Real Analysis Ph.D. Qualifying Exam
Mathematics, Temple University
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Part I. (Do 3 problems)

1. Let $f_n(x) = n \sin\left(\frac{x}{n}\right)$. Prove that:
   (a) f_n converges uniformly on any finite interval.
   Hint: $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$ for all x.
   (b) f_n does not converge uniformly on $\mathbb{R}$.
   (c) f_n does not converge in measure on $\mathbb{R}$.
   Hint: show that the interval $(n\pi, (n+1)\pi)$ is contained in the set $\{x : |f_n(x) - x| > \epsilon\}$.

2. Let $E \subset \mathbb{R}^n$. The function $f : E \to \mathbb{R}$ is upper semicontinuous at $x_0 \in E$ if for each $\epsilon > 0$ there exists $\delta > 0$ such that $f(x) \leq f(x_0) + \epsilon$ for all $|x - x_0| < \delta$, $x \in E$.

   If f is upper semicontinuous in E compact, then prove that
   (a) f is bounded above in E,
   (b) f attains its maximum in E, that is there exists $x_0 \in E$ such that $f(x_0) = \sup_{x \in E} f(x)$.

3. Let μ be a Borel measure in $\mathbb{R}$ with $\mu(\mathbb{R}) < \infty$. Define $f(x) = \mu((-\infty, x])$ for $x \in \mathbb{R}$. Prove that
   (a) f is monotone increasing
   (b) $\mu((a, b]) = f(b) - f(a)$; for $a < b$
   (c) f is continuous from the right
   (d) $\lim_{x \to -\infty} f(x) = 0$.

4. Let $|E| < \infty$ and let X denote the class of Lebesgue measurable functions in E. Given $f, g \in X$ define
$$d(f, g) = \int_E \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|}\, dx.$$
   Prove that
   (a) (X, d) is a metric space
   (b) $f_k \to f$ in (X, d) if and only if $f_k \to f$ in measure in E.

   HINT: the function $\alpha(x) = \frac{x}{1+x}$ is increasing in $[0, \infty)$ and satisfies $\alpha(x + y) \leq \alpha(x) + \alpha(y)$.

Part II. (Do 2 problems)

1. Let $1 \le r < p < \infty$ and $0 < \lambda < 1$ with $\dfrac{\lambda}{r} + \dfrac{1-\lambda}{s} = \dfrac{1}{p}$. Prove the inequality

$$\|f\|_p \le \|f\|_r^{\lambda}\, \|f\|_s^{1-\lambda},$$

   where $\|\cdot\|_p$ denotes the L^p-norm of f over E.

2. Let $f \in L^1(E)$ with $|E| < \infty$ and define the sequence

$$f_n(x) = \frac{1}{n}\log\left(1 + e^{nf(x)}\right).$$

   Prove that

$$\lim_{n\to\infty}\int_E f_n(x)\,dx = \int_E f^+(x)\,dx.$$

   Proceed in steps as follows proving:

   (a) $0 \le f_n(x) \le |f(x)| + \log 2$. Hint: $0 < e^{nf(x)} \le e^{n|f(x)|}$ and $e^{n|f(x)|} \ge 1$

   (b) if $f(x) \ge 0$, then $f_n(x) \to f(x)$. Hint: write $f_n(x) = \dfrac{1}{n}\log\left(e^{nf(x)}\left(1 + e^{-nf(x)}\right)\right)$

   (c) if $f(x) < 0$, then $f_n(x) \to 0$.

   (d) the function $|f(x)| + \log 2 \in L^1(E)$, and conclude using Lebesgue dominated convergence theorem.

3. If λ, μ are measures on a sigma algebra of sets Σ, recall that λ is absolutely continuous with respect to μ, written $\lambda \ll \mu$, if $A \in \Sigma$ and $\mu(A) = 0$ then $\lambda(A) = 0$; and λ is singular with respect to μ, written $\lambda \perp \mu$, if there exists $A \in \Sigma$ with $\lambda(A) = 0$ and $\mu(A^c) = 0$.

   Let λ_1, λ_2 and μ be measures on a sigma algebra Σ. Prove that

   (a) If $\lambda_1 \ll \mu$ and $\lambda_2 \ll \mu$, then $\lambda_1 + \lambda_1 \ll \mu$

   (b) If $\lambda_1 \ll \mu$ and $\lambda_2 \perp \mu$, then $\lambda_1 \perp \lambda_2$

   (c) If $\lambda_1 \ll \mu$ and $\lambda_1 \perp \mu$, then $\lambda_1 = 0$.