Real Analysis Ph.D. Qualifying Exam
Temple University
August 2020

Part I (Do three problems)

I.1. Let g be an absolutely continuous monotone function on $[0,1]$, and $E \subseteq [0,1]$ a set of Lebesgue measure 0. Prove that $g(E)$ has measure 0.

I.2. Prove that Lebesgue measure λ on $\mathbb{R}$ is translation invariant: if A is a Lebesgue measurable subset of $\mathbb{R}$, then for each $u \in \mathbb{R}$, $u + A$ is also Lebesgue measurable and $\lambda(u + A) = \lambda(A)$.

I.3. Let $f \in L^1(\mathbb{R})$. Evaluate

$$\lim_{y\to\infty} \int_{\mathbb{R}} |f(x+y) - f(x)|dx.$$

Hint: Note that the limit is NOT as $y \to 0$.

I.4. A function is said to be **lower semi-continuous** if

$$f(x) \le \liminf_{n\to\infty} f(x_n)$$

whenever $\lim_{n\to\infty} x_n = x$. Show that every lower semi-continuous function is Borel measurable.

Part II (Do two problems)

II.1. Let (X, Σ, μ) be a measure space and $\{f_n\}$ a sequence in $L^1(d\mu)$ which converges a.e. to $f \in L^1(d\mu)$. Prove $f_n \to f$ in $L^1(d\mu)$ if and only if $\int |f_n| d\mu \to \int |f| d\mu$.

Hint: Apply Fatou's lemma to $|f| + |f_n| - |f - f_n|$.

II.2. Let (X, Σ, μ) be a finite measure space. If f is integrable, compute and justify the limit

$$\lim_{n\to\infty} \int_X |f(x)|^{1/n} d\mu(x).$$

II.3. Show that the function defined by

$$\phi(t) = \int_{-\infty}^{\infty} \frac{\sin(xt)}{1+x^4} dx \quad (t \in \mathbb{R})$$

is well-defined and differentiable on $(0, \infty)$.