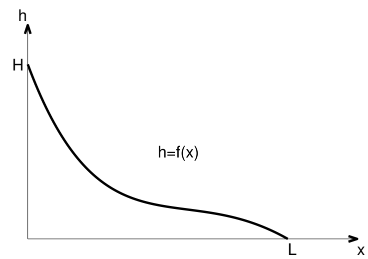
Applied Mathematics Qualifying Written Exam (January 2024)

1 Part I: do 3 of 4

1. A point mass m slides down along a frictionless slide, whose shape is described by the equation $h = f(x)$, where h is the height above the ground level of the point on the slide at the distance x from the origin, as in the figure. The function $f : [0, L] \to [0, H]$ is a smooth monotone decreasing function.
   (a) Introduce the degrees of freedom for this mechanical system and write down the action functional
   (b) Write down the Euler-Lagrange equation (equation of motion) for this mechanical system. **You don't need to solve this equation.**



2. Consider the problem of optimal control of the rocket railroad car, governed by the equation
$$m\ddot{x}(t) = \alpha(t), \quad |\alpha(t)| \le a_{\max},$$
where the goal is to bring the car to a complete stop at the origin in shortest time.
   (a) Write down the control theory Hamiltonian for this problem
   (b) Formulate the Pontryagin's maximum principle for this problem. **Your formulation of the Pontryagin's maximum principle should be self-contained, i.e. it should include the primary and adjoint state dynamics. You don't need to solve any equations or analyze the consequences of the maximum principle.**

3. Eliminate all parameters in Fisher's reaction-diffusion equation
$$\frac{\partial u}{\partial t} = ru\left(1 - \frac{u}{K}\right) + D\frac{\partial^2 u}{\partial x^2},$$
by appropriate scaling, where $r > 0$, $K > 0$ and $D > 0$.

4. What is the appropriate inner product for the Sturm-Liouville eigenvalue problem?
$$-xy'' + (x-1)y' = \lambda y, \qquad x \in (0, +\infty).$$

2 Part II: do 2 of 3

1. Use dimensional analysis to give a formula for the period of revolution of a single planet around a star. Assume that the planet is in a circular orbit.

2. Consider a singularly perturbed differential equation

$$\epsilon(1+x^2)y''(x) + y'(x) + y(x) = 0, \quad y(0) = 1,\ y(1) = 2,$$

   whose solution has a boundary layer at $x = 0$, as $\epsilon \to 0^+$.

   (a) Find the outer solution $y_{\text{out}}(x)$.

   (b) Find an inner solution $y_{\text{in}}(x)$.

   (c) Find the uniform approximation $Y(x)$ for the solution on $[0, 1]$ as $\epsilon \to 0^+$.

3. Find the first two non-zero terms of the asymptotic expansion for the integral

$$f(x) = \int_x^{\infty} e^{-t^3}\, dt$$

   as $x \to +\infty$.