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SOME RESULTS IN RAMSEY THEORY

A Dissertation

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the Temple University Graduate Board

in Partial Fulfillment

of the Requirements for the Degree

DOCTOR OF PHILOSOPHY

by

Aaron Robertson

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ABSTRACT

SOME RESULTS IN RAMSEY THEORY

by Aaron Robertson

Doctor of Philosophy

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Advisor: Dr. Doron Zeilberger

In this dissertation we find several new results in the field of Ramsey Theory. In the second chapter we find lower bounds for some classical Ramsey numbers, give two general lower bounds for multicolored Ramsey numbers, and define the concept of difference Ramsey numbers. Extending techniques due to Greenwood, Gleason, and Chung we are able to establish eleven new lower bounds for multicolored Ramsey numbers. These are $R(3,3,6) \geq 60, R(3,3,9) \geq 91$, $R(3,3,11) \geq 141$, $R(5,5,5) \geq 242$, $R(6,6,6) \geq 692$, $R(3,3,3,4) \geq 91$, $R(3,3,3,5) \geq 137$, $R(3,3,3,6) \geq 183$, $R(3,3,3,7) \geq 220$, $R(3,3,3,9) \geq 336$, and $R(3,3,3,11) \geq 431$.

In the third chapter we turn to Schur numbers, which are closely related to the Ramsey numbers. In this area we show that the minimum number, asymptotically, of monochromatic Schur triples in any 2-coloring of $[1,n]$ is $n^2/22 + O(n)$. We then use this result to establish an upper bound for the minimum number, asymptotically, of monochromatic Schur triples in any r-coloring of $[1, n]$. After establishing this bound, we define Issai numbers, a new generalization of the Schur numbers. We show that these numbers extend very naturally from difference Ramsey numbers. After presenting some values for small Issai numbers we turn to another Ramsey-type number, the van der Waerden number. For this number we determine that the minimum number, asymptotically, of van der Waerden triples is bounded between $n^2/38 + O(n)$ and $n^2/16 + O(n)$.

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On a more personal note, I would like to thank all of my friends at Temple. In particular I would like to thank Hans Johnston, Kurt Ludwick, Paul Nekoranik, James Palermo, and Judith Vogel for many years of happiness.

Most importantly I would like to thank my wife, Elisa, and my parents Doug and Pearl Robertson. They offered their unconditional support during my trip through graduate school.

DEDICATION

With all of my love, I dedicate this work to my wife Elisa. A truly outstanding individual, she wholeheartedly supports me in all of my endeavors.

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CHAPTER 1
INTRODUCTION

1.1 Early Results

Until Andrew Wiles proved Fermat's Last Theorem, almost every mathematician who ever lived dreamed of proving the deceptively simple statement:

Fermat's Last Theorem: *For $n \geq 3$, $x^n + y^n = z^n$ has no nontrivial solution in the integers.*

The lure to prove this theorem which had not been cracked for so many years drew Issai Schur in. Of course, we know that Schur did not prove it. He did, however, prove the following theorem.

Theorem: *For $n \geq 1$, $x^n + y^n = z^n$ has a solution in the integers mod p, for a sufficiently large prime, p.*

To prove this result, Schur had to prove an intermediate result, which, in most mathematicians' view, greatly overshadows the above theorem. The lemma he used is know today as Schur's Theorem. Schur's Theorem is widely held as the first Ramsey-type theorem to spark research activity. As an aside, the first Ramsey-type theorem is generally believed to be a work of David Hilbert. However, Hilbert's work went unnoticed as far as opening up the field of Ramsey Theory. Schur, in 1916, proved the following.

Schur's Theorem: *Given r, there exists an integer $N = N(r)$ such that any r-coloring of the integers 1 through N must admit a monochromatic solution to $x + y = z$.*

Schur is speculated to have worked on further applications of his "lemma." However, no published work in this direction appears. We now jump forward to 1927 when the next major Ramsey-type theorem was proved. B.L. van der Waerden proved the following theorem, first conjectured by Schur.

Van der Waerden's Theorem: *If the positive integers are 2-colored, then there must be an arbitrarily long monochromatic arithmetic progression.*

Three years later, in 1930, Frank Ramsey proved his eponymous theorem. This theorem epitomizes the flavor of the subject, and is a beautiful result.

Ramsey's Theorem: *Let $k_i \geq 2$, $i = 1, 2, \ldots, r$, be given. Then there exists a minimal positive integer $N = R(k_1, k_2, \ldots, k_r)$ such that any r-coloring of the edges of the complete graph on N vertices, K_N, must admit a j-colored complete graph on k_j vertices for some $j \in \{1, 2, \ldots, r\}$.*

This result caught the interest of many mathematicians, and also introduced many of them to the field of combinatorics. However the result was not widely publicized. Consider as proof the fact that Erdős and Szekeres discovered an equivalent form of Ramsey's Theorem independently *five* years later. To their dismay, this beautiful result had already been discovered.

It was at this point that Ramsey Theory was given its name by Erdős. However, Ramsey Theory was still not exploding with activity. An even more striking fact pertaining to the small amount of activity in Ramsey Theory is that it was not until 1955 that the first nontrivial Ramsey number was discovered by Greenwood and Gleason ([GG]). This result came 25 years after the publication of Ramsey's Theorem. However, in the past 40 years or so, Ramsey Theory has experienced a flurry of activity. Thanks in large part to the travelling mathematician Paul

Erdős, Ramsey Theory has become a vast field of many interesting, appealing, and beautiful results. Paul Erdős used to say that all of the best proofs of all theorems are stored in a book, that only a higher power has access to. He would call this book, The Book. Erdős himself, is sure to have many proofs appear in the book, many of which are results from Ramsey Theory. For a survey of some proofs which should be included in The Book, see [AZ].

1.2 Recent Results

Ramsey Theory has continued to be a very active area of research. The passing of Paul Erdős in 1996, was the end of an era. But his legacy will live on in the problems he left behind, many of which are in Ramsey Theory. A good collection of some of his open problems can be found in [CGr].

Currently, Ramsey Theory is as active as it ever was. For example, in 1995 J. H. Kim stunned the mathematical community when he proved that the Ramsey number $R(3,n)$ has order of magnitude $n^2/log(n)$ ([Kim]). Another groundbreaking result was the determination of the Ramsey number $R(4,5)$ by B. McKay and S. Radziszowski ([MR]). Together with many computers they calculated it to be 25.

With the computer technology and speed available today, Ramsey Theory can be furthered in directions untouchable by human methods. Many problems in Ramsey Theory are *finite* problems. Thus, it is conceivable that computers will someday soon be quick enough to tackle such enormous problems, given that there are quick and intelligent algorithms to use.

The research that follows in this dissertation melds constuctive, algorithmic, and computational combinatorics to gives *Some Results in Ramsey Theory.*

CHAPTER 2
CLASSICAL RAMSEY NUMBERS

2.1 Introduction

At any party of six people, either three of them mutually know each other, or three of them mutually do not know each other.

The above fact is known as the puzzle problem or the party problem. Its roots are very old, but it is a classical example of a problem from the field of Ramsey Theory. Ramsey Theory is a very elegant and intriguing field of mathematics. The problems are quite often very easy to pose and understand, but difficult to solve. The methods used to solve these problems often require creative and innovative ideas.

In this chapter we will concentrate on Ramsey's Theorem ([Ram]). Since Ramsey's result in 1930, Ramsey Theory has flourished and produced a vast number of beautiful results and proofs. The determination of Ramsey numbers (the numbers whose existence are proved in Ramsey's Theorem) has been one subarea whose research has been long lasting. In this chapter we will extend results of Greenwood, Gleason, and Chung, and employ some computer algorithms to obtain some new results.

Now we will restate Ramsey's Theorem for edgewise colorings:

Ramsey's Theorem: *Let $k_i \geq 2$, $i = 1, 2, \ldots, r$, be given. Then there exists a minimal positive integer $N = R(k_1, k_2, \ldots, k_r)$ such that any r-coloring of the edges of the complete graph on N vertices must admit a j-colored complete graph on k_j vertices for some $j \in \{1, 2, \ldots, r\}$.*

With regard to the above theorem, we will make the following definition which will be called upon many times throughout this dissertation.

Definition 2.1 *Ramsey Property:* An edgewise r-colored complete graph on N vertices, K_N, is said to have the Ramsey Property if there exist j, $1 \leq j \leq r$, such that a monochromatic j-colored complete graph on k_j vertices is a subgraph of the r-colored K_N.

Much effort has gone into finding the exact values of these Ramsey numbers. For a current survey, see Stanislaw Radziszowski's Dynamic Survey of Small Ramsey Numbers [Rad]. As of this writing, only 10 Ramsey numbers for edgewise colorings of complete graphs are known. On the other hand, there has been some good progress made with finding lower bounds for these hard-to-compute numbers. In the sections to follow we will obtain new lower bounds for several multicolored Ramsey numbers. We will also develop a recursive algorithm which will find the best possible lower bound of edgewise colored complete graphs with a certain structure imposed on them. Using this algorithm we can find good lower bounds for some Ramsey numbers *automatically.*

2.2 The Finite Field Method

In this first section we add two more lower bounds to Radziszowski's Dynamic Survey [Rad] on the subject. We show, by using the finite field technique in [GG], that $R(5,5,5) \geq 242$ and $R(6,6,6) \geq 692$. The previous best lower bound for $R(5,5,5)$ was 169 given by Song [S], who more generally shows that $R(\underbrace{5,5,\ldots,5}_{r\ times}) \geq 4(6.48)^{r-1}+1$ holds for all r. For $R(6,6,6)$ there was no established nontrivial lower bound.

Consider the number $R(5,5,5)$. To find a lower bound, L, we are searching for a three coloring of K_L which avoids a monochromatic K_5. We use an argument of Greenwood and Gleason, which is reproduced here for the sake of completeness.

Let L be prime and consider the field of L elements, numbered from 0 to $L-1$. Associate each field element with a vertex of K_L. We require that 3 divides $L-1$. Now consider the cubic residues of the multiplicative group $\mathbf{Z}_L^* = \mathbf{Z}_L \setminus \{0\}$, which form a coset of $\mathbf{Z}_L^*$. Since 3 divides $L-1$, there must be 2 other cosets.

Let i and j be two vertices of K_L. Color the edges of K_L as follows: If $j-i$ is a cubic residue color the edge connecting i and j red, if it is in the second coset, color the edge blue, and if it is the third coset, color the edge green. (Note that the order of differencing is immaterial since -1 is a cubic residue.)

Now suppose that a monochromatic K_5 exists in this coloring. Without loss of generality we may call the five vertices 0, a, b, c, and d, with $0<a<b<c<d$. Then the set of edges, $E=\{a,b,c,d,b-a,c-a,d-a,c-b,d-b,d-c\}$, must be a subset of one of the cosets. Since $a \neq 0$, multiplication by a^{-1} is allowed. Set $B=ba^{-1}$, $C=ca^{-1}$, and $D=da^{-1}$. Then the set $a^{-1}E = \{1,B,C,D,B-1,C-1,D-1,C-B,D-B,D-C\}$ must be a subset of the cubic residues. Hence if we find an L for which there does not exist B,C, and D such that $a^{-1}E$ is a subset of the cubic residues, then we can conclude that

$R(5,5,5) > L$. Of course, this argument holds for $R(\underbrace{t,t,\ldots,t}_{k\ times})$ for any k, and any t provided -1 is a k^{th} residue and $k \mid L-1$.

To search for these prime lower bounds, we employ the Maple package RES (available from the author's website[1]). We only achieved results when we restricted our search to fields of prime order (although any finite field can be explored using RES (or at least easily modified to do so)). Since we are considering the number $R(5,5,5)$, reject any prime, q, for which 3 does not divide $q-1$. This can be accomplished automatically by using the procedure `pryme`. By using the procedure `res` we produce all of the cubic residues of $\mathbf{Z}_p^*$, for a given prime, p. We then use the procedure `siv` to discard any residue, R, for which $R-1$ is not a residue. We now have a much more manageable list to search. Calling the procedure `diffcheck`, we check all possible 3-sets (for B, C, and D) to determine whether a 3-set with all differences between any two elements *all* being cubic residues exists or not. If such a 3-set exists, `diffcheck` will output the first 3-set it finds. However, in the event that no such 3-set exists, `diffcheck` will output 1, meaning a lower bounds has been established.

RES can also be used to search finite fields whose order is not prime. For example, to verify that the field on 2^4 elements, avoids a monochromatic triangle by using cubic residues (this fact was proven in [GG]), type `GalField3(2,4,3)`.

By using RES we were able to find the following lower bounds: $R(5,5,5) \geq 242$ and $R(6,6,6) \geq 692$. These are obtained by the following colorings: (Since -1 is a cubic residue it suffices to list only entries up to 120 for $R(5,5,5)$ and 345 for $R(6,6,6)$.)

[1] www.math.temple.edu/~aaron/

$R(5,5,5) > 241$:

Color 1: 1, 5, 6, 8, 17, 21, 23, 25, 26, 27, 28, 30, 33, 36, 40, 41, 43, 44, 47, 48, 57, 61, 64, 73, 76, 79, 85, 87, 91, 93, 98, 101, 102, 103, 105, 106, 111, 115, 116, 117

Color 2: 2, 7, 9, 10, 11, 12, 16, 19, 29, 31, 34, 35, 37, 39, 42, 45, 46, 50, 52, 54, 55, 56, 59, 60, 66, 67, 71, 72, 80, 82, 83, 86, 88, 89, 94, 95, 96, 113, 114, 119

Color 3: 3, 4, 13, 14, 15, 18, 20, 22, 24, 32, 38, 49, 51, 53, 58, 62, 63, 65, 68, 69, 70, 74, 75, 77, 78, 81, 84, 90, 92, 97, 99, 100, 104, 107, 108, 109, 110, 112, 118, 120

$R(6,6,6) > 691$:

Color 1: 1, 2, 4, 5, 8, 10, 16, 19, 20, 21, 25, 27, 31, 32, 33, 38, 39, 40, 42, 50, 51, 54, 62, 64, 66, 67, 69, 71, 73, 76, 78, 80, 83, 84, 87, 89, 95, 100, 102, 105, 107, 108, 109, 123, 124, 125, 128, 132, 134, 135, 138, 139, 142, 146, 149, 151, 152, 155, 156, 160, 163, 165, 166, 168, 173, 174, 178, 179, 181, 190, 191, 195, 199, 200, 204, 210, 214, 216, 218, 246, 248, 250, 255, 256, 259, 263, 264, 268, 270, 271, 276, 278, 283, 284, 291, 292, 293, 298, 301, 302, 304, 309, 310, 311, 312, 320, 326, 329, 330, 332, 333, 335, 336, 343, 345

Color 2: 7, 9, 11, 13, 14, 17, 18, 22, 23, 26, 28, 29, 34, 35, 36, 41, 44, 45, 46, 52, 55, 56, 58, 65, 68, 70, 72, 82, 85, 88, 90, 92, 97, 103, 104, 110, 111, 112, 115, 116, 127, 129, 130, 131, 133, 136, 140, 141, 144, 145, 147, 159, 164, 167, 170, 171, 175, 176, 177, 180, 183, 184, 189, 194, 197, 205, 206, 208, 209, 217, 220, 222, 224, 225, 227, 229, 230, 231, 232, 233, 237, 241, 243, 247, 251, 254, 257, 258, 260, 262, 266, 272, 273, 275, 279, 280, 281, 282, 288, 290, 294, 297, 303, 313, 318, 323, 325, 328, 331, 334, 337, 339, 340, 341, 342

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2.3 On $R(3,3,3,k_1,\ldots,k_r)$

We now turn our attention to finding a general lower bound for the Ramsey numbers in the title of this section. Let $N = R(k_1, k_2, \ldots, k_r)$. The Ramsey Property implies that there must exist a graph on $N-1$ vertices which avoids the Ramsey Property. Using such a graph, along with the construction in [C], we will extend Fan Chung's result to prove that, for any natural number r and for any $k_i \geq 3$, $i = 1, 2, \ldots r$,

$$R(3,3,3,k_1,k_2,\ldots,k_r) \geq 3R(3,3,k_1,k_2,\ldots,k_r) + R(k_1,k_2,\ldots,k_r) - 3.$$

We prove the above inequality via a construction. Fix $r \geq 1$ and $k_i \geq 3$ for $i = 1, 2, \ldots r$. Let $M = R(3,3,k_1,k_2,\ldots k_r) - 1$. Ramsey's Theorem proves the existence of a graph, S, on M vertices which avoids the Ramsey Property. Call the incidence matrix of this graph $T_{r+2} = T_{r+2}(x_0, x_1, x_2, \ldots, x_{r+2})$. The x_i, for $i = 1, 2, \ldots r+2$, are the $r+2$ colors. Since any diagonal entry in T_{r+2} does not represent an edge of K_M, we place x_0 along the diagonal of T_{r+2} and nowhere else. We note that the order of the x_i, for $i = 0, 1, 2, \ldots, r+2$, is extremely important; by the definition of S, and the order of the x_i's we mean that there is no x_1-colored nor x_2-colored triangles, and no x_{i+2}-colored K_{k_i}, for $i = 1, 2, \ldots r$.

Let the colors be $1, 2, \ldots, r+3$. Consider the following slightly modified construction from [C] in Figure 2.1 below. By permuting the colors of the $(r+2)$-colored graph S we can construct T_{r+3},

$$T_{r+3}(0,1,2,\ldots,r+3) = \begin{array}{cccc} A & & & \\ D & B & & \\ E & F & C & \\ 1,\ldots,1 & 2,\ldots,2 & 3,\ldots,3 & \\ \vdots & \vdots & \vdots & G \\ 1,\ldots,1 & 2,\ldots,2 & 3,\ldots,3 & \end{array}$$

Figure 2.1: Block Incidence Matrix Associated With $R(3,3,3,k_1,\ldots,k_r)$

the incidence matrix of a $(r+3)$-colored graph H on $3M + R(k_1, k_2, \ldots, k_r) - 1$ vertices, where

$$A = T_{r+2}(0,2,3,4,5,\ldots,r+3)$$
$$B = T_{r+2}(0,3,1,4,5,\ldots,r+3)$$
$$C = T_{r+2}(0,1,2,4,5,\ldots,r+3)$$
$$D = T_{r+2}(3,2,1,4,5,\ldots,r+3)$$
$$E = T_{r+2}(2,1,3,4,5,\ldots,r+3)$$
$$F = T_{r+2}(1,3,2,4,5,\ldots,r+3)$$

and G is any matrix on $R(k_1, k_2, \ldots, k_r) - 1$ vertices in the colors 4 through $r+3$ which avoids the Ramsey Property.

Using Fan Chung's result in [C] where she proves that $R_s(3) \geq 3R_{s-2}(3) + R_{s-3}(3) - 3$ (where $R_s(3) = R(3,3,\ldots,3)$ with s 3's) we see that the $(r+3)$-colored graph H avoids a 1-colored, 2-colored, and 3-colored triangle. We now argue that no $(j+3)$-colored K_{k_j} exists in H for $j = 1, 2, \ldots, r$: Assume there exists a J-colored K_{k_J} in H, for some J between 4 and $r+3$. Then there must exist $\binom{k_J}{2}$ entries in H all of value J which represent the edges of the

J-colored K_{k_J}. First note that if one of these entries is in G, then *all* entries must be in G as there are no entries of value J in the columns to the left of G. Hence the $\binom{k_J}{2}$ entries are in the submatrix Q, consisting of the first $3M$ rows and the first $3M$ columns of T_{r+3}. However, by the construction of T_{r+3} we see that *all* of these entries' coordinates can be taken modulo M, since the entries of value J in each block are in exactly the same relative positions as in the upper left block A. We further note that if (s,t) and (u,v) are two of the entries' coordinates in question, then $(s,t) \not\equiv (u,v) \pmod{M}$ (componentwise). If we had $s \equiv u \pmod{M}$, then since (u,s) must also have the same value as (u,v) we would have $(u,s) \equiv (u,u) \pmod{M}$. This implies that the entry J is on the diagonal of $A = T_{r+2}(0,2,3,4,\ldots,r+3)$, a contradiction. Hence, if we have a J-colored K_{k_J} in the submatrix Q then there must be a J-colored K_{k_J} in $A = T_{r+2}(0,2,3,4,\ldots,r+3)$, contradicting the definition of T_{r+2}.

Remark 2.1 *Up to the renaming of colors and vertices, the above permutation configuration of colors which defines T_4 is the* **only** *configuration which will avoid monochromatic triangles. This was shown by an exhaustive search of all permutations of four colors.*

Using the above observation we give 6 new lower bounds for multicolored Ramsey numbers. Currently in Radziszowski's Dynamic Survey [R], we have that $R(3,3,3,4) \geq 87$, due to Exoo [E2]. By applying the result of this section to $R(3,3,3,4)$ and using the fact that $R(3,3,4) \geq 30$ [K], we get the new lower bound: $R(3,3,3,4) \geq 91$. Using the bound $R(3,3,5) \geq 45$ [E3,KLR], we get the bound $R(3,3,3,5) \geq 137$. Finally, using $R(3,3,6) \geq 54$, $R(3,3,7) \geq 72$, $R(3,3,9) \geq 110$, and $R(3,3,11) \geq 138$ all from [SLZL], we get $R(3,3,3,6) \geq 165$, $R(3,3,3,7) \geq 220$, $R(3,3,3,9) \geq 336$, and $R(3,3,3,11) \geq 422$.

We also note that since $R(3,3,3,4) \leq R(3,3,R(3,4)) = R(3,3,9)$ (see [B]) we have $R(3,3,9) \geq 91$ which beats the previous best lower bound of 90 [LS].

2.4 On $R(3,k,l)$

We now extend Fan Chung's construction in [C] in a different direction. Inspired by her incidence matrix construction, we search for other constructions which give lower bounds for general multicolored Ramsey numbers. The progress made in this direction so far is summed up in the following theorem.

Theorem 2.1 $R(3,k,l) \geq 4R(k,l-2) - 3$

Proof: Let $M = R(k,l-2) - 1$ and let G be a 2-coloring of K_M which contains no red K_k and no blue K_{l-2}. Further, let $T(x_0, x_1, x_2)$ be the incidence matrix for G, where the x_0 entries are only along the diagonal, and there is no x_1-colored K_k and no x_2-colored K_{l-2}.

We will prove the above theorem via a construction. We will show that the matrix in Figure 2.2, S, is an incidence matrix on $4M$ vertices which avoids a 1-colored K_3, a 2-colored K_k, and a 3-colored K_l. We then conclude that $R(3,k,l) \geq 4R(k,l-2) - 3$. Define S as:

$$S = \begin{array}{llll} \widehat{A} & & & \\ B & A & & \\ C & C & A & \\ \widehat{C} & C & \widehat{B} & A \end{array}$$

Figure 2.2: Block Incidence Matrix Associated with $R(3,k,l)$

where $A = \widehat{A} = T(0,2,3)$, $B = \widehat{B} = T(3,2,1)$, and $C = \widehat{C} = T(1,2,3)$. The argument below shows that S avoids the above-mentioned monochromatic complete graphs.

No 1-colored K_3: To have a 1-colored triangle we must have three entries' coordinates (i,j), (k,j), and (k,i) all of value 1. Without loss of generality we may assume $j < i < k$. For any 1-colored triangle we would have either $(i,j) \equiv (k,j) \pmod M$ (componentwise), while $k \not\equiv i \pmod M$, or $(k,i) \equiv (k,j) \pmod M$, while $i \not\equiv j \pmod M$; both clear contradictions.

No 2-colored K_k: Since all 2 entries are in the same relative position, any $\binom{k}{2}$ entries' coordinates representing the edges of a K_k can be taken modulo M. Hence, there exists a 2-colored K_k in S if and only if there exists a 2-colored K_k in $T(0,2,3)$. By definition of $T(0,2,3)$, no 2-colored K_k can exist in S.

No 3-colored K_l: Without loss of generality we can consider the structure of the $\binom{l}{2}$ entries representing the edges of a K_l as pictured in Figure 2.3 below.

$$
\begin{array}{lllll}
\bullet(i_0,j) & & & & \\
\bullet(i_1,j) & \bullet(i_1,i_0) & & & \\
\bullet(i_2,j) & \bullet(i_2,i_0) & \bullet(i_2,i_1) & & \\
\vdots & \vdots & \vdots & \ddots & \\
\bullet(i_{l-2},j) & \bullet(i_{l-2},i_0) & \bullet(i_{l-2},i_1) & \dots & \bullet(i_{l-2},i_{l-3})
\end{array}
$$

Figure 2.3: Structure of a K_l Subgraph

If there is no entry in neither B nor $\widehat{B}$, then by taking all entries' coordinates modulo M we would have a 3-colored K_l in $\widehat{A}$, a contradiction of the definition of $\widehat{A}$. If we have entries in either B or $\widehat{B}$, or both, then we have three cases:

Case I: $(i,j) \in B$ but no $(k,l) \in \widehat{B}$. Delete the entire column of 3's containing (i,j) (the leftmost column in the picture above). By taking the remaining $\binom{l-1}{2}$ 3 entries' coordinates modulo M, we get a 3-colored $K_{l-1} \in \widehat{A}$, again a contradiction of the definition of $\widehat{A}$.

Case II: $(i,j) \in \widehat{B}$ but no $(k,l) \in B$. Delete the entire row of 3's containing (i,j) (the bottom row in the picture above). By taking the remaining $\binom{l-1}{2}$ 3 entries' coordinates modulo M, we get a 3-colored $K_{l-1} \in \widehat{A}$, a contradiction of the definition of $\widehat{A}$.

Case III: $(i,j) \in \widehat{B}$ and $(k,l) \in B$. Delete the entire column of 3's containing (i,j) and also the entire row of 3's containing (k,l). By taking the remaining $\binom{l-2}{2}$ 3 entries' coordinates modulo M, we get a 3-colored $K_{l-2} \in \widehat{A}$, again a contradiction of the definition of $\widehat{A}$.

This completes the proof, thereby giving the general bound $R(3,k,l) \geq 4R(k,l-2)-3$. Since $R(3,9)=36$, this general result shows that $R(3,3,11) \geq 141$, which beats the previous best bound of 138 found in [SLZL]. Coupling this new result with the general bound in section 1.3 we see that $R(3,3,3,11) \geq 431$.

Corollary 2.1 $R(3,k_1,k_2,\ldots,k_r) \geq 4R(k_1-2,k_2,\ldots,k_r)-3$

Proof: Let $A=\widehat{A}=T(0,2,3,4,5,\ldots,r+1)$, $B=\widehat{B}=T(3,2,1,4,5,\ldots,r+1)$ and $C=\widehat{C}=T(1,2,3,4,5,\ldots,r+1)$. Using the same construction as found in the proof of Theorem 2.1, we argue that there is no j-colored K_{k_j}, for $j = 4,5,\ldots,r+1$, by appealing to the argument proving no 2-colored K_k.

2.5 Difference Ramsey Numbers

Here we will find good lower bounds for the classical Ramsey numbers. To accomplish this, we find edgewise colorings of complete graphs which avoid the Ramsey Property. Our approach is to construct a recursive algorithm to find the best possible colorings among those colorings we search. Since searching *all* possible colorings of a complete graph on any nontrivial number of vertices is not feasible by today's computing standards, we must restrict the class of colored graphs to be searched. The class of graphs we will search will be the class of *difference graphs.*

Definition 2.2 *Difference Graph:* Consider the complete graph on n vertices, K_n. Number the vertices 1 through n. Let $i < j$ be two vertices of K_n. Let B_n be a set of arbitrary integers between 1 and $n-1$. Call B_n the set of blue differences on n vertices. We now color the edges of K_n as follows: if $j - i \in B_n$ then color the edge connecting i and j blue, otherwise color the edge red. The resulting colored graph will be called a difference graph.

Given k and l, a difference graph with the maximal number of vertices which avoids both a blue K_k and a red K_l will be called a *maximal difference Ramsey graph.* Let the number of vertices of a maximal difference Ramsey graph be V. Then we will define the *difference Ramsey number*, denoted $D(k, l)$, to be $V + 1$.

Definition 2.3 *Difference Ramsey Numbers:* The difference Ramsey number, denoted $D(k, l)$, is the minimal integer such that any difference graph must either contain a red K_k subgraph or a blue K_l subgraph.

Since the class of difference graphs is a subclass of all two-colored complete graphs, we have that $D(k, l) \leq R(k, l)$. Hence, by finding the difference Ramsey numbers, we are finding lower bounds for the classical Ramsey numbers.

Before we present the computational aspect of these difference Ramsey numbers, we establish that the recursive inequality $D(k, l) \leq D(k-1, l) + D(k, l-1)$, which is analogous to the upper bound derived from Ramsey's proof [GRS p. 3], does *not* easily follow from Ramsey's proof.

To see this consider the difference Ramsey number $D(3, 3) = 6$. Let the set of red differences be $R_6 = \{1, 2, 4\}$ (and thus the set of blue differences is

$B_6 = \{3,5\}$). Call this difference graph D_6. In Ramsey's proof, a vertex v is isolated. The next step is to notice that, regardless of the choice of v, the number of red edges from v to $D_6 \setminus \{v\} \geq D(2,3) = 3$. Let G be the graph which has each vertex connected to the vertex v by a red edge. If $v \in \{1,6\}$ then G has 3 vertices, otherwise it has 4 vertices. Either way, the number of vertices of G is at least $D(2,3) = 3$.

In order for Ramsey's argument to work in our difference graph situation, we must show that G is isomorphic to a difference graph. Assume there exists an isomorphism, $\phi : \{1,2,3,4,5,6\} \longrightarrow \{1,2,3,4,5,6\}$, such that the vertex set of G, is mapped onto $\{1,2,3\}$ or $\{1,2,3,4\}$ (depending on the number of vertices of G), and the edge coloring is preserved. Then $\phi(G)$ would be a difference graph. Notice now that $\phi(\{v\}) \in \{4,5,6\}$. For any choice of $\phi(\{v\})$ we obtain the contradiction that the difference 1 must be both red and blue (for different edges). Hence, no such isomorphism can exist, and thus we cannot use the difference Ramsey number property to conclude that the inequality holds.

However, the difference Ramsey numbers seem to be, for small values, quite close to the Ramsey numbers. This may just be a case of the Law of Small Numbers, but numerical evidence from this section leads us to make the following conjecture.

Conjecture 1: $D(k,l) \leq D(k-1,l) + D(k,l-1)$

The set of difference graphs is a superclass of the often searched cyclic (or circular) graphs (see the survey [CG] by Chung and Grinstead), which are similarly defined. The distinction is that, using the notation above, for a graph to be cyclic we require that if $b \in B_n$, then we must have $n - b \in B_n$. By removing this cycliç condition, we remove from the coloring the dependence on n, and can thereby construct a *recursive* algorithm to find the set of maximal difference Ramsey graphs.

The recursive step in the algorithm is described as follows. A difference graph on n vertices consists of B_n, the set of blue differences, and R_n, the set of red differences. Thus $B_n \cup R_n = \{1, 2, 3, \ldots, n-1\}$. To obtain a difference graph on $n+1$ vertices, we consider the difference $d = n$. If $B_n \cup \{d\}$ avoids a red clique, then we have a difference graph on $n+1$ vertices where $B_{n+1} = B_n \cup \{d\}$ and $R_{n+1} = R_n$. (Note that now $B_{n+1} \cup R_{n+1} = \{1, 2, 3, \ldots, n\}$.) Likewise, if $R_n \cup \{d\}$ avoids a blue clique, then we have a different difference graph on $n+1$ vertices with $B_{n+1} = B_n$ and $R_{n+1} = R_n \cup \{d\}$. Hence, we have a simple recursion which is not possible with cyclic colorings. (By increasing the number of vertices from n to $n+1$, a cyclic graph may become noncyclic (for some $b \in B_n$, we have that $n - b \notin B_n$)). We can now use our recursive algorithm to find *automatically* (and we must note theoretically due to time and memory constraints, but **much** less time and memory than would be required to search all graphs) all maximal difference Ramsey graphs for any given k and l.

2.5.1 About the Maple Package AUTORAMSEY

AUTORAMSEY is a Maple package that *automatically* computes all difference graphs with the maximum number of vertices that avoid both a blue K_k and a red K_l. Hence, this package *automatically* finds lower bounds for the Ramsey number $R(k, l)$. In the spirit of automation, and to take another step towards AI, AUTORAMSEY can create a verification Maple program tailored to the maximal graph(s) calculated in AUTORAMSEY (that can be run at your leisure) and can *write* a LaTeX paper giving the lower bound for the Ramsey number $R(k,l)$ along with a maximal difference graph that avoids both a blue K_k and a red K_l.

The computer generated program is a straightforward program that can be used to (double) check that the results obtained in `AUTORAMSEY` do indeed avoid both a blue K_k and a red K_l. Further, this program can be easily altered (with instructions on how to do so) to search two-colored complete graphs for k-cliques and l-anticliques.

`AUTORAMSEY` has also been translated into Fortran77 as `DF.f` to speed up the algorithm implementation. The code for the translated programs (dependent upon the clique sizes we are trying to avoid) is available for download from the author's webpage[2].

2.5.2 The Algorithm

Below, we give the pseudocode for finding the maximal difference Ramsey graph(s). In turn, it will also find the exact value of the difference Ramsey numbers $D(k,l)$. Because the number of difference graphs is of order 2^n as compared to $2^{n^2/2}$ for all colored graphs, the algorithm can feasibly work on larger Ramsey numbers.

Let D_n be the class of difference graphs on n vertices. Let `GoodSet` be the set of difference graphs that avoid both a blue K_k and a red K_l.

Set $\mathrm{m} = \min(\mathrm{k}, \mathrm{l})$

Find $\mathrm{D_{m-1}}$, our starting point.

Set GoodSet $= \mathrm{D_{m-1}}$.

Set $\mathrm{j} = \mathrm{m} - 1$

WHILE flag $\neq 0$ do

[2]www.math.temple.edu/~aaron/

FOR i from 1 to | GoodSet | do

Take T ∈ GoodSet, where T is of the form $T = [B_j, R_j]$

where B_j and R_j are the blue and red difference sets on j vertices

Consider $S_B = [B_j \cup \{j\}, R_j]$ and $S_R = [B_j, R_j \cup \{j\}]$

If S_B avoids both a blue K_k and a red K_l then

NewGoodSet := NewGoodSet ∪ S_B

If S_R avoids both a blue K_k and a red K_l then

NewGoodSet := NewGoodSet ∪ S_R

Repeat FOR loop with a new T

If | NewGoodSet |= 0 then RETURN GoodSet and set flag = 0

Else, set GoodSet = NewGoodSet, NewGoodSet = {}, and j = j + 1

Repeat WHILE loop

For this algorithm to be efficient we must have the subroutine which checks whether or not a monochromatic clique is avoided be very quick. We use the following lemma to achieve quick results in the Fortran77 code. (The Maple code is mainly for separately checking (with a different, much slower, but more straightforward, algorithm) the Fortran77 code for small cases.)

Lemma 2.1 *Define the binary operation* $*$ *to be* $x*y = |x-y|$. *Let* D *be a set of differences. If* D *contains a* k*-clique, then there exists* $K \subset D$, *with* $|K| = k-1$, *such that for all* $x, y \in K$, $x*y \in D$.

Proof: We will prove the contrapositive. Let $K = \{d_1, d_2, \ldots, d_{k-1}\}$. Order and rename the elements of K so that $d_1 < d_2 < \ldots < d_{k-1}$. Let $v_0 < v_1 < \ldots < v_{k-1}$ be the vertices of a k-set where $d_i = v_i - v_0$. By supposition, there exists $I < J$

such that $d_J * d_I = d_J - d_I \notin D$. This is the edge connecting v_J with v_I. Since this edge is not in D, D contains no k-clique.

By using this lemma we need only check *pairs* of elements in a k-set, rather than constructing all possible colorings using the k-set. Further, we need not worry about the ordering of the pairs; the operation $*$ is commutative.

2.5.3 Some Results

It is easy to find lower bounds for $R(k,l)$, so we must show that the algorithm gives "good" lower bounds. Below are two tables of the difference Ramsey number results obtained so far. The first table is of the difference Ramsey number values. The second table is of the number of maximal difference Ramsey graphs found. For these we make no claim of nonisomorphism. If we are considering the diagonal Ramsey number $R(k,k)$, then the number of maximal difference graphs takes into account the symmetry of colors; i.e. we do not count a reversal of colors as a different difference graph. Where lower bounds are listed we have made constraints on the size of the set `GoodSet` in the algorithm due to memory and/or (self-imposed) time restrictions.

When we compare our test results to the well known maximal Ramsey graphs for $R(3,3)$, $R(3,4)$, $R(3,5)$, $R(4,4)$ [GG], and $R(4,5)$ [MR], we find that the program has found the critical colorings for all of these numbers. The classical coloring in [GRS] for $R(3,4)$ is not a difference graph, and hence is not found by the program. More importantly, however, is that the program does find a difference graph on 8 vertices that avoids both a blue K_3 and a red K_4. Hence, for the Ramsey numbers found by Gleason and Greenwood [GG], and for $R(4,5)$ found by McKay and Radziszowski [MR] we have found critical Ramsey graphs which are also difference graphs.

Table 2.1: Difference Ramsey Numbers

k \ l	3	4	5	6	7	8	9	10	11
3	6	9	14	17	22	27	36	39	46
4		18	25	34	47	≥ 53	≥ 62		
5			42	≥ 57					

Table 2.2: Number of Maximal Difference Ramsey Graphs

k \ l	3	4	5	6	7	8	9	10	11
3	1	2	3	7	13	13	4	21	6
4		1	6	24	21	n/a	n/a		
5			11	n/a					

The algorithm presented above can be trivially extended to search difference graphs with more than two colors. The progress made so far in this direction follows.

2.5.4 Multicolored Difference Ramsey Numbers

The algorithm presented here can be applied to an arbitrary number of colors. To change from two to three colors, the recursive step in the algorithm simply becomes the addition of the next difference to each of the three color sets B_n, R_n, and G_n (G for green). Everything else remains the same. Hence, the alteration of the program to any number of colors is a simple one. The main hurdle encountered while searching difference graphs of more than two colors is that the size of the set `GoodSet` in the algorithm grows *very* quickly. In fact, the system's memory while fully searching all difference graphs was consumed within seconds for most multicolored difference Ramsey numbers. However, we have been able to obtain the following 3-colored Ramsey number results:

$$D(3,3,3) = 15 \qquad D(3,3,4) = 30$$
$$D(3,3,5) = 42 \qquad D(3,3,6) \geq 60$$

We note here that $D(3,3,6) \geq 60$ implies that $R(3,3,6) \geq 60$, which is a new result. The previous best lower bound was 54 [SLZL]. The coloring on 59 vertices is cyclic, hence we need only list the differences up to 29:

Table 2.3: Critical Coloring Which Shows $R(3,3,6) \geq 60$

Color Number	Differences Colored
1	5,12,13,14,16,20,22
2	10,15,19,24,26,27
3	1,2,3,4,6,7,8,9,11,17,18,21,23,25,28,29

Coupling the result $R(3,3,6) \geq 60$ with the general bound from section 1.3 we find that $R(3,3,3,6) \geq 183$.

2.5.5 Future Directions

Currently the algorithm which searches for the maximal difference Ramsey graphs is a straightforward search. If the memory requirement exceeds the space in the computer, the algorithm will only return a lower bound. In the future this algorithm should be adapted to backtrack searches or network searching. For a backtrack search we would keep track of the difference for which the memory barrier is reached and then start splitting up the search. This would create a tree structure. We then check all leaves on this tree and choose the maximal graph.

For network searching, the same type of backtrack algorithm would be used except that difference branches of the tree would be sent to different computers. This would be much quicker, but of course would cost much more in computer facilities.

CHAPTER 3
SCHUR TRIPLES AND ASSOCIATED NUMBERS

3.1 Introduction

In 1892, Hilbert proved what is perhaps the first result in Ramsey theory. His result, however, did not spark interest in the field. It was not until 1916, when Issai Schur proved the following theorem that the spark of activity in Ramsey Theory was ignited.

Schur's Theorem: *Given r, there exists an integer $N = N(r)$ such that any r-coloring of the integers 1 through N must admit a monochromatic solution to $x + y = z$.*

In this chapter we investigate the asymptotic properties of monochromatic Schur triples, as well as some related numbers. These related numbers include van der Waerden triples and Issai numbers, an extension of the Schur numbers.

3.2 On the Asymptotic Behavior of Schur Triples

Schur's theorem assures us that any r-coloring of the integers must have a minimal N_r so that $[1, N_r]$, is *guaranteed* to contain a *monochromatic* Schur triple

$\{i, j, i+j\}$. Since monochromatic Schur triples are unavoidable, one may still want to be able to color $[1, n]$ in such a way as to *minimize* their number. In this article we answer a question raised in [GRR], by showing that, asymptotically, the minimum number of monochromatic Schur triples that can occur in any 2-coloring of the integers from 1 to n is $n^2/22 + O(n)$. It is easy to see that this is achieved by coloring $[1, 4n/11]$ and $[10n/11, n]$ blue while coloring the rest, i.e. $(4n/11, 10n/11)$ red. This coloring was discovered by Zeilberger [Z].

3.2.1 Formulation

Color the integers 1 through n either red or blue. If an integer, m, is colored blue we shall say $color(m) = 0$, if it is colored red we shall say $color(m) = 1$. We are now searching for integers $1 \leq x < y < z \leq n$ such that $x + y = z$ and either x, y, and z are all colored blue or x, y, and z are all colored red.

Define the following events on $[n] = \{1, 2, \ldots, n\}$ for $(i, j, k) \in \{0, 1\}^3$:

$$\alpha_{i,j,k} = \mid \{(a, b, c) \in [n] : a + b = c \text{ and } color(a) = i, color(b) = j, color(c) = k\} \mid .$$

Our goal here is to find an expression for $\alpha_{0,0,0} + \alpha_{1,1,1}$. To this end we count the following sums of the above events: $\alpha_{i,j,0} + \alpha_{i,j,1}$, $\alpha_{i,0,j} + \alpha_{i,1,j}$, and $\alpha_{0,i,j} + \alpha_{1,i,j}$ with $(i, j) \in \{0, 1\}^2$.

We proceed by calculating upper and lower bounds for the sums of events defined above. Using the programs LBeqns and UBeqns in the maple package SCHUR (available from the author's website[1]). the following bounds can be verified. Let $x_i = 0$ if $color(i) = 0$, $x_i = 1$ if $color(i) = 1$, and define $\bar{x_i} = 1 - x_i$.

1. $\sum_{i=1}^{\frac{n-1}{2}} \sum_{j=i+1}^{n-i} \bar{x}_i \bar{x}_j \leq \alpha_{0,0,0} + \alpha_{0,0,1} \leq \sum_{i=1}^{\frac{n}{2}} \sum_{j=i+1}^{n-i+1} \bar{x}_i \bar{x}_j$
2. $\sum_{i=1}^{\frac{n-1}{2}} \sum_{j=i+1}^{n-i} \bar{x}_i x_j \leq \alpha_{0,1,0} + \alpha_{0,1,1} \leq \sum_{i=1}^{\frac{n}{2}} \sum_{j=i+1}^{n-i+1} \bar{x}_i x_j$

[1] www.math.temple.edu/~aaron/

3. $\sum_{i=1}^{\frac{n-1}{2}} \sum_{j=i+1}^{n-i} x_i\bar{x}_j \leq \alpha_{1,0,0} + \alpha_{1,0,1} \leq \sum_{i=1}^{\frac{n}{2}} \sum_{j=i+1}^{n-i+1} x_i\bar{x}_j$

4. $\sum_{i=1}^{\frac{n-1}{2}} \sum_{j=i+1}^{n-i} x_i x_j \leq \alpha_{1,1,0} + \alpha_{1,1,1} \leq \sum_{i=1}^{\frac{n}{2}} \sum_{j=i+1}^{n-i+1} x_i x_j$

5. $\sum_{i=3}^{n-1} \sum_{j=i+1}^{min(2(i-1),n)} \bar{x}_i\bar{x}_j \leq \alpha_{0,0,0} + \alpha_{1,0,0} \leq \sum_{i=1}^{n-1} \sum_{j=i+1}^{min(2i,n)} \bar{x}_i\bar{x}_j$

6. $\sum_{i=3}^{n-1} \sum_{j=i+1}^{min(2(i-1),n)} \bar{x}_i x_j \leq \alpha_{0,0,1} + \alpha_{1,0,1} \leq \sum_{i=1}^{n-1} \sum_{j=i+1}^{min(2i,n)} \bar{x}_i x_j$

7. $\sum_{i=3}^{n-1} \sum_{j=i+1}^{min(2(i-1),n)} x_i\bar{x}_j \leq \alpha_{0,1,0} + \alpha_{1,1,0} \leq \sum_{i=1}^{n-1} \sum_{j=i+1}^{min(2i,n)} x_i\bar{x}_j$

8. $\sum_{i=3}^{n-1} \sum_{j=i+1}^{min(2(i-1),n)} x_i x_j \leq \alpha_{0,1,1} + \alpha_{1,1,1} \leq \sum_{i=1}^{n-1} \sum_{j=i+1}^{min(2i,n)} x_i x_j$

9. $\sum_{i=1}^{\frac{n-1}{2}} \sum_{j=2i+1}^{n} \bar{x}_i\bar{x}_j \leq \alpha_{0,0,0} + \alpha_{0,1,0} \leq \sum_{i=1}^{\frac{n}{2}} \sum_{j=2i-1}^{n} \bar{x}_i\bar{x}_j$

10. $\sum_{i=1}^{\frac{n-1}{2}} \sum_{j=2i+1}^{n} \bar{x}_i x_j \leq \alpha_{0,0,1} + \alpha_{0,1,1} \leq \sum_{i=1}^{\frac{n}{2}} \sum_{j=2i-1}^{n} \bar{x}_i x_j$

11. $\sum_{i=1}^{\frac{n-1}{2}} \sum_{j=2i+1}^{n} x_i\bar{x}_j \leq \alpha_{1,0,0} + \alpha_{1,1,0} \leq \sum_{i=1}^{\frac{n}{2}} \sum_{j=2i-1}^{n} x_i\bar{x}_j$

12. $\sum_{i=1}^{\frac{n-1}{2}} \sum_{j=2i+1}^{n} x_i x_j \leq \alpha_{1,0,1} + \alpha_{1,1,1} \leq \sum_{i=1}^{\frac{n}{2}} \sum_{j=2i-1}^{n} x_i x_j$

We now note that for each of the above 12 inequality ranges the difference between the lower bound and the upper bound is $O(n)$. Hence we can conclude that the following equations must hold:

E_1. $\alpha_{0,0,0} + \alpha_{0,0,1} = \sum_{i=1}^{\frac{n-1}{2}} \sum_{j=i+1}^{n-i} \bar{x}_i\bar{x}_j + O(n)$

E_2. $\alpha_{0,1,0} + \alpha_{0,1,1} = \sum_{i=1}^{\frac{n-1}{2}} \sum_{j=i+1}^{n-i} \bar{x}_i x_j + O(n)$

E_3. $\alpha_{1,0,0} + \alpha_{1,0,1} = \sum_{i=1}^{\frac{n-1}{2}} \sum_{j=i+1}^{n-i} x_i\bar{x}_j + O(n)$

E_4. $\alpha_{1,1,0} + \alpha_{1,1,1} = \sum_{i=1}^{\frac{n-1}{2}} \sum_{j=i+1}^{n-i} x_i x_j + O(n)$

E_5. $\alpha_{0,0,0} + \alpha_{1,0,0} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{min(2(i-1),n)} \bar{x}_i\bar{x}_j + O(n)$

E_6. $\alpha_{0,0,1} + \alpha_{1,0,1} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{min(2(i-1),n)} \bar{x}_i x_j + O(n)$

E_7. $\alpha_{0,1,0} + \alpha_{1,1,0} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{min(2(i-1),n)} x_i\bar{x}_j + O(n)$

E_8. $\alpha_{0,1,1} + \alpha_{1,1,1} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{min(2(i-1),n)} x_i x_j + O(n)$

E_9. $\alpha_{0,0,0} + \alpha_{0,1,0} = \sum_{i=1}^{\frac{n-1}{2}} \sum_{j=2i+1}^{n} \bar{x}_i\bar{x}_j + O(n)$

E_{10}. $\alpha_{0,0,1} + \alpha_{0,1,1} = \sum_{i=1}^{\frac{n-1}{2}} \sum_{j=2i+1}^{n} \bar{x}_i x_j + O(n)$

E_{11}. $\alpha_{1,0,0} + \alpha_{1,1,0} = \sum_{i=1}^{\frac{n-1}{2}} \sum_{j=2i+1}^{n} x_i\bar{x}_j + O(n)$

E_{12}. $\alpha_{1,0,1} + \alpha_{1,1,1} = \sum_{i=1}^{\frac{n-1}{2}} \sum_{j=2i+1}^{n} x_i x_j + O(n)$

We now have a system of 12 equations with 8 unknowns. Written in matrix form we have $Ax = b$ with $x = [\alpha_{0,0,0}\ \alpha_{0,0,1}\ \alpha_{0,1,0}\ \alpha_{1,0,0}\ \alpha_{0,1,1}\ \alpha_{1,0,1}\ \alpha_{1,1,0}\ \alpha_{1,1,1}]^T$ and $b = [rhs(E_1)\ rhs(E_2)\ rhs(E_3)\ ...\ rhs(E_{12})]^T$, where $rhs(E_i)$ is the right hand side of equation E_i. Solving this system for the number of monochromatic Schur triples, i.e. $\alpha_{0,0,0}+\alpha_{1,1,1}$, we find a particular solution, $x_{par} = rhs(E_1)-rhs(E_6)+rhs(E_{12})$. Hence, all solutions of this system are $x = x_{par} + c(kernel(A))$, where c is any constant. Since we want $\alpha_{0,0,0}+\alpha_{1,1,1}$ and we find that the kernel of A is $[-1\ 1\ 1\ 1\ -1\ -1\ -1\ 1]^T$ we see that $\alpha_{0,0,0}+\alpha_{1,1,1} = rhs(E_1) - rhs(E_6) + rhs(E_{12}) - c + c = rhs(E_1) - rhs(E_6) + rhs(E_{12})$ for any choice of c. Hence we have found an expression for the number of monochromatic Schur triples of any 2-coloring of [n]:

$$\sum_{i=1}^{\frac{n-1}{2}} \sum_{j=i+1}^{n-i} \bar{x_i}\bar{x_j} - \sum_{i=1}^{n-1} \sum_{j=i+1}^{min(2(i-1),n)} \bar{x_i}x_j + \sum_{i=1}^{\frac{n-1}{2}} \sum_{j=2i+1}^{n} x_i x_j + O(n)$$

We will now derive an equivalent expression for the number of monochromatic Schur triples using the expression above. By symmetry of colors we can let $x_i = \bar{x_i}$. Keeping in mind that all simplification can be done modulo $O(n)$, a routine calculation shows that the above expression is equivalent to the following:

$$\sum_{i=1}^{\frac{n}{2}} \sum_{j=i+1}^{n-i} x_i x_j - \sum_{i=1}^{\frac{n}{2}} \sum_{j=2i+1}^{n} x_j + \sum_{i=1}^{n} \sum_{j=i+1}^{n} x_i x_j - \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} x_i + \frac{n^2}{4} + O(n)$$

Let $k = \sum_{i=1}^{n} x_i$. Using the facts that $\sum_{i=1}^{n}\sum_{j=i+1}^{n} x_i x_j = \binom{k}{2}$ and $\sum_{i=1}^{\frac{n}{2}}\sum_{j=2i+1}^{n} x_j = \sum_{i=1}^{n} \lfloor \frac{i-1}{2} \rfloor x_j$ we see that the above is equivalent to:

$$\sum_{i=1}^{\frac{n}{2}} \sum_{j=i+1}^{n-i} x_i x_j + \sum_{i=1}^{n} \left\lfloor \frac{i}{2} \right\rfloor x_j + \frac{k^2}{2} - kn + \frac{n^2}{4} + O(n)$$

3.2.2 Reformulation

We can reformulate the problem of minimizing the number of monochromatic Schur triples into the following 'Advanced Discrete Calculus' problem.

Let

$$F(x_1, \ldots, x_n) = \sum_{i=1}^{\frac{n}{2}} \sum_{j=i+1}^{n-i} x_i x_j + \sum_{i=1}^{n} \left\lfloor \frac{i}{2} \right\rfloor x_j + \frac{k^2}{2} - kn + \frac{n^2}{4}.$$

Find the global minimum of F over the n-dimensional (discrete) unit cube, i.e. over $\{0,1\}^n$. To this end, we make the following definition.

Definition 3.1 *Discrete partial derivative of* $f(x_1, \ldots, x_n)$*:* For any function, $f(x_1, \ldots, x_n)$, the discrete partial derivative $\partial_r f$ is defined to be

$$\partial_r f(x_1, \ldots, x_r, \ldots, x_n) := f(x_1, \ldots, x_r, \ldots, x_n) - f(x_1, \ldots, 1 - x_r, \ldots, x_n)\,.$$

In this situation, a global minimum must also be a local minimum (in the Hamming metric). Let $(z_1, \ldots, z_n)$ be a local minimum. Then obviously we have the n inequalities

$$\partial_r F(z_1, \ldots, z_n) \leq 0 \quad \text{for } 1 \leq \text{r} \leq \text{n} \quad .$$

Noting that $(2x_r - 1)^2 \equiv 1$ and $(2x_r - 1)x_r \equiv x_r$ on $\{0,1\}^n$, a routine calculation shows that for $1 \leq r \leq n$,

$$\partial_r F(x_1, \ldots, x_n) = (2x_r - 1)\left\{\sum_{i=1}^{n} x_i + \sum_{i=1}^{n-r} x_i - (n - \lfloor \frac{r}{2} \rfloor) - \frac{1}{2} I_{\{r \leq \frac{n}{2}\}}\right\} - \frac{1}{2} I_{\{r \leq \frac{n}{2}\}} - \frac{1}{2}$$

where I_S is the indicator function on the set S. Fix k, $1 \leq k \leq n$, and focus on those $(z_1, \ldots, z_n)$ for which $\sum_{i=1}^{n} z_i = k$. By symmetry we may assume that $k \geq n/2$.

Since at a local minimum $(z_1, \ldots, z_n)$, $\partial_r G(z_1, \ldots, z_n) \leq 0$ precisely when

$$(2z_r - 1)\left\{k - n + \lfloor r/2 \rfloor + \sum_{j=1}^{n-r} z_j - 1/2\right\} \leq 1 \qquad \text{for } 1 \leq r \leq n/2$$
$$(2z_r - 1)\left\{k - n + \lfloor r/2 \rfloor + \sum_{j=1}^{n-r} z_j\right\} \leq 1/2 \qquad \text{for } n/2 < r \leq n$$

it follows that any local minimum $(z_1, \ldots, z_n)$ satisfies the following Ping-Pong Recurrence.

3.2.3 Ping-Pong Recurrence

Choose $a, b \in \{0, 1\}$ arbitrarily each time $\widehat{H}$ or $\widetilde{H}$ is used, where $\widehat{H}$ and $\widetilde{H}$ are the following functions:

$$\widehat{H}(y) := \begin{cases} 0, & \text{if } y > 1/2; \\ 1, & \text{if } y < 0; \\ a, & \text{if } 0 \leq y \leq 1/2. \end{cases}$$
$$\widetilde{H}(y) := \begin{cases} 0, & \text{if } y > 1; \\ 1, & \text{if } y < -1; \\ b, & \text{if } -1 \leq y \leq 1. \end{cases}$$

Then we must have, for $r = n, n-1, \ldots, n - \lfloor n/2 \rfloor + 1$,

$$z_r = \widehat{H}\left(k - n + \left\lfloor \frac{r}{2} \right\rfloor + \sum_{j=1}^{n-r} z_j\right), \qquad \text{(Right Volley)}$$
$$z_{n-r+1} = \widehat{H}\left(2k - n - 1/2 + \left\lfloor \frac{n-r+1}{2} \right\rfloor - \sum_{j=r}^{n} z_j\right), \qquad \text{(Left Volley)}$$

and if n is odd then $z_{(n+1)/2} = \widehat{H}(k - n + \lfloor \frac{n+1}{4} \rfloor + \sum_{j=1}^{(n-1)/2} z_j)$.

These equations determine a solution (depending upon the choices of the a's and b's made along the way), z (if it exists), in the order $z_n, z_1, z_{n-1}, z_2, \ldots$. When we solve the Ping-Pong recurrence we forget the fact that $\sum_{i=1}^{n} z_i = k$. Most of the time a solution will not satisfy this last condition, but when it does, we have a genuine local minimum. Note that *any* local minimum must show up in this way, and hence any global minimum as well.

By analyzing the output of the Maple routine `ptor2` in the Maple package `RON` (available from the author's website[2]), we are able to find the following solutions, *for n sufficiently large*, to the Ping-Pong recurrence. Stealing notation from the theory of formal languages, for any word (or letter) W, W^m means 'W repeated m times'.

Let $w = 2k - n$, $k \neq n/2$ (this case must be dealt with separately). By symmetry we may assume that $k \geq n/2$. Then $0 < w \leq n$. If $w \geq n/2$ then the only solution is 0^n. If $w < n/2$, then let s be the unique integer $0 \leq s < \infty$, that satisfies $n/(12s + 14) \leq w < n/(12s + 2)$.

Case I: If $n/8 \leq w < n/2$ then the solutions are $0^{\lfloor n/2 \rfloor} 1^{n - \lfloor n/2 \rfloor - w - c_1} 0^{w + c_1}$, where $c_1 \in \{-1, 0, 1\}$.

[2] www.math.temple.edu/~aaron/

Case II: If $n/(12s+8) \leq w < n/(12s+2)$ then the solutions are

$$\begin{cases} 0^{4w+c_1} 1^{\lfloor n/2 \rfloor - 4w - c_1} 0^{n - \lfloor n/2 \rfloor - 7w - (c_2+c_3+c_4)} 1^{6w+c_3} 0^{w+c_4} & \text{for } s = 1; \\ 0^{4w+c_4} (1^{6w+c_5^{s_i}} 0^{6w+c_6^{s_i}})^{s/2} Q (0^{6w+c_7^{s_i}} 1^{6w+c_8^{s_i}})^{s/2} 0^{w+c_9} & \text{for } s > 1. \end{cases}$$

where the c_j's and $c_j^{s_i}$'s are bounded constants (independent of n) and Q can be an (almost) arbitrary mix of r zeroes and ones (where r is the unique integer such that the length of this interval is n). Further, the number of ones in Q is at most $12w$. Notation: (1) the $c_j^{s_i}$'s can be different constants with i ranging from 1 to $s/2$; (2) if s is odd $(ab)^{s/2}$ is $(ab)^{(s-1)/2}a$.

Case III: If $n/(12s+14) \leq w < n/(12s+8)$ then the solutions are

$$\begin{cases} 0^{4w+d_1} 1^{n-5w-(d_1+d_2)} 0^{w+d_2} & \text{for } s = 0; \\ 0^{4w+d_3} (1^{6w+d_4^{s_i}} 0^{6w+d_5^{s_i}})^{s/2} Q (0^{6w+d_6^{s_i}} 1^{6w+d_7^{s_i}})^{s/2} 0^{w+d_8} & \text{for } s > 0. \end{cases}$$

where the d_j's and $d_j^{s_i}$'s are bounded constants and Q can be an (almost) arbitrary mix of r zeroes and ones, with the number of ones in Q at most $6w$.

Case IV: if $w = 0$ (i.e. $s = \infty$), the solutions are:

$$0^{g_1} (1^{g_2^{n_i}} 0^{g_3^{n_i}})^{n/(2G_1)} Q (0^{g_4^{n_i}} 1^{g_5^{n_i}})^{n/(2G_2)}$$

where $g_1 \in \{0, 1, 2\}$, the other g_i's and $g_i^{n_i}$'s are bounded between 3 and 11, Q is an (almost) arbitrary mix of r zeroes and ones with the number of ones bounded between 0 and 22, $G_1 = \sum_i (g_2^{n_i} + g_3^{n_i})$, and $G_2 = \sum_i (g_4^{n_i} + g_5^{n_i})$.

The verification that these are in fact the solutions is a routine verification. All that is required is to plug the answers back in, and verify that they satisfy the Ping-Pong recurrence.

Now it is time to impose the extra condition that $\sum_{i=1}^{n} z_i = k$ $(= (w+n)/2)$. With Cases I and II a routine calculation yields a contradiction of the applicable range of w when n is sufficiently large. For Case III, a routine calculation yields a local minimum of $w = n/11$ if $s = 0$. If $s > 0$ argue as follows. Let t be the number of 1's in Q. Recall that r is the total number of 0's and 1's in Q. Let $w_c(s) = n/(12s + c)$ where we must have $8 \leq c \leq 14$. Since we need $\sum_{i=1}^{n} z_i = k$ $(= (w+n)/2)$, we see that $6w_c(s)s+t = n(12s+c+1)/(24s+2c)$ gives $t = (c+1)w_c(s)/2$. Further, since the number of 1's in Q is bounded by $6w_c(s)$, we find that we must have $c \leq 11$. We also must have $r = n - w_c(s)(12s+5)$, by the definition of r. Using the simple inequality $r \geq t$, we have $n - w_c(s)(12s + 5) \geq (c + 1)w_c(s)/2$. From this deduce that $c \geq 11$. Hence we must have $c = 11$ at a local minimum. Thus the local minima for Case III, $s > 0$, are $w_s = n/(12s+11)$. Case IV gives infinitely many local minima.

We are now ready to give the colorings which will give a locally minimum value for the number of monochromatic Schur triples. Since the global minimum must occur as a local minimum, we need only determine the minimum of these local minima. The local minima are asymptotically equivalent (mod $O(n)$) to

$$\begin{cases} Z_s := 0^{4w_s}(1^{6w_s}0^{6w_s})^{\frac{s}{2}}1^{6w_s}(0^{6w_s}1^{6w_s})^{\frac{s}{2}}0^{w_s} & \text{for } 0 \leq s < \infty \text{ (with } w_s := \frac{n}{12s+11}), \\ Z_\infty^t = (0^t1^t)^{n/(2t)} & \text{for } 3 \leq t \leq 11 \end{cases}$$

We now evaluate F to calculate the number of monochromatic Schur triples at each of these Z_j. Recall that

$$F(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n/2} \sum_{j=i+1}^{n-i} x_i x_j + \sum_{i=1}^{n} \left\lfloor \frac{i}{2} \right\rfloor x_j + \frac{k^2}{2} - kn + \frac{n^2}{4} + O(n)$$

For $0 \leq s < \infty$ we break F down as follows:

$$\sum_{i=1}^{n/2} \sum_{j=i+1}^{n-i} x_i x_j = \sum_{i=1}^{\lfloor \frac{s+1}{2} \rfloor} \sum_{j=(12i-8)w_s}^{(12i-2)w_s} \left\{ \sum_{l=j+1}^{(12i-2)w_s} 1 + 6w_s(s-(2i-1)) + \sum_{l=n-(12i-5)w_s}^{max(n-j, n-(12i-5)w_s)} 1 \right\}$$

$$+2\left(\frac{j+1}{2} - \left\lfloor \frac{j+1}{2} \right\rfloor\right) \sum_{i=(n-3)w_s/2}^{n/2} \sum_{j=i+1}^{n-i} 1 .$$

A routine calculation shows that $\sum_{i=1}^{n/2} \sum_{j=i+1}^{n-i} x_i x_j = \frac{9}{4}(4s^2 + 5s + 1)w_s^2 + O(n)$. Using $\sum_{i=a}^{b} \lfloor \frac{i}{2} \rfloor = (b^2 - a^2)/4 + O(n)$ we see that

$$\sum_{i=1}^{n} \left\lfloor \frac{i}{2} \right\rfloor x_j = \sum_{i=1}^{s} \sum_{j=(12i-8)w_s}^{(12i-2)w_s} \left\lfloor \frac{j}{2} \right\rfloor + \sum_{i=n-7w_s}^{n-w_s} \left\lfloor \frac{i}{2} \right\rfloor$$

is equal to $(18s^2+39s+21)w_s^2+O(n)$. Since $k = (w_s+n)/2$, a routine simplification shows that for $0 \leq s < \infty$

$$F(Z_j) = \frac{12s+8}{16(12s+11)} n^2 + O(n) \quad ,$$

which is strictly increasing in s. At $s = 0$ we get $n^2/22 + O(n)$ (and as $s \to \infty$ we get $n^2/16 + O(n)$). Further, another routine calculation shows that $Z_\infty^t = n^2/16 + O(n)$ for any natural number t (which, of course, is the above limit).

3.2.4 The Answer

The above arguments of this section have shown that the coloring found by Zeilberger is in fact the global minimum. In our notation,

$$Z_0 = 0^{4n/11}\, 1^{6n/11}\, 0^{n/11}$$

sets the world-record minimum value of $n^2/22 + O(n)$ monochromatic Schur triples.

3.2.5 A Lower Bound for the r-color Case

Here we show that our result implies a good upper bound for the general r-coloring of the first n integers. If we r-color the integers (with colors $C_1 \dots C_r$) from 1 to n then the minimum number of monochromatic Schur triples is bounded above by

$$\frac{n^2}{2^{2r-3}11} + O(n).$$

This comes from the following coloring:

$$\begin{cases} Color(i) = C_j & \text{if } \frac{n}{2^j} < i \leq \frac{n}{2^{j-1}} \quad \text{for } 1 \leq j \leq r-2, \\ Color(i) = C_{r-1} & \text{if } 1 \leq i \leq \frac{4n}{2^{r-2}11} \text{ or } \frac{10n}{2^{r-2}11} < i \leq \frac{n}{2^{r-2}}, \\ Color(i) = C_r & \text{if } \frac{4n}{2^{r-2}11} < i \leq \frac{10n}{2^{r-2}11}. \end{cases}$$

The coloring above is constructed to insure that there are no C_j-colored Schur triples for $1 \leq j \leq r-2$, and that there is a minimal number of C_{r-1} and C_r colored Schur triples in the interval $[1, n/2^{r-2}]$.

3.3 Issai Numbers

We may extend Schur's theorem to the following theorem:

Theorem 3.1 *Given r and k, there exists an integer $N = N(r,k)$ such that any r-coloring of the integers 1 through N must admit a monochromatic solution to $\sum_{i=1}^{k-1} x_i = x_k$.*

This is not a new theorem. In fact it is a special case of Rado's Theorem [GRS p. 56]. We will, however, present a simple proof which relies only on Ramsey's Theorem and the notions already presented in this dissertation.

Proof: Consider the r-colored difference Ramsey number $N = D(\underbrace{k,k,\ldots,k}_{r\ times})$. Then any r-coloring of K_N must have a monochromatic K_k subgraph. Let the vertices of this subgraph be $\{v_0, v_1, \ldots, v_{k-1}\}$, with the differences $d_i = v_i - v_0$. By ordering and renaming we may assume that $d_1 < d_2 < \ldots < d_{k-1}$. Since K_k is monochromatic, we have that the edges $\overline{v_{i-1}v_i}$, for $i = 1, 2, \ldots, k-1$, and $\overline{v_{k-1}v_0}$ must all be the same color. Since the r-colored K_N is a difference graph we have that d_1, d_{k-1}, and $(d_{i+1} - d_i)$, for $i = 1, 2, \ldots, k-1$, must all be assigned the same color. Hence we have the monochromatic solution $d_1 + \sum_{i=1}^{k-2} (d_{i+1} - d_i) = d_{k-1}$.

Using this theorem we will define *Issai numbers.* But first, another definition is in order.

Definition 3.2 *Schur k-tuple:* We will call a k-tuple, $(x_1, x_2, \ldots, x_k)$, a Schur k-tuple if $\sum_{i=1}^{k-1} x_i = x_k$.

In the case where $k = 3$, the 3-tuple $(x, y, x+y)$ is called a Schur triple. In Schur's Theorem the only parameter is r, the number of colors. Hence, a Schur number is defined to be the minimal integer $S = S(r)$ such that any r-coloring of the integers 1 through S must contain a monochromatic Schur triple. It is known that $S(2) = 5$, $S(3) = 14$, and $S(4) = 45$. The Schur numbers have been generalized in [BB] and [Sch] in directions different from what will be presented here. We will extend the Schur numbers in the same fashion as the Ramsey numbers were extended from $R(k,k)$ to $R(k,l)$.

Definition 3.3 *Issai Number:* Let $S = S(k_1, k_2, \ldots, k_r)$ be the minimal integer such that any r-coloring of the integers from 1 to S must have a monochromatic Schur k_i-tuple, for some $i \in \{1, 2, \ldots, r\}$. S will be called an Issai number.

The existence of these Issai numbers is implied by the existence of the difference Ramsey numbers $D(k_1, k_2, \ldots, k_r)$. In fact, we have the following result:

Lemma 3.1 $S(k_1, k_2, \ldots, k_r) \leq D(k_1, k_2, \ldots, k_r) - 1$

Proof: By definition of a difference Ramsey number, there exists a minimal integer $N = D(k_1, k_2, \ldots, k_r)$ such that any r-coloring of K_N must contain a monochromatic K_{k_i}, for some $i \in \{1, 2, \ldots, r\}$. Using the same reasoning as in the proof of Theorem 1 and the fact that the differences in the difference graph are $1, 2, \ldots, N-1$, we have the stated inequality.

Using this new definition and notation, it is already known that $S(3,3) = 5$, $S(3,3,3) = 14$, and $S(3,3,3,3) = 45$. We note here that since $D(3,3,3) = 15$ we immediately have $S(3,3,3) \leq 14$, whereas before, since $R(3,3,3) = 17$, we had only that $S(3,3,3) \leq 16$.

Attempts to find a general bound for $S(k,l)$ have thus far been unsuccessful. The values in Table 3.1 (below) leads us to make the following seemingly trivial conjecture:

Conjecture 2: $S(k-1, l) \leq S(k, l)$

The difficulty here is that a monochromatic Schur k-tuple in no way implies the existence of a monochromatic Schur $(k-1)$-tuple. To see this, consider the following coloring of $\{1,2,\ldots,9\}$. Color $\{1,3,5,9\}$ red, and the other integers blue. Then we have the red Schur 4-tuple $(1,3,5,9)$. However no red Schur triple exists in this coloring.

Below we give some Issai number values and exceptional colorings. We used the Maple package ISSAI (available from the author's webpage[3]) to calculate the exact values as well as an exceptional coloring given below. ISSAI is written for two colors, but can easily be extended to any number of colors. The value $S(3,3) = 5$ has been known since before Schur proved his theorem. The value $S(4,4) = 11$ follows from Beutelspacher and Brestovansky in [BB], who more generally show that $S(k,k) = k^2 - k - 1$. The remaining values are new.

Table 3.1: Issai Numbers

k \ l	3	4	5	6	7
3	5	7	11	13	≥ 17
4		11	14		

The exceptional colorings found by ISSAI are as follows. Let $S(k,l)$ denote the minimal number such that any 2-coloring of the integers from 1 to $S(k,l)$ must contain either a red Schur k-tuple or a blue Schur l-tuple. It is enough to list only those integers colored red:

[3]www.math.temple.edu/~aaron/

Table 3.2: Critical Colorings of Some Issai Numbers

Issai Number	Lower Bound	Integers Colored Red
S(3,4)	7	1,6
S(3,5)	11	1,3,8,10
S(4,4)	11	1,2,9,10
S(3,6)	13	1,3,10,12
S(4,5)	14	1,2,12,13
S(3,7)	17	1,3,5,12,14,16

3.4 On the Asymptotic Behavior of van der Waerden Triples

In this last section we turn our attention briefly to a related triple, which we have called a van der Waerden triple, named after B. van der Waerden who, in 1927, published the following theorem, conjectured by Schur.

Van der Waerden's Theorem: *For all natural numbers k and r, there exists a minimal integers $W(k,r)$ such that if $\{1,2,\ldots,W(k,r)\}$ is r-colored, then there must exist a monochromatic k-term arithmetic progression.*

Very few exact values for these van der Waerden numbers are known. In this section we will give upper and lower bounds for the asymptotic minimum number of monochromatic 3-term arithmetic progressions, which we call *van der Waerden triples*, in any 2-coloring of the first n natural numbers.

First we note that the minimum number of monochromatic van der Waerden triples is bounded above by $n^2/16 + O(n)$. This comes from the coloring $(0011)^{n/4}$ (where, again, W^m means W repeated m times). For this, the only monochromatic van der Waerden triples we have are of the form $(i, i+4d, i+8d)$. Hence we have $\sum_{i=1}^{n-8}\sum_{d=1}^{(n-i)/8} 1$ monochromatic van der Waerden triples, which gives the desired upper bound.

Turning our attention to finding a lower bound for the minimum number of monochromatic van der Waerden triples, consider the complete graph on n vertices, K_n. Color the edges of K_n two colors. Goodman [G] proved that this colored graph must contain $n^3/24 + O(n^2)$ monochromatic triangles. Let (a, b, c) be a van der Waerden triple. Using a technique in [GRR], we define a map, γ, from the set of triangles to the set of vdw-triples:

$$\gamma : \{i < j < k\} \longmapsto \begin{cases} (k + i - 2j, k - j, k - i) & \text{if } j - i < k - j \\ (2j - k - i, j - i, k - i) & \text{if } k - j < j - i \end{cases}$$

provided that $k - j \neq j - i$.

Observe that the van der Waerden triple (a, b, c) can be associated to at most $2(n - c)$ triangles. Further, we have that for each (a, b, c) there are $c/2$ van der Waerden triples with last term c. These facts, coupled with Goodman's result, imply that there are strictly more than $n^2/38 + O(n)$ monochromatic van der Waerden triples.

Hence, the minimum number, asymptotically, of monochromatic van der Waerden triples in any 2-coloring of $\{1, 2, \ldots, n\}$ is bounded between $n^2/38+O(n)$ and $n^2/16 + O(n)$.

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