Real Analysis Ph.D. Qualifying Exam
Temple University
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Part I. (Do 3 problems)

1. Let x_k be a sequence in a metric space (X, d) such that $\sum_{k=1}^{\infty} d(x_k, x_{k+1}) < \infty$. Prove that x_k is a Cauchy sequence.

2. Prove that the function
$$F(x) = \int_0^{+\infty} \frac{\cos(x\,t^2)}{1+t^2}\,dt$$
is well defined and is continuous for all $x \in \mathbb{R}$.

3. Let $E \subset \mathbb{R}^n$. The function $f : E \to \mathbb{R}$ is upper semicontinuous at $x_0 \in E$ if for each $\epsilon > 0$ there exists $\delta > 0$ such that $f(x) \le f(x_0) + \epsilon$ for all $|x - x_0| < \delta$, $x \in E$.

   Prove that if f is upper semicontinuous in E compact, then f is bounded above in E.

4. Prove Dini's theorem: Let X be a compact topological space. If $f_n : X \to \mathbb{R}$ is a sequence of continuous functions such that $f_n(x) \to 0$ for each $x \in X$ and $f_n(x) \ge f_{n+1}(x)$ for all x and n, then $f_n \to 0$ uniformly in X.

   HINT: for $\epsilon > 0$ consider $F_n = \{x \in X : f_n(x) < \epsilon\}$.

Part II. (Do 2 problems)

1. Let $f \in L^1(E)$. Prove that for each $\epsilon > 0$ there exists $\delta > 0$ such that for any $A, B \subset E$ measurable with $|A \triangle B| < \delta$ we have
$$\left| \int_A f(x)\,dx - \int_B f(x)\,dx \right| < \epsilon.$$

2. Suppose $f_k \to f$ a.e. on $\mathbb{R}^n$, f_k measurable. Prove that for each $\epsilon > 0$ there exist a sequence of disjoint measurable sets E_j of finite measure such that $|\mathbb{R}^n \setminus \cup_{j=1}^{\infty} E_j| < \epsilon$ and $f_k \to f$ uniformly on each E_j.

3. Let $0 < p \le q < \infty$, $f \in L^q(X, \mu)$, and $E \subset X$ with $0 < \mu(E) < \infty$. Prove that
$$\left(\frac{1}{\mu(E)} \int_E |f(x)|^p\,d\mu(x) \right)^{1/p} \le \left(\frac{1}{\mu(E)} \int_E |f(x)|^q\,d\mu(x) \right)^{1/q}.$$