Real Analysis Ph D Qualifying Exam
Temple University
January 2020

Part I (Do three problems)

I.1. Give an example of an f which is not Lebsgue integrable on $\mathbb{R}$, but whose improper Riemann integral exists and is finite. Prove your answer.

I.2. Show that if $g : \mathbb{R} \to \mathbb{R}$ is continuous and compactly supported, and $f, \{f_n\} \in L^1(\mathbb{R})$ are such that $\int_{\mathbb{R}} |f_n - f| dx \to 0$ as $n \to \infty$, then $gf \in L^1(\mathbb{R})$ and $\int_{\mathbb{R}} |gf_n - gf| dx \to 0$ as $n \to \infty$.

I.3. Let $f : [a, b] \to \mathbb{R}$ be continuous. Prove that

$$\lim_{n\to\infty} \left(\int_a^b |f(x)|^n \, dx \right)^{1/n} = \sup_{x\in[a,b]} |f(x)|.$$

I.4. Consider a sequence of real-valued functions $\{f_n(x)\}$. Recall that f_n converges in measure to f if for every $\delta > 0$,

$$\lim_{n\to\infty} |\{x \in \mathbb{R} : |f_n(x) - f(x)| > \delta\}| = 0.$$

(a) Prove that if f_n converges to f in L^1 then f_n converges to f in measure.
(b) Does convergence in measure imply convergence in L^1? Justify your answer.
(c) Do there exist sequences f_n defined on $[0, 1]$ that converge in measure, to the function 0, but do not converge a.e.?

Part II (Do two problems)

II.1. Let $A \subset [0, 1]$ be a Borel set such that $0 < m(A \cap I) < m(I)$ for every interval I in $[0, 1]$. Here m denotes the Lebesgue measure. Let $F(x) = m([0, x] \cap A)$. Prove that F is absolutely continuous and strictly increasing on $[0, 1]$ and that $F' = 0$ on a set of positive measure.

II.2. Let $f(x, y)$, $0 \le x, y \le 1$, satisfy the following conditions: for each x, $f(x, y)$ is an integrable function of y, and $(\partial f(x, y)/\partial x)$ is a bounded function of (x, y). Show that $(\partial f(x, y)/\partial x)$ is a measurable function of y for each x and

$$\frac{d}{dx} \int_0^1 f(x, y) dy = \int_0^1 \frac{\partial}{\partial x} f(x, y) dy.$$

II.3. Let $\Gamma(y) = \int_0^\infty e^{-x} x^{y-1} dx$, $y > 0$.
(a) Show that Γ is continuous on $(0, \infty)$, without using Part (b).
(b) Show that $\Gamma'(y) = \int_0^\infty e^{-x} x^{y-1} \ln x dx$, $y > 0$.