Real Analysis Ph.D. Qualifying Exam
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- Justify your answers thoroughly.
- You are allowed to rely on a previous part of a multi-part problem even if you do not work out the previous part.
- For any theorem that you wish to cite, you should either give its name or a statement of the theorem.
- $\mathbb{N}$ and $\mathbb{R}$ stand for the set of natural numbers, and the set of real numbers, respectively.

Part I (Do three problems)

I.1.

(a) Show that there exists a constant $c > 0$ such that $(1 + x/n)^n \geq 1 + cx^2$ for all $x \geq 0$ and all $n \in \mathbb{N}$, $n \geq 2$.

(b) Compute $\lim_{n\to\infty} \int_0^\infty (1 + x/n)^{-n} \sin(x/n)\, dx$ and justify the calculation. Here dx denotes integration with respect to the Lebesgue measure in $\mathbb{R}$.

I.2. Let $a, b \in \mathbb{R}$ such that $a < b$, and consider a function $f : [a, b] \longrightarrow \mathbb{R}$. Show that $V(f; [a, b])$, the total variation of f on $[a, b]$, satisfies $V(f; [a, b]) = f(b) - f(a)$ if and only if f is monotonically increasing on $[a, b]$.

I.3. Let $(X, \mathfrak{M}, \mu)$ be a measure space (with the typical convention that μ is positive). Suppose f_n, g_n, for $n \in \mathbb{N}$, and f, g are real valued integrable functions on X satisfying

(a) $f_n \to f$ and $g_n \to g$, as $n \to \infty$, μ-a.e. on X,

(b) $|f_n| \leq g_n$ on X for each $n \in \mathbb{N}$,

(c) $\int_X g_n\, d\mu \to \int_X g\, d\mu$ as $n \to \infty$.

Prove that $\int_X f_n\, d\mu \to \int_X f\, d\mu$. Hint: Use Fatou's Lemma for $f_n + g_n$ and for $-f_n + g_n$.

I.4. Let $(X, \mathfrak{M}, \mu)$ be a measure space (with the typical convention that μ is positive). A collection of functions $\{f_\alpha\}_{\alpha\in A}$ in $L^1(X, \mathfrak{M}, \mu)$ is called uniformly integrable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\left|\int_E f_\alpha\, d\mu\right| < \varepsilon$ for all $\alpha \in A$ whenever $E \in \mathfrak{M}$ satisfies $\mu(E) < \delta$. Show that

(a) Any finite subset of $L^1(X, \mathfrak{M}, \mu)$ is uniformly integrable.

(b) Prove that if $\{f_n\}_{n\in\mathbb{N}}$ is a sequence in $L^1(X, \mathfrak{M}, \mu)$ that converges in the L^1 metric to $f \in L^1(X, \mathfrak{M}, \mu)$, then the collection $\{f_n\}_{n\in\mathbb{N}}$ is uniformly integrable.

Part II (Do two problems)

II.1. Let $(X, \mathfrak{M}, \mu)$ be a measure space (with the typical convention that μ is positive) and assume that $1 < p < \infty$, and that $f \in L^p(X, \mathfrak{M}, \mu)$. Suppose D is a dense subset of $L^q(X, \mathfrak{M}, \mu)$, where $1/p + 1/q = 1$. Prove that $f = 0$ μ-a.e. on X if and only if $\int_X fg\, d\mu = 0$ for all $g \in D$.

II.2. Consider the measure space $(\mathbb{R}, \mathfrak{M}, \mathcal{L}^1)$, where $\mathfrak{M}$ is the σ-algebra of Lebesgue measurable sets in $\mathbb{R}$ and $\mathcal{L}^1$ is the Lebesgue measure on $(\mathbb{R}, \mathfrak{M})$.

(a) Prove that if $f \in L^p(\mathbb{R}, \mathfrak{M}, \mathcal{L}^1)$ and $g \in L^q(\mathbb{R}, \mathfrak{M}, \mathcal{L}^1)$ for $1 \le p < \infty$ with $1/p + 1/q = 1$, then the function $h : \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$h(x) = (f * g)(x) = \int_{\mathbb{R}} f(x-y)g(y)\, d\mathcal{L}^1(y)$$

is uniformly continuous.

(b) Show that if $A \in \mathfrak{M}$ is such that $\mathcal{L}^1(A) > 0$ then $\mathbf{1}_A * \mathbf{1}_A \neq 0$ $\mathcal{L}^1$-a.e. by computing its integral (you are allowed to change order of integration without having to justify it.) Here $\mathbf{1}_A$ denotes the characteristic function of the set A.

(c) Let $A \in \mathfrak{M}$ be such that $\mathcal{L}^1(A) > 0$. Prove that the set

$$A + A = \{x \mid \exists\, a \in A, b \in A,\ x = a + b\}$$

contains an open interval.

II.3. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^1$ bijective function. Define $\nu(A) := |\varphi^{-1}(A)|$ for each Borel measurable set $A \subseteq \mathbb{R}$. Prove that ν is a Borel measure in $\mathbb{R}$ and that, with $\mathcal{L}^1$ denoting the Lebesgue measure in $\mathbb{R}$, the Radon-Nikodym derivative $d\nu/d\mathcal{L}^1$ equals $|\varphi'|$.