Real Analysis Ph.D. Qualifying Exam
Temple University
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Part I. (Do 3 problems)

1. Let $f : [a, b] \to \mathbb{R}$ be a bounded function and set

$$M = \sup_{[a,b]} f(x), \quad m = \inf_{[a,b]} f(x), \quad M^* = \sup_{[a,b]} |f(x)|, \quad m^* = \inf_{[a,b]} |f(x)|.$$

   Prove that $M^* - m^* \leq M - m$.

2. Let $E \subset \mathbb{R}^n$. The function $f : E \to \mathbb{R}$ is upper semicontinuous at $x_0 \in E$ if for each $\epsilon > 0$ there exists $\delta > 0$ such that $f(x) \leq f(x_0) + \epsilon$ for all $|x - x_0| < \delta$, $x \in E$.

   If f is upper semicontinuous in E compact, then f bounded above in E.

3. Let $f(x) = x^2 \sin(1/x^3)$ for $x \in [-1, 1]$, $x \neq 0$, and $f(0) = 0$. Show that f is differentiable on $[-1, 1]$ but f' is unbounded on $[-1, 1]$.

4. If $f \in C[0, +\infty)$, $f(x) \to L$ as $x \to +\infty$, then prove that $\dfrac{1}{t}\int_0^t f(s)\, ds \to L$ as $t \to +\infty$.

Part II. (Do 2 problems)

1. Let μ^* be an outer measure on the subsets of X. Prove that $E \subset X$ is Carathèodory measurable if and only if for each $\epsilon > 0$ there exists a Carathèodory measurable set $F \subset E$ such that $\mu^*(E \setminus F) < \epsilon$.

2. Let f, f_k be measurable functions in $\mathbb{R}$ such that $f_k \to f$ a.e. Suppose there exist $g, g_k \in L^1(\mathbb{R})$ such that $|f_k| \leq g_k$, $g_k \to g$, a.e., and $\lim_{k\to\infty} \int_{\mathbb{R}} g_k = \int_{\mathbb{R}} g$. Prove that

$$\lim_{k\to\infty} \int_{\mathbb{R}} |f_k - f| = 0.$$

   Hint: $|f_k - f| \leq g_k + |f|$, write $\int_{\mathbb{R}} \liminf_{k\to\infty} (g_k + |f| - |f_k - f|)\, dx$ and use Fatou's Lemma.

3. Let $f \in L^1(\mathbb{R}^n)$ with $\|f\|_1 = \int_{\mathbb{R}^n} |f(x)|\, dx = r < 1$. Define $f_k = f \star \cdots \star f$ where the convolution is taken k times. Prove that

   (a) $f_k \in L^1(\mathbb{R}^n)$ for all k,

   (b) $f_k \to 0$ in $L^1(\mathbb{R}^n)$ as $k \to \infty$,

   (c) $g(x) := \sum_k |f_k(x)|$ belongs to $L^1(\mathbb{R}^n)$, and conclude that $f_k(x) \to 0$ a.e.