Real Analysis Ph.D. Qualifying Exam

Temple University

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- Justify your answers thoroughly.
- You are allowed to rely on a previous part of a multi-part problem even if you do not work out the previous part.
- Notation: $\mathbb{R}$ and $\mathbb{N}$ denote the set of real numbers and the set of natural numbers, respectively, and dx indicates integration with respect to the Lebesgue measure on $\mathbb{R}$.

Part I. (Do 3 problems):

I.1. Consider the sequence $\{f_n\}_{n\in\mathbb{N}}$, where for each $n \in \mathbb{N}$ the function $f_n : [0,1] \to \mathbb{R}$ is absolutely continuous and satisfies $f_n(0) = 13$ and

$$\int_{[0,1]} |f_n'|^4 \, dx \leq 7.$$

Prove that $\{f_n\}_{n\in\mathbb{N}}$ has a subsequence that converges uniformly to a continuous function on $[0,1]$.

I.2. Let $(X, \mathfrak{M}, \mu)$ be a measure space and $f : (X, \mathfrak{M}) \longrightarrow [0, \infty]$ be a measurable function such that $\displaystyle\int_X f \, d\mu = 3$. For each $n \in \mathbb{N}$ consider the function $g_n : (X, \mathfrak{M}) \longrightarrow [0, \infty]$ given by

$$g_n(x) := n \cdot \ln\Big(1 + \frac{f(x)}{n}\Big), \qquad \forall\, x \in X.$$

Show that g_n is a measurable function for each $n \in \mathbb{N}$ and that

$$\lim_{n\to\infty} \int_X g_n \, d\mu = 3.$$

I.3. Consider the sequence $\{f_n\}_{n\in\mathbb{N}}$ where for each $n \in \mathbb{N}$ the function $f_n : \mathbb{R} \longrightarrow \mathbb{R}$ is given by:

$$f_n(x) := \begin{cases} |x|, & \text{if } \frac{1}{n} \leq |x| \leq 1, \\ \frac{n}{2}x^2 + \frac{1}{2n}, & \text{if } |x| \leq \frac{1}{n}, \end{cases} \qquad x \in \mathbb{R}.$$

(1) Show that for each $\alpha > 0$ the sequence $\big\{(f_n)^\alpha\big\}_{n\in\mathbb{N}}$ converges uniformly on the interval $[-1, 1]$, and find the limit function.

(2) Show that

$$\lim_{n\to\infty} \int_{[0,1]} \big(f_n(x)\big)^\alpha \, dx,$$

exists and evaluate the limit.

(3) Show that for each $n \in \mathbb{N}$ the function f_n is differentiable on the interval $(-1, 1)$, and that the sequence $\{f_n'\}_{n\in\mathbb{N}}$ converges pointwise but not uniformly on $(-1, 1)$.

I.4. Show that if $f : [0,1] \longrightarrow \mathbb{R}$ is a Lebesgue measurable function, then given $\varepsilon > 0$, there exists a continuous function $g : [0,1] \longrightarrow \mathbb{R}$ such that $|\{x \in [0,1] \, : \, f(x) \neq g(x)\}| < \varepsilon$ (where if $E \subseteq \mathbb{R}$ is a Lebesgue measurable set then $|E|$ denotes its Lebesgue measure).

Part II. (Do 2 problems):

II.1. Let $f : [0,1] \to \mathbb{R}$ be a bounded, Lebesgue measurable function which satisfies

$$\int_{[0,1]} f(x) x^k \, dx = \frac{1}{(k+2)(k+3)} = \frac{1}{k+2} - \frac{1}{k+3}, \qquad \text{for each} \qquad k \in \mathbb{N} \cup \{0\}.$$

Show that $f(x) = x - x^2$ almost everywhere (with respect to the Lebesgue measure) on $[0,1]$.

II.2. Let $(X, \mathfrak{M}, \mu)$ be a measure space and consider $f \in L^1(X, \mathfrak{M}, \mu)$. Show that for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\displaystyle\int_E |f| \, d\mu < \varepsilon$ whenever $E \in \mathfrak{M}$ is such that $\mu(E) < \delta$.

II.3 Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of functions such that $f_n : \mathbb{R} \longrightarrow \mathbb{R}$ is Lebesgue measurable for each $n \in \mathbb{N}$. Assume that $f \in L^1(\mathbb{R})$ is such that $|f_n(x)| \leq f(x)$ for each $n \in \mathbb{N}$ and for almost every $x \in \mathbb{R}$ (with respect to the Lebesgue measure). Show that if for any bounded interval $I \subseteq \mathbb{R}$ one has $\displaystyle\lim_{n \to \infty} \int_I f_n \, dx = 0$ then $\displaystyle\lim_{n \to \infty} \int_{\mathbb{R}} f_n \, dx = 0$.