Real Analysis Ph.D. Qualifying Exam

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- Justify your answers thoroughly.
- You are allowed to rely on a previous part of a multi-part problem even if you do not work out the previous part.
- For any theorem that you wish to cite, you should either give its name or a statement of the theorem.

Part I. (Do 3 problems):

I.3. A map $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is called Lipschitz if there exists a constant $c > 0$ such that $||Tx - Ty|| \leq c||x - y||$ for all x and y in $\mathbb{R}^n$. Here, as usual, $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$.

(a) Show that Lipschitz maps take Lebesgue measurable sets to Lebesgue measurable sets.

(b) Show by example that continuity is not enough to insure this property.

I.2. Let $(X, \mathfrak{M}, \mu)$ be a measure space such that $\mu(X) < \infty$. Show that if $0 < p_1 < p_2 < \infty$ then any $f \in L^{p_2}(X, \mathfrak{M}, \mu)$ satisfies $f \in L^{p_1}(X, \mathfrak{M}, \mu)$ and

$$\|f\|_{p_1} \leq \|f\|_{p_2} \cdot \big(\mu(X)\big)^{\frac{1}{p_1} - \frac{1}{p_2}}.$$

I.3. Let $(X, \mathfrak{M}, \mu)$ be again a finite measure space and fix $E \in \mathfrak{M}$. For each $n \in \mathbb{N}$ define the function $f_n : (X, \mathfrak{M}) \longrightarrow \mathbb{R}$ given by

$$f_n(x) := \begin{cases} \chi_E(x), & \text{if } n \text{ is odd,} \\ 1 - \chi_E(x), & \text{if } n \text{ is even,} \end{cases} \qquad \text{for each } x \in X.$$

Compute $\displaystyle\int_X \liminf_{n\to\infty} f_n \, d\mu$ and $\displaystyle\liminf_{n\to\infty} \int_X f_n \, d\mu$.

I.4. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of Lebesgue measurable function on $[0,1]$ and suppose that

$$\sum_{n=1}^{\infty} |\{x \in [0,1] \,|\, f_n(x) > 1\}| < \infty$$

where $|\cdot|$ denotes Lebesgue measure on $[0,1]$. Prove that $\displaystyle\limsup_{n\to\infty} f_n(x) \leq 1$ for almost every $x \in [0,1]$.

Part II. (Do 2 problems):

II.1. Let a $E \subset \mathbb{R}^2$ be Lebesgue measurable and suppose that each point of $[0,1] \times [0,1]$ is a point of density of E. Show that $|E| \geq 1$ where now $|\cdot|$ denotes Lebesgue measure on $\mathbb{R}^2$.

II.2. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $L^2[0,1]$ with $\sup_{n\geq 1} ||f_n||_{L^2[0,1]} \leq 1$ and let $K(\cdot,\cdot)$ be a continuous function on $[0,1] \times [0,1]$. Define

$$g_n(x) := \int_0^1 K(x,y) f_n(y) dy.$$

Prove that $\{g_n\}_{n\in\mathbb{N}}$ has a uniformly convergent subsequence.

II.3. Let $K(\cdot)$ be a continuous nonnegative function of compact support on $\mathbb{R}^n$. Suppose also that $\int_{\mathbb{R}^n} K(x)\,dx = 1$ and, for each $\epsilon > 0$ set $K_\epsilon(x) := \epsilon^{-n} K(x/\epsilon)$. For any $f \in L^p(\mathbb{R}^n)$ with $p \in [1,\infty)$ and for each $\epsilon > 0$ define

$$f_\epsilon(x) = (f * K_\epsilon)(x) = \int_{\mathbb{R}^n} K_\epsilon(x-y) f(y) dy.$$

Show that f_ϵ is uniformly continuous on $\mathbb{R}^n$ and that $f_\epsilon \to f$ in $L^p(\mathbb{R}^n)$ as $\epsilon \to 0^+$ (here the background measure for the L^p spaces in discussion is the Lebesgue measure in $\mathbb{R}^n$).