Real Analysis Ph.D. Qualifying Exam
Temple University
January 11, 2019

Part I. (Do 3 problems)

1. Suppose $f_n \to f$ uniformly in E where f_n are continuous. Prove that if $x_0 \in E$ and $x_n \to x_0$ with $x_n \in E$, then $f_n(x_n) \to f(x_0)$.

2. Let $f_n(x) = n \sin\left(\dfrac{x}{n}\right)$. Prove that:

   (a) f_n converges uniformly on any finite interval.
   Hint: $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \dots$ for all x.

   (b) f_n does not converge uniformly on $\mathbb{R}$.

   (c) f_n does not converge in measure on $\mathbb{R}$. Hint: the interval $(n\pi, (n+1)\pi)$ is contained in the set $|f_n(x) - x| > \epsilon$.

3. Prove that the upper lim is sub additive and lower lim is super additive:

$$\limsup_{k\to\infty} (a_k + b_k) \leq \limsup_{k\to\infty} a_k + \limsup_{k\to\infty} b_k$$
$$\liminf_{k\to\infty} (a_k + b_k) \geq \liminf_{k\to\infty} a_k + \liminf_{k\to\infty} b_k.$$

   To avoid operations with $\pm\infty$ assume the sequences are bounded.

4. Prove that on $C[0,1]$ the norms $\|f\|_\infty = \max_{x\in[0,1]} |f(x)|$ and $\|f\|_1 = \int_0^1 |f(x)|\,dx$ are not equivalent.

Part II. (Do 2 problems)

1. Let $f \in L^p(E,\mu)$, $1 \leq p < \infty$, and $E = \cup_{j=1}^{\infty} E_j$ with $E_j \subset E_{j+1}$. Prove that $f\chi_{E_j} \to f$ in $L^p(E,\mu)$.

2. Let μ be a Borel measure in $\mathbb{R}$ with $\mu(\mathbb{R}) < \infty$. Define $f(x) = \mu((-\infty, x])$ for $x \in \mathbb{R}$. Prove that

   (a) f is monotone increasing

   (b) $\mu((a,b]) = f(b) - f(a)$; for $a < b$

   (c) f is continuous from the right

   (d) $\lim_{x\to-\infty} f(x) = 0$.

3. Let $f : [a,b] \to \mathbb{R}$ integrable. Prove that the functions $f_n(x) = \dfrac{1}{(n-1)!}\displaystyle\int_a^x (x-t)^{n-1} f(t)\,dt$ are well defined for $a \leq x \leq b$, $n = 1, 2, \cdots$ and satisfy $\displaystyle\int_a^x f_n(t)\,dt = f_{n+1}(x)$.