R) = 1$.

Proof: Put $t = \frac{S}{R}$ in Proposition 3.2 and use the fact that $\Gamma_t \supset \Delta_t$.

In the mid-1930's, Erich Hecke established a correspondence theory on the Hecke group $H(\lambda, 1)$, in particular, on the modular group $\lambda = 1$ (see [5],[6]). Roughly 30 years later, André Weil developed a theory of correspondence on the subgroup $\Gamma_0(S)$ of the modular group (see [20]). Our goal here follows a similar track. In Theorem 2.2 of Chapter 2, we established the Hecke correspondence on the generalized Hecke group $H(\lambda, \mu)$, in particular, on the modular analogue $H\left(\frac{R}{S}, \frac{S^2}{R^2}\right)$. We now aim to establish the Hecke correspondence on the subgroup $\Gamma_0(S) \cap \Gamma^0(R)$ of $H\left(\frac{R}{S}, \frac{S^2}{R^2}\right)$. This is the subject of Chapter 5. Note that $\Gamma_0(S) \cap \Gamma^0(1) = \Gamma_0(S)$. By using the context of the generalized Hecke group, and following the techniques of Weil, we will extend Weil's results on $\Gamma_0(S)$ to the group $\Gamma_0(S) \cap \Gamma^0(R)$. Before doing this, we study properties of the group $\Gamma_0(S) \cap \Gamma^0(R)$. This is the topic of the next chapter.

CHAPTER 4

THE CONGRUENCE GROUP $\Gamma(R,S)$

After defining the congruence group $\Gamma(R,S)$, we discuss $\Gamma(R,S)$ as a conjugate of $\Gamma_0(RS)$, and comment briefly on a fundamental region for $\Gamma(R,S)$, including some information about the cusps. Lastly, we discuss the index of $\Gamma(R,S)$ in the modular group.

4.1 Definition

Let $N \in \mathbf{Z}^+$. Recall that

$$\Gamma(N) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(1) \mid a \equiv d \equiv 1 \bmod N,\ b \equiv c \equiv 0 \bmod N\}$$

is called the ***principal congruence group of level*** N. Any subgroup of $\Gamma(1)$ which contains $\Gamma(N)$ is called a *congruence group of level* N. For our purposes, we consider the following special congruence groups of level N:

$$\Gamma_0(N) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(1) \mid c \equiv 0 \bmod N\},$$

$$\Gamma^0(N) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(1) \mid b \equiv 0 \bmod N\}.$$

In [18], Bruno Schoeneberg defines a deeper congruence group of level N:

$$\Gamma_0^0(N) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(1) \mid b \equiv c \equiv 0 \bmod N\},$$

that is, $\Gamma_0^0(N) = \Gamma^0(N) \cap \Gamma_0(N)$. In [3], Emil Grosswald describes the algebraic structure of $\Gamma_0^0(p)$ for any prime p, indicating the set of independent generators and relations defining $\Gamma_0^0(p)$.

We now generalize $\Gamma_0^0(N)$ to the less-known congruence group $\Gamma(R, S)$ with two integer variables R and S. To my knowledge, this group appears only in [15] in which Morris Newman discusses conditions for congruence groups to be free.

Definition 4.1 *Let $R,S \in \mathbb{Z}^+$. We define*

$$\Gamma(R, S) := \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(1) \mid b \equiv 0 \bmod R, \ c \equiv 0 \bmod S \}.$$

Note that $\Gamma(R, S)$ is a congruence group of level *l.c.m.*$[R, S]$. If R and S are coprime, then $\Gamma(R, S)$ is a congruence group of level RS. Obviously, $\Gamma(R, S) = \Gamma^0(R) \cap \Gamma_0(S)$. Note also that $\Gamma(R, 1) = \Gamma^0(R)$ and $\Gamma(1, S) = \Gamma_0(S)$. To be consistent with the earlier notation, we write $\Gamma(N, N)$ for $\Gamma_0^0(N)$.

In analogy with Weil's extended group of $\Gamma_0(S)$ denoted $\Gamma_0^*(S)$ $\Big($see [20], where $\Gamma_0^*(S) = \left\langle \Gamma_0(S), \left(\begin{smallmatrix} 0 & -1 \\ S & 0 \end{smallmatrix}\right) \right\rangle = \left\langle \Gamma_0(S), T(S) \right\rangle \Big)$, we define the extended group of $\Gamma(R, S)$ as follows:

Definition 4.2 *For $R,S \in \mathbb{Z}^+$,*

$$\Gamma^*(R, S) := \left\langle \Gamma(R, S), \left(\begin{smallmatrix} 0 & -R \\ S & 0 \end{smallmatrix}\right) \right\rangle = \left\langle \Gamma(R, S), T\left(\tfrac{S}{R}\right) \right\rangle.$$

4.2 Conjugates

In this section, we show that the group $\Gamma(R, S)$ can be realized as a conjugate of $\Gamma_0(RS)$. This was noted briefly, without proof, by Morris Newman in [15]. Moreover, we carry over this result to the extended groups $\Gamma^*(R, S)$ and $\Gamma_0^*(RS)$.

Proposition 4.1 *Put* $A = \begin{pmatrix} \sqrt{R} & 0 \\ 0 & \frac{1}{\sqrt{R}} \end{pmatrix}$. *We have* $A^{-1}\Gamma(R,S)A = \Gamma_0(RS)$, *i.e.,* $\Gamma(R,S)$ *and* $\Gamma_0(RS)$ *are conjugate subgroups in the group of real 2x2 matrices of determinant 1.*

Proof: To see this, first consider $\begin{pmatrix} a & Rb \\ Sc & d \end{pmatrix} \in \Gamma(R,S)$. Then we have

$$\begin{aligned} A^{-1}\begin{pmatrix} a & Rb \\ Sc & d \end{pmatrix}A &= \begin{pmatrix} \frac{1}{\sqrt{R}} & 0 \\ 0 & \sqrt{R} \end{pmatrix}\begin{pmatrix} a & Rb \\ Sc & d \end{pmatrix}\begin{pmatrix} \sqrt{R} & 0 \\ 0 & \frac{1}{\sqrt{R}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{a}{\sqrt{R}} & \sqrt{R}b \\ \sqrt{R}Sc & \sqrt{R}d \end{pmatrix}\begin{pmatrix} \sqrt{R} & 0 \\ 0 & \frac{1}{\sqrt{R}} \end{pmatrix} \\ &= \begin{pmatrix} a & b \\ RSc & d \end{pmatrix}, \end{aligned}$$

an element of $\Gamma_0(RS)$, so $A^{-1}\Gamma(R,S)A \subset \Gamma_0(RS)$.

Now consider $\begin{pmatrix} a & b \\ RSc & d \end{pmatrix} \in \Gamma_0(RS)$. Then we have

$$\begin{aligned} A\begin{pmatrix} a & b \\ RSc & d \end{pmatrix}A^{-1} &= \begin{pmatrix} \sqrt{R} & 0 \\ 0 & \frac{1}{\sqrt{R}} \end{pmatrix}\begin{pmatrix} a & b \\ RSc & d \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{R}} & 0 \\ 0 & \sqrt{R} \end{pmatrix} \\ &= \begin{pmatrix} a\sqrt{R} & b\sqrt{R} \\ \sqrt{R}Sc & \frac{d}{\sqrt{R}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{R}} & 0 \\ 0 & \sqrt{R} \end{pmatrix} \\ &= \begin{pmatrix} a & bR \\ Sc & d \end{pmatrix}, \end{aligned}$$

an element of $\Gamma(R,S)$, so $A\Gamma_0(RS)A^{-1} \subset \Gamma(R,S)$. Therefore, we have $A^{-1}\Gamma(R,S)A = \Gamma_0(RS)$ and Proposition 4.1 follows.

Corollary 4.1 *The groups* $\Gamma(R,S)$ *and* $\Gamma^0(RS)$ *are conjugate under* $AT^{-1} = \begin{pmatrix} 0 & \sqrt{R} \\ \frac{-1}{\sqrt{R}} & 0 \end{pmatrix}$.

Proof: This follows immediately by the well-known fact that $\Gamma_0(N)$ and $\Gamma^0(N)$ are conjugate under T. More precisely, $T^{-1}\Gamma^0(RS)T = \Gamma_0(RS)$ and Proposition 4.1 imply that $\Gamma(R,S) = A\Gamma_0(RS)A^{-1} = AT^{-1}\,\Gamma^0(RS)\,TA^{-1}$.

Corollary 4.2 *The groups* $\Gamma^*(R,S)$ *and* $\Gamma_0^*(RS)$ *are conjugate under* $A = \begin{pmatrix} \sqrt{R} & 0 \\ 0 & \frac{1}{\sqrt{R}} \end{pmatrix}$.

Proof: By Proposition 4.1, it suffices to check that $A^{-1}\begin{pmatrix} 0 & -R \\ S & 0 \end{pmatrix}A = \begin{pmatrix} 0 & -1 \\ RS & 0 \end{pmatrix}$. We calculate:

$$\begin{aligned}
A^{-1}\begin{pmatrix} 0 & -R \\ S & 0 \end{pmatrix}A &= \begin{pmatrix} \frac{1}{\sqrt{R}} & 0 \\ 0 & \sqrt{R} \end{pmatrix}\begin{pmatrix} 0 & -R \\ S & 0 \end{pmatrix}\begin{pmatrix} \sqrt{R} & 0 \\ 0 & \frac{1}{\sqrt{R}} \end{pmatrix} \\
&= \begin{pmatrix} 0 & -\sqrt{R} \\ S\sqrt{R} & 0 \end{pmatrix}\begin{pmatrix} \sqrt{R} & 0 \\ 0 & \frac{1}{\sqrt{R}} \end{pmatrix} \\
&= \begin{pmatrix} 0 & -1 \\ RS & 0 \end{pmatrix}.
\end{aligned}$$

We note the slight correction of [15] in which Newman states that $\Gamma(R,S)$ and $\Gamma_0(RS)$ are conjugate subgroups in the group of *rational* 2x2 matrices of determinant 1. However, if R is square-free, then requiring determinant 1 forces the conjugating matrix to have only *real* entries. Newman then mentions, without proof, that the matrix for conjugation can be chosen to have *integral* entries if and only if R and S are coprime. We prove one direction here.

Proposition 4.2 *If R and S are coprime, then $\Gamma(R,S)$ and $\Gamma_0(RS)$ are conjugate subgroups in the group of integral 2x2 matrices of determinant 1.*

Proof: R and S coprime implies that $\exists\ x, y \in \mathbf{Z}$ such that $B = \begin{pmatrix} R & x \\ S & y \end{pmatrix} \in SL(2,\mathbf{Z})$. Consider $\begin{pmatrix} a & Rb \\ Sc & d \end{pmatrix} \in \Gamma(R,S)$. Then

$$\begin{aligned}
B^{-1}\begin{pmatrix} a & Rb \\ Sc & d \end{pmatrix}B &= \begin{pmatrix} y & -x \\ -S & R \end{pmatrix}\begin{pmatrix} a & Rb \\ Sc & d \end{pmatrix}\begin{pmatrix} R & x \\ S & y \end{pmatrix} \\
&= \begin{pmatrix} * & * \\ S(Rc-a) & R(d-Sb) \end{pmatrix}\begin{pmatrix} R & x \\ S & y \end{pmatrix} \\
&= \begin{pmatrix} * & * \\ RS[(Rc-a)+(d-Sb)] & * \end{pmatrix},
\end{aligned}$$

an element of $\Gamma_0(RS)$, so $B^{-1}\Gamma(R,S)B \subset \Gamma_0(RS)$. For the opposite inclusion, let $\begin{pmatrix} a & b \\ RSc & d \end{pmatrix} \in \Gamma_0(RS)$. Then we have

$$\begin{aligned}
B\begin{pmatrix} a & b \\ RSc & d \end{pmatrix}B^{-1} &= \begin{pmatrix} R & x \\ S & y \end{pmatrix}\begin{pmatrix} a & b \\ RSc & d \end{pmatrix}\begin{pmatrix} y & -x \\ -S & R \end{pmatrix} \\
&= \begin{pmatrix} R(a+xSc) & Rb+xd \\ S(a+yRc) & Sb+yd \end{pmatrix}\begin{pmatrix} y & -x \\ -S & R \end{pmatrix} \\
&= \begin{pmatrix} * & R[-(a+xSc)+(Rb+xd)] \\ S[(a+yRc)y-(Sb+yd)] & * \end{pmatrix},
\end{aligned}$$

an element of $\Gamma(R, S)$, so $B\Gamma_0(RS)B^{-1} \subset \Gamma(R, S)$. Therefore, we have that $B^{-1}\Gamma(R, S)B = \Gamma_0(RS)$ and so $\Gamma(R, S)$ and $\Gamma_0(RS)$ are conjugate subgroups in the group of integral 2x2 matrices of determinant 1.

Remark: It is important to note that conjugation by $A = \left(\begin{smallmatrix} R & x \\ S & y \end{smallmatrix}\right) \in SL(2, \mathbb{Z})$ does not take the inversion in $\Gamma^*(R, S)$ to the inversion in $\Gamma_0^*(RS)$. As we will see in the next chapter, to achieve a Hecke-Weil correspondence on $\Gamma(R, S)$, the correct matrix for conjugation is $A = \left(\begin{smallmatrix} \sqrt{R} & 0 \\ 0 & \frac{1}{\sqrt{R}} \end{smallmatrix}\right)$. In light of Chapter 2, case $c = 0$, conjugation by this matrix is very natural for Hecke correspondence.

Corollary 4.3 *If R and S are coprime, then $\Gamma(R, S)$ and $\Gamma^0(RS)$ are conjugate subgroups in the group of integral 2x2 matrices of determinant 1 .*

Proof: This follows immediately from Proposition 4.2 and the fact that $\Gamma_0(RS)$ and $\Gamma^0(RS)$ are conjugate under T.

4.3 A Fundamental Region

It is a standard result that if $\mathcal{R}$ is a fundamental region (F.R) for the group Γ, $[\Gamma(1){:}\Gamma] < \infty$, then M($\mathcal{R}$) is a fundamental region for the conjugate group $M\Gamma M^{-1}$ for any $M \in$GL(2, $\mathbb{Q}$). We now fix a F.R. $\mathcal{R}$ for $\Gamma_0(RS)$. Therefore, by Proposition 4.1, A($\mathcal{R}$) is a F.R. for $\Gamma(R, S)$, $A = \left(\begin{smallmatrix} \sqrt{R} & 0 \\ 0 & \frac{1}{\sqrt{R}} \end{smallmatrix}\right)$. Thus, if q is a parabolic cusp for $\Gamma_0(RS)$, then A(q) $= \left(\begin{smallmatrix} \sqrt{R} & 0 \\ 0 & \frac{1}{\sqrt{R}} \end{smallmatrix}\right)$q = Rq is a parabolic cusp for $\Gamma(R, S)$. Therefore, results of J-M Deshouillers and H. Iwaniec in [2] can easily be applied to describe explicitly the cusps of $\Gamma(R, S)$, including their number. They are of the form

$$\frac{Ru}{w},\ (u, w) = 1 \text{ with } u, w > 0 \text{ such that } w \mid RS.$$

Two cusps $\frac{Ru_1}{w_1}$, $\frac{Ru_2}{w_2}$ of the above type are $\Gamma(R,S)$-equivalent if and only if $w_1 = w_2$ and $u_1 \equiv u_2 \pmod{(w_1, \frac{RS}{w_1})}$. Furthermore, the number of inequivalent cusps of $\Gamma(R,S)$ equals the number of inequivalent cusps of $\Gamma_0(RS)$. From [2], this number is

$$h(\Gamma(R,S)) = \textstyle\sum_{w|RS} \phi((w, \frac{RS}{w})).$$

As a special case, we note that if R $= p^{n_1}$ and S $= p^{n_2}$, p a prime, then a straightforward calculation shows that

$$h(\Gamma(p^{n_1}, p^{n_2})) = \begin{cases} p^{\frac{n_1+n_2}{2}}(1+\frac{1}{p}) & : \; n_1+n_2 \; even \\ 2p^{\frac{n_1+n_2-1}{2}} & : \; n_1+n_2 \; odd \end{cases}$$

4.4 Index in the Modular Group

In the last section, we saw that certain congruence groups are conjugate to others. We exploit this and use results of Bruno Schoeneberg in [18] to calculate the index of $\Gamma(R,S)$ in $\Gamma(1)$.

To illustrate the generalization to $\Gamma(R,S)$, we first outline the results for $\Gamma_0(N)$, $\Gamma^0(N)$, and $\Gamma(N,N)$ given in [18]. Since $\Gamma_0(N)$ and $\Gamma^0(N)$ are conjugate in $\Gamma(1)$, their index in $\Gamma(1)$ is the same. Schoeneberg calculates the index using two prior calculations. First, he shows through elementary means that

$$[\Gamma(1) : \Gamma(N)] = N^3 \prod_{p|N}(1 - \tfrac{1}{p^2}).$$

Similarly, he shows that $[\Gamma_0(N) : \Gamma(N)] = N\phi(N)$ by noting that the index $[\Gamma_0(N) : \Gamma(N)] = |\, \Gamma_0(N)/\Gamma(N) \,| = N\phi(N)$, since if $c \equiv 0 \pmod N$, then the congruence $ad - bc \equiv 1 \pmod N$ has exactly $N\phi(N)$ solutions.

With these two indices, he calculates $[\Gamma(1) : \Gamma_0(N)]$ as follows:

$$\begin{aligned}
[\Gamma(1) : \Gamma_0(N)] &= \frac{[\Gamma(1) : \Gamma(N)]}{[\Gamma_0(N) : \Gamma(N)]} \\
&= \frac{N^3 \prod_{p|N}(1 - \frac{1}{p^2})}{N\phi(N)} \\
&= \frac{N^3 \prod_{p|N}(1 - \frac{1}{p})(1 + \frac{1}{p})}{N \cdot N \prod_{p|N}(1 - \frac{1}{p})} \\
&= N \prod_{p|N}(1 + \frac{1}{p}).
\end{aligned}$$

This method also shows that $[\Gamma(N, N) : \Gamma(N)] = \phi(N)$ by noting that the index $[\Gamma(N, N) : \Gamma(N)] = \mid \Gamma(N, N)/\Gamma(N) \mid = \phi(N)$, since if $b \equiv c \equiv 0 \pmod N$, then the congruence $ad - bc \equiv 1 \pmod N$ has exactly $\phi(N)$ solutions. Thus, as above, we have

$$\begin{aligned}
[\Gamma(1) : \Gamma(N, N)] &= \frac{[\Gamma(1) : \Gamma(N)]}{[\Gamma(N, N) : \Gamma(N)]} \\
&= \frac{N^3 \prod_{p|N}(1 - \frac{1}{p^2})}{\phi(N)} \\
&= \frac{N^3 \prod_{p|N}(1 - \frac{1}{p})(1 + \frac{1}{p})}{N \prod_{p|N}(1 - \frac{1}{p})} \\
&= N^2 \prod_{p|N}(1 + \frac{1}{p}).
\end{aligned}$$

We note that, by similar reasoning, Svetlana Katok also obtains this result (see [7]). However, neither Katok nor Schoeneberg calculates the index of

$\Gamma(R,S)$ in $\Gamma(1)$. We provide it here. In doing this, instead of following the method above, we use the fact that $\Gamma(R,S)$ is conjugate to $\Gamma_0(RS)$. Therefore, we have

$$[\Gamma(1) : \Gamma(R,S)] = [\Gamma(1) : \Gamma_0(RS)] = RS \prod_{p|RS} (1+\tfrac{1}{p}).$$

Note that this is consistent with the calculation for $[\Gamma(1) : \Gamma(N,N)]$.

If $(R,S) = 1$, then we have

$$\begin{aligned} [\Gamma(1):\Gamma(R,S)] &= RS \prod_{p|RS} (1+\frac{1}{p}) \\ &= R\prod_{p|R}(1+\frac{1}{p})\ S\prod_{p|S}(1+\frac{1}{p}) \\ &= [\Gamma(1):\Gamma_0(R)]\ [\Gamma(1):\Gamma_0(S)] \\ &= [\Gamma(1):\Gamma^0(R)]\ [\Gamma(1):\Gamma_0(S)]. \end{aligned}$$

By Proposition 4.2 and Corollary 4.3, if $(R,S) = 1$, we obtain the following:

$$[\Gamma(1):\Gamma_0(RS)] = [\Gamma(1):\Gamma^0(RS)] = [\Gamma(1):\Gamma^0(R)]\ [\Gamma(1):\Gamma_0(S)].$$

CHAPTER 5

WEIL CORRESPONDENCE ON $\Gamma(R, S)$

5.1 Hecke Correspondence and the Twisted Mellin Transform

Recall that $\Gamma(1, S) = \Gamma_0(S)$ and $\Gamma(R, 1) = \Gamma^0(R)$. In [20], André Weil proved a Hecke correspondence on $\Gamma_0(S)$. The main result here is an extension of that theorem to the group $\Gamma(R, S)$. Furthermore, whereas Weil dealt with cusp forms, our framework is the less restrictive case of entire modular forms. The presentation follows closely that of [20]. (Note: To avoid notational problems, we will denote translations by the blackboard character $\mathbb{S}$.)

In keeping with the notation of Weil, we make a minor change in our notation of Chapter 2 by writing characters as subscripts in Dirichlet series. Also, the constant C appearing here has a slighlty different meaning which will be made clear in context. More noticeable is the absence of the multiplier system in the definition of the slash operator, although a multiplier system will be defined. Throughout, we assume that the weight $k \in \mathbb{Z}^+$. We have

Definition 5.1 *If* $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_+(2, \mathbb{R})$, *define the* **slash operator** "|" *as*

$$F|M \equiv (F\mid_k M)(z) = (detM)^{k/2}(cz+d)^{-k}F(Mz). \tag{5.1}$$

We note that this definition is consistent with our earlier one upon normalization of $M \in SL(2,\mathbb{R})$, absent the multiplier.

Definition 5.2 *Let $\{a_n\}$, $\{b_n\}$ be two sequences in $\mathbb{C}$. For some $\sigma > 0$, suppose $a_n = \mathcal{O}(n^\sigma)$, $b_n = \mathcal{O}(n^\sigma)$. Define the following:*

$$\begin{aligned} F(z) &= \sum_{n=0}^{\infty} a_n e^{2\pi i n z/R_1} & G(z) = \sum_{n=0}^{\infty} b_n e^{2\pi i n z/R_2} \\ \omega_F(s) &= \sum_{n=1}^{\infty} a_n n^{-s} & \omega_G(s) = \sum_{n=1}^{\infty} b_n n^{-s} \\ \Omega_F(s) &= \left(\frac{2\pi}{R}\right)^{-s} \Gamma(s)\omega_F(s) & \Omega_G(s) = \left(\frac{2\pi}{R}\right)^{-s} \Gamma(s)\omega_G(s) \end{aligned}$$

Lemma 5.1 *Let $\mu > 0$. TFAE:*

Ω_F, Ω_G can be continued meromorphically to the entire s-plane, remain bounded in every LVS, and satisfy the functional equation

$$\Omega_F(s) = C\mu^{k/2-s}\Omega_G(k-s). \tag{5.2}$$

For $z \in \mathbb{H}$,

$$[G \mid T(\mu)]\,(z) = C^{-1} i^{-k} F(z). \tag{5.3}$$

Proof: Replacing s by $k-s$ in (5.2) gives $\Omega_F(k-s) = C\mu^{s-k/2}\Omega_G(s)$. By slashing both sides of (5.3), we obtain $[F \mid T(\mu)]\,(z) = i^k C(-1)^k G(z)$. Then, with $C = i^k$ and $R_i = \lambda_i$ $(i = 1,2)$, Lemma 5.1 is a restatement of Theorem 2.2.

As a corollary, set $a_n = b_n\ \forall n$ so $F = G$ and write $\Omega(s)$ for its associated Dirichlet series. This gives

Lemma 5.2 *Let $\mu > 0$. TFAE:*

$\Omega_F(s)$ can be continued meromorphically to the entire s-plane, remain bounded in every LVS, and satisfy the functional equation

$$\Omega_F(s) = C\mu^{k/2-s}\Omega_F(k-s). \tag{5.4}$$

For $z \in \mathbb{H}$,

$$[F \mid T(\mu)]\,(z) = C^{-1} i^{-k} F(z). \tag{5.5}$$

Note: By the same argument in Chapter 2, replacing s by $k-s$ in the functional equation shows that, in Lemma 5.2, $C = \pm 1$. To be consistent with Corollary 2.2, set $C = i^k C^*$, $C^* = v(T(\mu))$. Then $\pm 1 = i^k C^* \Rightarrow C^* = \pm i^{-k}$, which agrees with the multiplier for an inversion (see the end of Chapter 1).

We recall the concept of twisted series by Dirichlet characters as introduced by Weil in [20].

Definition 5.3 *Let c_n be a sequence in $\mathbb{C}$ (not all $c_n = 0$) s.t. $c_n = O(n^\sigma)$, for some $\sigma > 0$. Let $m \in \mathbb{Z}^+$ and χ a primitive Dirichlet character modulo m. Define*

$$\begin{aligned} L_\chi(s) &= \sum_{n=1}^{\infty} c_n \chi(n) n^{-s} \\ \Omega_\chi(s) &= \left(\frac{Rm}{2\pi}\right)^s \Gamma(s) L_\chi(s) \\ F_\chi(z) &= \sum_{n=1}^{\infty} c_n \chi(n) e^{2\pi i n z/R} \end{aligned}$$

If $m = 1$ (i.e., $\chi = 1$, the principal character), we write L, Ω, F.

Definition 5.4 *For $m > 1$, let $g(\chi)$ be the* **Gauss sum**

$$g(\chi) = \textstyle\sum_{a(mod m)} \chi(a) e^{2\pi i a/m}.$$

We will need the following:

$$\chi(-1)\chi(n) = \tfrac{g(\chi)}{m} \textstyle\sum_{a(mod m)} \bar{\chi}(a) e^{2\pi i n a/m},$$

a standard result on Gaussian Sums, valid only for *primitive* characters (see [1]). We note that in [20], Weil omits the factor $\chi(-1)$. This was pointed out by Knopp in [9]. As in [20], it follows that

$$\chi(-1) F_\chi = \tfrac{g(\chi)}{m} \textstyle\sum_{a(mod m)} \bar{\chi}(a) F \mid \mathbf{S}(\tfrac{Ra}{m}).$$

Suppose $(R, m) = 1$. Then as a runs through a complete residue system(C.R.S.) mod m, so does Ra and we can write the above equation as the following:

$$\bar{\chi}(R)\chi(-1)F_\chi = \frac{g(\chi)}{m} \sum_{Ra(\mathrm{mod} m)} \bar{\chi}(Ra)\, F \mid \mathbf{S}\left(\frac{Ra}{m}\right). \tag{5.6}$$

Define $\Omega^*_\chi(s) = \left(\frac{R}{2\pi}\right)^s \Gamma(s) L_\chi(s) = m^{-s}\, \Omega_\chi(s)$. Then we have

$$\begin{aligned} \Omega_\chi(s) &= C_\chi \mu^{k/2-s} \Omega_{\bar{\chi}}(k-s) \\ \Leftrightarrow m^s \Omega^*_\chi(s) &= C_\chi \mu^{k/2-s} m^{k-s} \Omega^*_{\bar{\chi}}(k-s) \\ \Leftrightarrow \Omega^*_\chi(s) &= C_\chi \mu^{k/2-s} m^{k-2s} \Omega^*_{\bar{\chi}}(k-s) \\ \Leftrightarrow \Omega^*_\chi(s) &= C_\chi (\mu m^2)^{k/2-s} \Omega^*_{\bar{\chi}}(k-s). \end{aligned}$$

By Lemma 5.1, this holds $\Leftrightarrow [F_{\bar{\chi}} \mid T(\mu m^2)]\ (z) = {C_\chi}^{-1} i^{-k}\, F_\chi(z)$,

$$\text{i.e.,} \Leftrightarrow F_\chi(z) = C_\chi i^k\, [F_{\bar{\chi}} \mid T(\mu m^2)]\ (z).$$

By (5.6), this holds $\Leftrightarrow$

$$\chi(R) g(\chi) \textstyle\sum_{Ra \ \mathrm{mod} m} \bar{\chi}(Ra)\, F \mid \mathbf{S}(\frac{Ra}{m}) = \bar{\chi}(R) C_\chi i^k g(\bar{\chi}) \textstyle\sum_{Ra \ \mathrm{mod} m} \chi(Ra)\, F \mid \mathbf{S}(\frac{Ra}{m}) T(\mu m^2).$$

This gives the following:

Lemma 5.3 *Let $m \in \mathbf{Z}^+, m > 1$ and fix χ, a primitive Dirichlet character modulo m. Then TFAE:*
$\Omega_\chi, \Omega_{\bar{\chi}}$ can be continued meromorphically to the entire s-plane, remain bounded in every LVS, and satisfy the functional equation

$$\Omega_\chi(s) = C_\chi \mu^{k/2-s} \Omega_{\bar{\chi}}(k-s). \tag{5.7}$$

For $z \in \mathbb{H}$,

$$\chi(R) g(\chi) \sum_{Ra \ \mathrm{mod} m} \bar{\chi}(Ra)\, F \mid \mathbf{S}\left(\frac{Ra}{m}\right) = \bar{\chi}(R) C_\chi i^k g(\bar{\chi}) \sum_{Ra \ \mathrm{mod} m} \chi(Ra)\, F \mid \mathbf{S}\left(\frac{Ra}{m}\right) T(\mu m^2). \tag{5.8}$$

Remarks:

(1) For $(Ra, m) \neq 1$, $\chi(Ra) = \bar{\chi}(Ra) = 0$, so we may assume that $(Ra, m) = 1$ on both sides of the equality in (5.8).

(2) Assume also that $(RS, m) = 1$ which is equivalent to $(R, m) = (S, m) = 1$. Thus, given a such that $(a, m) = 1$, we have $(RSa, m) = 1$ which implies $\exists b, n \in \mathbb{Z}$ such that $1 = mn - RSab$, i.e., $1 = (m)n - (Sa)Rb$. Therefore, there exists the matrix $\begin{pmatrix} m & -Rb \\ -Sa & n \end{pmatrix} \in \Gamma(R, S)$. We donote this matrix by $\gamma(a, Rb)$.

Claim 1: $\mathbf{S}(\frac{Ra}{m})T(\frac{S}{R}m^2) = T(\frac{S}{R})\gamma(a, Rb)\mathbf{S}(\frac{Rb}{m})\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$.

Proof of Claim 1:

$$\begin{aligned} LHS &= \begin{pmatrix} 1 & \frac{Ra}{m} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -R \\ Sm^2 & 0 \end{pmatrix} \\ &= \begin{pmatrix} RSam & -R \\ Sm^2 & 0 \end{pmatrix}. \end{aligned}$$

$$\begin{aligned} RHS &= \begin{pmatrix} 0 & -R \\ S & 0 \end{pmatrix}\begin{pmatrix} m & -Rb \\ -Sa & n \end{pmatrix}\begin{pmatrix} 1 & \frac{Rb}{m} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \\ &= \begin{pmatrix} RSa & -Rn \\ Sm & -RSb \end{pmatrix}\begin{pmatrix} 1 & \frac{Rb}{m} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \\ &= \begin{pmatrix} RSa & R^2S\frac{ab}{m} - Rn \\ Sm & 0 \end{pmatrix}\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \\ &= \begin{pmatrix} RSam & R^2Sab - Rmn \\ Sm^2 & 0 \end{pmatrix} \\ &= \begin{pmatrix} RSam & -R(mn - RSab) \\ Sm^2 & 0 \end{pmatrix} \\ &= \begin{pmatrix} RSam & -R \\ Sm^2 & 0 \end{pmatrix}. \end{aligned}$$

Claim 2: As Ra runs through a C.R.S. mod m, so does Rb.

Proof of Claim 2: First, we note that $(Ra, m) = 1 \Rightarrow (RSa, m) = 1$ (since $(S, m) = 1$) and hence, as above, $\exists$ b,n$\in \mathbb{Z}$ such that $1 = (m)n - (RSa)b$, i.e., $1 = (m)n - (Rb)Sa \Rightarrow (Rb, m) = 1$.

Second, we need to show that $Ra_1 \equiv Ra_2 (mod m) \Rightarrow Rb_1 \equiv Rb_2 (mod m)$. $Ra_1 \equiv Ra_2 (mod m) \Rightarrow 1 \equiv RSa_1b_1 \equiv RSa_2b_2 (mod m) \Rightarrow m \mid S(a_1Rb_1 - a_2Rb_2)$. Now $(S, m) = 1 \Rightarrow m \mid (a_1Rb_1 - a_2Rb_2)$. Now $m \mid (Ra_1 - Ra_2)$ and $(Ra_1 - Ra_2) \mid (Ra_1 - Ra_2)b_2 \Rightarrow m \mid (Ra_1b_2 - Ra_2b_2)$. Therefore,

$m \mid [(a_1Rb_1 - a_2Rb_2) - (Ra_1b_2 - Ra_2b_2)]$, i.e., $m \mid a_1(Rb_1 - Rb_2)$. But $(a, m) = 1$. Therefore, $m \mid (Rb_1 - Rb_2)$ or $Rb_1 \equiv Rb_2 (mod m)$. This proves Claim 2.

Note: Since $RSab \equiv -1(mod m)$, we have $\chi(-RSab) = \chi(1) = 1$ which implies $\chi(Ra)\chi(-Sb) = 1$. Thus, $\chi(Ra) = \frac{1}{\chi(-Sb)} = \bar{\chi}(-Sb)$. Set $\mu = \frac{S}{R}$.

By Claim 1 and Claim 2, (5.8) is equivalent to the following:

$$\chi(R)g(\chi) \sum_{Rb(mod m)} \bar{\chi}(Rb)\, F \mid \mathbf{S}(\tfrac{Rb}{m})$$
$$= \bar{\chi}(R)C_\chi i^k g(\bar{\chi}) \sum_{Rb(mod m)} \chi(Ra)\, F \mid \mathbf{S}(\tfrac{Ra}{m})T(\tfrac{S}{R}m^2)$$
$$= \bar{\chi}(R)C_\chi i^k g(\bar{\chi}) \sum_{Rb(mod m)} \bar{\chi}(-Sb)\, F \mid T(\tfrac{S}{R})\gamma(a, Rb)\mathbf{S}(\tfrac{Rb}{m})\left(\begin{smallmatrix} m & 0 \\ 0 & m \end{smallmatrix}\right)$$
$$= \bar{\chi}(R)C_\chi i^k g(\bar{\chi}) \sum_{Rb(mod m)} \bar{\chi}(b)\bar{\chi}(-S)\, F \mid T(\tfrac{S}{R})\gamma(a, Rb)\mathbf{S}(\tfrac{Rb}{m}).$$

The latter equality follows from Definition 5.1 since for any function h, $h \mid \left(\begin{smallmatrix} m & 0 \\ 0 & m \end{smallmatrix}\right) = (m^2)^{k/2}m^{-k}h(\frac{mz}{m}) = h$. Thus, by subtracting and factoring out $\mathbf{S}(\frac{Rb}{m})$, we have that (5.8) is equivalent to

$$\sum_{Rb(mod m)} \bar{\chi}(b)\, [F - C_\chi i^k \frac{g(\bar{\chi})}{g(\chi)}\bar{\chi}(-RS)\, F \mid T\left(\frac{S}{R}\right)\gamma(a, Rb)] \mid \mathbf{S}\left(\frac{Rb}{m}\right) = 0. \tag{5.9}$$

5.2 The Direct Theorem

Recall that $\Gamma^*(R, S)$ is the group generated by $\Gamma(R, S)$ and $T(\frac{S}{R})$.

Proposition 5.1 *$\Gamma^*(R, S)$ is discrete.*

Proof: Claim: $T(\frac{S}{R})$ is in the normalizer of $\Gamma(R, S)$. To see this, consider $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(R, S)$. Then we have

$$\begin{aligned} T\left(\frac{S}{R}\right)^{-1} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) T\left(\frac{S}{R}\right) &= \left(\begin{smallmatrix} 0 & \frac{1}{S} \\ \frac{-1}{R} & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -R \\ S & 0 \end{smallmatrix}\right) \\ &= \begin{pmatrix} \frac{c}{S} & \frac{d}{S} \\ \frac{-a}{R} & \frac{-b}{R} \end{pmatrix}\left(\begin{smallmatrix} 0 & -R \\ S & 0 \end{smallmatrix}\right) \\ &= \begin{pmatrix} d & \frac{-Rc}{S} \\ \frac{-Sb}{R} & a \end{pmatrix}, \end{aligned}$$

an element of $\Gamma(R,S)$ since $S \mid c$ and $R \mid b$. This proves the claim. Since $T(\frac{S}{R})$ is in the normalizer of $\Gamma(R,S)$ and of order 2 (as a linear fractional transformation), $\Gamma(R,S)$ is of index 2 in $\Gamma^*(R,S)$. Therefore, $\Gamma^*(R,S)$ is discrete.

Remark 5.1 *For the same reasons given in the remarks on page 5, the mapping $F \mapsto F|\ T(\frac{S}{R})$ preserves the finite-dimensional space of entire modular forms on $\Gamma(R,S)$ of fixed even integral weight (and M.S.), and furthermore, as an operator on the subspace of cusp forms on $\Gamma(R,S)$ of even integral weight (and M.S.), $T(\frac{S}{R})$ is self-adjoint with respect to the Petersson inner product, and thus, is normal. Therefore, a basis can be chosen such that for each basis element F, we have $F|\ T(\frac{S}{R}) = \lambda_F\ F$, where $\lambda_F \in \mathbb{C}$. Slashing both sides shows that $\lambda_F = \pm e^{\pi i k/2} = \pm 1$, the M.S. of even integral weight for an inversion. (See page 7.) Thus, we assume that modular forms, initially defined on $\Gamma(R,S)$, are actually on $\Gamma^*(R,S)$.*

Definition 5.5 *Suppose ϵ is a Dirichlet character modulo RS. We say that F is of* **Hecke type (k,R,S,ϵ)** *if F is regular at the cusps of $\Gamma(R,S)$ and*

$$F \mid M = \epsilon^{-1}(a)F,\ \forall M = \left(\begin{smallmatrix} a & Rb \\ Sc & d \end{smallmatrix}\right) \in \Gamma(R,S). \tag{5.10}$$

We now show that ϵ is real on $\Gamma(R,S)$. In doing this, we will use the following calculation:

$$\begin{aligned} T\left(\frac{S}{R}\right)\left(\begin{smallmatrix} a & Rb \\ Sc & d \end{smallmatrix}\right)T\left(\frac{S}{R}\right)^{-1} &= \left(\begin{smallmatrix} 0 & -R \\ S & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & Rb \\ Sc & d \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & \frac{1}{S} \\ \frac{-1}{R} & 0 \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} -RSc & -Rd \\ aS & RSb \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & \frac{1}{S} \\ \frac{-1}{R} & 0 \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} d & -Rc \\ -Sb & a \end{smallmatrix}\right). \end{aligned}$$

Suppose F is of Hecke type (k,R,S,ϵ) and $F \mid T(\frac{S}{R}) = C^{-1}i^{-k}F$. Then for $\left(\begin{smallmatrix} d & -Rc \\ -Sb & a \end{smallmatrix}\right) \in \Gamma(R,S)$,

$$\begin{aligned}
\epsilon^{-1}(d)F &= F \mid \left(\begin{smallmatrix} d & -Rc \\ -Sb & a \end{smallmatrix}\right) \\
&= F \mid T\left(\frac{S}{R}\right)\left(\begin{smallmatrix} a & Rb \\ Sc & d \end{smallmatrix}\right)T\left(\frac{S}{R}\right)^{-1} \\
&= C^{-1}i^{-k}F \mid \left(\begin{smallmatrix} a & Rb \\ Sc & d \end{smallmatrix}\right)T\left(\frac{S}{R}\right)^{-1} \\
&= C^{-1}i^{-k}\epsilon^{-1}(a)F \mid T\left(\frac{S}{R}\right)^{-1} \\
&= C^{-1}i^{-k}\epsilon^{-1}(a)Ci^{k}F \\
&= \epsilon^{-1}(a)F.
\end{aligned}$$

Therefore, if F is of Hecke type (k,R,S,ϵ) and $F \mid T(\frac{S}{R}) = C^{-1}i^{-k}F$, then $\epsilon(a) = \epsilon(d)$ if $ad \equiv 1 \pmod{RS}$, and thus for any $\left(\begin{smallmatrix} a & Rb \\ Sc & d \end{smallmatrix}\right) \in \Gamma(R,S)$. This, in turn, implies $\epsilon(a)^2 = \epsilon(a)\epsilon(d) = \epsilon(ad) = \epsilon(1) = 1$ which implies $\epsilon(a) = \pm 1$ and, in particular, that ϵ is real on $\Gamma(R,S)$.

Theorem 5.1 *(Direct Theorem) Let ϵ be a Dirichlet character modulo RS. Let $F(z) = \sum_{n=0}^{\infty} c_n e^{2\pi i n z/R}$ be of Hecke type (k,R,S,ϵ) such that*

$$F \mid T\left(\frac{S}{R}\right) = C^{-1}i^{-k}F.$$

Then $\Omega(s) = (2\pi)^{-s}\Gamma(s)\sum_{n=1}^{\infty} c_n n^{-s}$ can be continued meromorphically to the entire s-plane, remains bounded in every LVS, and satisfies the functional equation

$$\Omega(s) = C\left(\frac{S}{R}\right)^{k/2-s}\Omega(k-s). \tag{5.12}$$

Furthermore, for each primitive Dirichlet character χ with conductor $f_\chi = m$ such that $(m, RS) = 1$, the functions Ω_χ,$\Omega_{\bar{\chi}}$ can be continued meromorphically to the entire s-plane, remain bounded in every LVS, and satisfy the functional equation

$$\Omega_\chi(s) = C_\chi\left(\frac{S}{R}\right)^{k/2-s}\Omega_{\bar{\chi}}(k-s), \;\; C_\chi = C\epsilon(f_\chi)\frac{g(\chi)}{g(\bar{\chi})}\chi(-RS). \tag{5.13}$$

Proof: The first assertion follows immediately by Lemma 5.2. For the second assertion, we apply Lemma 5.3 and the remarks following it. By Lemma 5.3, (5.7) is equivalent to (5.8). By the remarks following Lemma 5.3, (5.8) is equivalent to (5.9). Now

$$\begin{aligned} F \mid T\left(\frac{S}{R}\right)\gamma(a, Rb) &= C^{-1} i^{-k} F \mid \left(\begin{smallmatrix} m & -Rb \\ -Sa & n \end{smallmatrix}\right) \\ &= C^{-1} i^{-k} \epsilon^{-1}(m) F. \end{aligned}$$

Therefore, $C i^k \epsilon(m)\, F \mid T(\frac{S}{R})\gamma(a, Rb) = F$. We rewrite this equality as the following:

$$C\epsilon(m)\tfrac{g(\chi)}{g(\bar{\chi})}\chi(-RS)\ i^k \tfrac{g(\bar{\chi})}{g(\chi)}\bar{\chi}(-RS)\ F \mid T(\tfrac{S}{R})\gamma(a, Rb) = F.$$

Thus, with $C_\chi = C\epsilon(f_\chi)\frac{g(\chi)}{g(\bar{\chi})}\chi(-RS)$, we have the following equality:

$$C_\chi i^k \tfrac{g(\bar{\chi})}{g(\chi)}\bar{\chi}(-RS)\ F \mid T(\tfrac{S}{R})\gamma(a, Rb) - F = 0$$

$$\Rightarrow [C_\chi i^k \tfrac{g(\bar{\chi})}{g(\chi)}\bar{\chi}(-RS)\ F \mid T(\tfrac{S}{R})\gamma(a, Rb) - F] \mid \mathbf{S}(\tfrac{Rb}{m}) = 0$$

$$\Rightarrow \textstyle\sum_{Rb (mod m)} \bar{\chi}(b)\ [C_\chi i^k \tfrac{g(\bar{\chi})}{g(\chi)}\bar{\chi}(-RS)\ F \mid T(\tfrac{S}{R})\gamma(a, Rb) - F] \mid \mathbf{S}(\tfrac{Rb}{m}) = 0.$$

This is (5.9) and Theorem 5.1 follows.

5.3 Character and Matrix Manipulations

Notation 5.1 *Let $\mathcal{M} = \{4\} \cup \{odd\ primes\}$.*

Note: *If $m \in \mathcal{M}$, then all nonprincipal characters modulo m are primitive. In general, primitive implies nonprincipal. Therefore, with respect to $\mathcal{M}$, a character is nonprincipal $\Leftrightarrow$ it is primitive.*

Lemma 5.4 *Suppose $m \in \mathcal{M}$, $\ni (m, RS) = 1$. Let $0 \neq C_m^* \in \mathbf{C}$. Fix χ, a primitive Dirichlet character modulo m. TFAE:*

$$\Omega_\chi(s) = C_\chi \left(\frac{S}{R}\right)^{k/2-s} \Omega_{\bar{\chi}}(k-s),\ C_m^* = C_\chi i^k \frac{g(\bar{\chi})}{g(\chi)}\bar{\chi}(-RS). \tag{5.14}$$

For all $Rb_1, Rb_2 \ni (Rb_1, m) = (Rb_2, m) = 1$, *we have, for* $(a, m) = 1$,

$$[F - C^*_m\, F \mid T\left(\frac{S}{R}\right)\gamma(a, Rb_1)] \mid \mathbf{S}\left(\frac{Rb_1}{m}\right) = [F - C^*_m\, F \mid T\left(\frac{S}{R}\right)\gamma(a, Rb_2)] \mid \mathbf{S}\left(\frac{Rb_2}{m}\right). \tag{5.15}$$

Proof: We first note that (5.14) is equivalent to (5.9):

$$\sum_{Rb(mod m)} \bar{\chi}(b)\,[F - C^*_m\, F \mid T\left(\frac{S}{R}\right)\gamma(a, Rb)] \mid \mathbf{S}\left(\frac{Rb}{m}\right) = 0. \tag{5.16}$$

We define $F \mid \lambda(b)$ as the following:

$$F \mid \lambda(b) = [F - C^*_m\, F \mid T\left(\frac{S}{R}\right)\gamma(a, Rb)] \mid \mathbf{S}\left(\frac{Rb}{m}\right).$$

Then (5.16) is equivalent to

$$\sum_{Rb(mod m)} \bar{\chi}(b) F \mid \lambda(b) = 0. \tag{5.18}$$

In proving Lemma 5.4, we first show that (5.15) $\Rightarrow$ (5.14), i.e., that (5.15) $\Rightarrow$ (5.18). Thus, suppose

$$F \mid \lambda(b_1) = F \mid \lambda(b_2) = F \mid \lambda$$

for all $Rb_1, Rb_2 \ni (Rb_1, m) = (Rb_2, m) = 1$. We must verify (5.18). But

$$\begin{aligned}\sum_{Rb(mod m)} \bar{\chi}(b)\, F \mid \lambda(b) &= \sum_{\substack{Rb(mod m)\\(Rb,m)=1}} \bar{\chi}(b)\, F \mid \lambda(b)\\ &= F \mid \lambda \sum_{\substack{Rb(mod m)\\(Rb,m)=1}} \bar{\chi}(b)\\ &= 0, \end{aligned} \tag{5.20}$$

since χ primitive $\Rightarrow$ χ nonprincipal $\Rightarrow \sum_{Rb(mod m)} \bar{\chi}(b) = \sum_{b(mod m)} \bar{\chi}(b) = 0$, the latter equality following from elementary number theory (see [16]).

Conversely, suppose (5.14) holds, i.e., suppose (5.18) holds for each primitive Dirichlet character χ (mod m). We must derive (5.15), that is,

$F \mid \lambda(b_1) = F \mid \lambda(b_2) = F \mid \lambda$ for all $Rb_1, Rb_2 \ni (Rb_1, m) = (Rb_2, m) = 1$, i.e., for all $b_1, b_2 \ni (b_1, m) = (b_2, m) = 1$. Now $(b_1, m) = (b_2, m) = 1$ implies $\exists\ b^{(1)}, t_1$ and $b^{(2)}, t_2$ such that $b_1 b^{(1)} - mt_1 = 1$ and $b_2 b^{(2)} - mt_2 = 1$, i.e.,

$$\exists b^{(1)}, b^{(2)} \ni b_1 b^{(1)} \equiv b_2 b^{(2)} \equiv 1 \pmod{m}. \tag{5.21}$$

Now $m \in \mathcal{M}$ implies that all Dirichlet characters except χ_0 are primitive. Therefore, by elementary number theory (see [16]), we have

$$\sum_{\chi\ primitive} \bar{\chi}(b) = \begin{cases} \phi(m) - 1 & : \quad b \equiv 1 \pmod{m} \\ -1 & : \quad b \not\equiv 1 \pmod{m} \end{cases}$$

Note that if $b = b_1$, then $b^{(1)} b = b^{(1)} b_1 \equiv 1 \pmod{m}$. Similarly, if $b = b_2$, then $b^{(2)} b = b^{(2)} b_2 \equiv 1 \pmod{m}$. Therefore, as above,

$$\sum_{\chi\ primitive} \bar{\chi}(b^{(1)} b) = \begin{cases} \phi(m) - 1 & : \quad b = b_1 \\ -1 & : \quad b \neq b_1 \end{cases}$$

and

$$\sum_{\chi\ primitive} \bar{\chi}(b^{(2)} b) = \begin{cases} \phi(m) - 1 & : \quad b = b_2 \\ -1 & : \quad b \neq b_2 \end{cases}$$

Thus,

$$\sum_{\chi\ primitive} \bar{\chi}(b^{(1)} b) - \sum_{\chi\ primitive} \bar{\chi}(b^{(2)} b) = \begin{cases} 0 & : \quad b \neq b_1, b_2 \\ \phi(m) & : \quad b = b_1 \\ -\phi(m) & : \quad b = b_2 \end{cases}$$

But from (5.18), we have

$$\sum_{\chi\ primitive} (\bar{\chi}(b^{(1)}) - \bar{\chi}(b^{(2)})) \sum_{Rb (mod m)} \bar{\chi}(b)\, F \mid \lambda(b) = 0.$$

Therefore,

$$\begin{aligned} LHS &= \sum_{Rb(mod m)} F \mid \lambda(b) \sum_{\chi primitive} (\bar{\chi}(b^{(1)}b) - \bar{\chi}(b^{(2)}b)) \\ &= \sum_{Rb(mod m)} F \mid \lambda(b) \left[\sum_{\chi \ primitive} \bar{\chi}(b^{(1)}b) - \sum_{\chi \ primitive} \bar{\chi}(b^{(2)}b) \right] \\ &= \phi(m) F \mid \lambda(b_1) + (-\phi(m)) F \mid \lambda(b_2) \ + \ 0 \cdot \sum_{\substack{Rb(mod m) \\ Rb \neq Rb_1, Rb_2}} F \mid \lambda(b) \\ &= \phi(m)(F \mid \lambda(b_1) - F \mid \lambda(b_2)), \end{aligned}$$

and so $\phi(m)(F \mid \lambda(b_1) - F \mid \lambda(b_2)) = 0$. Therefore, $F \mid \lambda(Rb_1) = F \mid \lambda(b_2)$.

Lemma 5.5 *Let* $k \in \mathbf{Z}^+$ *and* $\gamma(a,b) = \left(\begin{smallmatrix} m & -Rb \\ -Sa & n \end{smallmatrix}\right) \in \Gamma(R,S)$ *with* $m,n \in \mathcal{M}$. *Assume (5.14) holds for conductors* m *and* n, *i.e., for each primitive Dirichlet character* χ *modulo* m *(respectively modulo* n*) such that* $(m, RS) = 1$ *(respectively* $(n, RS) = 1$*),*

$$\Omega_\chi(s) = C_\chi \left(\frac{S}{R}\right)^{k/2-s} \Omega_{\bar{\chi}}(k-s), \; C_m^* = C_\chi i^k \frac{g(\bar{\chi})}{g(\chi)} \bar{\chi}(-RS). \qquad (5.23)$$

Suppose that $C_m^* C_n^* = (-1)^k$. *Assume also that* F *satisfies (5.4):*

$$\Omega(s) = C\mu^{k/2-s}\Omega(k-s).$$

Then $F \mid \gamma(a,b) = ({C_m^*}^{-1} C i^k)\, F$, *where* $C = \pm 1$ *is from Lemma 5.2.*

Proof of Lemma: Let $\gamma(a,b) = \left(\begin{smallmatrix} m & -Rb \\ -Sa & n \end{smallmatrix}\right)$, $\gamma(-a,-b) = \left(\begin{smallmatrix} m & Rb \\ Sa & n \end{smallmatrix}\right) \in \Gamma(R,S)$. In Lemma 5.4, we replace Rb_1 and Rb_2 by Rb and $R(-b)$ respectively. Now $Rb \not\equiv -Rb \pmod m$ since otherwise, $m \mid 2Rb$ and $(Rb, m) = 1$ imply $m \mid 2$, a contradiction since $m = 4$ or an odd prime. Therefore, (5.15) in Lemma 5.4 becomes

$$[F - C_m^* \, F \mid T\left(\frac{S}{R}\right)\gamma(a,b)] \mid \mathbf{S}\left(\frac{Rb}{m}\right) = [F - C_m^* \, F \mid T\left(\frac{S}{R}\right)\gamma(-a,-b)] \mid \mathbf{S}\left(\frac{-Rb}{m}\right). \qquad (5.25)$$

By assumption, F satisfies (5.4). Therefore, by Lemma 5.2, we have

$$Ci^k \, F \mid T\left(\frac{S}{R}\right) = F.$$

which implies

$$C_m^* \, F \mid T\left(\frac{S}{R}\right) = C_m^* C^{-1} i^{-k} \, F.$$

Thus, with $\varsigma = C_m^* C^{-1} i^{-k}$, (5.25) can be rewritten as

$$[F - \varsigma \, F \mid \gamma(a,b)] \mid \mathbf{S}\left(\frac{Rb}{m}\right) = [F - \varsigma \, F \mid \gamma(-a,-b)] \mid \mathbf{S}\left(\frac{-Rb}{m}\right).$$

i.e.,

$$[F - \varsigma \, F \mid \gamma(-a,-b)] = [F - \varsigma \, F \mid \gamma(a,b)] \mid \mathbf{S}\left(\frac{2Rb}{m}\right). \qquad (5.29)$$

On the other hand, since $[\gamma(a,b)]^{-1} = \left(\begin{smallmatrix} m & -Rb \\ -Sa & n \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} n & Rb \\ Sa & m \end{smallmatrix}\right)$ and $[\gamma(-a,-b)]^{-1} = \left(\begin{smallmatrix} m & Rb \\ Sa & n \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} n & -Rb \\ -Sa & m \end{smallmatrix}\right)$, and (5.15) in Lemma 5.4 holds for n as well as m, we have (as above), with $\varsigma^* = C_n^* C^{-1} i^{-k}$,

$$[F - \varsigma^* F \mid [\gamma(-a,-b)]^{-1}] \mid \mathbf{S}\left(\frac{Rb}{n}\right) = [F - \varsigma^* F \mid [\gamma(a,b)]^{-1}] \mid \mathbf{S}\left(\frac{-Rb}{n}\right).$$

i.e.,

$$[F - \varsigma^* F \mid [\gamma(-a,-b)]^{-1}] = [F - \varsigma^* F \mid [\gamma(a,b)]^{-1}] \mid \mathbf{S}\left(\frac{-2Rb}{n}\right). \quad (5.31)$$

By assumption, $C_m^* C_n^* = (-1)^k$, and by the note after Lemma 5.2, (5.4) implies $C = \pm 1$. Therefore, since k is an integer, we have

$$\begin{aligned} \varsigma\varsigma^* &= [C_m^* C^{-1} i^{-k}][C_n^* C^{-1} i^{-k}] \\ &= C_m^* C_n^* C^{-2} i^{-2k} \\ &= (-1)^{2k} \\ &= 1, \end{aligned}$$

or $\varsigma^* = \varsigma^{-1}$. Hence, $-[F - \varsigma F \mid \gamma(-a,-b)] \mid \varsigma^{-1}[\gamma(-a,-b)]^{-1} = -\varsigma^{-1} F \mid [\gamma(-a,-b)]^{-1} + F \mid 1 = F - \varsigma^* F \mid [\gamma(-a,-b)]^{-1}$. Therefore, (5.31) becomes

$$-[F - \varsigma F \mid \gamma(-a,-b)] \mid \varsigma^{-1}[\gamma(-a,-b)]^{-1} = [F - \varsigma^* F \mid [\gamma(a,b)]^{-1}] \mid \mathbf{S}\left(\frac{-2Rb}{n}\right).$$

i.e.,

$$[F - \varsigma F \mid \gamma(-a,-b)] = -\varsigma\,[F - \varsigma^* F \mid [\gamma(a,b)]^{-1}] \mid \mathbf{S}\left(\frac{-2Rb}{n}\right) \gamma(-a,-b).$$

Now $\varsigma^*\varsigma = 1 \Rightarrow -\varsigma\,[F - \varsigma^* F \mid [\gamma(a,b)]^{-1}] = -\varsigma F + F \mid [\gamma(a,b)]^{-1} = [F - \varsigma F \mid \gamma(a,b)] \mid [\gamma(a,b)]^{-1}$

and so (5.31) becomes

$$[F - \varsigma F \mid \gamma(-a,-b)] = [F - \varsigma F \mid \gamma(a,b)] \mid [\gamma(a,b)]^{-1} \mathbf{S}\left(\frac{-2Rb}{n}\right) \gamma(-a,-b). \quad (5.34)$$

Combining (5.29) and (5.34), we have

$$[F - \varsigma F \mid \gamma(a,b)] \mid \mathbb{S}\left(\frac{2Rb}{m}\right) = [F - \varsigma F \mid \gamma(a,b)] \mid [\gamma(a,b)]^{-1}\mathbb{S}\left(\frac{-2Rb}{n}\right)\gamma(-a,-b).$$

or

$$[F - \varsigma F \mid \gamma(a,b)] \mid \mathbb{S}\left(\frac{2Rb}{m}\right) - [F - \varsigma F \mid \gamma(a,b)] \mid U\mathbb{S}\left(\frac{2Rb}{m}\right) = 0 \quad (5.36)$$

where $U = [\gamma(a,b)]^{-1}\ \mathbb{S}\ (\frac{-2Rb}{n})[\gamma(-a,-b)]\ \mathbb{S}(\frac{-2Rb}{m})$. We now calculate U:

$$\begin{aligned} U &= \begin{pmatrix} n & Rb \\ Sa & m \end{pmatrix}\begin{pmatrix} 1 & \frac{-2Rb}{n} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} m & Rb \\ Sa & n \end{pmatrix}\begin{pmatrix} 1 & \frac{-2Rb}{m} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} n & -Rb \\ Sa & \frac{-2SaRb}{n}+m \end{pmatrix}\begin{pmatrix} m & -Rb \\ Sa & \frac{-2SaRb}{m}+n \end{pmatrix} \\ &= \begin{pmatrix} mn - RSab & -nRb + \frac{2SaR^2b^2}{m} - nRb \\ 2Sam - \frac{2S^2a^2Rb}{n} & -SaRb + (\frac{-2SaRb+mn}{n})(\frac{-2SaRb+mn}{m}) \end{pmatrix}. \end{aligned}$$

Using $RSab = mn - 1$ and writing everything in terms of m and n, we obtain

$$U = \begin{pmatrix} 1 & \frac{-2Rb}{m} \\ \frac{2Sa}{n} & \frac{4}{mn} - 3 \end{pmatrix}.$$

Claim: U is elliptic of infinite order.

Proof of Claim: Trace(U) $= 1 + \frac{4}{mn} - 3 = \frac{4}{mn} - 2 = 2(\frac{2}{mn} - 1)$. But $m, n \in \mathcal{M} \Rightarrow 0 < \frac{2}{mn} < 1 \Rightarrow -2 <$ Trace(U) < 0. Therefore U is elliptic. To prove that U is of infinite order, we show that the eigenvalues of U are not roots of unity. (See [14].) The eigenvalues are the roots of the characteristic polynomial

$$x^2 - Trace(U)x + Determinant(U),$$

i.e.,

$$x^2 + 2(1 - \tfrac{2}{mn})x + 1,$$

a polynomial p(x) in $\mathbb{Q}[x]$ of negative discriminant. Therefore, the roots are complex and algebraic of degree 2. However, since p(x) is the monic irreducible polynomial for these roots in $\mathbb{Q}[x]$, and the coefficients of p(x) are not in $\mathbb{Z}$, these roots are not algebraic integers. In particular, they are not roots of unity. Therefore, U has infinite order and the claim follows.

Put $G = [F - \zeta\, F \mid \gamma(a,b)]$. Then by (5.36), we have the following: $[F - \zeta\, F \mid \gamma(a,b)] - [F - \zeta\, F \mid \gamma(a,b)] \mid U = 0$ or $G - G \mid U = 0$, and thus $G \mid U = G$. Since U is elliptic of infinite order, the group $< U >$ cannot be discrete, and hence there do not exist nonconstant automorphic forms on $< U >$ (see [8]). Therefore, G is a constant. Since $k \neq 0$ and the only constant automorphic form of nonzero weight is the 0 function, it follows that $G = 0$ and thus $\zeta\, F \mid \gamma(a,b) = F$. Therefore, $F \mid \gamma(a,b) = \zeta^{-1} F = (C_m^*)^{-1} C i^k F$ and the proof of Lemma 5.5 is complete.

5.4 The Converse Theorem

Theorem 5.2 *(The Converse Theorem) Let $k \in \mathbb{Z}^+$ and $\mathcal{M}^* \subset \mathcal{M}$ such that $\mathcal{M}^*$ contains elements in every arithmetic progression $\{an + b \mid (a,b) = 1\}$. (There exist such $\mathcal{M}^* \subset \mathcal{M}$ since $\mathcal{M}$ itself satisfies this by Dirichlet's theorem.) Suppose also that ϵ is a Dirichlet character modulo RS. Suppose that (5.4) of Lemma 5.2 holds for the Mellin transform of F. Suppose also that (5.7) of Lemma 5.3 holds for the twisted Mellin transform by each primitive Dirichlet character χ of conductor $f_\chi \in \mathcal{M}^*$, where $C_\chi = C\epsilon(f_\chi)\frac{g(\chi)}{g(\bar{\chi})}\chi(-RS)$.*

Then F is an entire modular form of Hecke type (k,R,S,ϵ) such that $F \mid T(\frac{S}{R}) = Ci^{-k}F$. Finally, if $L(s) = \sum_{n=1}^{\infty} c_n n^{-s}$ converges absolutely for $s = k - \delta$, $0 < \delta < k$, then F is a cusp form.

Proof: That $F \mid T(\frac{S}{R}) = Ci^{-k}F$ is Lemma 5.2 since $C = C^{-1}$. Now let $M = \begin{pmatrix} a & Rb \\ Sc & d \end{pmatrix} \in \Gamma(R,S)$. We consider two cases:

Case 1: $(b = 0)$ In this case, we have $a = d = \pm 1$. For the moment, we assume $a = d = 1$. Then $M = \begin{pmatrix} 1 & 0 \\ Sc & 1 \end{pmatrix} \in \Gamma(R,S)$. On the other hand,

$$
\begin{aligned}
T\left(\frac{S}{R}\right)\begin{pmatrix}1 & -Rc\\ 0 & 1\end{pmatrix}T\left(\frac{S}{R}\right)^{-1} &= \begin{pmatrix}0 & -R\\ S & 0\end{pmatrix}\begin{pmatrix}1 & -Rc\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & \frac{1}{S}\\ \frac{-1}{R} & 0\end{pmatrix}\\
&= \begin{pmatrix}0 & -R\\ S & -RSc\end{pmatrix}\begin{pmatrix}0 & \frac{1}{S}\\ \frac{-1}{R} & 0\end{pmatrix}\\
&= \begin{pmatrix}1 & 0\\ Sc & 1\end{pmatrix}\\
&= M.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
F \mid M &= F \mid T\left(\frac{S}{R}\right)\begin{pmatrix}1 & -Rc\\ 0 & 1\end{pmatrix}T\left(\frac{S}{R}\right)^{-1}\\
&= C^{-1}i^{-k}F \mid \begin{pmatrix}1 & -Rc\\ 0 & 1\end{pmatrix}T\left(\frac{S}{R}\right)^{-1}\\
&= C^{-1}i^{-k}F \mid T\left(\frac{S}{R}\right)^{-1}\\
&= C^{-1}i^{-k}Ci^{k}F\\
&= F\\
&= \epsilon^{-1}(1)F.
\end{aligned}
$$

This proves Case 1: $(b = 0)$ for $a = d = 1$.

Case 2: $(b \neq 0)$ Now $ad - RSbc = 1 \Rightarrow (a, RSb) = (d, RSb) = 1$. By the assumption on $\mathcal{M}^*, \exists$ integers $m = a + RSbs \in \mathcal{M}^*, n = d + RSbt \in \mathcal{M}^*$ for some s,$t \in \mathbf{Z}$. Then $(m, RS) = (n, RS) = 1$. Put $b^* = c + mt + ns - RSbst$ and $M^* = \begin{pmatrix}m & Rb\\ Sb^* & n\end{pmatrix}$. Then a calculation shows that $mn - RSbb^* = 1$, thence $M^* \in \Gamma(R, S)$.

By assumption, $C_\chi = C\epsilon(f_\chi)\frac{g(\chi)}{g(\bar{\chi})}\chi(-RS)$. In accordance with Lemma 5.5, set $C^*_m = C_\chi i^k \frac{g(\bar{\chi})}{g(\chi)}\bar{\chi}(-RS)$ for $\chi \pmod{m}$. Similarly, set $C^*_n = C_\chi i^k \frac{g(\bar{\chi})}{g(\chi)}\bar{\chi}(-RS)$ for $\chi \pmod{n}$. Then $C^*_m = Ci^k\epsilon(m)$ for $\chi \pmod{m}$ and $C^*_n = Ci^k\epsilon(n)$ for $\chi \pmod{n}$.

Claim: The assumptions of Lemma 5.5 hold for the conductors m and n, with $C^*_m = Ci^k\epsilon(m)$ and $C^*_n = Ci^k\epsilon(n)$.

Proof of Claim: Above, we showed that $(m, RS) = (n, RS) = 1$. Therefore, by the present assumptions of Theorem 5.2, it suffices to check that $C^*_m C^*_n = (-1)^k$. To do this, we calculate:

$$\begin{aligned} C^*_m C^*_n &= C^2 i^{2k} \epsilon(m)\epsilon(n) \\ &= (-1)^k \epsilon(m)\epsilon(n) \\ &= (-1)^k \epsilon(mn). \end{aligned}$$

But $mn - RSbb^* = 1 \Rightarrow mn \equiv 1 \pmod{RS}$. Therefore, $\epsilon(mn) = \epsilon(1) = 1$ and the claim follows.

Applying Lemma 5.5 gives

$$\begin{aligned} F \mid M^* &= {C^*_m}^{-1} C i^k \; F \\ &= (C^{-1} i^{-k} \epsilon^{-1}(m)) C i^k \; F \\ &= \epsilon^{-1}(m) \; F. \end{aligned}$$

Therefore, the transformation law holds for M^*. To prove it for $M = \left(\begin{smallmatrix} a & Rb \\ Sc & d \end{smallmatrix}\right)$, note that

$$\begin{aligned} \left(\begin{smallmatrix} 1 & 0 \\ -St & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} m & Rb \\ Sb^* & n \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ -Ss & 1 \end{smallmatrix}\right) &= \left(\begin{smallmatrix} m & Rb \\ -Stm+Sb^* & -RStb+n \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ -Ss & 1 \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} m & Rb \\ -Stm+Sb^* & d \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ -Ss & 1 \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} m & Rb \\ S(b^*-tm) & d \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ -Ss & 1 \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} m-RSbs & Rb \\ S(b^*-tm)-Sds & d \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} a & Rb \\ S(b^*-tm-ds) & d \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} a & Rb \\ S[b^*-tm-s(n-RSbt)] & d \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} a & Rb \\ S[b^*-tm-sn+RSbst)] & d \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} a & Rb \\ Sc & d \end{smallmatrix}\right) \\ &= M. \end{aligned}$$

Thus M is a product of elements in $\Gamma(R, S)$ for which the transformation law holds.

Claim: $v[(\begin{smallmatrix} a & Rb \\ Sc & d \end{smallmatrix})] = \epsilon(a)$ defines a multiplier system on $\Gamma(R,S)$.

Proof of Claim: Let $M_1 = (\begin{smallmatrix} a_1 & Rb_1 \\ Sc_1 & d_1 \end{smallmatrix})$, $M_2 = (\begin{smallmatrix} a_2 & Rb_2 \\ Sc_2 & d_2 \end{smallmatrix}) \in \Gamma(R,S)$ so $M_1M_2 = (\begin{smallmatrix} a_1a_2+RSb_1c_2 & * \\ * & * \end{smallmatrix})$. Then

$$\begin{aligned} v(M_1M_2) &= \epsilon(a_1a_2 + RSb_1c_2) \\ &= \epsilon(a_1a_2) \\ &= \epsilon(a_1)\epsilon(a_2) \\ &= v(M_1)v(M_2), \end{aligned}$$

i.e., v is a MS on $\Gamma(R,S)$ which is also a character on $\Gamma(R,S)$, as is required since any MS of integer weight is necessarily a character. We observe that ϵ is actually multiplicative on the product $\Gamma_0(S)\Gamma^0(R)$. Since ϵ defines a multiplier system and $m \equiv a \pmod{RS}$, the transformation law holds for M as well. This proves Case 2: $(b \neq 0)$.

To complete the proof, we consider the case $b = 0, a = d = -1$. Then $M = (\begin{smallmatrix} -1 & 0 \\ Sc & -1 \end{smallmatrix})$. Form

$$\begin{aligned} M\mathbf{S}(Ru) &= (\begin{smallmatrix} -1 & 0 \\ Sc & -1 \end{smallmatrix})(\begin{smallmatrix} 1 & Ru \\ 0 & 1 \end{smallmatrix}) \\ &= (\begin{smallmatrix} -1 & -Ru \\ Sc & RSuc-1 \end{smallmatrix}) \\ &= \widehat{M}. \end{aligned}$$

Applying the above case of b $\neq 0$ to $\widehat{M}$, we obtain
$F \mid \widehat{M} = \epsilon^{-1}(-1)F$. Finally, we have

$$\begin{aligned} F \mid M &= F \mid \widehat{M}\mathbf{S}(-Ru) \\ &= \epsilon^{-1}(-1)F \mid \mathbf{S}(-Ru) \\ &= \epsilon^{-1}(-1)F. \end{aligned}$$

To complete the proof of the theorem, we show that if $L(s) = \sum_{n=1}^{\infty} c_n n^{-s}$ converges absolutely for $s = k - \delta$, $0 < \delta < k$, then F is a cusp form. We have

$$F(x+iy) = \sum_{n=0}^{\infty} c_n e^{2\pi i n(x+iy)/R}$$

which implies

$$|F(x+iy)| \leq \sum_{n=0}^{\infty} |c_n| e^{-2\pi ny/R}.$$

Let the n^{th} partial sum $S_n = \sum_{v=0}^{n} |c_v|$. Then we make the following
Claim: $S_n = \mathcal{O}(n^{k-\delta})$, as $n \to +\infty$.

Assuming for the moment that the claim is true, we have

$$\lim_{n\to\infty} S_n e^{-2\pi ny/R} = 0.$$

Therefore, an application of Abel's Partial Summation gives

$$\begin{aligned} \sum_{n=0}^{\infty} |c_n| e^{-2\pi ny/R} &= \sum_{n=0}^{\infty} S_n (1 - e^{-2\pi y/R})(e^{-2\pi y/R})^n \\ &= (1 - e^{-2\pi y/R}) \sum_{n=0}^{\infty} S_n e^{-2\pi ny/R}. \end{aligned}$$

It is relatively easy to check that, if $f(x+iy) = \sum_{n=0}^{\infty} a_n e^{2\pi in(x+iy)/\lambda}$, and $a_n = \mathcal{O}(n^{\sigma})$, as $n \to +\infty$, then $f(x+iy) = \mathcal{O}(y^{-\sigma-1})$, uniformly in x, as $y \to 0^+$. Applying this above, we have

$$|F(x+iy)| \leq \frac{(1-e^{-2\pi y/R})}{y} C y^{\delta-k}, \text{ uniformly in } x, \text{ as } y \to 0^+$$

which implies, $\forall z = x+iy \in \mathbb{H}$,

$$|F(x+iy)| \leq C^* y^{\delta-k}, \text{ uniformly in } x.$$

This shows that the Fourier expansion for F vanishes at the cusp $i\infty$. This growth condition shows that F vanishes at all the finite cusps as well. To see this, let $q = A(i\infty)$ be a finite cusp for the S.F.R. $\mathcal{R}$ of $\Gamma(R,S)$ from section 4.3. The Fourier expansion at q is the following:

$$F(z) = (z-q)^{-k} \sum_{n+\kappa\geq 0} c_n(q) e^{2\pi i(n+\kappa)(A^{-1}z)/\lambda}$$

or

$$(z-q)^k F(z) = \sum_{n+\kappa\geq 0} c_n(q) e^{2\pi i(n+\kappa)(A^{-1}z)/\lambda},$$

valid for $Im(A^{-1}z) > y$, for some $y \geq 0$. Recall that as $z \to q$ within $\mathcal{R}$, $A^{-1}z \to i\infty$. Therefore, with $z = q + iy$, the RHS of the above expansion $\to c_0(q)$ as $y \to 0$. In terms of the LHS, this implies the following:

$$i^k y^k F(q + iy) \to c_0(q) \text{ as } y \to 0,$$

i.e.,

$$y^k F(q + iy) \to i^{-k} c_0(q) \text{ as } y \to 0.$$

Since $y^k F(z) = \mathcal{O}(y^\sigma)$, for some $\sigma > 0$, uniformly in x, as $y \to 0^+$, we have $c_0(q) = 0$ and so F vanishes at q. Therefore F is a cusp form. Now for the

Proof of Claim:

$$\begin{aligned} S_n = \sum_{v=0}^{n} |c_v| &\leq \sum_{v=0}^{n} |c_v| (\frac{n}{v})^{k-\delta} \\ &\leq \sum_{v=0}^{\infty} |c_v| (\frac{n}{v})^{k-\delta} \\ &= n^{k-\delta} \sum_{v=0}^{\infty} |c_v| \, v^{-(k-\delta)} \\ &= n^{k-\delta} \cdot constant, \end{aligned}$$

the latter equality following by assumption that $L(s) = \sum_{n=1}^{\infty} c_n n^{-s}$ converges absolutely for $s = k - \delta$, $0 < \delta < k$. This proves the claim and completes the proof of the theorem.

REFERENCES

[1] Apostol, T.M. 1990. *Modular functions and Dirichlet series in number theory.* Springer-Verlag, New York.

[2] Deshouillers, J.-M., and Iwaniec, H. 1982. *Kloosterman sums and Fourier coefficients of cusp forms.* Invent. Math. **70**, 219-288.

[3] Grosswald, E. 1950. *On the structure of some subgroups of the modular group.* Amer. Jour. Math. **lxxii**, no. 4: 809-834.

[4] Hawkins, J., and Knopp, M. 1993. *A Hecke-Weil correspondence theorem for automorphic integrals on $\Gamma_0(N)$, with arbitrary rational period functions.* In *A Tribute to Emil Grosswald: Number Theory and Related Analysis.* AMS Contemporary Math. **143**, 451-475.

[5] Hecke, E. 1936. *Über die Bestimmung Dirichletscher Reihen durch ihre Funktionalgleichung.* Math. Ann. **112**, 664-699.

[6] Hecke, E. 1938. *Lectures on Dirichlet series, modular functions and quadratic forms.* Edwards Bros. Inc., Ann Arbor.

[7] Katok, S. 1992. *Fuchsian Groups.* University of Chicago Press, Chicago.

[8] Knopp, M. 1966. *Polynomial automorphic forms and nondiscontinuous groups.* Tran. Amer. Math. Soc. **123**, 506-520.

[9] Knopp, M. 1985. *Modular integrals on $\Gamma_0(N)$ and Dirichlet series with functional equations, in Number theory.* Lecture Notes in Mathematics. no. 1135: 211-224. Springer-Verlag, New York.

[10] Knopp, M. 1993. *Modular functions in analytic number theory.* Chelsea, New York.

[11] Knopp, M., and Newman, M. 1993. *On groups related to the Hecke groups.* Proc. Amer. Math. Soc. **119**, no. 1: 77-80.

[12] Lebedev, N.N. 1972. *Special functions and their applications.* Dover, New York. (Transl. Richard A. Silverman).

[13] Lehner, J. 1964. *Discontinuous groups and automorphic functions.* American Mathematical Society, Providence.

[14] Lehner, J. 1966. *A short course in automorphic functions.* Holt, Rinehart and Winston, New York.

[15] Newman, M. 1964. *Free subgroups and normal subgroups of the modular group.* Illinois J. Math. **8**, no. 2: 262-265.

[16] Rademacher, H. 1964. *Lectures on elementary number theory.* Robert E. Krieger, Malabar.

[17] Razar, M.J. 1977. *Modular forms for $\Gamma_0(N)$ and Dirichlet series.* Trans. Amer. Math. Soc. **231**, no. 2: 489-495.

[18] Schoeneberg, B. 1974. *Elliptic modular functions.* Springer-Verlag, New York.

[19] Titchmarsh, E.C. 1951. *The theory of the Riemann zeta-function.* Oxford U. Press, London.

[20] Weil, A. 1967. *Über die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen.* Math. Annalen. **168**, 149-156.