# Ph.D. Comprehensive Examination
# Real Analysis
# August 2022

## Part I. Do three of these problems.

**I.1**. Prove that if $E$ is Lebesgue measurable subset of $\mathbb{R}^n$ with $|E| < \infty$ then for all $\epsilon > 0$ there exists a compact set $K$ such that $K \subset E$ and $|E \setminus K| < \epsilon$.

**I.2**. For each part, prove that there exists a sequence $f_n : [0,1] \to \mathbb{R}$ of measurable nonnegative functions with the properties

(a) $\displaystyle\int_0^1 f_n(x)\,dx = 1$ for all $n \in \mathbb{N}$ but $\displaystyle\lim_{n\to\infty} f_n(x) = 0$ for almost all $x \in [0,1]$;

(b) $\displaystyle\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 0$ but $\displaystyle\limsup_{n\to\infty} f_n(x) = 1$ for almost all $x \in [0,1]$.

Hint: For (b), consider using the characteristic functions of the 'cubes' $\frac{1}{2^m}[0,1] + \frac{k}{2^m}$, $m \in \mathbb{N}$, $k = 0, \ldots, 2^m - 1$.

**I.3**. Let $f : (1,\infty) \to \mathbb{R}$ be defined by

$$f(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Prove that $f$ is differentiable, and find its derivative.

**I.4**. Let $X$ be a set and let $\mathcal{M}$ be a $\sigma$-algebra of subsets of $X$. Suppose $\mu, \nu\, \lambda$ are $\sigma$-finite measures on $(X, \mathcal{M})$ with $\nu$ absolutely continuous with respect to $\mu$ and $\mu$ absolutely continuous with respect to $\lambda$.

(a) Prove that $\nu$ is absolutely continuous with respect to $\lambda$.

(b) Write the Radon-Nikodym derivative of $\nu$ with respect to $\lambda$ in terms of those of $\mu$ with respect to $\lambda$ and $\nu$ with respect to $\mu$.

## Part II. Do two of these problems.

**II.1**. (a) Let $\{f_n\}$ be a sequence of nonnegative measurable functions on $(a,b)$ such that $\sum_{n=1}^{\infty} \int_a^b f_n(x)\,dx < \infty$. Show that $\sum_{n=1}^{\infty} f_n(x)$ converges for a.e. $x$ in $(a,b)$.

(b) Suppose the series $\sum_{k=1}^{\infty} a_k$ converges absolutely. Let $g \in L^1(a,b)$ and let $\{x_k\}_{k=1}^{\infty}$ be a sequence in $(a,b)$. Show that

$$\sum_{k=1}^{\infty} a_k g(x - x_k)$$

converges absolutely for a.e. $x$ in $(a,b)$.

**II.2**. Suppose $f \in L^1(0,1)$.

(a) Let $0 < \delta < 1$. Prove that $(n+1)\int_0^{1-\delta} x^n f(x)\,dx \to 0$ as $n \to \infty$.

(b) Suppose $\lim_{x\to 1^-} f(x) = a$. Prove that $(n+1)\int_0^1 x^n f(x)\,dx \to a$ as $n \to \infty$.

**II.3**. Let $f \in L^1(\mathbb{R})$. Prove that $\int_{\mathbb{R}} \cos(nx)\, f(x)\,dx \to 0$ as $n \to \infty$.

For this problem you may use without justification integration by parts and the fact that $\{g \in C^1(\mathbb{R}) : g \text{ vanishes outside some bounded interval}\}$ is dense in $L^1(\mathbb{R})$

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Justify your answers thoroughly. For any theorem that you wish to cite, you should either give its name or a statement of the theorem.