Real Analysis Ph.D. Qualifying Exam
Temple University
August 24, 2012

Part I. (Do 3 problems)

1. Let $f \in C^2(a, b)$. Prove that

$$\lim_{h\to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x)$$

for each $x \in (a, b)$.

2. Let $f_n(x) = \dfrac{n\,x}{x^2 + n^2}$, $x \in \mathbb{R}$. Show that

   (a) f_n does not converge uniformly in $\mathbb{R}$;

   (b) f_n does not converge in measure in $\mathbb{R}$.

3. Let $f(x, y) = \dfrac{x - y}{(x + y)^3}$. Show that $f \notin L^1([1, \infty) \times [1, \infty))$. Justify your answer.

4. Find

$$\lim_{n\to\infty} \int_0^n \left(1 - \frac{x}{n}\right)^n e^{x/2}\, dx.$$

Justify your answer.

Part II. (Do 2 problems)

1. Let f be absolutely continuous on $[a, b]$. Prove that

   (a) if $E \subset [a, b]$ with $|E| = 0$, then $|f(E)| = 0$;

   (b) if E is measurable, then $f(E)$ is measurable.

2. Let $f_1, \cdots, f_k$ be continuous real valued functions on the interval $[a, b]$. Show that the set $\{f_1, \cdots, f_k\}$ is linearly dependent on $[a, b]$ if and only if the $k \times k$ matrix with entries

$$\langle f_i, f_j \rangle = \int_a^b f_i(x)\, f_j(x)\, dx$$

has determinant zero.

3. Let $f_n : E \to \mathbb{R}$ be a sequence of measurable functions and $a \in \mathbb{R}$. Suppose that

$$\sum_{n=1}^{\infty} |\{x \in E : f_n(x) > a\}| < \infty.$$

Prove that $\overline{\lim}_{n\to\infty} f_n(x) \le a$ for a.e. $x \in E$.

HINT: let $E_n = \{x \in E : f_n(x) > a\}$ and $S = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$. Show $|S| = 0$, and if $\overline{\lim}_{n\to\infty} f_n(x) > a$, then $x \in S$.