Real Analysis Ph.D. Qualifying Exam
Temple University
August 25, 2023

- Justify your answers thoroughly.
- You are allowed to rely on a previous part of a multi-part problem even if you do not work out the previous part.
- For any theorem that you wish to cite, you should either give its name or a statement of the theorem.

Part I (Do three problems)

I.1. Let $\{f_n\}$ be a sequence of measurable real-valued functions on a measure space $(X, \mathcal{A}, \mu)$.

(a) Suppose that $f_n \to f$ in measure and $|f_n| \leq g$ with $g \in L^1(d\mu)$. Show that $f_n \to f$ in $L^1(d\mu)$.
(b) Show that the result in (a) is false if the condition $|f_n| \leq g$ with $g \in L^1(d\mu)$ is omitted.

I.2. For $a > 0$, show that

$$\int_0^\infty e^{-ax} x^{-1} \sin x dx = \arctan(a^{-1})$$

by integrating $e^{-axy} \sin x$ with respect to x and y.

I.3. Let μ_* be an outer measure on the subsets of X. Prove that $E \subset X$ is Carathéodory measurable if and only if for every $\epsilon > 0$ there exists a Carathéodory measurable set O such that $E \subset O$ and $\mu_*(O \setminus E) < \epsilon$.

I.4. Let $f \in L^1(\mathbb{R})$. Define

$$F(x) = \frac{1}{2}\left(\int_{-1}^{x} f(t)dt - \int_{x}^{1} f(t)dt\right).$$

Is F absolutely continuous on $[-1, 1]$? Is $F' = f$ almost everywhere?

Part II (Do two problems)

II.1. Compute the limit $\lim_{n\to\infty} \int_0^\infty (1+(x/n))^{-n} \cos(x/n)\, dx$ and justify the computations.

II.2. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and in $L^1(\mathbb{R})$. For each of the parts (a) and (b) give a proof or a counterexample.

(a) Is it true that f is bounded on $\mathbb{R}$?
(b) Is it true that $f(x) \to 0$ as $x \to \infty$?

How do the result for (a) and (b) changes under the additional assumption that f' exists everywhere and is bounded?

II.3. Prove that if $A \subset \mathbb{R}^n$ is a measurable set with Lebesgue measure $0 < |A| < \infty$, then the set

$$A + A = \{x : \ \exists\, a, b \in A, \ x = a + b\}$$

contains an open ball.

Hint: Take for granted the following fact: if $1 \le p < \infty$, $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ with $\frac{1}{p} + \frac{1}{q} = 1$, then the convolution function $(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy$ is uniformly continuous. Take $f = g = \chi_A$, and show that $\int_{\mathbb{R}^n} (\chi_A * \chi_A)(x)\, dx > 0$; χ_A denotes the characteristic function of A.