Numerical Analysis Qualifying Written Exam (January 2024)

Part I: do 3 of 4

1. Interpolation.
(a) Describe Chebyshev polynomials and prove that they are orthogonal in an appropriate inner product.
(b) Using the roots of these polynomials (appropriately scaled for the interval of interest) is known to be better than using equidistant points.
In what sense is this better? Give at least two different reasons.
(c) Discuss some interpolation method with which you can use these non-uniformly distributed points. Provide formulas.

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2. Floating point arithmetic and error analysis.
(a) The QR factorization of a matrix is known to be backwards stable. What does that mean for $A = QR$?
(b) Show that the floating point addition of two real numbers is backward stable. Is the same true for the subtraction of two real numbers? Why or why not?
(c) Show that in floating point arithmetic, the sum is not associative. What about the product?

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3. Absolute stability and stiff problems.
(a) For the ODE initial value problem $u'(t) = f(u(t))$, $u(0) = u_0$, formulate (i) the forward Euler method, (ii) the backward Euler method, and (iii) the Crank-Nicolson method.
(b) Calculate the order of accuracy for each of these three methods.
(c) How is the region of absolute stability of a general Runge-Kutta method defined? Calculate and sketch the regions of absolute stability for each of the three methods above.
(d) Explain what a stiff problem is. Then, via the regions of absolute stability found above, explain how well-suited (or not) each of the methods above is for stiff problems.
(e) For the ODE

$$\begin{bmatrix} u \\ v \end{bmatrix}' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} u \\ v \end{bmatrix},$$

sketch and explain which qualitative behavior each of the three methods above will generate. Relate those behaviors to the regions of absolute stability. Argue which method is best-suited for the problem, and why.

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4. Runge-Kutta methods; L-stability.
Consider the method defined by the Butcher tableau

$$\begin{array}{c|cc} 1/4 & 1/4 & \\ 1 & 2/3 & 1/3 \\ \hline & 2/3 & 1/3 \end{array}\,.$$

(a) Write out one step of the method, i.e., given U^n at t_n, find U^{n+1} at $t_n + k$.
(b) Prove that this method is second order accurate (and not third order accurate).
(c) How is the growth factor $R(z)$ of a general Runge-Kutta method defined?
(d) Show that the growth factor $R(z)$ of the specific method above has the property $\lim_{z\to\infty} R(z) = 0$.
(e) Explain one way in which a method with $\lim_{z\to\infty} R(z) \neq 0$ could yield undesirable behavior. Provide the definition of L-stability.

Part II: do 2 of 3

1. Newton's method.
Consider the Newton Method for the solution of $F(x) = 0$, when $F : \mathbb{R}^n \to \mathbb{R}^n$, i.e., the n-dimensional case.
(a) What is the best possible convergence rate for this method? Under which conditions can this convergence rate be achieved. Give details conditions on F and on the initial vector x_0.
(b) Prove that given the conditions in (a), the method has indeed the convergence rate stated in (a).
(c) Indicate what happens if each of the conditions in (a) do not hold. Indicate in which cases there is no convergence, and in which cases the convergence is of a lower rate.

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2. Solution of nonlinear equations.
Let $F_i : \mathbb{R}^2 \to \mathbb{R}^2, i = 1, 2$ given by

$$F_1(x) = \begin{bmatrix} x_1 \cos x_2 \\ x_2 \cos x_1 \end{bmatrix}, \quad F_2(x) = \begin{bmatrix} x_1^2 - x_2^2 + 1 \\ x_1^2 + x_2^2 - 1 \end{bmatrix}.$$

(a) Give conditions on these particular F_i and on the possible initial vector x_0 so that Newton's method converges for these functions if the conditions are satisfied, or show that there may not be convergence.
(b) Describe the Bisection Method for the scalar case, that is, $f : \mathbb{R} \to \mathbb{R}$. Discuss its convergence properties.
(c) How would you generalize the Bisection Method to $\mathbb{R}^2$, so you could apply it to the functions of (a)?

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3. Multistep methods; zero-stability and convergence.
For the ODE initial value problem $u'(t) = f(u(t))$, $u(0) = u_0$, $t \in [0, T]$, consider a general linear r-step method

$$\sum_{j=0}^{r} \alpha_j U^{n+j} = k \sum_{j=0}^{r} \beta_j f(U^{n+j}) \ .$$

(a) Work out the local truncation error of the method, providing at least the 3 leading terms of the expansion in powers of the time step size k.
(b) Derive which conditions the coefficients α_j and β_j must satisfy to yield a consistent method.

(c) For the special class of methods which have $\beta_0 = \beta_1 = \cdots = \beta_{r-1} = 0$, derive values for the coefficients α_0, α_1, α_2, and β_2 for the 2-step scheme that is second order accurate.
(d) Provide the definition of zero-stability, and prove that the scheme derived in (c) is zero-stable.
(e) State the general relationship between consistency, convergence, and zero-stability. Argue rigorously whether the scheme $U^{n+2} - 3U^{n+1} + 2U^n = -kf(U^n)$ is convergent or not.