Real Analysis Ph.D. Qualifying Exam
Temple University
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Part I. (Do 3 problems)

1. Let $f_n(x) = n\,x\,e^{-n\,x^2}$ on $[0, +\infty)$. Prove that

   (a) f_n converges to zero pointwise in $[0, +\infty)$

   (b) f_n does not converge uniformly in $[0, +\infty)$

   (c) f_n converges in measure on $[0, +\infty)$

   (d) $\int_0^\infty f_n(x)\,dx = \dfrac{1}{2}$.

   HINT for (c): may use that $e^z \geq z^2/2$ for all $z \geq 0$.

2. Let $f \in C^1[0, +\infty)$ such that $f(x) \to 0$ as $x \to +\infty$. Prove that

$$\int_0^\infty f(x)^2\,dx \leq 2\left(\int_0^\infty x^2 f(x)^2\,dx\right)^{1/2}\left(\int_0^\infty f'(x)^2\,dx\right)^{1/2}.$$

   HINT: write $f(x)^2 = -\int_x^\infty \left(f(t)^2\right)'\,dt$.

3. Let E be a subset of a metric space (X, d) and let $E_k = \{x \in X : \text{dist}(x, E) < 1/k\}$. Let μ be a Borel measure on (X, d) such that $\mu(C) < \infty$ for each bounded Borel set C.

   (a) Prove that E_k is open.

   (b) If E is compact, then prove that $\mu(E) = \lim_{k\to\infty} \mu(E_k)$.

4. Let $f \in L^1(0, b)$ and $g(x) = \displaystyle\int_x^b \frac{f(t)}{t}\,dt$. Prove that $g \in L^1(0, b)$ and

$$\int_0^b g(x)\,dx = \int_0^b f(t)\,dt.$$

Part II. (Do 2 problems)

1. Given $\theta \in \mathbb{R}$, let $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the mapping given by

$$T_\theta(x, y) = (x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta).$$

   Prove that $\|f \circ T_\theta - f\|_p \to 0$, as $\theta \to 0$, for each $f \in L^p(\mathbb{R}^2)$, $1 \leq p < \infty$.

   HINT: prove it first for f continuous with compact support and then use the density of this class of functions in L^p.

2. Let f_k be absolutely continuous functions on $[a, b]$ and $f_k(a) = 0$. Suppose f_k' is a Cauchy sequence in $L^1([a, b])$. Show that there exists f absolutely continuous on $[a, b]$ such that $f_k \to f$ uniformly on $[a, b]$.

3. Let $\{E_j\}_{j=1}^\infty$ be a sequence of μ-measurable sets in a measure space (X, Σ, μ) such that $\mu(E_j \cap E_i) = 0$ for $j \neq i$. Prove that

$$\mu\left(\bigcup_{j=1}^\infty E_j\right) = \sum_{j=1}^\infty \mu(E_j).$$