Real Analysis Ph.D. Qualifying Exam

Temple University

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- Justify your answers thoroughly.
- You are allowed to rely on a previous part of a multi-part problem even if you do not work out the previous part.
- Notation: $\mathbb{R}$ and $\mathbb{N}$ denote the set of real numbers and the set of natural numbers, respectively, and λ stands for the Lebesgue measure on $\mathbb{R}$.
- For any theorem that you wish to cite, you should either give its name or a statement of the theorem.

Part I. (Do 3 problems):

I.1. Show that for any Lebesgue measurable set $E \subseteq \mathbb{R}$ with $\lambda(E) = 1$ there exists a Lebesgue measurable set $A \subseteq E$ with $\lambda(A) = \frac{1}{2}$.

I.2. Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be two continuous functions such that there exists $E \subseteq \mathbb{R}$ Lebesgue measurable such that $\lambda(E) = 0$ and $f\big|_{E^c} = g\big|_{E^c}$ (here the vertical bar denotes the restriction and the superscript "c" stands for complement). Show that $f(x) = g(x)$ for each $x \in \mathbb{R}$.

I.3. For the following: if your answer is "yes", provide an example; if your answer is "no", provide a proof.

(a) Does there exist a real-valued function of a real variable that is continuous at every rational and discontinuous at every irrational?

(b) Does there exist a real-valued function of a real variable that is continuous at every irrational and discontinuous at every rational?

I.4. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions of bounded variation on the interval $[0, 1]$ and assume that $f_n \longrightarrow f$ as $n \to \infty$, pointwise on $[0, 1]$, and that there exists a finite constant $M > 0$ such that

$$V[f_n; 0, 1] \leq M, \qquad \forall\, n \in \mathbb{N}.$$

Here, if g is some real-valued function on $[0, 1]$ then $V[g; 0, 1]$ denotes the total variation of g on $[0, 1]$. Show that f is of bounded variation on $[0, 1]$ and that $V[f; 0, 1] \leq M$.

Part II. (Do 2 problems):

II.1. Let $p \in (0, \infty)$ and recall that a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is said to be in weak-$L^p(\mathbb{R})$ if f is Lebesgue measurable and there exists $C(p) > 0$ (a constant depending only on p) such that

$$\lambda\Big(\{x : |f(x)| > t\}\Big) \leq C(p)t^{-p} \quad \text{for all} \quad t > 0.$$

Show that if f lies both in weak-$L^p(\mathbb{R})$ and weak-$L^q(\mathbb{R})$ for $1 \leq p < r < q < \infty$ then $f \in L^r(\mathbb{R})$.

II.2. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a Lebesgue integrable function. Show that

$$\lim_{t\to\infty} \int_{\mathbb{R}} f(x)\cos(xt)\, d\lambda(x) = \lim_{t\to\infty} \int_{\mathbb{R}} f(x)\sin(xt)\, d\lambda(x) = 0.$$

II.3. Suppose that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a Lebesgue integrable function. Show that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x)\sin^2(nx)\, d\lambda(x) = \frac{1}{2}\int_{\mathbb{R}} f(x)\, d\lambda(x).$$